Embeddings and Automorphisms in Affine Algebraic Geometry

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von

Dr. Immanuel Elias van Santen geb. Stampfli

aus

Günsberg SO

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To Anna, Johannes and Justus

Contents

Chapter 1

Introduction

It is a fundamental problem in mathematics to embed geometric objects into others and to study these embeddings. Fundamental guiding questions for geometric objects X, Y in this context are the following:

(A) (**Existence**) Does there exist an embedding of X into Y?

(B) (**Equivalence**) Are two embeddings $f, g\colon X \hookrightarrow Y$ related by some automorphism of the ambient space Y? More formally, having two embeddings $f, g\colon X \hookrightarrow Y$, one may ask for an automorphism φ of Y such that the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[d, "\varphi"] \\ X \arrow[ur, hook, "f"] \arrow[dr, hook, "g"'] & \\ & Y \end{tikzcd} .$$

The study of these questions has a long history. We will first briefly address some classical results concerning the existence of embeddings into the euclidean (projective) space in different contexts; this small survey is by no means complete.

A starting point of these embedding questions are the results obtained by Whitney. By the weak Whitney embedding theorem, every closed smooth manifold M can be smoothly embedded into the real euclidean space $\mathbb{R}^n$ as long as $n \geq 2 \dim M + 1$ [Whi36]. Based on the now called Whitney trick [Whi44], Whitney strengthened his result to the condition $n \geq 2 \dim M$. This is known today as Whitey's strong embedding theorem. In contrast, the real projective space of dimension 2^k for $k \geq 0$ yields an example of a 2^k-dimensional smooth manifold that does not embed into $\mathbb{R}^{2 \cdot 2^k - 1}$ [Pet57], and thus the dimension condition in the strong Whitney embedding theorem cannot be strengthened.

Based on Whitney's strong embedding theorem, Nash and Kuiper were able to prove that every Riemannian manifold M admits a continuously differentiable isometric embedding into $\mathbb{R}^n$ provided that $n \geq 2 \dim M + 1$ [Nas54, Kui55]. In case M is compact, the dimension condition $n \geq 2 \dim M$ is enough. In contrast, a compact locally flat Riemannian manifold M cannot be four times continuously differentiable isometrically embedded into the euclidean space of dimension $2 \dim M - 1$ [CK52].

In contrast to the embedding theorems due to Whitney, Nash and Kuiper there is a much weaker dimension condition for Stein manifolds. In fact, every Stein manifold M with $\dim M > 1$ admits a holomorphic embedding into the complex euclidean space $\mathbb{C}^n$ if $n > \frac{3}{2} \dim M$ by Eliashberg-Gromov and Schürmann [EG92, Sch97]. Examples of Forster show that this dimension condition cannot be improved [For70], when $\dim M > 1$. It is still an open problem whether every open Riemann surface can be holomorphically embedded into $\mathbb{C}^2$.

In algebraic geometry, there are the following existence results concerning embeddings into the complex projective space $\mathbb{P}^n(\mathbb{C})$ and into the complex affine space $\mathbb{C}^n$: If X is a smooth projective (affine) algebraic variety, then there exists an algebraic embedding into $\mathbb{P}^n(\mathbb{C})$ (into $\mathbb{C}^n$), provided that $n \geq 2\dim X + 1$ by Theorems due to Holme [Hol75], Kaliman [Kal91] and Srinivas [Sri91]. By Theorems of Horrocks-Mumford [HM73] and Van de Ven [VdV75] (in the projective case), and by a Theorem of Bloch-Murthy-Szpiro [BMS89] (in the affine case), these dimension conditions are also optimal. There are also versions for singular varieties due to Holme [Hol75], Kaliman [Kal91] and Srinivas [Sri91].

Concerning the equivalence of embeddings into euclidean space, one has the following classical results: By Kaliman [Kal91] and Srinivas [Sri91], two algebraic embeddings of a smooth affine variety into $\mathbb{C}^n$ are the same up to an algebraic automorphism of $\mathbb{C}^n$, provided that $n > 2\dim X + 1$. Analogous results hold as well in different settings, see e.g. [Jel09]: In particular, two embeddings of a smooth compact real manifold (compact real analytic manifold) into $\mathbb{R}^n$ are the same up to a diffeomorphism (real analytic automorphism) of $\mathbb{R}^n$, provided that $n > 2\dim X + 1$.

Focusing on more specific settings in affine algebraic geometry, the famous Abhyankar-Moh-Suzuki Theorem [AM75, Suz74] says that up to algebraic automorphisms of the affine plane $\mathbb{C}^2$ there exists exactly one algebraic embedding of the affine line $\mathbb{C}$ into $\mathbb{C}^2$. This result holds more generally for so-called cuspidal curves (i.e. the normalization is isomorphic to the affine line) instead of the affine line by a theorem due to Lin-Zaidenberg [ZL83]. Another example is the following: The union of all the coordinate hyperplanes in the affine space $\mathbb{C}^n$ has a unique embedding up to automorphisms of $\mathbb{C}^n$ by Jelonek [Jel97].

It is natural to ask the above embedding questions for more general targets than the euclidean space. The first part of my results I will present in this habilitation, concern exactly these questions in the context of affine algebraic geometry. I.e. the geometric objects under consideration are zero sets of polynomials in finitely many variables and the considered embeddings are given by polynomial maps. Mostly, I considered the case where the target is an algebraic group. This is joint work with my collaborators Peter Feller and Jérémy Blanc. Related to these embedding questions, in the algebraic context, I studied together with Stefan Maubach maximal $\mathbb{C}$-subalgebras of a given $\mathbb{C}$-algebra.

In order to attack these embedding questions, one needs to understand to a certain amount the automorphisms of the target space Y of an embedding $X \hookrightarrow Y$. In fact, the affine varieties I consider as targets have usually a huge automorphism group and it is a challenge for its own sake to understand these automorphism groups. For example, the group $\mathrm{Aut}(\mathbb{C}^n)$ of polynomial automorphisms is fairly good understood for $n = 1, 2$, whereas for $n \geq 3$ these groups are huge and still rather mysterious. The second part of my results cover my research on several questions concerning the above mentioned automorphism groups. Specifically, I studied the following fundamental problems:

(C) (**Characterization**) Understand to what extend the automorphism group $\mathrm{Aut}(X)$ of a geometric object X determines X itself.

(D) (**Dynamics**) Understand the dynamics of automorphisms of a geometric object X.

(E) (**Low degree**) Understand the automorphisms of a geometric object X that are small with respect to some measure.

Klein proposed in his famous Erlangen program from 1872 to study geometric objects X via their automorphisms in case $\mathrm{Aut}(X)$ is rather big. In several situations it turns out that $\mathrm{Aut}(X)$ completely determines X, i.e. if $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic, then X and Y are isomorphic. In particular, this happens for smooth manifolds, symplectic manifold or contact manifolds, see [Fil82, Ryb95, Ryb02]. In affine algebraic geometry, usually $\mathrm{Aut}(X)$ is small and hence it cannot determine X completely. However, for certain affine varieties where $\mathrm{Aut}(X)$ is big enough, the group $\mathrm{Aut}(X)$ still determines X. Concerning problem (D), together with Hanspeter Kraft and Andriy Regeta, I focused on exactly these problems in the context of affine algebraic geometry. More precisely, together with Hanspeter Kraft and Andriy Regeta, I considered the case $X = \mathbb{C}^n$ and together with Andriy Regeta, I considered the case when X is a quasi-affine spherical variety (i.e. there is a faithful algebraic action of a reductive algebraic group on the quasi-affine variety X such that a Borel subgroup acts with a dense orbit) under a stronger assumption on the group isomorphism between the automorphism groups under consideration.

The study of the dynamics of an automorphism f is the study of its iterates $f^i = f \circ \cdots \circ f$ when i goes to infinity. In algebraic geometry, one aspect that catches the dynamical behaviour of an automorphism is its dynamical degree. In case f is an automorphism of $\mathbb{C}^n$ one may define it as the limit of the numbers $(\deg(f^i))^{\frac{1}{i}}$ as i goes to infinity. I studied together with Jérémy Blanc the question, which real numbers can arise as dynamical degrees of automorphisms of $\mathbb{C}^n$. We developed a technique to compute dynamical degrees in certain cases. In particular, we gave all dyanmical degrees of all so-called affine-triangular automorphisms of $\mathbb{C}^3$, i.e. automorphisms that are a composition of an affine linear automorphism of $\mathbb{C}^3$ with an automorphism of $\mathbb{C}^3$ of the form $(x, y, z) \mapsto (p(x), q(x, y), r(x, y, z))$ where p, q, r are polynomials over $\mathbb{C}$.

Concerning problem (E) in affine algebraic geometry, and specifically for $X = \mathbb{C}^n$, a natural measure of the complexity is the degree of the automorphism. There is a conjecture due to Rusek [Rus88] which says, that every automorphism of degree ≤ 2 is a so-called tame automorphism, i.e. a finite composition of affine and triangular automorphisms of $\mathbb{C}^n$. Whereas the conjecture is confirmed in case $n \leq 5$ by results due to Fornæs and Wu [FW98], Meisters and Olech [MO91], and Sun [Sun14], it is an open problem, whether the same also holds for automorphisms of degree 3. In case $n \leq 2$, all automorphisms of $\mathbb{C}^2$ are tame. Motivated by this I studied together with Jérémy Blanc the next case, i.e. automorphisms of $\mathbb{C}^3$ of degree 3. In particular we were able to show that all such automorphisms are tame and we computed their dynamical degrees.

1.1 Articles and preprints written after my PhD

[1] Peter Feller and Immanuel van Santen.
Existence of Embeddings of Smooth Varieties into Linear Algebraic Groups.
Preprint, July 2020.
https://arxiv.org/abs/2007.16164.

[2] Peter Feller and Immanuel Stampfli.
Holomorphically equivalent algebraic embeddings.
Preprint, Oktober 2014.
http://arxiv.org/abs/1409.7319.

[3] Jérémy Blanc and Immanuel van Santen.
Dynamical degrees of affine-triangular automorphisms of the affine space.
accepted for publication in Ergodic Theory Dynam. Systems, August 2021.
https://arxiv.org/abs/1912.01324.

[4] Jérémy Blanc and Immanuel van Santen.
Automorphisms of the affine 3-space of degree 3.
accepted for publication in Indiana Univ. Math. J., May 2021.
https://arxiv.org/abs/1912.02144 or Official webpage.

[5] Andriy Regeta and Immanuel van Santen.
Characterizing quasi-affine spherical varieties via the automorphism group.
Journal de l'École polytechnique — Mathématiques. **8** (2021), 379–414.
https://arxiv.org/abs/1911.10896 or Official webpage.

[6] Hanspeter Kraft, Andriy Regeta and Immanuel van Santen.
Is the affine space determined by its automorphism group?
Int. Math. Res. Not. IMRN (2019).
https://arxiv.org/abs/1707.06883 or Official webpage.

[7] Peter Feller and Immanuel van Santen.
Uniqueness of Embeddings of the Affine Line into Algebraic Groups.
J. Algebraic Geom. **28** (2019), no. 4, 649–698.
https://arxiv.org/abs/1609.02113 or Official webpage.

[8] Jérémy Blanc and Immanuel van Santen.
Embeddings of Affine Spaces into Quadrics.
Trans. Amer. Math. Soc. **371** (2019), no. 12, 8429–8465.
https://arxiv.org/abs/1702.00779 or Official webpage.

[9] Stefan Maubach and Immanuel Stampfli.
On Maximal Subalgebras.
J. Algebra **483** (2017), 1-36.
http://arxiv.org/abs/1501.03753 or Official webpage.

[10] Immanuel Stampfli.
Algebraic embeddings of $\mathbb{C}$ into $SL_n(\mathbb{C})$.
Transform. Groups **22** (2017), no. 2, 525–535.
https://arxiv.org/abs/1505.03735 or Official webpage.

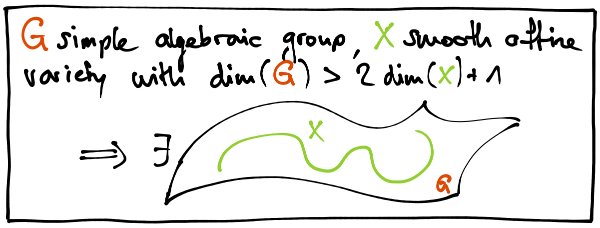
These are also my articles for the habilitation with the exception of the article [2]. After uploading this article on arXiv.org, we were informed that the main result was already proven by Kaliman in https://arxiv.org/abs/1309.3791 now published in [Proc. Amer. Math. Soc.].

1.2 Main results

My research can be manly divided up into the study of embeddings and into the study of automorphisms in affine algebraic geometry. Each of these can again be divided up into parts, according to the questions/problems posed in (A), (B) and in (C), (D), (E).

In the next section, I will survey the main results that I received after my PhD at the University of Basel in 2013.

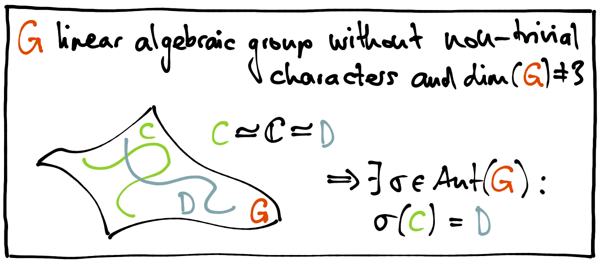
Existence questions about embeddings



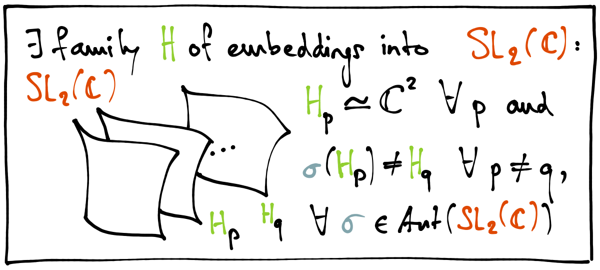
As already mentioned, every smooth affine variety X admits an algebraic embedding into $\mathbb{C}^n$ provided that $n \geq 2\dim X+1$ by the Holme-Kaliman-Srinivas embedding theorem. Together with Peter Feller, I was able to prove an analogous theorem where we replaced the affine space by any simple algebraic group G under the dimension condition $\dim G > 2\dim X + 1$; see [1]. Moreover, we were able to show that there exists for each algebraic group G of dimension n and for each integer $d \geq \frac{n}{2}$ a smooth irreducible affine variety X of dimension d that does not admit an embedding into G by adopting the strategy of Bloch-Murthy-Szpiro [BMS89]. In particular, the dimension condition $\dim G > 2\dim X + 1$ may be improved at most by one in case $\dim(G)$ is odd and the dimension condition $\dim G > 2\dim X + 1$ is optimal in case $\dim(G)$ is even.

Equivalence questions about embeddings

Together with Peter Feller, I studied algebraic embeddings of smooth affine varieties X into the complex affine space $\mathbb{C}^n$ up to holomorphic automorphisms [2]. We were able to weaken the classical dimension bound given by Kaliman and Srinivas for algebraic embeddings and algebraic automorphisms to $n \geq 2\dim X + 1$ in this more relaxed setting. After we put a first version of this manuscript on arXiv, we were informed that the main result was already established by Kaliman [Kal15]. Therefore, I will not report on this work here.

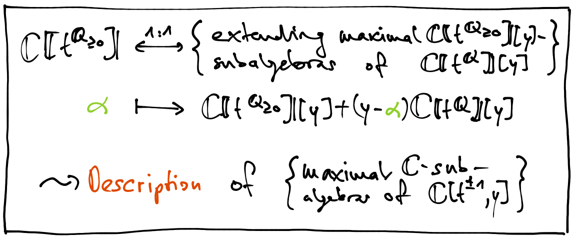


In 2013 there was a break-through in the understanding of so-called flexible varieties (informally speaking these are varieties with "a lot" of additive group actions): The flexibility of an irreducible smooth affine variety Y of dimension ≥ 2 is equivalent to the transitivity of the natural action of $\mathrm{SAut}(Y)$ on Y and also to the m-transitivity of this action for each $m \geq 1$ where $\mathrm{SAut}(Y)$ denotes the subgroup of the algebraic automorphisms $\mathrm{Aut}(Y)$ that are induced by additive group actions. In particular, connected linear algebraic groups without non-trvial characters of dimension ≥ 2 are flexible. In [10], I proved that all algebraic embeddings of $\mathbb{C}$ into $\mathrm{SL}_n(\mathbb{C})$ are the same up to an algebraic automorphism for $n \geq 3$ and up to a holomorphic automorphism for $n \geq 2$. Together with Peter Feller, I was able to generalize the first part of this result to linear algebraic groups without non-trivial characters of dimension different from 3; see [7].



Together with Jérémy Blanc, I studied algebraic embeddings of the affine plane $\mathbb{C}^2$ into the special linear group $\mathrm{SL}_2(\mathbb{C})$ [8]. While it is a long standing open problem, whether all algebraic embeddings of $\mathbb{C}^2$ into $\mathbb{C}^3$ (or even into $\mathbb{C}^4$) are the same up to algebraic automorphisms (or even holomorphic automorphisms), we were able to provide huge families of algebraic embeddings of $\mathbb{C}^2$ into $\mathrm{SL}_2(\mathbb{C})$, where different members of that family are not the same up to algebraic automorphisms of $\mathrm{SL}_2(\mathbb{C})$.

Maximal subalgebras



Together with Stefan Maubach, I classified the so-called extending maximal $\mathbb{C}[[t^{\mathbb{Q}_{\geq 0}}]][y]$-subalgebras of $\mathbb{C}[[t^{\mathbb{Q}}]][y]$, where $\mathbb{C}[[t^{\mathbb{Q}}]]$ denotes the field of Hahn series over $\mathbb{C}$ with exponents in the rational numbers $\mathbb{Q}$. Using this classification result, we were able to describe the maximal $\mathbb{C}$-subalgebras of the polynomal ring over the ring of Laurent polynomials $\mathbb{C}[t^{\pm 1}, y]$ [9]. This was the first such classification result for a commutative algebra of dimension > 1.

Characterization of varieties via their automorphisms

Together with Andriy Regeta and Hanspeter Kraft, I could prove that the abstract group $\mathrm{Aut}(\mathbb{C}^n)$ of algebraic automorphisms determines $\mathbb{C}^n$ (up to isomorphisms) within the class of n-dimensional smooth irreducible quasi-projective varieties with a finite Picard group and non-vanishing Euler characteristic [6]. Furthermore, together with Andriy Regeta, I was able to partially generalize the above result in the following sense: If X is a smooth affine G-spherical variety where G is a connected reductive algebraic group, and if Y is an affine irreducible normal variety such that there is a group isomorphism $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ that preserves algebraic group actions, then X and Y are isomorphic as G-varieties [5].

G connected linear reductive alg. group
X G-spherical, smooth, affine
Y irreducible, normal, affine
If $\exists\ \mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ that preserves
alg. group actions $\Rightarrow X \underset{G}{\simeq} Y$

Dynamics and low degree automorphisms of the affine space

Together with Jérémy blanc, I developped a technique in order to calculate dynamical degrees of algebraic endomorphisms of $\mathbb{C}^n$ under certain assumptions on these endomorphisms. Using this result, we were able to compute the dynamical degree of every composition of an affine automorphism and a triangular automorphism of $\mathbb{C}^3$ [3]. Moreover, we described all algebraic automorphisms of $\mathbb{C}^3$ up to composition with affine automorphisms at the source and target and as an application of this description, we were able to compute all dynamical degrees of them, using the above mentioned technique [4].

Description of the dynamical degrees:

$$\left\{ \lim_{i\to\infty} \deg(\underbrace{\mathbb{C}^3 \xrightarrow{f} \cdots \xrightarrow{f} \mathbb{C}^3}_{i\text{-times}})^{1/i} \;\middle|\; \begin{array}{l} f = p\circ t,\ p\in \mathrm{Aff}(\mathbb{C}^3), \\ t\in \mathrm{Aut}(\mathbb{C}^3) \text{ triangular} \end{array} \right\}$$

Chapter 2

Embedding questions

As mentioned in the beginning, I studied embedding questions in the category of affine varieties over the complex numbers $\mathbb{C}$. I.e. the geometric objects are common zero sets of polynomials in the affine space $\mathbb{C}^n$ endowed with the Zariski topology. The embeddings under consideration are morphisms $f\colon X \to Y$ of affine varieties such that $f(X)$ is closed in Y and f induces an isomorphism $X \simeq f(X)$ of affine varieties.

In affine algebraic geometry, there is a purely algebraic description of the embeddings. In fact, denoting by $\mathbb{C}[Z]$ the coordinate ring of an affine variety Z, then a morphism $f\colon X \to Y$ of affine varieties is an embedding, if and only if the comorphism $f^*\colon \mathbb{C}[Y] \to \mathbb{C}[X]$, $p \mapsto p \circ f$ is surjective. Beside this purely algebraic characterization, there is also the following geometric description, that turns out to be very useful (and in fact it holds not only for affine varieties): A morphism $f\colon X \to Y$ of varieties is an embedding if and only if

- f is proper,
- f is injective,
- f is immersive, i.e. for all $x \in X$, the differential $\mathrm{d}_x f : T_x X \to T_{f(x)} Y$ is injective

(see Appendix B in [1]). In the following sections, I report on my articles concerning the embedding questions (A) and (B). Although many results work in more generality, I choose for the ground field always the field of complex numbers in order to make the exposition as simple as possible.

2.1 Existence of embeddings of smooth varieties into linear algebraic groups

In this section, I discuss the paper [1]. This is joint work with Peter Feller. Having the classical Holme-Kaliman-Srinivas embedding theorem for the target the affine space $\mathbb{C}^n$ in mind, it is natural to look for more general algebraic groups as targets of embeddings. More precisely, X will be a smooth affine variety and Y will be the underlying affine variety of an algebraic group. One of our main result is the following.

Theorem 2.1.1 (cf. Theorem A in [1])**.** *Let G be a simple algebraic group and let X be a smooth affine variety. If the dimension condition $\dim G > 2\dim X + 1$ holds, then X admits an embedding into G.*

We got also an analogous result for semi-simple algebraic groups, however, with a stronger dimension condition, depending on the number of minimal connected normal subgroups. Concerning the optimality of the dimension condition in Theorem 2.1.1, we were able to prove the following.

Proposition 2.1.2 (cf. Proposition B in [1])**.** *For every non-finite algebraic group G and every $d \geq \frac{\dim G}{2}$, there exists an irreducible smooth affine variety of dimension d that does not admit an embedding into G.*

So in case the simple algebraic group G in Theorem 2.1.1 is of even dimension, Proposition 2.1.2 implies that the dimension condition is optimal, whereas for an odd dimensional G, the dimension condition can be possibly improved at most to $\dim G \geq 2\dim X + 1$. The proof of Proposition 2.1.2 is a generalization of a Chow-group-based argument due to Bloch, Murthy, and Szpiro [BMS89]; in fact, they showed Proposition 2.1.2 in the special case, when G is the affine space.

Next, I will report on the strategy of the proof of the existence result, Theorem 2.1.1. We fix a smooth affine variety X of dimension d. Let us first recall the strategy for the classical embedding theorem, where the target is the affine space. There exists $N \geq 2d + 1$ such that X is a closed subset of $\mathbb{C}^N$. As long as $N > 2d + 1$, for a generic linear projection $\pi\colon \mathbb{C}^N \to \mathbb{C}^{N-1}$, the restriction $\pi|_X\colon X \to \mathbb{C}^{N-1}$ is proper, injective and immersive, i.e. it is an embedding. The result is then established by induction.

In case the target is not the affine space, this strategy doesn't work because of the absence of generic projections. The idea is, to construct the embedding from "bottom up". More precisely, this idea goes back to Eliashberg-Gromov [EG92] where they construct embeddings of Stein manifolds into affine spaces. One starts with a finite surjective morphism $X \to \mathbb{C}^d$ (which exists due to Noether normalization). Now, one needs a "nice" morphism $\pi\colon Y \to \mathbb{C}^d$ that allows to lift $X \to \mathbb{C}^d$ to an injective and immersive morphism $f\colon X \to Y$. By construction, f is then also proper and thus it is an embedding. The problem lies in the existence of the "nice" morphism $\pi\colon Y \to \mathbb{C}^d$. One setting where this strategy works is the following (which constitutes our main embedding tool).

Theorem 2.1.3 (cf. Theorem 2.5 in [1])**.** *Let Y be a smooth irreducible affine variety such that:*

a) There is a principal $\mathbb{G}_a$-bundle $\rho\colon Y \to Q$;
b) There is a smooth morphism $\pi\colon Y \to P$ such that there are "enough" algebraic group actions on Y that fix π;
c) There is a morphism $\eta\colon Q \to P$ that admits a section and satisfies $\eta \circ \rho = \pi$.

If there exists a smooth affine variety X with $\dim Y \geq 2\dim X + 1$ and a finite surjective morphism $r\colon X \to P$, then there exists an embedding $f\colon X \to Y$ with $r = \pi \circ f$.

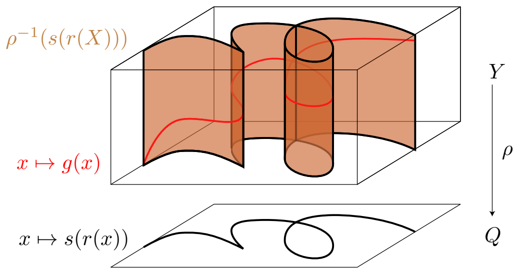
A precise statement of condition b) can be found in Theorem 2.5 in [1]. The next diagram illustrates the situation of Theorem 2.1.3.

$$\begin{tikzcd}
Y \arrow[r, "\text{principal } \mathbb{G}_a\text{-bundle } \rho"] \arrow[dr, "\pi"] & Q \arrow[d, "\eta \text{ with a section}"] \\
X \arrow[u, hook, "\exists \text{ embedd. } f"] \arrow[r, "\text{finite, surjective } r"'] & P
\end{tikzcd} .$$

Roughly, the idea of the proof of Theorem 2.1.3 is the following. We define for every morphism $f\colon X \to Z$ its θ-invariant by

$$\theta_f := \max\{\dim(Z \times_X Z \setminus \Delta_X)\,,\ \dim(\ker(\mathrm{d}f) \setminus 0_X)\}$$

where Δ_X denotes the diagonal inside the fibre product $Z \times_X Z$ with respect to f, $\ker(\mathrm{d}f)$ denotes the kernel of the differential $\mathrm{d}f\colon TX \to TZ$ and $0_X \subset TX$ denotes the zero section of the tangent bundle $TX \to X$. The θ-invariant is a measure for the injectivity and immersivity of f. In particular, if $\theta_f < 0$, then f is injective and immersive. Now, one starts with the morphism $s \circ r\colon X \to Q$ where $s\colon P \to Q$ is a section of $\eta\colon Q \to P$. Then one uses the fact that ρ restricts to a trivial $\mathbb{G}_a$-bundle $\rho^{-1}((s \circ r)(X)) \to (s \circ r)(X)$ in order to get a morphism $g\colon X \to Y$ with $\rho \circ g = s \circ r$ and $\theta_g < \theta_{s \circ r}$. The next picture illustrates the morphism $g\colon X \to Y$ over $s \circ r\colon X \to Q$ in the case, when X is a curve:



Now, one uses a parametric transversality result for flexible affine varieties due to Kaliman [Kal20] in order to get an automorphism of Y that fixes π such that $\theta_g = \theta_{\rho \circ \alpha \circ g}$. Note that the θ-invariant of $\rho \circ \alpha \circ g\colon X \to Q$ is smaller than that one of $s \circ r\colon X \to Q$. Repeating this process, we find a morphism $f\colon X \to Y$ with negative θ-invariant, i.e. it is injective and immersive. Moreover, $\pi \circ f = r$, thus f is finite and hence $f\colon X \to Y$ is our desired embedding.

In order to apply Theorem 2.1.3 to construct an embedding of a smooth affine variety X of dimension d into the simple algebraic group G as in Theorem 2.1.1, we choose a hypersurface in G that is isomorphic to $Q \times \mathbb{C}^k$ for some algebraic group Q and some $k \geq 0$. In fact, using the classification of parabolic subgroups in simple algebraic groups, we can choose k and Q in such a way that $\dim Q - 1 \leq k$ and that Q is generated by unipotent elements (see Propositions 3.8, 3.9 in [1]). Since $\dim Q + k = \dim G - 1 \geq 2d + 1$, we get thus $k \geq d$. We then set $F := Q \times \mathbb{C}^{k-d}$ and choose some closed subgroup $U \subset F$ that is isomorphic to $\mathbb{G}_a$. Applying Theorem 2.1.3 to the canonical projections

$$\pi\colon F \times \mathbb{C}^d \xrightarrow{\rho} (F/U) \times \mathbb{C}^d \xrightarrow{\eta} \mathbb{C}^d$$

yields then an embedding of X into $F \times \mathbb{C}^d$, hence also into G.

While embedding X into $F \times \mathbb{C}^d \simeq Q \times \mathbb{C}^k$, we lost possibly one dimension in the dimension condition. In order to strengthen Theorem 2.1.1, it is natural to try to apply the embedding tool (Theorem 2.1.3) directly to $Y = G$ and some "nice" morphism $\pi \colon G \to P$. It seems, that the only natural candidates for such a π are algebraic quotient morphisms $G \to G/H =: P$ for some closed algebraic subgroup H of G. However, in general, there is no finite surjective morphism $X \to G/H$, due to the following result.

Proposition 2.1.4 (cf. Proposition 5.1 in [1])**.** *Let G be a simple algebraic group, $H \subsetneq G$ a proper closed subgroup and X an irreducible smooth affine variety with the rational homology of a point and such that X is simply connected. Then, there exists no finite surjective algebraic morphism $X \to G/H$.*

The proof of Proposition 2.1.4 is based on a purely topological fact, namely on Hopf's theorem on the Umkehrungshomomorphismus from algebraic topology (cf. Theorem A.1 in [1]). In fact this topological fact implies that for any proper and dominant morphism $f \colon X \to Z$ between complex n-dimensional smooth varieties, the induced homomorphism in $\mathbb{Q}$-homology $f_k \colon H_k(X, \mathbb{Q}) \to H_k(Z, \mathbb{Q})$ is surjective for all non-negative integers k. Then we use the knowledge of the rational homology groups of complex simple algebraic groups to deduce Proposition 2.1.4.

However, if the dimension of the target algebraic group is small, we are able to get the existence of embeddings with an optimal dimension condition:

Proposition 2.1.5 (cf. Proposition 3.11 in [1])**.** *Let G be an algebraic group without nontrivial characters such that $\dim G \leq 8$. If X is a smooth affine variety with $2 \dim X + 1 \leq \dim G$, then X admits an embedding into G.*

Recently, Kaliman put a preprint on arXiv, which shows in particular the optimality of the dimension bound, when the algebraic group is a product of the form $\prod_{i=1}^{N} \mathrm{SL}_{k_i}(\mathbb{C})$:

Theorem 2.1.6 (cf. [Kal21, Theorem 1.1])**.** *Let G be a semisimple algebraic group such that its Lie algebra is a product of Lie algebras of special linear groups. Then every smooth affine variety Z with $2 \dim Z + 1 \leq \dim G$ admits an embedding into G.*

I will finish this section, by reporting on the existence of embeddings into algebraic groups in the *holomorphic* setting. In fact, Andrist-Forsternič-Ritter-Wold proved that every Stein manifold X admits a holomorphic embedding into every Stein manifold Y that satisfies the (volume) density property, provided that $\dim Y \geq 2 \dim X + 1$ [AFRW16]. Using that every connected algebraic group G without non-trivial characters satisfies the density property or $G \simeq \mathbb{G}_a$ by Donzelli-Dvorsky-Kaliman [DDK10], we get that every smooth affine variety X with $2 \dim X + 1 \leq \dim G$ admits a *holomorphic* embedding into G.

2.2 Algebraic embeddings of $\mathbb{C}$ into $\mathrm{SL}_n(\mathbb{C})$

In this section, I will report on the article [10]. Recall that by results of Kaliman and Srinivas and in fact also Jelonek [Jel87, Theorem 1.1], the complex line $\mathbb{C}$

admits one algebraic embedding into $\mathbb{C}^n$ up to algebraic automorphisms of $\mathbb{C}^n$, provided that $n \geq 4$. By Abhyankar-Moh [AM75] and Suzuki [Suz74], this statement also holds for $n = 2$ and it is widely open for $n = 3$. However, if one studies algebraic embeddings of $\mathbb{C}$ into $\mathbb{C}^3$ up to holomorphic automorphisms of $\mathbb{C}^3$, then there is exactly one by Kaliman [Kal92]. In fact, one may even replace the complex line $\mathbb{C}$ by any affine curve [Kal15]. It is natural to study algebraic embeddings into more general targets and I chose as a first example the special linear group $\mathrm{SL}_k(\mathbb{C})$. In fact, the following holds:

Theorem 2.2.1 (cf. Main Theorem in [10]). *The complex affine line admits a unique algebraic embedding into*

a) $\mathrm{SL}_k(\mathbb{C})$ *up to algebraic automorphisms of* $\mathrm{SL}_k(\mathbb{C})$ *for* $k \geq 3$*;*
b) $\mathrm{SL}_2(\mathbb{C})$ *up to holomorphic automorphisms of* $\mathrm{SL}_2(\mathbb{C})$.

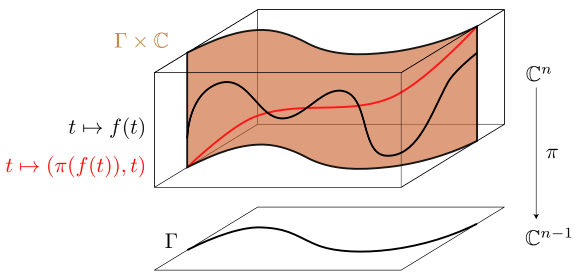
First, I will report on the proof of Theorem 2.2.1a). For this, we recall the classical argument, that every embedding $f\colon \mathbb{C} \hookrightarrow \mathbb{C}^n$, $n \geq 4$ is linear up to an *algebraic* automorphism of $\mathbb{C}^n$. In fact, for a generic linear map $\alpha \in \mathrm{GL}_n(\mathbb{C})$, the composition $r \coloneqq \pi \circ \alpha \circ f\colon \mathbb{C} \to \mathbb{C}^{n-1}$ is still an embedding, where $\pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ denotes the projection to the first $n-1$ coordinates:

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "f"] \arrow[drr, "\text{embedding } r"'] & \mathbb{C}^n \arrow[r, "\alpha"] & \mathbb{C}^n \arrow[d, "\pi"] \\ & & \mathbb{C}^{n-1} \end{tikzcd} .$$

After replacing f by $\alpha \circ f$, we may assume that $r = \pi \circ f$ is an embedding. Let $\Gamma \coloneqq r(\mathbb{C}) \subset \mathbb{C}^{n-1}$ and let $\rho\colon \mathbb{C}^n \to \mathbb{C}$ be the projection to the last coordinate. Then, the morphism $\Gamma \to \mathbb{C}$, $v \mapsto r^{-1}(v) - \rho(f(r^{-1}(v)))$ extends to a morphism $h\colon \mathbb{C}^{n-1} \to \mathbb{C}$. Now, consider the automorphism

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n\,, \quad (v, s) \mapsto (v, s + h(v))\,.$$

Then, $(\rho \circ \varphi \circ f)(t) = t$ for all $t \in \mathbb{C}$. In fact, the automorphism φ moves the embedding $t \mapsto f(t)$ into the embedding $t \mapsto (\pi(f(t)), t)$. The next picture illustrates the curves $t \mapsto f(t)$ and $t \mapsto (\pi(f(t)), t)$ over Γ:



Hence, after replacing f with $\varphi \circ f$, we may assume that $\rho(f(t)) = t$. Now, we consider the automorphism

$$\psi\colon \mathbb{C}^n \to \mathbb{C}^n\,, \quad (v, s) \mapsto (v - \pi(f(s)), s)\,.$$

Then $(\psi \circ f)(t) = (0, t) \in \mathbb{C}^{n-1} \times \mathbb{C}$ for all $t \in \mathbb{C}$, i.e. $\psi \circ f \colon \mathbb{C} \to \mathbb{C}^n$ is our desired linear embedding and this finishes the classical argument.

The idea in the proof of Theorem 2.2.1b) is to move a fixed embedding $f \colon \mathbb{C} \to \mathrm{SL}_n(\mathbb{C})$ into the embedding $\mathbb{C} \to \mathrm{SL}_n(\mathbb{C})$, $t \mapsto \mathrm{E}_{n1}(t)$ via an automorphism of $\mathrm{SL}_n(\mathbb{C})$, where $\mathrm{E}_{n1}(t)$ denotes the elementary matrix with $n1$-th entry equal to t. Similarly, as in the classical argument, we may assume that $\pi \circ f \colon \mathbb{C} \to \mathrm{M}_{n,n-1}(\mathbb{C})$ is again an embedding, where $\pi \colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{M}_{n,n-1}(\mathbb{C})$ denotes the projection onto the first $n-1$ columns. Furthermore, one may achieve then, that the first column of $f(t)$ is given by the transpose of $(1, 0, \ldots, t)$. Denoting by $\pi_1 \colon \mathrm{SL}_n(\mathbb{C}) \to \mathbb{C}^n$ the projection to the first column, one gets thus $\pi_1(f(t)) = \pi_1(\mathrm{E}_{n1}(t))$ for all $t \in \mathbb{C}$. Note that π_1 is a principal bundle and, analogously to the classical case, one can move the embedding $t \mapsto f(t)$ into $t \mapsto \mathrm{E}_{n1}(t)$ via the π_1-automorphism

$$\varphi \colon \mathrm{SL}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C}) \, , \quad A \mapsto A \cdot f(A_{n1})^{-1} \cdot \mathrm{E}_{n1}(A_{n1})$$

where A_{n1} denotes the $n1$-th -entry of the matrix A.

Now, I will report on the proof of Theorem 2.2.1b). Before, let me explain the argument of Kaliman [Kal92], that every embedding $f \colon \mathbb{C} \hookrightarrow \mathbb{C}^3$ is linear up to a *holomorphic* automorphism of $\mathbb{C}^3$. Similarly, as before, one studies the morphism $r = f \circ \alpha \circ \pi \colon \mathbb{C} \to \mathbb{C}^2$, where $\alpha \in \mathrm{GL}_3$ and $\pi \colon \mathbb{C}^3 \to \mathbb{C}^2$ is the projection on the first two coordinates. From dimension reasons, one cannot expect that $r \colon \mathbb{C} \to \mathbb{C}^2$ is an embedding. However, for a well chosen α, the morphism $r \colon \mathbb{C} \to \mathbb{C}^2$ is birational onto its image $\Gamma \subset \mathbb{C}^2$ and Γ has at worst finitely many simple normal crossing singularities. We replace f by $\alpha \circ f$. Now, after applying a certain *holomorphic* π-automorphism we may assume that

$$\rho(f(t_{i,1})) - \rho(f(t_{i,2}) = t_{i,1} - t_{i,2} \quad \text{for all } i \in \{1, \ldots, s\}$$

where $\rho \colon \mathbb{C}^3 \to \mathbb{C}$ denotes the projection to the last coordinate, $v_1, \ldots, v_s \in \mathbb{C}^2$ denotes the simple normal crossing singularities of Γ and $f^{-1}(v_i) = \{t_{1,i}, t_{2,i}\}$ for all $i \in \{1, \ldots, s\}$. Thus, one may choose a holomorphic map $h_0 \colon \Gamma \to \mathbb{C}$ such that $h_0(r(t)) = t - \rho(f(t))$ for all $t \in \mathbb{C}$. Since Γ is closed in $\mathbb{C}^2$, one may extend h_0 to a holomorphic map $h \colon \mathbb{C}^2 \to \mathbb{C}$. As before, one considers the automorphism $\varphi \colon \mathbb{C}^3 \to \mathbb{C}^3$ given by $\varphi(v, s) = (v, s + h(v))$ and one receives $(\rho \circ \varphi \circ f)(t) = t$ for all $t \in \mathbb{C}$. Similarly, as before, one may then move $t \mapsto f(t)$ into the desired linear embedding $t \mapsto (0, t) \in \mathbb{C}^2 \times \mathbb{C}$ via a holomorphic automorphism of $\mathbb{C}^3$.

The idea of the proof of Theorem 2.2.1(b)) is in some sense similar to the argument above. One studies the principal $\mathbb{G}_a$-bundle $\pi \colon \mathrm{SL}_2 \to \mathbb{C}^2 \setminus \{(0,0)\}$ and one may achieve as before, that $\pi \circ f \colon \mathbb{C} \to \mathbb{C}^2 \setminus \{(0,0)\}$ is briational onto its image Γ and Γ has at worst simple normal crossing singularities. As replacement of the projection $\mathbb{C}^3 \to \mathbb{C}$ to the last coordinate, one considers the morphism

$$\rho \colon \mathrm{SL}_2 \to \mathbb{C} \, , \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto y \, .$$

The idea is then to move the embedding f in such a way, that $\rho(f(t)) = t$ for all $t \in \mathbb{C}$. This is the main bulk of the whole argument. Let $p, q \colon \mathbb{C} \to \mathbb{C}$ be holomorphic maps such that $p(t)t = x(t) - x(0)$ and $q(t)t = w(t) - w(0)$ for all $t \in \mathbb{C}$. Consider the automorphism

$$\varphi \colon \mathrm{SL}_2 \to \mathrm{SL}_2 \, , \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -q(y) & 1 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -p(y) & 1 \end{pmatrix}$$

of SL_2. Using that $x(0)w(0) = 1$, one receives

$$(\varphi \circ f)(t) = \begin{pmatrix} x(0) & t \\ 0 & w(0) \end{pmatrix} \quad \text{for all } t \in \mathbb{C}\,.$$

Up to an algebraic automorphism of SL_2, the embedding $t \mapsto (\varphi \circ f)(t)$ is equal to the embedding $t \mapsto E_{12}(t)$, where $E_{12}(t)$ denotes the elementary matrix with 12-th entry equal to t.

After giving the idea of the proof of Theorem 2.2.1, let me finish this section with the following partial generalizations concerning embeddings into $\mathrm{SL}_n(\mathbb{C})$ due to Kaliman [Kal20] (in order to simplify the notation, I formulate them only in the smooth case):

Theorem 2.2.2 ([Kal20, Theorem 0.5, Theorem 0.4]). *Let $\varphi\colon Y_1 \to Y_2$ be an isomorphism of two closed smooth subvarieties of $\mathrm{SL}_n(\mathbb{C})$.*

a) If $n \geq 3$, $Y_i \simeq \mathbb{A}^k$ and $k \leq \frac{n}{3} - 1$ (or $k = 1$), then φ extends to an algebraic *automorphism of $\mathrm{SL}_n(\mathbb{C})$.*

b) If one of the following cases occur:

- *$3 \dim Y_1 + 1 \leq n - 2$, $H_i(Y_1, \mathbb{Z}) = 0$ for $i \geq 3$ and $H_2(Y_1, \mathbb{Z})$ is a free abelian group; or*
- *Y_1 is a curve and $n \geq 5$; or*
- *Y_1 is a curve with only one place at infinity and $n \geq 3$,*

then φ extends to a holomorphic *automorphism of $\mathrm{SL}_n(\mathbb{C})$.*

2.3 Uniqueness of embeddings of the affine line into algebraic groups

In this section, I report on the article [7]. This is joint work with Peter Feller. In the last section, I explained that two embeddings of the affine line into $\mathrm{SL}_n(\mathbb{C})$ are the same up to an algebraic automorphism of $\mathrm{SL}_n(\mathbb{C})$ provided that $n \geq 3$. It is natural to ask, whether this holds for more general algebraic varieties. The following is the main result, we got in this setting:

Theorem 2.3.1 (cf. Theorem 1.1 in [1]). *Let G be an algebraic group without nontrivial characters of dimension $\neq 3$. Then two embeddings of the affine line are the same up to an automorphism of G.*

Without loss of generality, we may and will assume that the group G in Theorem 2.3.1 is in addition also connected. If G is of dimension 2, then G is isomorphic to the affine plane $\mathbb{C}^2$ and then Theorem 2.3.1 follows from the Abhyankar-Moh-Suzuki Theorem [AM75, Suz74]. Hence, we may and will assume that the dimension of G is ≥ 4. I will now report on the idea of the proof.

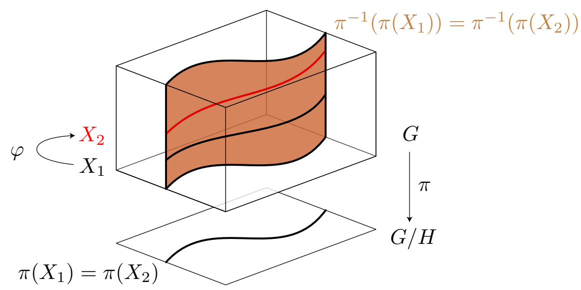
In the case $G = \mathrm{SL}_n(\mathbb{C})$ from the last section, we used explicit coordinates (i.e. we used the standard representation) to show that two embeddings are the same up to an algebraic (a holomorphic) automorphism of $\mathrm{SL}_n(\mathbb{C})$. Now, in the general case, we do not have such explicit coordinates and thus we need a replacement. Roughly, the idea is to replace the quotients $\mathrm{SL}_n(\mathbb{C}) \to \mathrm{M}_{n,r}$

that map a matrix to the first r columns by algebraic quotients $G \to G/H$ for algebraic subgroups H in G such that G/H is affine or just quasi-affine.

In fact, there is an easy class of embedding of $\mathbb{C}$ into G, namely the unipotent one-parameter subgroups $U \subset G$, i.e. one-dimensional algebraic subgroups of G that are isomorphic to $\mathbb{G}_a$. In fact one can show that all these embeddings are the same up to automorphisms of G. The idea is then first to reduce the problem to the case, when G is a simple algebraic group and thus we assume that G is simple.

We fix a curve $X \subset G$ that is isomorhic to the complex line. The goal is now, to give an idea, how one can move X into a unipotent one-parameter subgroup of G. For doing this, we need a tool to move a curve inside G. The picture after the result illustrates the setting.

Proposition 2.3.2 (cf. Proposition 5.1 in [7]). *Let $H \subseteq G$ be a closed subgroup such that the quotient G/H is quasi-affine and let $\pi\colon G \to G/H$ be the quotient morphism. If $X_1, X_2 \subset G$ are close curves that are isomorphic to $\mathbb{C}$ with $\pi(X_1) = \pi(X_2)$ and such that $\pi|_{X_i}\colon X_i \to \pi(X_i)$ is an isomorphism for $i = 1, 2$, then there exists an automorphism φ of G such that $\varphi(X_1) = X_2$.*



For this, we fix a Borel subgroup $B \subset G$ and we choose a maximal parabolic subgroup $P \subset G$ that lies over B. In the quotient G/P, there is a unique one-dimensional B-orbit and we denote by $E \subset G$ the preimage of the closure of this one-dimensional B-orbit under the natural projection $G \to G/P$. We choose now an opposite parabolic subgroup P^- to P, i.e. $P^- \cap P$ is a common Levi factor of P and P^- and we denote by $R_u(P^-)$ the unipotent radical of P^-. Now, we use heavily the fact, that the restriction to E of the natural projection $\pi\colon G \to G/R_u(P^-)$ yields a locally trivial $\mathbb{C}$-bundle and the image $\pi(E)$ is a big open subset of $G/R_u(P^-)$, i.e. the complement of $\pi(E)$ in $G/R_u(P^-)$ has codimension at least 2. After composing f with an automorphism of the form $G \to G$, $g \mapsto g_0 g$ (for a well-chosen $g_0 \in G$), we may assume that $\pi(X)$ is contained in $\pi(E)$ as $\pi(E)$ is a big open subset of $G/R_u(P^-)$. Since $\pi|_E\colon E \to \pi(E)$ is a locally trivial $\mathbb{C}$-bundle, it has a section X' over $\pi(X)$. Using Proposition 2.3.2 one can thus move X into $X' \subset E$. This furnishes the main step in the proof. Further, the idea is then to move X into a algebraic subgroup G that is not the whole of G. Having this, one can then move X into a one-parameter unipotent subgroup of G.

Based on Theorem 2.3.1, Kaliman and Udumyan proved the following generalization below. Recall, the following terminology: For any closed subvariety Z in an affine variety X and any $k \geq 1$, the closed subscheme $\mathrm{Spec}(\mathbb{C}[X]/I_X(Z)^k)$

of X is called the k-th infinitesimal neighbourhood of Z in X, where $I_X(Z)$ denotes the vanishing ideal of Z inside the coordinate ring $\mathbb{C}[X]$.

Theorem 2.3.3 (cf. Theorem 0.1 in [KU20])**.** *Let G be an algebraic group without nontrivial characters of dimension ≥ 4. If $C_1, C_2 \subseteq G$ are isomorphic to the affine line, then for any $k \geq 1$, each isomorphism of the k-th infinitesimal neighbourhoods of C_1 and C_2 in G where the determinant of the Jacobian is equal to 1 extends to an automorphism of G.*

2.4 Embeddings of affine spaces into quadrics

In this section, I report on the paper [8]. This is joint work with Jérémy Blanc. We studied embeddings of affine spaces into quadrics. The later are smooth hypersurfaces of affine spaces that are given by one polynomial of degree 2. The motivation of this study was, that until now, it is an open problem whether all embeddings of $\mathbb{C}^2$ into $\mathbb{C}^3$ (or even into $\mathbb{C}^4$) are the same up to algebraic (or even holomorphic) automorphisms. Amongst others, we studied embeddings of the plane $\mathbb{C}^2$ into the following quadric

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} x & t \\ s & y \end{pmatrix} \;\middle|\; xy - st = 1 \right\} \subseteq \mathbb{C}^4$$

instead of $\mathbb{C}^3$.

An example of a familiy of such embeddings is the following (where $\lambda \in \mathbb{C}^*$):

$$\rho_\lambda \colon \mathbb{C}^2 \to \mathrm{SL}_2(\mathbb{C}), \quad (s,t) \mapsto \begin{pmatrix} 1 & t \\ \lambda s & 1 + \lambda st \end{pmatrix}.$$

The images of these embeddings in $\mathrm{SL}_2(\mathbb{C})$ are all the same and they are given by the condition $x = 1$. Already this simple family produces many embeddings that are distinct up to automorphisms of $\mathrm{SL}_2(\mathbb{C})$:

Theorem 2.4.1 (cf. Theorem 2 in [8])**.** *For $\lambda, \lambda' \in \mathbb{C}^*$, the emebeddings ρ_λ and $\rho_{\lambda'}$ are the same up to an automorphism of $\mathrm{SL}_2(\mathbb{C})$ if and only if $\lambda = \pm\lambda'$.*

In order to proof the above theorem we established the following extension result for automorphisms of $\mathbb{C}^2$:

Theorem 2.4.2 (cf. Theorem 2 in [8])**.** *An automorphism φ of $\mathbb{C}^2$ extends to an automorphism of $\mathrm{SL}_2(\mathbb{C})$ via ρ_1 if and only if the determinant of the Jacobian $\det(\mathrm{Jac}(\varphi))$ is equal to ± 1.*

Note that $\rho_\lambda(s,t) = \rho_1(\lambda s, t)$ for each $\lambda \in \mathbb{C}^*$. We fix $\lambda, \lambda' \in \mathbb{C}^*$ and denote by φ the automorphism of $\mathbb{C}^2$ given by $(s,t) \mapsto (\lambda(\lambda')^{-1}s, t)$. Now, ρ_λ and $\rho_{\lambda'}$ are the same up to an automorphism of $\mathrm{SL}_2(\mathbb{C})$ if and only if there exists an automorphism ψ of $\mathrm{SL}_2(\mathbb{C})$ with $\rho_1(\lambda s, t) = \psi(\rho_1(\lambda' s, t)$. The latter condition is equivalent to the fact that $\rho_1 \circ \varphi = \psi \circ \rho_1$. Using Theorem 2.4.2, we get now that ρ_λ and $\rho_{\lambda'}$ are the same up to an automorphism of $\mathrm{SL}_2(\mathbb{C})$ if and only if $\det(\mathrm{Jac}(\varphi)) = \pm 1$. This amounts to say that $\lambda = \pm\lambda'$ and gives Theorem 2.4.1.

In order to proof Theorem 2.4.2, let me mention that the automorphisms of $\mathbb{C}^2$ are generated by the group of linear automorphisms $\mathrm{GL}_2(\mathbb{C})$ and the automorphisms φ_p that are given by $(s,t) \mapsto (s, t + p(s))$ where p runs over all polynomials in $\mathbb{C}[s]$, as all automorphisms of $\mathbb{C}^2$ are tame (see [Jun42]). Hence the subgroup of automorphisms in $\mathrm{Aut}(\mathbb{C}^2)$ where the determinant of the Jacobian is equal to ± 1 is generated by the linear involution ι given by $(s,t) \mapsto (t,s)$ and the automorphisms φ_p, $p \in \mathbb{C}[s]$. However ι and φ_p extend to the automorphisms

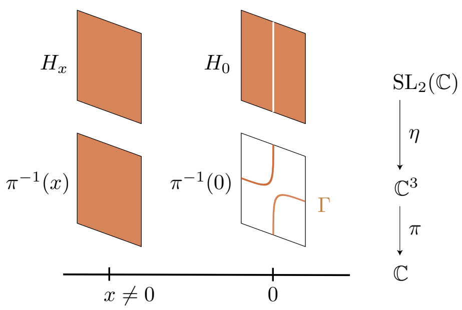
$$\begin{pmatrix} x & t \\ s & y \end{pmatrix} \mapsto \begin{pmatrix} x & s \\ t & y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & t \\ s & y \end{pmatrix} \mapsto \begin{pmatrix} x & t + xp(s) \\ s & y + sp(s) \end{pmatrix}$$

of $\mathrm{SL}_2(\mathbb{C})$ via ρ_1. This gives one implication in Theorem 2.4.2.

In order to establish the other implication, we give a certain geometric interpretation of $\mathrm{SL}_2(\mathbb{C})$ using $\mathbb{C}^3$. This interpretation turns out to be very useful also for future investigations. We consider the morphisms

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{C}) & \xrightarrow{\eta} & \mathbb{C}^3 \\ \begin{pmatrix} x & t \\ s & y \end{pmatrix} & \mapsto & (x,t,s) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{C}^3 & \xrightarrow{\pi} & \mathbb{C} \\ (x,t,s) & \mapsto & x \end{array} .$$

Note that η restricts to an isomorphism $\eta^{-1}(U) \to U$, where $U \subseteq \mathbb{C}^3$ denotes the complement of the plane $\pi^{-1}(0)$. Denote $H_x = \eta^{-1}(\pi^{-1}(x)) \subseteq \mathrm{SL}_2(\mathbb{C})$ for $x \in \mathbb{C}$. Note that η maps H_0 surjectively onto the hyperbola Γ in $\pi^{-1}(0)$ that is given by $st + 1 = 0$ and η restricts to a trivial $\mathbb{C}$-bundle $H_0 \to \Gamma$. In fact, $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to the complement of the strict transform of $\pi^{-1}(0)$ inside the blow-up $\mathrm{Bl}_\Gamma(\mathbb{C}^3)$ of $\mathbb{C}^3$ with center Γ. The following picture illustrates the fibres of $\pi \circ \eta \colon \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}$:



This establishes the existence of the following injective group homomorphism

$$\begin{array}{ccc} \left\{ \theta \in \mathrm{Aut}(\mathbb{C}^3) \;\middle|\; \begin{array}{c} \pi \circ \theta = \pi\,, \\ \theta(\Gamma) = \Gamma \end{array} \right\} & \xrightarrow{\Upsilon} & \{ \psi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C})) \mid \pi \circ \eta \circ \psi = \pi \circ \eta \} \\ \theta & \mapsto & \eta^{-1} \circ \theta \circ \eta\,. \end{array}$$

The key step is now the following:

Proposition 2.4.3 (cf. Proposition 4.5 in [8]). *The group homomorphism Υ is an isomorphism.*

Indeed, we get now the other implication in Theorem 2.4.2: The image of ρ is the hypersurface H_1 in $\mathrm{SL}_2(\mathbb{C})$. If ψ is an automorphism of $\mathrm{SL}_2(\mathbb{C})$ with $\psi(H_1) = H_1$, then ψ permutes the fibres of $\pi \circ \eta \colon \mathrm{SL}_2(\mathbb{C} \to \mathbb{C}$, i.e. it permutes the H_x for $x \in \mathbb{C}$. Since $H_x \simeq \mathbb{C}^2$ for $x \neq 0$ and $H_0 \simeq \mathbb{C} \times \mathbb{C}^*$, we get

$\psi(H_0) = H_0$. Together with $\psi(H_1) = H_1$, we get $\psi(H_x) = H_x$ for each $x \in \mathbb{C}$, i.e. $\pi \circ \eta \circ \psi = \pi \circ \eta$. By Proposition 2.4.3 there is now an automorphism θ of $\mathbb{C}^3$ with $\psi = \eta^{-1} \circ \theta \circ \eta$, θ fixes the fibres of π and $\theta(\Gamma) = \Gamma$. One can see, that the latter condition gives $\det(\mathrm{Jac}(\theta|_{\pi^{-1}(0)})) = \pm 1$. In summary we get now the result:

$$\det(\mathrm{Jac}(\psi|_{H_1})) = \det(\mathrm{Jac}(\theta|_{\pi^{-1}(1)})) = \det(\mathrm{Jac}(\theta|_{\pi^{-1}(0)})) = \pm 1\,.$$

The proof of Proposition 2.4.3 is a deformation argument based on work of Furter [Fur02]. Indeed, let $\psi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C}))$ such that $\pi \circ \eta \circ \psi = \pi \circ \eta$ and let $\theta := \eta \circ \psi \circ \eta^{-1}$. Then θ restricts to an automorphism of $U = \mathbb{C}^3 \setminus \pi^{-1}(0)$ and it is given by $\theta(x, s, t) = (x, \theta_1(x, s, t), \theta_2(x, s, t))$ for all $(x, s, t) \in U$ where θ_1, θ_2 are regular functions on U. Is is enough to establish that θ_1, θ_2 extend to $\mathbb{C}^3$. If this is not the case, then, say $\theta_1 \notin \mathbb{C}[x, s, t]$. There exists now $l > 0$ such that $f = x^l \theta_1 \in \mathbb{C}[x, s, t]$, but $f|_{\pi^{-1}(0)} \neq 0$. Note that $C_x = \{ (s, t) \in \mathbb{C}^2 \mid f(x, s, t) = 0 \} \subseteq \mathbb{C}^2$ is isomorphic to $\mathbb{C}$ for each $x \neq 0$, as $\theta|_{\pi^{-1}(x)}$ is an automorphism of $\pi^{-1}(x) = \mathbb{C}^2$ for each $x \neq 0$. Using that $f|_{\pi^{-1}(0)} \neq 0$, by the deformation argument [Fur02, Theorem 4], C_0 is isomorphic to a finite number of copies of $\mathbb{C}$. On the other hand, writing $p \colon \mathbb{C}^3 \to \mathbb{C}$, $(x, s, t) \mapsto x^l s$, we get

$$f(\Gamma) = f(\eta(H_0)) = p(\eta(\psi(H_0))) = p(\eta(H_0)) = p(\Gamma) = \{0\}$$

as $l > 0$. Hence $\Gamma \subseteq \{0\} \times C_0$. Since Γ is a closed curve in $\pi^{-1}(0) = \{0\} \times \mathbb{C}^2$ that is isomorphic to $\mathbb{C}^*$ and since $C_0 \simeq \mathbb{C}$ is a closed curve in $\mathbb{C}^2$, we arrive at a contradiction.

So far, we established that there are many distinct embeddings of $\mathbb{C}^2$ into $\mathrm{SL}_2(\mathbb{C})$. However, all these embeddings have the same image in $\mathrm{SL}_2(\mathbb{C})$. We proved also that there are many copies of $\mathbb{C}^2$ inside $\mathrm{SL}_2(\mathbb{C})$ that cannot be mapped onto each other by an automorphism of $\mathrm{SL}_2(\mathbb{C})$:

Theorem 2.4.4 (cf. Theorem 3, the proof of Proposition 5.8(1) and Lemma 5.10 in [8])**.** *Let $f \in \mathbb{C}[x, s, t]$ be an irreducible polynomial, let*

$$C_x := C_{f,x} := \{ (s, t) \in \mathbb{C}^2 \mid f(x, s, t) = 0 \} \subseteq \mathbb{C}^2 \quad \textit{for } x \in \mathbb{C}\,,$$

let $Z_f \subseteq \mathbb{C}^3$ be given by f and let $H_f := \eta^{-1}(Z_f) \subseteq \mathrm{SL}_2(\mathbb{C})$. Then we have:

- *a) If $C_x \simeq \mathbb{C}$ for at least one $x \neq 0$, then: $H_f \simeq \mathbb{C}^2$ if and only if $C_x \simeq \mathbb{C}$ for all $x \neq 0$ and $f(0, s, t) \in \{\mu s^m(s - \lambda), \mu t^m(t - \lambda)\}$ for some $\mu, \lambda \in \mathbb{C}^*$ and $m \geq 0$.*
- *b) If $f(0, s, t) = \mu \prod_{i=1}^k (t - \lambda_i)$ for some $k \geq 2$, $\mu \in \mathbb{C}^*$ and pairwise distinct $\lambda_1, \ldots, \lambda_k \in \mathbb{C}^*$, and if $C_x \simeq \mathbb{C}$ for all $x \neq 0$, then $\pi \circ \eta|_{H_f}$ is the only morphism $H_f \to \mathbb{C}$ with general fibre isomorphic to $\mathbb{C}$ up to automorphisms of the target $\mathbb{C}$.*
- *c) There is an uncountable set $F \subseteq \mathbb{C}[x, s, t]$ such that $H_f \simeq \mathbb{C}^2$ for each $f \in F$ and for $f_1 \neq f_2$ in F there is no $\psi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C}))$ with $\psi(H_{f_1}) = H_{f_2}$.*

Let me explain the very rough idea of the construction of the distinct copies of $\mathbb{C}^2$ in $\mathrm{SL}_2(\mathbb{C})$ up to automorphisms of $\mathrm{SL}_2(\mathbb{C})$ claimed in c). Whereas it is an open problem, whether there are distinct copies of $\mathbb{C}^2$ in $\mathbb{C}^3$, the so-called Danielewski surfaces inside $\mathbb{C}^3$ give many examples of pairwise distinct surfaces up to automorphisms of $\mathbb{C}^3$, see [DP09]. The idea is now to choose f in such a way, that Z_f is a Danielewski surface, but H_f is isomorphic to $\mathbb{C}^2$. One shows then that the existence of an automorphism of $\mathrm{SL}_2(\mathbb{C})$ that sends H_{f_1} onto some H_{f_2} would give an automorphism of $\mathbb{C}^3$ that sends Z_{f_1} onto Z_{f_2}.

We give now more details of the proof of Theorem 2.4.4. For this, we state the following generalization of the Abhyankar-Moh-Suzuki Theorem:

Theorem 2.4.5 (cf. [Bha88, Theorem 3.9]). *Let B be an algebraic variety and let $f\colon B \times \mathbb{C}^2 \to \mathbb{C}$ be a morphism such that there is a B-isomorphism $\xi\colon f^{-1}(0) \to B \times \mathbb{C}$ with respect to the natural projections to B. Then there exists a B-automorphism ϑ of $B \times \mathbb{C}^2$ such that $(f \circ \vartheta)(b, x, y) = x$ for all $(b, x, y) \in B \times \mathbb{C}^2$.*

a): By assumption, one non-zero fibre of $\pi|_{Z_f}\colon Z_f \to \mathbb{C}$ is isomorphic to $\mathbb{C}$ and therefore the same holds for $(\pi \circ \eta)|_{H_f}\colon H_f \to \mathbb{C}$ as well. Now, if $H_f \simeq \mathbb{C}^2$, then by the Abhyankar-Moh-Suzuki Theorem [AM75, Suz74] all fibres of the morphism $(\pi \circ \eta)|_{H_f}\colon H_f \to \mathbb{C}$ are isomorphic to $\mathbb{C}$. This gives $C_x \simeq \mathbb{C}$ for all $x \neq 0$ and Γ intersects C_0 transversally in exactly one point. Applying Theorem 2.4.5 to $B = \mathbb{C}^*$ and $f|_{\mathbb{C}^* \times \mathbb{C}^2}\colon \mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}$ and using the deformation argument [Fur02, Theorem 4], it follows that C_0 is isomorphic to a finite number of copies of $\mathbb{C}$ (see also Lemma 3.8 in [4]). The only possibilities of curves in $\mathbb{C}^2$ that are isomorphic to $\mathbb{C}$ and have no intersection with Γ are given by $s = 0$ or $t = 0$. Using an analysis at the points at infinity, one can see that the only curves in $\mathbb{C}^2$ that are isomorphic to $\mathbb{C}$ and intersect Γ transversally in one point are given by $t = \lambda$ or $s = \lambda$ for some $\lambda \in \mathbb{C}^*$. This implies then one direction. For the other implication, note that the assumptions give that all fibres of $(\pi \circ \eta)|_{H_f}\colon H_f \to \mathbb{C}$ are isomorphic to $\mathbb{C}$ and then $H_f \simeq \mathbb{C}^2$ e.g. by [Asa87, Corollary 3.2] and [BCW77].

b): All non-zero fibres of $\pi|_{Z_f}\colon Z_f \to \mathbb{C}$ are isomorphic to $\mathbb{C}$, whereas the zero-fibre consists of k copies of $\mathbb{C}$. As Γ intersects each of these k copies of $\mathbb{C}$ transversally in one point, $\eta|_{H_f}\colon H_f \to Z_f$ is the open subset of the blow-up of Z_f in these intersection points, where the strict transforms of these k copies of $\mathbb{C}$ are removed. In particular all non-zero fibres of $\pi \circ \eta|_{H_f}\colon H_f \to \mathbb{C}$ are isomorphic to $\mathbb{C}$, whereas the zero fibre consists of k copies of $\mathbb{C}$. In particular, Z_f and H_f are smooth. Moreover, using the geometry of $\eta|_{H_f}\colon H_f \to Z_f$ one can in fact construct a minimal smooth projective completion $\overline{H_f}$ of H_f such that the boundary $\overline{H_f} \setminus H_f$ is not a linear chain of projective lines $\mathbb{P}^1$. This gives then the claim by [Giz71] or [Ber83, Théorème 1.8].

c): Let $f_i \in \mathbb{C}[x, s, t]$, $i = 1, 2$ such that $C_{f_i,x} \simeq \mathbb{C}$ for each $x \neq 0$ and $f_i(0, s, t) = \mu t^m (t - 1)$ for some $\mu \in \mathbb{C}^*$, $m \geq 1$. In particular $H_{f_i} \simeq \mathbb{C}^2$ by a).

Applying Theorem 2.4.5 to $B = \mathbb{C}^*$ and $f_i|_{\mathbb{C}^* \times \mathbb{C}^2}\colon \mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}$, it follows that $\pi|_{Z_{f_i - a}}\colon Z_{f_i - a} \to \mathbb{C}$ is a trivial $\mathbb{C}$-bundle over $\mathbb{C}^*$ for all $a \in \mathbb{C}$. Moreover, $f_i(0, s, t) - a \in \mathbb{C}[t]$ has $m + 1$ distinct roots in $\mathbb{C}^*$ for general $a \in \mathbb{C}$. By b), for general $a \in \mathbb{C}$,

$$p_{i,a} \coloneqq \pi \circ \eta|_{H_{f_i - a}} H_{f_i - a} \to Z_{f_i - a} \to \mathbb{C}$$

is the only morphism $H_{f_i - a} \to \mathbb{C}$ with general fibre isomorphic to $\mathbb{C}$ up to automorphisms of the target $\mathbb{C}$. If $\psi \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C}))$ with $\psi(H_{f_1}) = H_{f_2}$, then there exists a $\mu \in \mathbb{C}^*$ such that $\psi(H_{f_1 - a}) = H_{f_2 - \mu a}$ for all $a \in \mathbb{C}$ and hence, it follows that $p_{2,\mu a} \circ \psi$ is the same as $p_{1,a}$ up to an automorphism of the target

$\mathbb{C}$ for general $a \in \mathbb{C}$. As the zero-fibre of $p_{i,a}$ is the only fibre that is non-isomorphic to $\mathbb{C}$, it follows that $\psi(p_{1,a}^{-1}(0)) = p_{2,\mu a}^{-1}(0)$ for general $a \in \mathbb{C}$. This gives thus $\psi((\pi \circ \eta)^{-1}(0)) = (\pi \circ \eta)^{-1}(0)$. Therefore we find $\lambda \in \mathbb{C}^*$ such that

$$\pi \circ \eta \circ \psi = \lambda \cdot (\pi \circ \eta) = \pi \circ \eta \circ \vartheta_\lambda \,,$$

where $\vartheta_\lambda \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C}))$ is given by

$$\vartheta_\lambda \begin{pmatrix} x & t \\ s & y \end{pmatrix} = \begin{pmatrix} \lambda x & t \\ s & \lambda^{-1} y \end{pmatrix} .$$

Applying Proposition 2.4.3 we find $\theta \in \mathrm{Aut}(\mathbb{C}^3)$ with $\eta^{-1} \circ \theta \circ \eta = \psi \circ (\vartheta_\lambda)^{-1}$. Hence $\eta^{-1} \circ \theta \circ \theta_\lambda \circ \eta = \eta^{-1} \circ \theta \circ \eta \circ \vartheta_\lambda = \psi$, where $\theta_\lambda \in \mathrm{Aut}(\mathbb{C}^3)$ is given by $\theta_\lambda(x,t,s) = (\lambda x, t, s)$. As $\psi(H_{f_1}) = H_{f_2}$, we find that

$$(\theta \circ \theta_\lambda)(Z_{f_1}) = Z_{f_2} \,.$$

So we traced back the problem of finding distinct $\mathbb{C}^2$ in $\mathrm{SL}_2(\mathbb{C})$ up to automorphisms to the problem of finding distinct surfaces of the form Z_{f_i} in $\mathbb{C}^3$ up to automorphisms.

However, the hypersurfaces Z_{f_i} of $\mathbb{C}^3$ are particular examples of so-called Danielewski surfaces. These surfaces are widely studied for example in [DP09]. Explicitly we may choose

$$F = \{\, x^{\deg(r)+3} s - (t-x) \cdot (t - 1 - x^2 r(x)) \mid r \in \mathbb{C}[x] \setminus \{0\} \,\}$$

by [DP09, Proposition 3.6] (in fact, in the formula in Lemma 5.10 in [8] there is a typo).

We studied also embeddings of $\mathbb{C}$ into the quadric surface

$$Q = \{\, (x,y,z) \in \mathbb{C}^3 \mid xy = z(z+1) \,\} \subseteq \mathbb{C}^3 \,.$$

While there is only one embedding of $\mathbb{C}$ into $\mathbb{C}^2$ up to automorphisms of $\mathbb{C}^2$ by the Abhyankar-Moh-Suzuki Theorem, there are many distinct embeddings of $\mathbb{C}$ into Q up to automorphisms of Q:

Theorem 2.4.6 (Theorem 1 in [8])**.** *There is are uncountably many distinct embedding of $\mathbb{C}$ into Q up to automorphisms of Q.*

In fact, we studied the family of embeddings

$$\nu_p \colon \mathbb{C} \to Q \,, \quad t \mapsto (t(1+tp(t)), p(t), tp(t)) \quad \text{where } p \in \mathbb{C}[t] \,.$$

We use the fact, that Q is equal to the complement of the diagonal Δ in $\mathbb{P}^1 \times \mathbb{P}^1$ and showed the following (this is the key step): if there is an automorphism $\alpha \in \mathrm{Aut}(Q) = \mathrm{Aut}((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta)$ such that $\nu_q = \alpha \circ \nu_p$ for polynomials $p, q \in \mathbb{C}[t]$ of degree ≥ 3, then α extends to an automorphism $\hat{\alpha} \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. Using the fact that $\hat{\alpha}$ maps the diagonal Δ onto itself and $\nu_q = \hat{\alpha} \circ \nu_p$, one gets $p = q$.

2.5 On maximal subalgebras

In this section I report on the work [9]. It is joint work with Stefan Maubach. The guiding problem was to classify maximal subrings of a given ring. So let me settle the definitions first:

Definition 2.5.1. A ring extension $A \subseteq R$ is called *minimal*, if $A \neq R$ and there exists no subring B of R such that $A \subsetneq B \subsetneq R$. In this case A is called a *maximal subring* of R and R is called a *minimal overring* of A.

Geometrically, this says the following: Having a dominant morphism between affine schemes $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$, then $A \subseteq R$ is a minimal ring extension, if there exists no affine scheme Z such that $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ factorizes as into two dominant morphisms $\mathrm{Spec}(R) \to Z \to \mathrm{Spec}(A)$. So, intuitively, $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ is "not decomposable" and serves as a "minimal block" in a composition.

There is the following fundamental result concerning minimal ring extensions.

Theorem 2.5.2 ([FO70, Théorème 2.2], cf. also Theorem 1.0.1 in [9])**.** *Let $A \subseteq R$ be a minimal ring extension and let $\varphi \colon \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ be the corresponding morphism of affine schemes. Then there exists a unique maximal ideal $\mathfrak{m} \subseteq A$, called* crucial maximal ideal*, such that*

$$\mathrm{Spec}(R) \setminus \varphi^{-1}(\mathfrak{m}) \xrightarrow[\simeq]{\varphi} \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$$

is an isomorphism. Moreover, the following statements are equivalent:

i) φ *is surjective*
ii) R *is a finite* A*-module*
iii) $\mathfrak{m} = \mathfrak{m} R$.

In order to make the further exposition simpler, we assume that R is a finitely generated $\mathbb{C}$-algebra which is also an integral domain. Moreover, we assume that every maximal subring A of R is a $\mathbb{C}$-subalgebra of R and we call it then a *maximal subalgebra* of R. According to the result above, there is a dichotomy between the maximal subalgebras A of R: Either the corresponding morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ is finite (i.e. R is a finite A-module) and we call this the *non-extending case* or the corresponding morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ is an embedding of $\mathrm{Spec}(R)$ onto an open subset of $\mathrm{Spec}(A)$ and we call it the *extending case*.

In some situations the following tool can be used to construct maximal subrings:

Lemma 2.5.3 ([FO70, Lemme 1.4])**.** *Let $A \subseteq R$ be a ring extension and let $\mathfrak{a} \subseteq A$ be an ideal that is also an ideal in R. Then $A \subseteq R$ is a minimal ring extension if and only if $A/\mathfrak{a} \subseteq R/\mathfrak{a}$ is a minimal ring extension.*

2.5.1 The non-extending case

Let $A \subseteq R$ be a subalgebra. By Theorem 2.5.2 and Lemma 2.5.3, A is a non-extending maximal subalgebra of R if and only if there exists a maximal ideal

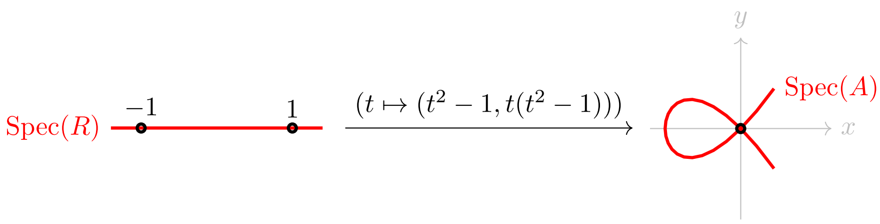
$\mathfrak{m} \subsetneq A$ such that $\mathfrak{m} = R\mathfrak{m}$ and $A/\mathfrak{m} \subsetneq R/\mathfrak{m}$ is a minimal ring extension. However, in this case $A/\mathfrak{m} \subsetneq R/\mathfrak{m}$ is finite and thus $A/\mathfrak{m}$ is then isomorphic to $\mathbb{C}$ and due to [FO70, Lemme 1.2] the minimal ring extensions $\mathbb{C} \subsetneq R'$ are of the form

$$\mathbb{C} \to \mathbb{C} \times \mathbb{C}\,, \; z \mapsto (z, z) \quad \text{or} \quad \mathbb{C} \hookrightarrow \mathbb{C}[\varepsilon]/(\varepsilon^2)\,.$$

If we are in the first case, then the fibre of the morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(A)$ over $\mathfrak{m}$ contains exactly two closed points. Intuitively, $\operatorname{Spec}(A)$ corresponds to the glueing of these two points in $\operatorname{Spec}(R)$ in such a way that the images of the corresponding tangent spaces form a direct sum of the tangent space of $\operatorname{Spec}(A)$ at $\mathfrak{m}$. A prominent example is the following:

$$\mathfrak{m} = (t^2 - 1, t(t^2 - 1)) \subsetneq A = \mathbb{C}[t^2 - 1, t(t^2 - 1)] \subsetneq R = \mathbb{C}[t]$$

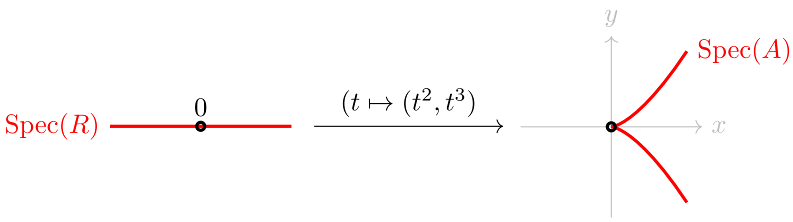
and the corresponding morphism can be illustrated as follows:



If we are in the second case, then the fibre of the morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(A)$ over $\mathfrak{m}$ contains exactly one closed point $\mathfrak{m}_0 \in \operatorname{Spec}(R)$ and the fibre over it is schematically non-reduced. Intuitively $\operatorname{Spec}(A)$ corresponds to deleting one tangent direction of $\operatorname{Spec}(R)$ at $\mathfrak{m}_0$, i.e. the differential of $\operatorname{Spec}(R) \to \operatorname{Spec}(A)$ at $\mathfrak{m}_0$ has exactly a on-dimensional kernel. A prominent example is the following:

$$\mathfrak{m} = (t^2, t^3) \subsetneq A = \mathbb{C}[t^2, t^3] \subsetneq R = \mathbb{C}[t]$$

and the corresponding morphism can be illustrated as follows:



More details about the non-extending case can be found in [MS17, §2]. In fact, one has in this non-extending case a rather clear picture of the situation.

2.5.2 The extending case

Much more difficult is the extending case. If $A \subsetneq R$ is an extending maximal subalgebra, then the corresponding morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(A)$ is an open embedding and the complement of the image consists of exactly one (closed) point $\mathfrak{m} \in \operatorname{Spec}(A)$. So these morphisms are embeddings, but contrary to our convention, these are open and not closed embeddings.

In case $\operatorname{Spec}(R)$ is one-dimensional, we have a full classification:

Theorem 2.5.4 (cf. Theorem 3.0.13 and Lemma 3.0.12 in [9])**.** *Let R be a finitely generated $\mathbb{C}$-algebra that is also an integral domain of dimension one, let $X \coloneqq \mathrm{Spec}(R)$ and let $\overline{X}$ be a projective closure of X such that $\overline{X}$ is smooth at every point of $\overline{X} \setminus X$. If $\overline{X} \setminus X$ consists only of one point, then R has no extending maximal subalgebra. Otherwise, for each $p \in \overline{X} \setminus X$,*

$$A = \{\, f \in R \mid f \text{ is defined at } p \,\}$$

is an extending maximal subalgebra and every extending maximal subalgebra of R is of this form. Moreover, each such A is a finitely generated $\mathbb{C}$-algebra.

Assume now that R is a finitely generated $\mathbb{C}$-algebra, that is also an integral domain and assume that the dimension is ≥ 2. If $A \subsetneq R$ is an extending maximal subalgebra, then A cannot be a finitely generated $\mathbb{C}$-algebra, since otherwise the localization $A_{\mathfrak{m}}$ at the crucial maximal ideal $\mathfrak{m}$ would be a Noetherian integral domain of dimension two and thus it would not be a discrete valuation ring, contradicting [FO70, Corollaire 3.4]. So this gives an indication, that the extending maximal subalgebras of R are much more difficult to understand than the non-extending ones.

As a first exploration in the two dimensional case, we considered the algebra $R = \mathbb{C}[t^{\pm 1}, y]$, i.e. R is the polynomial ring in the variable y over the Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$.

Let us consider a first example.

Example 2.5.5. For each polynomial $p \in \mathbb{C}[t]$, the algebra

$$A = \mathbb{C}[t, y] + (y - p)\mathbb{C}[t^{\pm 1}, y]$$

is an extending maximal subalgebra of $R = \mathbb{C}[t^{\pm 1}, y]$ with crucial maximal ideal

$$\mathfrak{m} = t\mathbb{C}[t, y] + (y - p)\mathbb{C}[t^{\pm 1}, y]\,.$$

Indeed, $\mathfrak{a} = (y - p)\mathbb{C}[t^{\pm 1}, y]$ is an ideal in A and in R and the ring extension

$$\mathbb{C}[t] = A/\mathfrak{a} \subsetneq R/\mathfrak{a} = \mathbb{C}[t^{\pm 1}, p] = \mathbb{C}[t^{\pm 1}]$$

is minimal. Thus by Lemma 2.5.3 $A \subsetneq R$ is a mimimal ring extension. As $t \in R$ is invertible and as $t \in \mathfrak{m}$, there exists no prime ideal $\mathfrak{p} \subseteq R$ with $\mathfrak{m} = A \cap \mathfrak{p}$. Hence $\mathfrak{m}$ is the crucial maximal ideal of the minimal ring extension $A \subsetneq R$ and $A \subsetneq R$ is extending.

Let me give now a slightly more involved example.

Example 2.5.6. For each $n \geq 1$, the algebra

$$A = \mathbb{C}[t, y] + (y^n - t)\mathbb{C}[t^{\pm 1}, y]$$

is an extending maximal subalgebra of $R = \mathbb{C}[t^{\pm 1}, y]$. Indeed,

$$\mathfrak{a} = (y^n - t)\mathbb{C}[t^{\pm 1}, y]$$

is an ideal in both rings A and R and the ring extension

$$\mathbb{C}[y] = A/\mathfrak{a} \subsetneq R/\mathfrak{a} = \mathbb{C}[y^{\pm n}, y] = \mathbb{C}[y^{\pm 1}]$$

is minimal. Thus again by Lemma 2.5.3 we conclude that the ring extension $A \subseteq R$ is minimal. As in Example 2.5.5, one can see that

$$\mathfrak{m} = t\mathbb{C}[t, y] + (y^n - t)\mathbb{C}[t^{\pm 1}, y]$$

is a maximal ideal in A such that there is no $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{m} = \mathfrak{p} \cap A$. Therefore the minimal ring extension $A \subseteq R$ is extending.

Now, there is also a different description of A. In fact, we extend the scalars $\mathbb{C}[t]$ by $\mathbb{C}[t^{1/n}]$ and consider the subalgebra

$$A' = \mathbb{C}[t^{1/n}, y] + (y - t^{1/n})\mathbb{C}[t^{\pm 1/n}, y]$$

of $R' = \mathbb{C}[t^{1/n}] \otimes_{\mathbb{C}[t]} R = \mathbb{C}[t^{\pm 1/n}, y]$. Using Example 2.5.5 (where we replace t by $t^{1/n}$) one concludes that $A' \subseteq R'$ is minimal and since $(y^n - t)\mathbb{C}[t^{\pm 1}, y] \subseteq (y - t^{1/n})\mathbb{C}[t^{\pm 1/n}, y]$ one gets $A \subseteq A'$. Moreover, as $A \subseteq R$ is minimal and as t^{-1} lies not inside $A' \cap \mathbb{C}[t^{\pm 1}, y]$, we get

$$A = A' \cap \mathbb{C}[t^{\pm 1}, y] \subsetneq R\,.$$

Thus we found a new description of A: it is the intersection of A' with $\mathbb{C}[t^{\pm 1}, y]$. The advantage of A' over A lies in the fact, that the ideal $(y - t^{1/n})\mathbb{C}[t^{\pm 1/n}, y]$ is generated by a linear polynomial in y, whereas $\mathfrak{a} = (y^n - t)\mathbb{C}[t^{\pm 1}, y]$ is generated by a degree n polynomial in y.

So, this last example illustrates, that extending the scalars can simplify the situation. We used exactly this idea. In order to formulate the results, let us consider the field of so-called *Hahn-series*:

$$K := \mathbb{C}[[t^{\mathbb{Q}}]] := \left\{ \sum_{s \in \mathbb{Q}} a_s t^s \;\middle|\; a_s \in \mathbb{C} \text{ and } \operatorname{supp}\left(\sum_{s \in \mathbb{Q}} a_s t^s \right) \subseteq \mathbb{Q} \text{ is well-ordered} \right\}$$

where

$$\operatorname{supp}\left(\sum_{s \in \mathbb{Q}} a_s t^s \right) := \{\, s \in \mathbb{Q} \mid a_s \neq 0 \,\}\,.$$

The field K is algebraically closed and complete with respect to the metric induced by the valuation $\nu\colon K \to \mathbb{Q}$, $\alpha \mapsto \min(\operatorname{supp}(\alpha))$. The valuation ring of K with respect to ν, we denote by

$$K^+ := \mathbb{C}[[t^{\mathbb{Q}_{\geq 0}}]] := \{\, \alpha \in \mathbb{C}[[t^{\mathbb{Q}}]] \mid \nu(\alpha) \geq 0 \,\}\,.$$

It turned out that extending the scalars $\mathbb{C}[t]$ to K^+ is very useful: the extending maximal subalgebras of $K[y]$ that contain $K^+[y]$ have in fact a fairly easy description:

Theorem 2.5.7 (cf. Theorem 1.0.2 in [9]). *The following map*

$$K^+ \xrightarrow{\alpha \mapsto A_\alpha} \left\{ \begin{array}{c} \textit{maximal extending subalgebras} \\ \textit{of } K[y] \textit{ that contain } K^+[y] \end{array} \right\},$$

is a bijection, where $A_\alpha := K^+[y] + (y - \alpha)K[y] \subseteq K[y]$.

Using the notation of the theorem above, we can now describe the extending maximal subrings of $\mathbb{C}[t^{\pm 1}, y]$ that contain $\mathbb{C}[t, y]$:

Theorem 2.5.8 (cf. Theorem 1.0.3 in [9]). *Let*

$$\mathcal{S} := \left\{ \alpha \in K^+ \;\middle|\; \begin{array}{l} \operatorname{supp}(\alpha) \textit{ is contained in a strictly} \\ \textit{increasing sequence of } \mathbb{Q} \end{array} \right\} .$$

Then we have a bijection

$$\mathcal{S}/G \xrightarrow{\alpha \mapsto A_\alpha \cap \mathbb{C}[t^{\pm 1}, y]} \left\{ \begin{array}{l} \textit{maximal extending subalgebras} \\ \textit{of } \mathbb{C}[t^{\pm 1}, y] \textit{ that contain } \mathbb{C}[t, y] \end{array} \right\} ,$$

where $G = \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{C}^*)$ and the action of G on $\mathcal{S}$ is given by

$$g \cdot \sum_{s \in \mathbb{Q}} a_s t^s := \sum_{s \in \mathbb{Q}} g(s) a_s t^s .$$

In general, there is the following dichotomy:

Proposition 2.5.9 (cf. Theorem 1.0.4 in [9]). *Let $A \subseteq \mathbb{C}[t^{\pm 1}, y]$ be a maximal extending subalgebra. Then exactly one of the following cases occur:*

i) There exists an automorphism σ of $\mathbb{C}[t^{\pm 1}, y]$ such that $\sigma(A)$ contains $\mathbb{C}[t, y]$.
ii) A contains $\mathbb{C}[t^{\pm 1}]$.

The first case is covered by Theorem 2.5.8 and thus we are left with the problem of the classification of all extending maximal subalgebras of $\mathbb{C}[t^{\pm 1}, y]$ that contain $\mathbb{C}[t^{\pm 1}]$. For this let us introduce the following notation:

$$\mathcal{M} := \{\text{extending maximal subalgebras of } \mathbb{C}[t, y] \text{ that contain } \mathbb{C}[t]\}$$

and

$$\mathcal{N} := \left\{ \begin{array}{c} \text{extending maximal subalgebras } A \text{ of} \\ \mathbb{C}[t, y^{\pm 1}] \text{ that contain } \mathbb{C}[t, y^{-1}] \text{ and such} \\ \text{that } A \to \mathbb{C}[t, y^{\pm 1}]/(t - \lambda) \text{ is surjective} \end{array} \right\} ,$$

where $\lambda \in \mathbb{C}$ is the unique complex number such that $t - \lambda$ lies in the crucial maximal ideal of the minimal ring extension $A \subseteq \mathbb{C}[t, y^{\pm 1}]$. The set $\mathcal{N}$ can be described using Theorem 2.5.8. The extending maximal subalgebras of $\mathbb{C}[t^{\pm 1}, y]$ that contain $\mathbb{C}[t^{\pm 1}]$ may now be described by the following result:

Theorem 2.5.10 (cf. Theorem 7.0.1 and Proposition 6.0.2 in [9]). *There exist bijections Θ and Φ*

$$\mathcal{N} \xrightarrow[1:1]{\Theta} \mathcal{M} \supset \left\{ \begin{array}{c} B \in \mathcal{M} \textit{ s.t. the crucial} \\ \textit{maximal ideal of } B \\ \textit{does not contain } t \end{array} \right\} \xleftarrow[1:1]{\Phi} \left\{ \begin{array}{c} \textit{extending maximal} \\ \textit{subalgebras of } \mathbb{C}[t^{\pm 1}, y] \\ \textit{that contain } \mathbb{C}[t^{\pm 1}] \end{array} \right\} .$$

where $\Theta(A) = A \cap \mathbb{C}[t, y]$ and $\Phi(A') = A' \cap \mathbb{C}[t, y]$.

Chapter 3

Automorphisms of affine varieties

In this part, I report on my research concerning the automorphism group $\mathrm{Aut}(X)$ of an affine variety X. As in the previous part, I will write the results for simplicity over the field of complex numbers $\mathbb{C}$, if not mentioned explicitly otherwise.

The first two papers concern question (C): specifically, I investigated the determination of the affine space $\mathbb{C}^n$ and spherical varieties by their automorphism groups. In order to state the results, let me mention the following terminology: We say that a group homomorphism of automorphism groups

$$\theta \colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$$

preserves algebraic group actions, if for each faithful algebraic group action $G \times X \to X$ of an algebraic group G, the action $G \times Y \to Y$, $(g, y) \mapsto \theta(g)(y)$ is again a faithful algebraic group action. If θ is a group isomorphism, then we say that it *preserves algebraic group actions* if this holds for θ and θ^{-1}. This notion coincides with the notion of "preserving algebraic subgroups" (see §5 in [5] and Theorem 9 in [6]).

The third paper addresses question (D): specifically, in this article I investigated the dynamical degree of a certain class of polynomial automorphisms of $\mathbb{C}^n$; the *dynamical degree* $\lambda(f)$ of a polynomial automorphism f is defined to be the number

$$\lambda(f) \coloneqq \lim_{i \to \infty} \left(\deg(f^i)\right)^{\frac{1}{i}} \in \mathbb{R},$$

where for any automorphism $g \in \mathrm{Aut}(\mathbb{C}^n)$ with corresponding coordinate functions $g_1, \ldots, g_n \colon \mathbb{C}^n \to \mathbb{C}$ we define the *degree* $\deg(g)$ of g by

$$\deg(g) = \max_{i=1,\ldots,n} \deg(g_i).$$

In the last paper, I investigated question (E): specifically I classified automorphisms f of $\mathbb{C}^3$ with $\deg(f) \leq 3$ up to composition with affine automorphisms (i.e. automorphisms of degree 1) at the source and target and I computed their dynamical degrees $\lambda(f)$.

3.1 Is the affine space determined by its automorphism group?

In this section, I will report on the article [6] joint with Hanspeter Kraft and Andriy Regeta. If there is a group isomorphism $\theta\colon \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(X)$ that preserves algebraic group actions and if X is a connected affine variety, then X and $\mathbb{C}^n$ are isomorphic by a result due to Kraft [Kra17, Theorem 1.1]. Motivated by this result, we studied to what extend one could neglect the hypothesis that θ preserves algebraic group actions:

Theorem 3.1.1 (cf. Main Theorem in [6]). *Let X be an irreducible quasi-projective n-dimensional variety such that there exists a group isomorphism $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(\mathbb{C}^n)$. Then $X \simeq \mathbb{C}^n$ if one of the following conditions holds.*

1) X is smooth, the Euler characteristic $\chi(X)$ is nonzero and the Picard group $\mathrm{Pic}(X)$ is finite;
2) X is toric and quasi-affine.

As an immediate consequence we get that $\mathrm{Aut}(\mathbb{C}^n \setminus S)$ and $\mathrm{Aut}(\mathbb{C}^n)$ cannot be isomorphic for each closed subvariety $S \subseteq \mathbb{C}^n$ with $\chi(S) \neq 1$, as in this case $\chi(\mathbb{C}^n \setminus S) = \chi(\mathbb{C}^n) - \chi(S) \neq 0$. In particular this applies to all finite subsets S of $\mathbb{C}^n$ with more than one element.

In order to point out the key steps in the proof of Theorem 3.1.1, let us introduce the following terminology for the automorphism group $\mathrm{Aut}(X)$ of a variety X. A map $\nu\colon A \to \mathrm{Aut}(X)$ is called a *morphism* if the associated map $A \times X \to X$, $(a, x) \mapsto \nu(a, x)$ is a morphism (of varieties). A subgroup $G \subseteq \mathrm{Aut}(X)$ is called an *algebraic subgroup* if there exists an algebraic group H and a group isomorphism $\nu\colon H \to G \subseteq \mathrm{Aut}(X)$ such that ν is a morphism. If $G \subseteq \mathrm{Aut}(X)$ is any subgroup, then we define *the identity component* by

$$G^\circ := \left\{ g \in G \;\middle|\; \begin{array}{l} \text{there exists a morphism } \nu\colon A \to \mathrm{Aut}(X) \text{ of an} \\ \text{irreducible variety } A \text{ such that } g, \mathrm{id}_X \in \nu(A) \subseteq G \end{array} \right\}$$

and its *dimension* by

$$\dim G := \sup \left\{ \dim A \;\middle|\; \begin{array}{l} \text{there exists an injective morphism} \\ \nu\colon A \to \mathrm{Aut}(X) \text{ with image in } G \end{array} \right\}.$$

For a subgroup $G \subseteq \mathrm{Aut}(X)$ the following statements are equivalent (see Theorem 2.9 in [6]):

- G is an algebraic subgroup of $\mathrm{Aut}(X)$;
- $\dim G$ is finite and G° has finite index in G;
- There exists a morphism $A \to \mathrm{Aut}(X)$ with image G.

In order to show that X and $\mathbb{C}^n$ are isomorphic, our main tool is the following result:

Proposition 3.1.2 (cf. Proposition 4.1 in [6]). *Let W be an irreducible quasi-affine variety and let $\theta\colon \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(W)$ be a group isomorphism such that θ maps the standard n-dimensional torus $T \subset \mathrm{Aut}(\mathbb{C}^n)$ onto an algebraic subgroup of $\mathrm{Aut}(W)$. Then W and $\mathbb{C}^n$ are isomorphic.*

The main steps in the proof of Proposition 3.1.2 are the following: T normalizes the subgroup of translations $\mathrm{Tr} \subseteq \mathrm{Aut}(\mathbb{C}^n)$ and acts on it with an open orbit O that satisfies $\mathrm{Tr} = O \circ O$. For a fixed $v_0 \in O$, the set O consists of all elements of the form $t \circ v_0 \circ t^{-1}$, $t \in T$. As $\theta(T)$ is an algebraic subgroup of $\mathrm{Aut}(W)$, there exists an algebraic group S together with a group isomorphism $\nu \colon S \to \theta(T) \subseteq \mathrm{Aut}(X)$ that is a morphism. Then

$$\begin{array}{rcl} S \times S & \to & \theta(\mathrm{Tr}) = \theta(O) \circ \theta(O) \\ (s_1, s_2) & \mapsto & \nu(s_1) \circ \theta(v_0) \circ \nu(s_1)^{-1} \circ \nu(s_2) \circ \theta(v_0) \circ \nu(s_2)^{-1} \end{array}$$

is surjective and the composition with the inclusion $\theta(\mathrm{Tr}) \subseteq \mathrm{Aut}(X)$ yields a morphism $S \times S \to \mathrm{Aut}(X)$. Thus $\theta(\mathrm{Tr})$ is an algebraic subgroup of $\mathrm{Aut}(W)$. As $\theta(\mathrm{Tr})$ is commutative and contains no element of finite order except id_X, it follows that $\theta(\mathrm{Tr})$ is unipotent. As W is quasi-affine, all orbits of $\theta(\mathrm{Tr})$ in W are closed. As W is irreducible and quasi-affine, one may then show that the algebraic subgroup $\theta(\mathrm{Tr})$ of $\mathrm{Aut}(W)$ acts with a dense orbit on W. Hence $\theta(\mathrm{Tr})$ acts transitively on W and this implies that W is an affine space $\mathbb{C}^m$. In terms of subgroups of $\mathrm{Aut}(\mathbb{C}^d)$ one may characterize d as the maximal number k such that $\mathrm{Aut}(\mathbb{C}^d)$ contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$. This implies thus $m = n$ and gives the statement of the proposition.

The following theorem is our main result in order to apply Proposition 3.1.2:

Theorem 3.1.3 (cf. Theorem 1.1 in [6]). *Let Y and Z be irreducible quasi-projective varieties, and let $\vartheta \colon \mathrm{Aut}(Y) \to \mathrm{Aut}(Z)$ be an group isomorphism. Assume that $n := \dim Y \geq \dim Z$ and that the following conditions are satisfied:*

i) Y is quasi-affine and toric;
ii) Z is smooth, $\chi(Z) \neq 0$, and $\mathrm{Pic}(Z)$ is finite.

Then $\dim Z = n$, and for each n-dimensional torus $T \subseteq \mathrm{Aut}(Y)$, the identity component of the image $\vartheta(T)^\circ$ is an algebraic subgroup of $\mathrm{Aut}(Z)$, isomorphic to a torus of dimension n. Furthermore, Z is quasi-affine.

In order to deduce Theorem 3.1.1 under the assumptions 1), we apply Theorem 3.1.3 to $Y := \mathbb{C}^n$, $Z := X$ and $\vartheta := \theta^{-1} \colon \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(X)$ and thus we get for the standard n-dimensional torus $T \subseteq \mathrm{Aut}(\mathbb{C}^n)$ that $\vartheta(T)^\circ$ is an algebraic subgroup of $\mathrm{Aut}(X)$ and moreover that X is quasi-affine. One can then show that in fact $\vartheta(T)^\circ = \vartheta(T)$. Hence the assumptions of Proposition 3.1.2 are satisfied for ϑ and we get $X \simeq \mathbb{C}^n$.

In order to deduce Theorem 3.1.1 under the assumptions 2), we apply Theorem 3.1.3 to $Y := X$, $Z := \mathbb{C}^n$ and $\vartheta := \theta \colon \mathrm{Aut}(X) \to \mathrm{Aut}(\mathbb{C}^n)$. Then for a fixed n-dimensional torus $T \subseteq \mathrm{Aut}(X)$, the identity component $\vartheta(T)^\circ$ is an algebraic subgroup of $\mathrm{Aut}(\mathbb{C}^n)$, isomorphic to an n-dimensional torus. One can again show, that $\vartheta(T)^\circ = \vartheta(T)$. Now, all n-dimensional tori in $\mathrm{Aut}(\mathbb{C}^n)$ are conjugated by a result due to Białynicki-Birula [BB66]. Thus there exists a $\varphi \in \mathrm{Aut}(\mathbb{C}^n)$ such that $\varphi \circ \vartheta(T) \circ \varphi^{-1}$ is the standard n-dimensional torus in $\mathrm{Aut}(\mathbb{C}^n)$. Now,

$$\vartheta^{-1}(\varphi \circ \vartheta(T) \circ \varphi^{-1}) = \vartheta^{-1}(\varphi) \circ T \circ \vartheta^{-1}(\varphi)^{-1}$$

is an algebraic subgroup of $\mathrm{Aut}(X)$ and thus again, Proposition 3.1.2 yields $X \simeq \mathbb{C}^n$.

The proof of Theorem 3.1.3 is the main bulk of the whole article. As Y is toric and $T \subset \mathrm{Aut}(Y)$ is a torus of maximal dimension, the centralizer $\mathrm{Cent}_{\mathrm{Aut}(Y)}(T)$ of T in $\mathrm{Aut}(Y)$ is equal to T itself. If p is a prime number, then there is a unique finite subgroup $\mu_p \subset T$ that is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ and we have $T \subseteq \mathrm{Cent}_{\mathrm{Aut}(Y)}(\mu_p)$. Thus we get $\theta(T) \subseteq \mathrm{Cent}_{\mathrm{Aut}(Z)}(\theta(\mu_p)) =: C$. We may choose p in such a way, that p does not divide $\chi(Z) \neq 0$. Hence, one can see that the fixed point set $Z^{\theta(\mu_p)}$ is non-empty. As Z is smooth one may even find an isolated point $z_0 \in Z^{\theta(\mu_p)}$. Using that $\mathrm{Pic}(Z)$ is finite and using the tangent representation $C^\circ \to T_{z_0}Z$, one may find C°-semi-invariant regular functions $f_1, \ldots, f_n \colon Z \to \mathbb{C}$ such that $z_0 \in \bigcap_{i=1}^n f_i^{-1}(0) \subseteq Z^{\theta(\mu_p)}$. Let $\chi_1, \ldots, \chi_n \colon C^\circ \to \mathbb{C}$ be the corresponding characters of the semi-invariants $f_1, \ldots, f_n$. Then one may show that the homomorphism

$$\chi \colon C^\circ \to (\mathbb{C}^*)^n\,, \quad g \mapsto (\chi_1(g), \ldots, \chi_n(g))$$

is *regular* in the sense that for each morphism $\nu \colon A \to \mathrm{Aut}(Z)$ with image in C°, the composition $\chi \circ \nu \colon A \to (\mathbb{C}^*)^n$ is a morphism. The morphism $f := (f_1, \ldots, f_n) \colon Z \to \mathbb{C}^n$ is C°-equivariant, when C° acts via $g \cdot (x_1, \ldots, x_n) = (\chi_1(g)x_1, \ldots, \chi_n(g)x_n)$ on $\mathbb{C}^n$. As z_0 is an isolated point of $f^{-1}(0)$, the morphism $f \colon Z \to \overline{f(Z)} =: W$ has finite degree, i.e. the field extension $f^* \colon \mathbb{C}(W) \to \mathbb{C}(Z)$ is finite. Now, the kernel $\ker(\chi)$ acts faithfully on $\mathbb{C}(Z)$ (as C° does) and leaves $\mathbb{C}(W)$ fixed. Thus $\ker(\chi)$ embeds into the finite group $\mathrm{Aut}_{\mathbb{C}(W)}(\mathbb{C}(Z))$ and hence $\ker(\chi)$ is finite. Now, if $\nu \colon A \to C^\circ$ is an injective morphism, then the composition $\chi \circ \nu \colon A \to (\mathbb{C}^*)^n$ has finite fibers and thus $\dim A \leq n$. Hence, by definition $\dim C^\circ \leq n$. As $C^\circ = (C^\circ)^\circ$, we get that C° is an algebraic subgroup of $\mathrm{Aut}(Z)$.

One may now show that C° is an n-dimensional torus in $\mathrm{Aut}(Z)$ and since $\dim Z \leq n$, we get $\dim(Z) = n$. The smoothness of Z implies thus that Z is a toric variety and since $\mathrm{Pic}(Z)$ is finite (and hence trivial), one can show that Z is quasi-affine.

Let me finish this section with the following generalization of Theorem 3.1.1 due to Cantat, Regeta and Xie, which shows that the assumptions 1) and 2) are supperfluous, if one is interested only in affine varieties:

Theorem 3.1.4 (cf. [CRX19, Theorem A]). *Let Y be a connected affine variety such that there exists a group isomorphism $\mathrm{Aut}(Y) \to \mathrm{Aut}(\mathbb{C}^n)$. Then $Y \simeq \mathbb{C}^n$.*

3.2 Characterizing smooth affine spherical varieties via the automorphism group

I will report on the article [5] joint with Andriy Regeta. If X is an affine toric variety different from the torus and if Y is an irreducible normal affine variety such that there is a group isomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ that preserves algebraic group actions, then $X \simeq Y$ by a result due to Liendo, Regeta and Urech [LRU19, Theorem 1.4]. Spherical varieties are natural generalizations of toric varieties. The aim of this paper was, to study how much information of a (quasi)-affine spherical variety X one receives via its automorphism group $\mathrm{Aut}(X)$.

In order to fix notation and to state the main result, let G be a connected reductive algebraic group and $B \subseteq G$ a Borel subgroup. For any algebraic group H, a variety X together with a faithful algebraic H-action is called an *H-variety*. Recall that an irreducible normal G-variety X is called *G-spherical* if B acts with a dense orbit on X. We denote by $\mathfrak{X}(B)$ the *character group* of B, i.e. the group of all regular homomorphisms $B \to \mathbb{C}^*$ and we denote by $\Lambda^+(X)$ the *weight monoid of* X, i.e.

$$\Lambda^+(X) = \left\{ \lambda \in \mathfrak{X}(B) \;\middle|\; \begin{array}{l} \text{there exists a regular function } f\colon X \to \mathbb{C} \text{ such} \\ \text{that } f(b^{-1}x) = \lambda(b)f(x) \text{ for all } b \in B,\, x \in X \end{array} \right\}.$$

Our main result is the following:

Theorem 3.2.1 (cf. Main Theorem A in [5]). *Let X, Y be irreducible normal quasi-affine varieties and let $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ be a group isomorphism that preserves algebraic group actions. If X is G-spherical and not isomorphic to a torus, then the following holds:*

(1) Y is G-spherical for the induced G-action via θ;
(2) the weight monoids $\Lambda^+(X)$ and $\Lambda^+(Y)$ inside $\mathfrak{X}(B)$ are the same;
(3) if X and Y are smooth and affine, then X and Y are isomorphic as G-varieties.

In order to formulate the strategy of the proof, we introduce some terminology. Let H be an algebraic group and let $m \geq 1$. A faithful algebraic group action of the additive group $\mathbb{C}^m$ on an H-variety X is *H-homogeneous of weight* $\lambda \in \mathfrak{X}(H)$ if

$$h \circ \rho(v) \circ h^{-1} = \rho(\lambda(h)v) \quad \text{for all } h \in H \text{ and all } v \in \mathbb{C}^m,$$

where $\rho\colon \mathbb{C}^m \to \mathrm{Aut}(X)$ denotes the group homomorphism induced by the $\mathbb{C}^m$-action on X. This notion will be crucial for the whole proof.

Main steps for the proof of Theorem 3.2.1(1). Assume that H is a connected solvable algebraic group, non-isomorphic to a torus. Then for a quasi-affine irreducible H-variety Z, the following statements are equivalent:

i) H acts with a dense orbit on Z;
ii) There exists a constant C such that for each faithful H-homogeneous $\mathbb{C}^m$-action on Z we have $m \leq C$.

Assume that H acts with a dense orbit on Z and let a faithful H-homogeneous $\mathbb{C}^m$-action on Z be given. From the faithfulness of the $\mathbb{C}^m$-action on Z one can see that the linear map

$$T_0\mathbb{C}^m \to \mathrm{Vec}(Z), \quad w \mapsto \xi_w := (z \mapsto (\mathrm{d}_0\mu_z)w)$$

is injective, where $\mathrm{Vec}(Z)$ denotes the vector space of all sections of the tangent bundle $TZ \to Z$ and $\mu_z\colon \mathbb{C}^m \to Z$, $v \mapsto v \cdot z$ denotes the orbit map associated to z. As the $\mathbb{C}^m$-action is H-homogeneous with respect to a certain weight $\lambda \in \mathfrak{X}(H)$, one may see for all $w \in T_0\mathbb{C}^m$ that

$$(d\varphi_h)\xi_w((\varphi_h)^{-1}(z)) = \lambda(h)\xi_w(z) \quad \text{for all } h \in H \text{ and all } z \in Z, \tag{3.1}$$

where $\varphi_h \colon Z \to Z$ denotes the automorphism $z \mapsto hz$. Since H acts with a dense orbit Hz_0 on Z it follows from (3.1) that ξ_w is completely determined by $\xi_w(z_0)$ for each $w \in T_0\mathbb{C}^m$. Hence the composition

$$T_0\mathbb{C}^m \xrightarrow{w \mapsto \xi_w} \mathrm{Vec}(Z) \xrightarrow{\xi \mapsto \xi(z_0)} T_{z_0}Z$$

is injective and thus we get the estimate $m = \dim T_0\mathbb{C}^m \leq \dim T_{z_0}Z$. This shows $i) \implies ii)$.

Now, assume that H doesn't act with a dense orbit on Z. As Z is quasi-affine, there exist H-semi-invariant regular functions $f_1, f_2 \colon Z \to \mathbb{C}$ of the same weight in $\mathfrak{X}(H)$ with $f_2 \neq 0$ such that $f := f_1/f_2$ is a non-constant rational H-invariant map on Z. Now $p(f_1, f_2) \neq 0$ for all non-zero homogeneous polynomials $p \in \mathbb{C}[T_1, T_2]$, as otherwise the non-constant function f would be algebraic over the algebraically closed field $\mathbb{C}$. As H is not a torus, there exists a one-dimensional unipotent subgroup $U \subseteq H$ that is normalized by H. Denoting by $\rho \colon \mathbb{C} \times Z \to Z$ the corresponding H-homogeneous $\mathbb{C}$-action on Z induced by U, we get that

$$\mathbb{C}^{m+1} \times Z \to Z\,, \quad ((t_0, \ldots, t_m), z) \mapsto \rho\left(\sum_{i=0}^{m} t_i f_1^i(z) f_2^{m-i}(z), z\right)$$

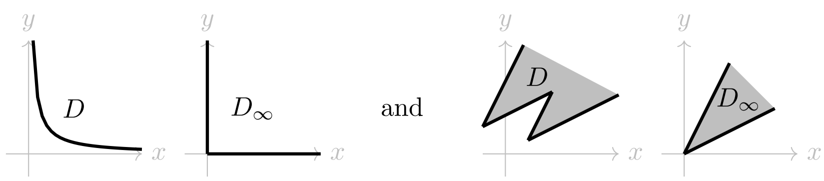
is a faithful H-homogeneous $\mathbb{C}^{m+1}$-action on Z for each $m \geq 1$. This shows $ii) \implies i)$.

If G is not a torus, then B is also not a torus. The above characterization of the existence of a dense B-orbit is in fact preserved under group isomorphisms $\mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ that preserve algebraic group actions. In order to get Theorem 3.2.1(1) we are left with the case when G and B are equal to a torus T. Hence, T acts faithfully on Y and thus $\dim Y \geq \dim T = \dim X$. Since $G = T$, it is enough to show that $\dim Y \leq \dim T$. As X is not a torus, one can show, that there exists a faithful action of a connected solvable group H on X such that H is not a torus and $\dim H = \dim T$. Using again the characterization of the existence of a dense H-orbit above, we get as before, that H acts faithfully with a dense orbit on Y. Thus $\dim Y \leq \dim H = \dim T$.

Main steps for the proof of Theorem 3.2.1(2). To any subset D of a finite dimensional non-zero Euclidean $\mathbb{R}$-vector space V with scalar product $(\cdot,\cdot) \colon V \times V \to \mathbb{R}$ and associated norm $\|\cdot\| \colon V \to \mathbb{R}$ one may associate the so-called *asymptotic cone* in V:

$$D_\infty := \left\{ x \in V \setminus \{0\} \;\middle|\; \begin{array}{l} \text{there exists a sequence } (x_i)_i \text{ in } D \setminus \{0\} \text{ with} \\ \|x_i\| \to \infty \text{ such that } x_i/\|x_i\| \to x/\|x\| \end{array} \right\} \cup \{0\}\,.$$

The asymptotic cone is indeed a cone, i.e. for each $x \in D_\infty$ we have $tx \in D_\infty$ for all real $t \geq 0$ and D_∞ is non-empty. The following picture illustrates two examples in the Euclidean plane $\mathbb{R}^2$:



The following subset of the character group $\mathfrak{X}(B)$ plays a prominent rôle for the proof

$$D(X) := \left\{ \lambda \in \mathfrak{X}(B) \;\middle|\; \begin{array}{l} \text{there exists a faithful } B\text{-homogeneous} \\ \mathbb{C}\text{-action of weigth } \lambda \text{ on } X \end{array} . \right\}$$

The set $D(X)$ is contained in the lattice $\mathfrak{X}(B)$ of the $\mathbb{R}$-vector space $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$ and we fix once and for all a scalar product on the $\mathbb{R}$-vector space $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$.

Since there is a group isomorphism $\operatorname{Aut}(X) \to \operatorname{Aut}(Y)$ that preserves algebraic group actions, it follows that $D(X) = D(Y)$. Now, if G is not a torus, then the set $D(X)$ (the set $D(Y)$) determines the weight monoid $\Lambda^+(X)$ (the weight monoid $\Lambda^+(Y)$) and thus we get $\Lambda^+(X) = \Lambda^+(Y)$:

Theorem 3.2.2 (cf. Main Theorem B in [5])**.** *Assume that X is a quasi-affine G-spherical variety that is non-isomorphic to a torus. If G is not a torus, then*

$$\Lambda^+(X) = \operatorname{Conv}(D(X)_\infty) \cap \operatorname{Span}_{\mathbb{Z}}(D(X)),$$

where the asymptotic cone and the linear span are taken inside $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$.

If G is a torus and X, Y are affine, then Theorem 3.2.1(2) may be retrieved from [LRU19, Theorem 1.4]. However, in case G is a torus and $\operatorname{Spec}(\mathcal{O}(X)) \not\simeq \mathbb{C} \times (\mathbb{C} \setminus \{0\})^{\dim X - 1}$, the above formula in Theorem 3.2.2 still holds. This can be retrieved with similar methods as in the case when G is not a torus. The case, when G is a torus and $\operatorname{Spec}(\mathcal{O}(X)) \simeq \mathbb{C} \times (\mathbb{C} \setminus \{0\})^{\dim X - 1}$ can be done separately. However we will not consider these two special cases here and illustrate the methods in the case when G is not a torus.

The main steps for the proof of Theorem 3.2.2 are the following. Let $T \subseteq B$ be a maximal torus and let $U \subseteq B$ be the unipotent radical of B. There is a natural B-action on the vector space $\operatorname{Vec}(X)$ given by

$$B \times \operatorname{Vec}(X) \to \operatorname{Vec}(X), \quad (b, \xi) \mapsto \big(x \mapsto (d\varphi_b)\xi((\varphi_b)^{-1}(x))\big) \quad ,$$

where $\varphi_b \colon X \to X$ denotes the automorphism $x \mapsto bx$. This action turns $\operatorname{Vec}(X)$ into a B-module. Now, the fixed points $\operatorname{Vec}^U(X)$ under the subgroup U of B form a B-submodule of $\operatorname{Vec}(X)$. Moreover, $\operatorname{Vec}^U(X)$ has a natural structure of an $\mathcal{O}(X)^U$-module, where $\mathcal{O}(X)^U$ denotes the subring of U-invariants of $\mathcal{O}(X)$. It turns out, that $\operatorname{Vec}^U(X)$ is a finitely generated $\mathcal{O}(X)^U$-module (by using the so-called transfer principle; see Corollary 4.8 in [5]) and thus we have a surjective $\mathcal{O}^U(X)$-module homomorphism

$$\pi \colon \bigoplus_{i=1}^{k} \mathcal{O}^U(X)\xi_i \to \operatorname{Vec}^U(X), \quad (f_1\xi_1, \ldots, f_k\,\xi_k) \mapsto f_1\xi_1 + \ldots + f_k\xi_k$$

for finitely many B-homogeneous $\xi_1, \ldots, \xi_k \in \operatorname{Vec}^U(X)$ of weights $\lambda_1, \ldots \lambda_k \in \mathfrak{X}(B)$. Moreover π is a B-module homomorphism. Now, if $\lambda \in \mathfrak{X}(B)$ is the weight of a B-homogeneous $\xi \in \operatorname{Vec}^U(X)$, then one may see that there is a B-homogeneous $\eta \in \bigoplus_{i=1}^k \mathcal{O}^U(X)\xi_i$ such that $\pi(\eta) = \xi$ and thus λ is the weight of η. Hence we have the following inclusion of subsets in $\mathfrak{X}(B)$

$$D'(X) := \left\{ \lambda \in \mathfrak{X}(B) \;\middle|\; \begin{array}{l} \text{there exists a } B\text{-homo-} \\ \text{geneous vector field} \\ \text{of weight } \lambda \text{ in } \operatorname{Vec}^U(X) \end{array} \right\} \subseteq \bigcup_{i=1}^{k} (\lambda_i + \Lambda^+(X)),$$

as $\Lambda^+(X)$ are precisely the B-weights of $\mathcal{O}^U(X)$. There exists a faithful B-homogeneous $\mathbb{C}$-action $\rho\colon \mathbb{C} \times X \to X$ induced from a certain one dimensional unipotent subgroup of the center of U (here we use the fact that U is non-trivial, which is implied by the fact that G is not a torus). Let $\lambda \in \mathfrak{X}(B)$ be the weight of ρ. Then for each B-semi-invariant $r \in \mathcal{O}^U(X)$ of weight $\lambda' \in \mathfrak{X}(B)$, the morphism $\mathbb{C} \times X \to X$, $(t, x) \mapsto \rho(r(x)t, x)$ is a faithful B-homogeneous $\mathbb{C}$-action on X of weight $\lambda + \lambda'$ and thus $\lambda + \lambda' \in D(X)$. This shows

$$\lambda + \Lambda^+(X) \subseteq D(X) \subseteq D'(X) \subseteq \bigcup_{i=1}^{k} (\lambda_i + \Lambda^+(X)) .$$

Taking asymptotic cones in $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$ yields

$$\Lambda^+(X)_\infty = D(X)_\infty .$$

Now, a certain quotient torus T' of the torus T acts faithfully on $\operatorname{Spec}(\mathcal{O}^U(X))$ and turns it into an affine toric variety (by the so-called transfer principle; see Proposition 4.6 in [5]). Note that there is a natural embedding $\mathfrak{X}(T') \subseteq \mathfrak{X}(T) = \mathfrak{X}(B)$. As $\operatorname{Spec}(\mathcal{O}^U(X))$ is an affine T'-toric variety, one may see that the convex cone generated by $\Lambda^+(X)$ in $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$ satisfies

$$\Lambda^+(X) = \operatorname{Conv}(\Lambda^+(X)) \cap \mathfrak{X}(T')$$

Furthermore, one may see that $\operatorname{Span}_{\mathbb{Z}}(D(X)) = \mathfrak{X}(T')$. Hence, in total

$$\begin{aligned} \Lambda^+(X) &= \operatorname{Conv}(\Lambda^+(X)) \cap \mathfrak{X}(T') \\ &= \operatorname{Conv}(\Lambda^+(X)_\infty) \cap \operatorname{Span}_{\mathbb{Z}}(D(X)) \\ &= \operatorname{Conv}(D(X)_\infty) \cap \operatorname{Span}_{\mathbb{Z}}(D(X)) , \end{aligned}$$

where the second equality follows from the fact, that $\Lambda^+(X)_\infty$ is equal to the convex cone spanned by $\Lambda^+(X)$ inside $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$.

The proof of Theorem 3.2.1(3) The statement is a direct consequence of Theorem 3.2.1(1), (2), as for smooth affine G-spherical varieties X, the weight monoid $\Lambda^+(X)$ determines the G-variety X due to a beautiful result of Losev [Los09, Theorem 1.3] (which confirmed Knop's conjecture).

3.3 Dynamical degrees of affine-triangular automorphisms of affine spaces

I will report in this section on the joint article with Jérémy Blanc, [3]. As an exception, we formulate in this section all results over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic. Although all results would work over an arbitrary field, we stick for simplicity to the assumption that the ground field is algebraically closed.

If $f_1, \ldots, f_n \in \mathbf{k}[x_1, \ldots, x_n]$ are polynomials, we write the endomorphism

$$\mathbf{k}^n \to \mathbf{k}^n , \quad x \mapsto (f_1(x), \ldots, f_n(x))$$

shortly by $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbf{k}^n)$ and we call $f_1, \ldots, f_n$ the *coordinate functions of* f or *components of* f. Moreover, we define the degree of an endomorphism $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbf{k}^n)$ by

$$\deg(f) = \max_{1 \le i \le n} \deg(f_i)$$

where $\deg(f_i)$ is the maximum of all the numbers $\sum_{i=1}^n a_i$ where $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$ runs over all n-tuples such that the coefficient of the monomial $x_1^{a_1} \cdots x_n^{a_n}$ in the polynomial f_i is non-zero.

The aim of this article was to study the dynamical degree $\lambda(f)$ of automorphisms $f \in \mathrm{Aut}(\mathbf{k}^n)$, where

$$\lambda(f) \coloneqq \lim_{i \to \infty} \left(\deg(f^i)\right)^{\frac{1}{i}} \in \mathbb{R} \quad \text{and} \quad f^i \coloneqq \underbrace{f \circ \ldots \circ f}_{i \text{ times}}$$

and more precisely to give the possible numbers in $\mathbb{R}$ that are dynamical degrees of automorphisms of $\mathbf{k}^n$. A very interesting feature of the dynamical degree is the fact, that it is invariant under conjugation of automorphisms of $\mathbf{k}^n$ (and even birational maps of $\mathbf{k}^n$). In case $n = 1$, the dynamical degree is always one and for $n = 2$, the dynamical degree is an integer (which can be deduced from a famous Theorem due to Jung [Jun42] and van der Kulk [vdK53] that describes $\mathrm{Aut}(\mathbf{k}^2)$ as a certain amalgamated product). Hence, the question starts to be interesting for $n \ge 3$ and already in this case the question is difficult.

One of the starting points of our work was the following unpublished result due to Jonsson:

Theorem 3.3.1 (Jonsson (unpublished))**.** *For each $n \ge 2$ and each polynomial $p \in \boldsymbol{k}[x_1, \ldots, x_{n-1}]$ of degree ≥ 2, let $f \in \mathrm{Aut}(\boldsymbol{k}^n)$ be the automorphism*

$$f = (x_n + p(x_1, \ldots, x_{n-1}), x_1, \ldots, x_{n-1}) \in \mathrm{Aut}(\boldsymbol{k}^n)\,.$$

Let $I \subset \mathbb{N}_0^{n-1}$ be the finite subset of indices of the monomials of p. Then

$$\lambda(f) = \max \left\{ \lambda \in \mathbb{R} \;\middle|\; \lambda^{n-1} = \sum_{j=1}^{n-1} i_j \lambda^{n-1-j} \quad \textit{for some } (i_1, \ldots, i_{n-1}) \in I \right\}.$$

The automorphisms f in Theorem 3.3.1 are special cases of so-called *affine-triangular automorphisms* of $\mathbf{k}^n$. These are automorphisms of $\mathbf{k}^n$ of the form $\alpha \circ \tau$, where α is an *affine automorphism* of $\mathbf{k}^n$, i.e. $\alpha \in \mathrm{Aut}(\mathbf{k}^n)$ and $\deg(\alpha) = 1$, and $\tau = (\tau_1, \ldots, \tau_n)$ is a *triangular automorphism* of $\mathbf{k}^n$, i.e. $\tau \in \mathrm{Aut}(\mathbf{k}^n)$ and $\tau_i \in \mathbf{k}[x_1, \ldots, x_i]$ for all $1 \le i \le n$.

Our main result concerning dynamical degrees of affine-triangular automorphisms is the following:

Theorem 3.3.2 (cf. Theorem 1 in [3])**.** *For each integer $d \ge 2$, the set of dynamical degrees of all affine-triangular automorphisms of $\boldsymbol{k}^3$ of degree $\le d$ is equal to*

$$\left\{ \frac{a + \sqrt{a^2 + 4bc}}{2} \;\middle|\; (a, b, c) \in \mathbb{N}_0^3,\ a + b \le d,\ c \le d \right\} \setminus \{0\}\,.$$

In order to write down the strategy of the proof, let me explain the technique we used to compute dynamical degrees. For any $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$, the *$\mu$-degree* of a polynomial $p \in \mathbf{k}[x_1, \dots, x_n]$ is defined by

$$\deg_\mu(p) = \max \left\{ \sum_{i=1}^n a_i \mu_i \;\middle|\; \begin{array}{l} \text{the coefficient of the monomial} \\ x_1^{a_1} \cdots x_n^{a_n} \text{ in } p \text{ is non-zero} \end{array} \right\}.$$

Note that for $\mu = (1, \dots, 1)$ we get back the classical degree. Moreover, we call a polynomial $p \in \mathbf{k}[x_1, \dots, x_n]$ *μ-homogeneous of degree ϑ* if p is a finite sum of monomials of μ-degree equal to ϑ. Thus we may write each $p \in \mathbf{k}[x_1, \dots, x_n]$ uniquely as

$$p = \sum_{\vartheta \in \mathbb{R}_{\geq 0}} p_\vartheta$$

where p_ϑ is μ-homogeneous of degree ϑ and only finitely many of the p_ϑ are non-zero. If $p \neq 0$, then the element $p_{\deg_\mu(p)}$ is called the *μ-leading part* of p.

For any $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$, the *$\mu$-degree* of an endomorphism $f = (f_1, \dots, f_n) \in \operatorname{End}(\mathbf{k}^n)$ is defined by

$$\deg_\mu(f) = \inf\{\, \vartheta \in \mathbb{R} \mid \deg_\mu(f_i) \leq \vartheta \mu_i \text{ for all } 1 \leq i \leq n \,\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

We call an endomorphism $f \in \operatorname{End}(\mathbf{k}^n)$ *μ-algebraically stable* if $\deg_\mu(f) < \infty$ and $\deg_\mu(f^i) = (\deg_\mu(f))^i$ for all $i \geq 1$. If all the components $f_1, \dots, f_n$ of f are non-zero, we define the *μ-leading part of f* to be the endomorphism $g = (g_1, \dots, g_n) \in \operatorname{End}(\mathbf{k}^n)$, where g_i is the μ-leading part of f_i for all $1 \leq i \leq n$.

We may associate to an endomorphism $f \in \operatorname{End}(\mathbf{k}^n)$ in a natural manner certain square matrices and we have then a notion of a maximal eigenvector and maximal eigenvalue of f with respect to these square matrices. These concepts turn out to be very fruitful in order to compute dynamical degrees.

Definition 3.3.3. Let $f = (f_1, \dots, f_n) \in \operatorname{End}(\mathbf{k}^n)$ such that $f_i \neq 0$ for all $1 \leq i \leq n$. We say that an $n \times n$ square matrix $M = (m_{ij})$ with coefficients in $\mathbb{N}_0$ is contained in f, if for each i, the coefficient of the monomial $x_1^{m_{i1}} \cdots x_n^{m_{in}}$ in f_i is non-zero. The *maximal eigenvalue of f* is then defined by

$$\theta_f \coloneqq \{\, |\xi| \in \mathbb{R}_{\geq 0} \mid \xi \text{ is an eigenvalue of a matrix that is contained in } f \,\}.$$

Moreover, we say that a non-zero $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ is a *maximal eigenvector of f* if

$$\deg_\mu(f_i) = \theta_f \mu_i \quad \text{for all } 1 \leq i \leq n.$$

Now, we may state our main result in order to compute dynamical degrees of endomorphisms of $\mathbf{k}^n$:

Proposition 3.3.4 (cf. Proposition B in [3]). *Let $f = (f_1, \dots, f_n) \in \operatorname{End}(\boldsymbol{k}^n)$ be a dominant endomorphism. Then the following holds:*

(1) There exists a maximal eigenvector of f.

(2) For all maximal eigenvectors $\mu = (\mu_1, \dots, \mu_n)$ of f we have $\theta_f = \deg_\mu(f)$ and the following statements hold:

(i) If f is μ-algebraically stable, then $\lambda(f) = \theta_f$.

(ii) Assume that $\theta_f > 1$ and denote by $g \in \mathrm{End}(\boldsymbol{k}^n)$ the μ-leading part of f. Then f is μ-algebraically stable, if and only if for each $r \geq 1$ there is $1 \leq i \leq n$ (depending on r) such that $\mu_i > 0$ and the i-th component of g^r is non-zero.

The main bulk of the proof of Theorem 3.3.2 lies in the proof of the following technical lemma, whose proof is heavily based on Proposition 3.3.4(2)(ii):

Lemma 3.3.5 (cf. Lemma 4.3.3 in [3]). *Let $f = \sigma \circ \tau \in \mathrm{Aut}(\boldsymbol{k}^3)$, where $\sigma \in \mathrm{Aut}(\boldsymbol{k}^3)$ is a permutation of the coordinates and $\tau \in \mathrm{Aut}(\boldsymbol{k}^3)$ is triangular. Suppose that the maximal eigenvalue $\theta := \theta_f$ is bigger than 1 and let μ be a maximal eigenvector of f such that f is not μ-algebraically stable. Then, one of the following cases holds:*

(i) $f = (\xi_3 x_3 + p_3(x_1, x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1))$ where ξ_2, $\xi_3 \in \boldsymbol{k}^*$, $p_1, p_2 \in \boldsymbol{k}[x_1]$, $p_3 \in \boldsymbol{k}[x_1, x_2]$, $\deg(p_1) = 1$, and $\deg(p_2) = \theta^2 > 1$. Moreover, there exists $s \in \boldsymbol{k}[x_2]$ such that the conjugation of f by $(x_1, x_2, x_3 + s(x_2))$ does not increase the degree of p_3 and (strictly) decreases the degree of p_2;

(ii) $f = (\xi_2 x_2 + p_2(x_1), \xi_3 x_3 + p_3(x_1, x_2), p_1(x_1))$ where ξ_2, $\xi_3 \in \boldsymbol{k}^*$, $p_1, p_2 \in \boldsymbol{k}[x_1]$, $p_3 \in \boldsymbol{k}[x_1, x_2]$, $\deg(p_1) = 1$, and $\deg(p_2) = \theta > 1$. Moreover, there exists $s \in \boldsymbol{k}[x_1]$ such that the conjugation of f by $(x_1, x_2 + s(x_1), x_3)$ (strictly) decreases the degrees of p_2 and p_3.

Now, using this lemma, I will explain how Theorem 3.3.2 can be deduced. Assume that $f = \alpha \circ \nu \in \mathrm{Aut}(\mathbf{k}^3)$ where α is affine and ν is triangular, and denote by $d = \deg(f)$ the degree of f. We may even assume that α is linear (as the translation part of α can be composed with ν and hence this composition is triangular). Then, there exist linear triangular automorphisms $\beta, \gamma \in \mathrm{Aut}(\mathbf{k}^3)$ and an automorphism $\sigma \in \mathrm{Aut}(\mathbf{k}^3)$ that permutes the coordinates such that $\alpha = \beta \circ \sigma \circ \gamma$ by the Bruhat decomposition. For the triangular automorphism $\tau = \gamma \circ \nu \circ \beta \in \mathrm{Aut}(\mathbf{k}^3)$ we have now

$$f = \alpha \circ \nu = \beta \circ \sigma \circ \gamma \circ \gamma^{-1} \circ \tau \circ \beta^{-1} = \beta \circ \sigma \circ \tau \circ \beta^{-1} .$$

As dynamical degrees are invariant under conjugation and as β is linear, we may assume $f = \sigma \circ \tau$ (the degree doesn't change). By Proposition 3.3.4(1), there exists a maximal eigenvector $\mu \in (\mathbb{R}_{\geq 0})^n$ of f. Now, we may apply Lemma 3.3.5 finitely many times and thus we may assume that either the maximal eigenvalue θ_f is one or f is μ-algebraically stable (after each application the sum of the degrees of the components of f decreases). If $\theta_f > 1$, then f is μ-algebraically stable and thus Proposition 3.3.4(2)(i) implies that $\lambda(f) = \theta_f$. However, a direct computation (by inspecting all the matrices contained in f) shows that there are $(a, b, c) \in \mathbb{N}_0^3$ with $a + b \leq d$, $c \leq d$ such that

$$\theta_f = \frac{a + \sqrt{a^2 + 4bc}}{2} \neq 0 .$$

On the other hand, let $d \geq 1$ be given, and let $(a, b, c) \in \mathbb{N}_0^3$ such that $a + b \leq d$, $c \leq d$ and $(a + \sqrt{a^2 + 4bc})/2 > 1$. Consider the affine-triangular automorphism

$$g = (x_3 + x_1^a x_2^b, x_2 + x_1^c, x_1) \in \mathrm{Aut}(\mathbf{k}^3) .$$

Then $\deg(g) \leq d$. Now, g is the composition of an automorphism of $\mathbf{k}^3$ that permutes the coordinates and a triangular automorphism of $\mathbf{k}^3$. A direct computation shows that the maximal eigenvalue θ_g of g is equal to $(a + \sqrt{a^2+4bc})/2 > 1$. As g is not of the form given in the two cases in Lemma 3.3.5, it follows that g is μ-algebraically stable for each maximal eigenvector μ of g. By Proposition 3.3.4(1), there exists a maximal eigenvector $\mu \in \mathbb{R}^n_{\geq 0}$ of f. Now, Proposition 3.3.4 (2)(i) shows that the dynamical degree $\lambda(g)$ is equal to $(a+\sqrt{a^2+4bc})/2$. This gives Theorem 3.3.2.

Let me also mention the following generalization of the unpublished result due to Jonsson (Theorem 3.3.1). For this, recall that a positive real number is called a *Handelman number* if it is the root of a monic integral polynomial $T^d + \sum_{i=0}^{d-1} c_i T^i$ where $c_i \leq 0$ for all $0 \leq i \leq d-1$. In particular, the dynamical degrees appearing in Theorem 3.3.1 are Handelman numbers.

Proposition 3.3.6 (cf. Proposition C in [3])**.** *Let $f \in \mathrm{Aut}(\boldsymbol{k}^n)$ be an automorphism of the form $f = \sigma \circ e$, where $\sigma \in \mathrm{Aut}(\boldsymbol{k}^n)$ is a permutation of the coordinates and e is an automorphism of the form*

$$e = (x_1, \ldots, x_{n-1}, x_n + p(x_1, \ldots, x_{n-1})) \in \mathrm{Aut}(\boldsymbol{k}^n)$$

where $p \in \boldsymbol{k}[x_1, \ldots, x_{n-1}]$ is any polynomial. If the maximal eigenvalue θ_f of f is bigger than 1, then there exists a maximal eigenvector μ of f such that f is μ-algebraically stable. In particular, the dynamical degree $\lambda(f)$ is equal to the maximal eigenvalue θ_f. This maximal eigenvalue θ_f is a Handelman number.

Let me finish this section with the following recent general result due to Dang and Favre concerning dynamical degrees of automorphisms of $\mathbf{k}^3$:

Theorem 3.3.7 ([DF21, Corollary 3])**.** *Dynamical degrees of polynomial automorphisms of $\boldsymbol{k}^3$ are algebraic numbers whose degree over the field of rational numbers $\mathbb{Q}$ is at most 6.*

3.4 Automorphisms of the affine 3-space of degree 3

In this section, I report on the joint article with Jérémy Blanc [8]. As an exception, we formulate in this section all results over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic.

We use the classical degree for polynomials $\mathbf{k}[x_1, \ldots, x_n]$ and automorphisms of $\mathbf{k}^n$, introduced in the last section. Moreover, we also identify each morphism $f \colon \mathbf{k}^n \to \mathbf{k}^m$ with its m-tuple of coordinate functions $(f_1, \ldots, f_m)$ and write then shortly $f = (f_1, \ldots, f_m)$ (analogous to the last section).

Two prominent subgroups of $\mathrm{Aut}(\mathbf{k}^n)$ are the so-called *affine automorphisms*

$$\mathrm{Aff}(\mathbf{k}^n) := \{\, \alpha \in \mathrm{Aut}(\mathbf{k}^n) \mid \deg(\alpha) = 1 \,\}$$

and the so-called *triangular automorphisms*

$$\mathrm{Triang}(\mathbf{k}^n) := \left\{ \tau = (\tau_1, \ldots, \tau_n) \in \mathrm{Aut}(\mathbf{k}^n) \;\middle|\; \begin{array}{l} \tau_i \in \mathbf{k}[x_1, \ldots, x_i] \\ \text{for all } 1 \leq i \leq n \end{array} \right\}.$$

The group generated by $\mathrm{Aff}(\mathbf{k}^n)$ and $\mathrm{Triang}(\mathbf{k}^n)$ inside the automorphism group $\mathrm{Aut}(\mathbf{k}^n)$ is called the subgroup of *tame automorphisms* and we denote it by $\mathrm{Tame}(\mathbf{k}^n)$. In case $n = 1$, we have $\mathrm{Tame}(\mathbf{k}) = \mathrm{Aut}(\mathbf{k})$ and by a famous result due to Jung and van der Kulk [Jun42, vdK53] we have also $\mathrm{Tame}(\mathbf{k}^2) = \mathrm{Aut}(\mathbf{k}^2)$. For a long time it was conjectured that the so-called Nagata automorphism

$$(x - 2y(zx + y^2) - z(zx + y^2)^2, y + z(zx + y^2), z) \in \mathrm{Aut}(\mathbf{k}^3)$$

is not tame and eventually this conjecture was proven by Shestakov and Umirbaev in a landmark paper [SU04] in case the characteristic of $\mathbf{k}$ is zero. Note that the degree of the Nagata automorphism is 5. The least degree of a non-tame automorphism (until now) in $\mathbf{k}^3$ (if $\mathrm{char}(\mathbf{k}) = 0$) is also 5. Amongst other things, we proved that all automorphisms of degree 3 of $\mathbf{k}^3$ are tame (see Theorem 3.4.1). It is still an open problem, whether all automorphisms of degree 4 in $\mathrm{Aut}(\mathbf{k}^3)$ are tame.

For stating our main result, we introduce the following equivalence relation on $\mathrm{Aut}(\mathbb{C}^n)$: Two automorphisms $f, g \in \mathrm{Aut}(\mathbb{C}^n)$ are called *equivalent*, if there exist $\alpha, \beta \in \mathrm{Aff}(\mathbf{k}^n)$ such that $g = \alpha \circ f \circ \beta$.

Theorem 3.4.1 (cf. Theorem 1 in [4]). *Each automorphism of $\boldsymbol{k}^3$ of degree ≤ 3 is either equivalent to a triangular automorphism or to an automorphism of the form*

$$(x + yz + za(x, z), y + a(x, z) + r(z), z) \in \mathrm{Aut}(\boldsymbol{k}^3)\,, \tag{3.2}$$

where $a \in \boldsymbol{k}[x, z] \setminus \boldsymbol{k}[z]$ is homogeneous of degree 2 and $r \in \boldsymbol{k}[z]$ is of degree ≤ 3.

Using Theorem 3.4.1 we also calculated the set of dyanmical degrees of all automorphisms of degree ≤ 3 of $\mathbf{k}^3$; see the last section for the definition of the dynamical degree of an automorphism of the affine space. In fact, the dynamical degrees of all automorphisms of $\mathbb{C}^3$ of degree 2 were calculated by Maegawa [Mae01, Theorem 3.1] and the list is given by $\{1, \sqrt{2}, (1 + \sqrt{5})/2, 2\}$. This list stays the same over $\mathbf{k}$ and in fact we came up with the following result:

Theorem 3.4.2 (cf. Theorem 2 in [4]). *We denote by Λ_d the set of all dynamical degrees of all automorphisms of $\boldsymbol{k}^3$ of degree d. Then we have:*

$$\begin{aligned}
\Lambda_1 &= \{1\} \\
\Lambda_2 &= \{1, \sqrt{2}, \tfrac{1+\sqrt{5}}{2}, 2\} \\
\Lambda_3 &= \{1, \sqrt{2}, \tfrac{1+\sqrt{5}}{2}, \sqrt{3}, 2, \tfrac{1+\sqrt{13}}{2}, 1 + \sqrt{2}, \sqrt{6}, \tfrac{1+\sqrt{17}}{2}, \tfrac{3+\sqrt{5}}{2}, 1 + \sqrt{3}, 3\}\,.
\end{aligned}$$

In fact, if $f \in \mathrm{Aut}(\mathbf{k}^3)$ and if $\deg(f) \leq 3$, then either f is equivalent to a triangular automorphism or to an automorphism of the from (3.2) by Theorem 3.4.1. In the first case, f is conjugated (via an affine automorphism) to an affine-triangular automorphism (i.e. a composition of an affine automorphism with a triangular automorphism) and since the dynamical degree is invariant under conjugation, this case is covered by the general result Theorem 3.3.2 from the last section. In particular this applies to all automorphisms of degree ≤ 2 of $\mathbf{k}^3$ (which follows again from Theorem 3.4.1, as equivalent automorphisms have the same degree). The main bulk of the proof of Theorem 3.4.2 lies in studying the case when f is equal to $\alpha \circ g$, where α is affine and g is an automorphism of the form (3.2). We proceeded then the computation of the dynamical degrees via the method from Proposition 3.3.4 .

In fact, by inspecting Theorem 3.3.2, one can see, that $\frac{3+\sqrt{5}}{2}$ is the only dynamical degree in Λ_3 that doesn't appear in the list of all dynamical degrees of affine-triangular automorphisms (of arbitrary degree) of $\mathbf{k}^3$. One may see (again using Proposition 3.3.4) that the automorphism

$$f = (y + xz, z, x + z(y + xz)) \in \mathrm{Aut}(\mathbf{k}^3)$$

from the last section has dynamical degree $\frac{3+\sqrt{5}}{2}$ (note that f is an automorphism of the form (3.2), where $a = xz$ and $r = 0$) composed with a cyclic permutation of the coordinates). As a consequence, we get that f cannot be conjugate via any automorphism of $\mathbf{k}^3$ (or even via any birational map of $\mathbf{k}^3$) to an affine-triangular automorphism of $\mathbf{k}^3$.

Let me explain, the main strategy of the proof ot Theorem 3.4.1. In fact it turned out that the following generalization of automorphisms was very fruitful in order to classify automorphisms of degree ≤ 3 of $\mathbf{k}^3$. We call a morphism

$$f\colon\ \mathbf{k}^d \to \mathbf{k}^n$$

an *affine linear system of affine spaces*, if for each affine hyperplane H in $\mathbf{k}^n$ the preimage $f^{-1}(H)$ is isomorphic to $\mathbf{k}^{d-1}$. This property is satisfied for automorphisms of $\mathbf{k}^n$ and it is preserved under compositions by affine automorphisms at the source and target. Moreover, the dimension of the target has to be smaller than the dimension of the source. If two affine linear systems of affine spaces $\mathbf{k}^d \to \mathbf{k}^n$ are the same up to composition with affine automorphisms at the source and target, we call them *equivalent* (which generalizes the notion introduced for automorphisms). Note that for each surjective affine linear map $\pi\colon\ \mathbf{k}^n \to \mathbf{k}^{n'}$ and for each affine linear system of affine spaces $f\colon\ \mathbf{k}^d \to \mathbf{k}^n$ the composition $\pi \circ f\colon\ \mathbf{k}^d \to \mathbf{k}^{n'}$ is again an affine linear system of affine spaces.

We proved in fact a generalization of Theorem 3.4.1. In order to formulate it, recall that *variables* are polynomials in $\mathbf{k}[x_1, \ldots, x_n]$ that are components of automorphisms of $\mathbf{k}^n$, that a ***k**-fibration* is a surjective morphism $f\colon X \to Y$ such that each fiber is (schematically) isomorphic to $\mathbf{k}$ and that a $\mathbf{k}$-fibration $f\colon X \to Y$ is called *trivial* if there exists an isomorphism $\varphi\colon Y \times \mathbf{k} \to X$ such that the composition $f \circ \varphi\colon Y \times \mathbf{k} \to Y$ is the projection onto the first factor.

Theorem 3.4.3 (cf. Theorem 3 in [4]). *Every affine linear system of affine spaces $\boldsymbol{k}^3 \to \boldsymbol{k}^n$ of degree ≤ 3 is equivalent to an element of the following eleven families. Case* I) *corresponds to $n = 1$, Cases* IIa) *and* IIb) *correspond to $n = 2$ and Case* III) *corresponds to $n = 3$.*

I) *variables of $\boldsymbol{k}[x, y, z]$:*

(1) $x + r_2(y, z) + r_3(y, z)$, *where $r_i \in \boldsymbol{k}[y, z]$ is homogeneous of degree i;*
(2) $xy + yr_2(y, z) + z$, *where $r_2 \in \boldsymbol{k}[y, z] \setminus \boldsymbol{k}[y]$ is homogeneous of degree 2;*
(3) $xy^2 + y(z^2 + az + b) + z$, *where $a, b \in \boldsymbol{k}$.*

IIa) *trivial **k**-fibrations:*

(4) $(x + p_2(y, z) + p_3(y, z), y + q_2z^2 + q_3z^3)$, *where $p_i \in \boldsymbol{k}[y, z]$ is homogeneous of degree i and $q_2, q_3 \in \boldsymbol{k}$;*

(5) $(yz + za_2(x,z) + x, y + a_2(x,z) + r_1 z + r_2 z^2 + r_3 z^3)$, *where* $a_2 \in \boldsymbol{k}[x,z] \setminus \boldsymbol{k}[z]$ *is homogeneous of degree* 2 *and* $r_i \in \boldsymbol{k}$;

(6) $(yz + za_2(x,z) + x, z)$, *where* $a_2 \in \boldsymbol{k}[x,z] \setminus \boldsymbol{k}[z]$ *is homogeneous of degree* 2;

(7) $(xy^2 + y(z^2 + az + b) + z, y)$, *where* $a, b \in \boldsymbol{k}$.

IIb) *non-trivial* $\boldsymbol{k}$*-fibrations:*

(8) $(x + z^2 + y^3, y + x^2)$, *where the characteristic of* $\boldsymbol{k}$ *is* 2;

(9) $(x + z^2 + y^3, z + x^3)$, *where the characteristic of* $\boldsymbol{k}$ *is* 3.

III) *automorphisms of* $\boldsymbol{k}^3$:

(10) $(x + p_2(y,z) + p_3(y,z), y + q_2 z^2 + q_3 z^3, z)$, *where* $p_i \in \boldsymbol{k}[y,z]$ *is homogeneous of degree* i *and* $q_2, q_3 \in \boldsymbol{k}$;

(11) $(yz + za_2(x,z) + x, y + a_2(x,z) + r_2 z^2 + r_3 z^3, z)$, *where* $a_2 \in \boldsymbol{k}[x,z] \setminus \boldsymbol{k}[z]$ *is homogeneous of degree* 2 *and* $r_2, r_3 \in \boldsymbol{k}$.

Let me give the main steps for the proof of Theorem 3.4.3. We call an affine linear system of affine spaces $f\colon \mathbf{k}^3 \to \mathbf{k}^n$ to be in *standard form* if there exist polynomials $p_1, \ldots, p_n \in \mathbf{k}[y]$ and $q_1, \ldots, q_n \in \mathbf{k}[y,z]$ such that

$$f = (xp_1(y) + q_1(y,z), \ldots, xp_n(y) + q_n(y,z))\colon \mathbf{k}^3 \to \mathbf{k}^n \ .$$

The first step was, to consider the affine linear systems of affine spaces $\mathbf{k}^3 \to \mathbf{k}$ of degree ≤ 3. Using a certain result about variables in $\mathbf{k}[x,y,z]$ due to Russell [Rus76, Theorem 2.3] we were able to show that they are always equivalent to an affine linear system of affine spaces in standard form and that they are even equivalent to the systems in Case I) of Theorem 3.4.3. In a second step we studied affine linear systems of affine spaces $\mathbf{k}^3 \to \mathbf{k}^n$ of degree ≤ 3 such that the homogenous parts of degree 3 of the components are all divisible by the same homogeneous polynomial of degree 2. In geometric terms, this means that the extension $\mathbb{P}^3 \dashrightarrow \mathbb{P}^n$ of $\mathbf{k}^3 \to \mathbf{k}^n$ has a conic in the base locus. It turned out that all these are equivalent to affine linear systems affine spaces in standard form. In a third step, we studied the affine linear systems of affine spaces $f\colon \mathbf{k}^3 \to \mathbf{k}^2$ of degree ≤ 3 and we showed that either f is equivalent to an affine linear system of affine spaces in standard form or f is equivalent to an affine linear system of affine spaces in Case IIb) of Theorem 3.4.3. This reduced then our study to the case of affine linear systems of affine spaces $\mathbf{k}^3 \to \mathbf{k}^n$ of degree ≤ 3 in standard form for $n = 2, 3$. This study then gave the Cases IIa) and III) in Theorem 3.4.3.

Let me finish this section by relating Theorem 3.4.3 to the Jacobian conjecture. One can in fact prove that the following implications hold for all $f \in \mathrm{End}(\mathbf{k}^n)$:

$$f \in \mathrm{Aut}(\mathbf{k}^n) \implies \begin{array}{l} f \text{ is an affine linear system} \\ \text{of affine spaces} \end{array} \implies \det \mathrm{Jac}(f) \in \mathbf{k}^* \, .$$

The Jacobian conjecture says that all the implications above are equivalences if the characteristic of $\mathbf{k}$ is zero. In case the characteristic of $\mathbf{k}$ is a prime p, then the second implication is wrong, as can be seen for example by the endomorphism $f = (x_1 + x_1^p, x_2, \ldots, x_n) \in \mathrm{End}(\mathbf{k}^n)$. If $n = 3$ and the degree of $f \in \mathrm{End}(\mathbf{k}^3)$ is at most 3, then Vistoli proved that the Jacobian conjecture holds [Vis99]. Theorem 3.4.3 says in particular that the first implication is an equivalence if $f \in \mathrm{End}(\mathbf{k}^3)$ and $\deg(f) \leq 3$ for algebraically closed fields of any characteristic.

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EXISTENCE OF EMBEDDINGS OF SMOOTH VARIETIES INTO LINEAR ALGEBRAIC GROUPS

PETER FELLER AND IMMANUEL VAN SANTEN

Abstract. We prove that every smooth affine variety of dimension d embeds into every simple algebraic group of dimension at least $2d+2$. We do this by establishing the existence of embeddings of smooth affine varieties into the total space of certain principal bundles. For the latter we employ and build upon parametric transversality results for flexible affine varieties due to Kaliman. By adapting a Chow-group-based argument due to Bloch, Murthy, and Szpiro, we show that our result is optimal up to a possible improvement of the bound to $2d+1$.

In order to study the limits of our embedding method, we use rational homology group calculations of homogeneous spaces and we establish a domination result for rational homology of complex smooth varieties.

Contents

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1. Introduction

In this text, *varieties* are understood to be (reduced) algebraic varieties over a fixed algebraically closed field $\mathbf{k}$ of characteristic zero endowed with the Zariski topology. We will focus on *affine* varieties—closed subvarieties of the affine space $\mathbb{A}^n$. A closed embedding, *embedding* for short, $f\colon Z \to X$ of an affine variety Z into an affine variety X is a morphism such that $f(Z)$ is closed in X and f induces an isomorphism $Z \simeq f(Z)$ of varieties.

A focus of this text lies on embeddings into the underlying varieties of affine algebraic groups. Recall that an affine algebraic group, an *algebraic group* for short, is a closed subgroup of the general linear group GL_k for some positive integer k. An algebraic group is *simple* if it has no non-trivial connected normal subgroup. We prove the following embedding theorem.

Theorem A (Theorem 3.7)**.** *Let G be the underlying affine variety of a simple algebraic group and Z be a smooth affine variety. If $\dim G > 2\dim Z + 1$, then Z admits an embedding into G.*

In case $\dim G$ is even, the dimension assumption on $\dim Z$ in terms of $\dim G$ from Theorem A is optimal; while in case $\dim G$ is odd, the dimension assumption can at best be relaxed by one, that is from $\dim G > 2\dim Z + 1$ to $\dim G \geq 2\dim Z + 1$. Indeed, we have the following.

Proposition B (Corollary 4.4)**.** *Let G be the underlying affine variety of an algebraic group of dimension $n \geq 1$. Then, for every integer $d \geq \frac{n}{2}$ there exists a smooth irreducible affine variety Z of dimension d that does not admit an embedding into G.*

Theorem A fits well in the context of classical embedding theorems in different categories. We provide this context in the next subsection and an outline of the proof of Theorem A in the subsection after that.

Before that, we discuss a domination result for the rational homology of smooth varieties, which we believe to be of independent interest. The connection to Theorem A comes from an application that explains one crucial obstacle to weakening the dimension assumption to $\dim G \geq 2\dim Z + 1$ in our proof of Theorem A; see Proposition D in the outline of the proof of Theorem A below. For this domination result we work over the field of complex numbers, and rational homology groups $H_*(\cdot;\mathbb{Q})$ are taken with respect to the Euclidean topology.

Theorem C. *Let $f\colon X \to Y$ be a proper surjective morphism between complex n-dimensional smooth varieties. Then the induced map on k-th rational homology $H_k(X;\mathbb{Q}) \to H_k(Y;\mathbb{Q})$ is a surjection for all integers $k \geq 0$.*

We will formulate a version of Theorem C (see Theorem A.2) in the category of complex manifolds that can be understood as a generalization of Gurjar's Theorem [Gur80] (see Remark A.4). We prove Theorem C via a version of Hopf's Theorem on the Umkehrungshomomorphismus for non-compact topological manifolds; see Appendix A.

Context: embedding theorems in various settings.

Holme-Kaliman-Srinivas embedding theorem. When considering affine varieties as closed subvarieties of the affine space $\mathbb{A}^n$, it is natural to wonder about their minimal embedding dimension in affine space. It turns out that every smooth affine variety Z embeds into $\mathbb{A}^n$ for $n \geq 2\dim Z + 1$; see Holme [Hol75], Kaliman [Kal91], and Srinivas [Sri91]. This can be understood as an analog of the following classical result in differential topology.

Whitney embedding theorem. The weak Whitney embedding theorem states that every closed smooth manifold M can be embedded into $\mathbb{R}^n$ for $n \geq 2\dim M + 1$ [Whi36]. The fact that Whitney's result also holds in case $n = 2\dim M$ is known as the strong Whitney embedding theorem, based on the so-called Whitney trick [Whi44]. Furthermore, if M is a closed smooth manifold such that $\dim M$ is not a power of 2, then Haefliger-Hirsch [HH63] proved that M embeds into $\mathbb{R}^{2\dim M-1}$. In contrast, the real projective space of dimension 2^k for $k \geq 0$ yields a 2^k-dimensional smooth manifold that does not embed into $\mathbb{R}^{2\cdot 2^k-1}$ [Pet57].

Holomorphic embeddings of Stein manifolds. Focusing on $\mathbf{k} = \mathbb{C}$ (hence $\mathbb{A}^n = \mathbb{C}^n$), it is natural to compare the Holme-Kaliman-Srinivas result with the holomorphic setup. It is known that every Stein manifold M of dimension at least 2 can be holomorphically embedded into $\mathbb{C}^n$ for $n > \frac{3}{2}\dim M$; see Eliashberg-Gromov [EG92] and Schürmann [Sch97]. Examples of Forster show that this dimension condition is optimal [For70].

Focusing on more general targets, Andrist, Forstnerič, Ritter, and Wold proved that for every Stein manifold X that satisfies the (volume) density property and every Stein manifold M such that $\dim X \geq 2\dim M + 1$, there exists a holomorphic embedding of M into X [AFRW16]. In particular, if G is a characterless algebraic group, then G satisfies the density property by Donzelli-Dvorsky-Kaliman [DDK10, Theorem A] or G is isomorphic to $\mathbb{C}$. Hence, every smooth affine variety Z with $2\dim Z + 1 \leq \dim G$ admits a holomorphic embedding into G. As far as the authors know, it remains open whether a dimension improvement à la Eliashberg-Gromov is possible.

Embeddings into projective varieties. Comparing with the projective setting, a further analog of the weak Whitney embedding theorem states that every smooth projective variety Z embeds into $\mathbb{P}^n$ provided $n \geq 2\dim Z + 1$; see Lluis [Llu55].

While the Holme-Kaliman-Srinivas embedding result concerning affine spaces generalizes to some, possibly all affine algebraic groups, the embedding result due to Lluis concerning projective spaces cannot generalize to projective algebraic groups, better known as abelian varieties. In fact, each rational map $Z \dashrightarrow A$ from a rationally connected variety Z into an abelian variety A is constant; see [Lan83, Corollary to Theorem 4, Ch. II].

Optimality of the dimension condition for algebraic embeddings. As seen above, in many categories, d-dimensional objects embed into the standard space of dimension $2d$, e.g. the strong Whitney embedding theorem, or even lower like in the case of the Eliashberg-Gromov result. In contrast, even the analog of the strong Whitney embedding theorem is known to fail for affine varieties. Indeed, by a result of Bloch-Murthy-Szpiro [BMS89], for every $d \geq 1$ there exists a d-dimensional smooth affine variety that does not embed into $\mathbb{A}^{2d}$. In fact, their argument (based on Chow group calculations) suffices to also yield Proposition B, as we will see in Section 4.

Incidentally, in the Lluis embedding theorem, the dimension bound is optimal in the sense that for every $d \geq 1$ there is a smooth projective variety of dimension d that does not admit an embedding into $\mathbb{P}^{2d}$; see Horrocks-Mumford [HM73] and Van de Ven [VdV75].

Proof strategy: an embedding method and its limits.

Proof strategy of the Holme-Kaliman-Srinivas theorem and an approach to more general targets. We recall the basic idea behind the Holme-Kaliman-Srinivas embedding theorem, which uses the same method as the proofs of the weak Whitney embedding theorem and the Lluis embedding theorem. To show that every smooth affine variety Z embeds into $X = \mathbb{A}^{2\dim Z+1}$, one starts from an arbitrary embedding $Z \subseteq \mathbb{A}^m$ for some large integer $m \gg 2\dim Z + 1$, and shows that the composition of the inclusion $Z \subseteq \mathbb{A}^m$ with a generic linear projection $\mathbb{A}^m \to \mathbb{A}^{2\dim Z+1}$ is still an embedding.

For more general targets X, one looses the availability of (many) projections from $\mathbb{A}^m$ to X. In contrast with the above strategy, instead, we consider a morphism $\pi \colon X \to \mathbb{A}^{\dim Z}$ and a finite morphism $Z \to \mathbb{A}^{\dim Z}$ (guaranteed to exist by Noether normalization) in order to build our embedding $Z \to X$ as a factorization of $Z \to \mathbb{A}^{\dim Z}$ through π. This approach is similar to the setup of Eliashberg-Gromov and their notion of relative embedding using their 'background map'; see [EG92, Section 2]. A strength of this approach lies in the following fact: checking that a morphism $f \colon Z \to X$ is an embedding (i.e. a proper injective morphism with everywhere injective differential), reduces to checking that f is injective and has everywhere injective differential, since any morphism that can be composed with another yielding a finite (in particular proper) morphism is proper. Sloppily speaking, one gets properness 'for free'.

Outline of the proof of Theorem A. More concretely, our approach to prove Theorem A can be understood in two steps. Step one involves finding a specific subvariety of a simple algebraic group using classical algebraic group theory. Using parametric transversality results, in step two we promote finite maps with target the base space of a principal bundle to embeddings into the total space. Here the total space is the subvariety constructed in step one. These two steps will be treated in detail in Sections 3 and 2, respectively.

We provide a short outline, where we fix a smooth affine variety Z and a simple algebraic group G with $\dim G > 2\dim Z + 1$.

Step one. We find a closed codimension one subvariety $X \subset G$ isomorphic to $\mathbb{A}^{\dim Z} \times H$, where H is a characterless closed subgroup of G. This will be achieved using a well-chosen maximal parabolic subgroup in G and constitutes the bulk of Section 3. It turns out that G itself cannot be a product of the form $\mathbb{A}^m \times H$ for any variety H underlying an algebraic group and $m > 0$; hence, the X we found has the largest possible dimension.

Link between the two steps. We note that step one reduces the proof of Theorem A to finding an embedding of Z into $\mathbb{A}^{\dim Z} \times H$. We set up a principal bundle together with a finite morphism from Z into the base. For the latter, denoting by $\mathbb{G}_a$ the underlying additive algebraic group of the ground field $\mathbf{k}$, we consider the principal $\mathbb{G}_a$-bundle $\rho\colon \mathbb{A}^{\dim Z} \times H \to \mathbb{A}^{\dim Z} \times H/U$, where U is a closed subgroup of H that is isomorphic to $\mathbb{G}_a$. Using Noether normalization, one has a finite morphism $Z \to \mathbb{A}^{\dim Z}$, which yields a morphism $r\colon Z \to \mathbb{A}^{\dim Z} \times H/U$ by composing with a section of the projection $\eta\colon \mathbb{A}^{\dim Z} \times H/U \to \mathbb{A}^{\dim Z}$ to the first factor. Writing $X \coloneqq \mathbb{A}^{\dim Z} \times H$ and $Q \coloneqq \mathbb{A}^{\dim Z} \times H/U$, we have the following commutative diagram

$$\begin{tikzcd} & X \arrow[d, "\rho"] \arrow[dr, "\pi"] & \\ Z \arrow[r, "r"] & Q \arrow[r, "\eta"] & \mathbb{A}^{\dim Z} \end{tikzcd} \quad . \tag{1}$$

Step two. We consider the following setup generalizing (1). This constitutes our embedding method mentioned earlier. Consider a principal $\mathbb{G}_a$-bundle $\rho\colon X \to Q$, where X is a smooth irreducible affine variety of dimension at least $2\dim Z + 1$, and a finite morphism $Z \to \mathbb{A}^{\dim Z}$ that is the composite of morphisms $r\colon Z \to Q$ and $\eta\colon Q \to \mathbb{A}^{\dim Z}$ such that the following holds. The composition $\pi \coloneqq \eta \circ \rho\colon X \to \mathbb{A}^{\dim Z}$ is a smooth morphism such that there are sufficiently many automorphisms of X that fix π (see Definition 2.1). Given this setup, we show that there exists an embedding of Z into X (see Theorem 2.5). This is done in Section 2 building on notions and results due to Kaliman [Kal20]. Next, we explain in broad strokes how we build such an embedding.

Note first that $\rho\colon X \to Q$ restricts to a trivial $\mathbb{G}_a$-bundle over any affine subvariety of Q. Hence, there exists a morphism

$$f_0\colon Z \to \rho^{-1}(r(Z)) \simeq r(Z) \times \mathbb{G}_a \subset X$$

such that $\rho \circ f_0 = r$. Then we use a generic automorphism φ of X that fixes π to construct an 'improved' morphism $f_1\colon Z \to X$ with $\rho \circ f_1 = \rho \circ \varphi \circ f_0$. 'Improved' means that f_1 and its differential are 'more injective' than f_0 and its differential, respectively. After finitely many, say k, such 'improvements', we get an injective morphism $f_k\colon Z \to X$ with everywhere injective differential. Note that by construction we have that $\pi \circ f_k =$

$\eta \circ r\colon Z \to \mathbb{A}^{\dim Z}$ is finite. This shows the properness of f_k, and thus f_k is an embedding of Z into X.

The case of small dimensions and other cases. While, in general, we do not know how to weaken the dimension assumption to the optimal $\dim G \geq 2\dim Z + 1$ in Theorem A, we are able to treat the case $\dim G \leq 8$: every smooth affine variety Z embeds in every characterless algebraic group G of dimension ≤ 8 if $2\dim Z + 1 \leq \dim G$; see Proposition 3.11.

From the method of the proof it is clear that Theorem A generalizes to products of a simple algebraic group with affine spaces (Theorem 3.7) and to products of a semisimple algebraic group with affine spaces but with a stronger dimension assumption (Theorem 3.10). In case the dimension of the affine space in the product is big enough, we get in fact the embedding result with the optimal dimension assumption; see Corollary 3.1. In particular, we give a new proof of the Holme-Kaliman-Srinivas embedding theorem; see Remark 3.2.

Our embedding method also yields that if a smooth affine variety Z embeds into a smooth affine variety X with $\dim X \geq 2\dim Z + 1$, then Z embeds into the target of every finite étale surjection from X, whenever X has sufficiently many automorphisms; see Corollary 2.26. In particular, Theorem A generalizes to homogeneous spaces of simple algebraic groups with finite stabilizer; see Proposition 2.13.

Limits of the method and relation to Theorem C. We end the introduction by coming back to a statement from earlier: the seemingly unrelated Theorem C explains a major obstacle to treating the case $\dim G = 2\dim Z + 1$. We explain this in terms of the above short two step outline. In fact, in step one we find $\pi\colon X \to \mathbb{A}^{\dim Z}$ by restricting the natural projection $p\colon G \to G/H$ for some closed subgroup H to $X \subseteq G$, i.e. $\pi \coloneqq p|_X\colon X \to p(X) \subseteq G/H$. However, by the dimension assumption that we need for step two, if we were to follow that strategy, we would have to choose $X \subseteq G$ of full dimension. Hence, assuming w.l.o.g. that G is irreducible, we would have to choose $X = G$ and would have to replace $\mathbb{A}^{\dim Z}$ with a homogeneous space G/H of dimension $\dim Z$ in diagram (1). For the embedding method from step two to work for G/H in place of $\mathbb{A}^{\dim Z}$ in diagram (1), we need in particular a finite morphism from Z to G/H; compare Theorem 2.5. However, there exist Z such that no finite morphism from Z to G/H exists. Concretely, working over $\mathbb{C}$, rational homology calculations for homogeneous spaces (see Proposition 5.2) and Theorem C yield the following result.

Proposition D (Proposition 5.1)**.** *Let Z be a simply-connected complex smooth algebraic variety with the rational homology of a point. If G/H is a $\dim Z$-dimensional complex homogeneous space of a complex simple algebraic group G, then there is no proper surjective morphism from Z to G/H.*

And indeed, we do not know whether such Z embed into simple algebraic groups of dimension $2\dim Z + 1$. Concretely, the authors cannot answer the following question, even over $\mathbb{C}$ and for contractible Z.

Question. *Does every 7-dimensional smooth affine variety embed into* SL_4*?*

Addendum: In a new arXiv preprint, Kaliman has answered this question affirmatively [Kal21, Theorem 1.1]. In fact, more generally, he proves that, if G is a semisimple algebraic group such that its Lie algebra is a product of Lie algebras of special linear groups, then every smooth affine variety Z with $2\dim Z + 1 \leq \dim G$ admits an embedding into G.

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2. Embeddings into the total space of a principal bundle

For the main result in this section the following definition will be useful:

Definition 2.1. Let X be a variety. A subgroup G of the group of algebraic automorphisms $\mathrm{Aut}(X)$ acts *sufficiently transitively on* X if the natural action on X is 2-transitive and the natural action on $(TX)^\circ$ is transitive, where $(TX)^\circ$ denotes the complement of the zero-section in the total space TX of the tangent bundle of X.

Let us recall the definition of an algebraic subgroup of an automorphism group which goes back to Ramanujam [Ram64].

Definition 2.2. Let X be a variety. A subgroup $H \subset \mathrm{Aut}(X)$ is called *algebraic subgroup* if there exists an algebraic group G and a faithful algebraic action $\rho\colon G \times X \to X$ such that H is the image of the homomorphism $f_\rho\colon G \to \mathrm{Aut}(X)$ induced by ρ.

Remark 2.3. Note that the algebraic group G in Definition 2.2 is uniquely determined by H in the following sense: if G' is another algebraic group with a faithful algebraic action ρ' on X such that $f_{\rho'}(G') = H$, then there exists an isomorphism of algebraic groups $\sigma\colon G' \to G$ such that $f_{\rho'} = f_\rho \circ \sigma$ [KRvS19, Theorem 9]. This allows us to identify G and H.

Moreover, we will use the following subgroups of the automorphism group of a variety:

Definition 2.4. Let X be a variety. Then $\mathrm{Aut}^{\mathrm{alg}}(X)$ denotes the subgroup of $\mathrm{Aut}(X)$ that is generated by all *connected* algebraic subgroups of $\mathrm{Aut}(X)$.

If X comes equipped with a morphism $\pi\colon X \to P$, then $\mathrm{Aut}_P(X)$ denotes the subgroup of $\mathrm{Aut}(X)$ that consists of the $\sigma \in \mathrm{Aut}(X)$ with $\pi \circ \sigma = \pi$. We define $\mathrm{Aut}_P^{\mathrm{alg}}(X)$ as the subgroup of $\mathrm{Aut}_P(X)$ that is generated by all connected algebraic subgroups of $\mathrm{Aut}(X)$ that lie in $\mathrm{Aut}_P(X)$.

The main result to construct embeddings in this article is the following theorem. Note that $\mathbb{G}_a$ denotes the underlying additive algebraic group of the ground field $\mathbf{k}$. The proof of the theorem is contained in Subsection 2.4.

Theorem 2.5. *Let X be a smooth irreducible affine variety such that:*

a) There is a principal $\mathbb{G}_a$-bundle $\rho\colon X \to Q$;
b) There is a smooth morphism $\pi\colon X \to P$ such that $\mathrm{Aut}_P^{\mathrm{alg}}(X)$ acts sufficiently transitively on each fiber of π;
c) There is a morphism $\eta\colon Q \to P$ that satisfies $\eta \circ \rho = \pi$.

If there exists a smooth affine variety Z such that $\dim X \geq 2\dim Z + 1$ and

d) there exists a morphism $r\colon Z \to Q$ such that $\eta \circ r\colon Z \to P$ is finite and surjective,

then there exists an embedding of Z into X.

Part of Theorem 2.5 can be illustrated by the following diagram

$$\begin{tikzcd} Z \arrow[rr, "\exists\text{ embedding}"] & & X \arrow[d, "\rho"] \arrow[dr, "\pi"] & \\ & Z \arrow[r, "r"] & Q \arrow[r, "\eta"] & P \end{tikzcd} .$$

Remark 2.6. Let X be a smooth affine irreducible variety and assume that conditions a), b), c) of Theorem 2.5 are satisfied. If Z is a smooth affine variety with $\dim X \geq 2\dim Z + 1$, $P = \mathbb{A}^{\dim Z}$, and $\eta\colon Q \to P$ has a section $s\colon P \to Q$, then condition d) is also satisfied. Indeed, in this case there exists a finite morphism $p\colon Z \to \mathbb{A}^{\dim Z}$ due to Noether's Normalization Theorem and one can choose $r \coloneqq s \circ p\colon Z \to Q$.

2.1. **Transversality results.** This subsection essentially amounts to collecting and rephrasing some material from [Kal20] that we need for the proof of Theorem 2.5.

Definition 2.7. Let $X \to P$ be a smooth morphism of smooth irreducible varieties and let $\mathcal{H} = (H_1, \ldots, H_s)$ be a tuple of connected algebraic subgroups $H_1, \ldots, H_s \subset \mathrm{Aut}_P(X)$. Then $\mathcal{H}$ is

(1) *big enough for proper intersection*, if for every morphism $f\colon Y \to X$ and every locally closed subvariety Z in X there is an open subset $U \subset H_1 \times \cdots \times H_s$ such that for every $(h_1, \ldots, h_s) \in U$ we have

$$\dim Y \times_X h_1 \cdots h_s \cdot Z \leq \dim Y \times_P Z + \dim P - \dim X . \tag{PI}$$

(2) *big enough for smoothness* if there exists an open dense subset $U \subset H_1 \times \cdots \times H_s$ such that the morphism

$$\Phi_{\mathcal{H}} \colon H_1 \times \cdots \times H_s \times X \to X \times_P X \,, \quad ((h_1, \ldots, h_s), x) \mapsto (h_1 \cdots h_s \cdot x, x)$$

is smooth on $U \times X$.

Proposition 2.8. *Let $X \to P$ be a smooth morphism of smooth irreducible varieties and let $\mathcal{H} = (H_1, \ldots, H_s)$ be a tuple of connected algebraic subgroups $H_1, \ldots, H_s$ in $\mathrm{Aut}_P(X)$. Then:*

(1) If $\mathcal{H}$ is big enough for smoothness, then $\mathcal{H}$ is big enough for proper intersection.

(2) If $\mathcal{H}$ is big enough for smoothness and $H_0, H_{s+1} \subset \mathrm{Aut}_P(Y)$ are two connected algebraic subgroups, then $(H_0, H_1, \ldots, H_s, H_{s+1})$ is big enough for smoothness.

Proof. (1): The proof closely follows [Kal20, Theorem 1.4]. By assumption, there is an open dense subset $U \subset H_1 \times \cdots \times H_s$ such that $\Phi_{\mathcal{H}}|_{U \times X} \colon U \times X \to X \times_P X$ is smooth. Let $f \colon Y \to X$ be a morphism and let Z be a locally closed subvariety of X. Let W be the fiber product of $Y \times_P Z \to X \times_P X$ and $\Phi_{\mathcal{H}}|_{U \times X}$:

$$\begin{array}{ccc} W & \longrightarrow & Y \times_P Z \\ \downarrow & & \downarrow \\ U \times X & \xrightarrow{\Phi_{\mathcal{H}}|_{U \times X}} & X \times_P X \,. \end{array}$$

By generic flatness [GW10, Theorem 10.84], we may shrink U and assume that $\pi \colon W \to U \times X \to U$ is flat. Take $h = (h_1, \ldots, h_s) \in U$. Then

$$\pi^{-1}(h)_{\mathrm{red}} \longrightarrow (Y \times_X h_1 \cdots h_s \cdot Z)_{\mathrm{red}} \,, \quad ((h, x), (y, z)) \to (y, h_1 \cdots h_s \cdot z)$$

is an isomorphism, since $(h_1 \cdots h_s \cdot x, x) = (f(y), z)$ for each $((h, x), (y, z)) \in \pi^{-1}(h)_{\mathrm{red}}$. If $\pi^{-1}(h)$ is empty, then (PI) from Definition 2.7 is satisfied (as by convention $\dim \varnothing = -\infty$) and thus we may assume that $\pi^{-1}(h)$ is non-empty and we get $\dim \pi^{-1}(h) \leq \dim W - \dim U$ by the flatness of π. By the smoothness of $\Phi_{\mathcal{H}}|_{U \times X}$ and the pullback diagram above, $W \to Y \times_P Z$ is smooth since smoothness is preserved under pullbacks. In particular, $\dim W \leq \dim Y \times_P Z + \dim U \times X - \dim X \times_P X$. In total we get

$$\begin{aligned} \dim Y \times_X h_1 \cdots h_s \cdot Z &= \dim \pi^{-1}(h) \\ &\leq \dim Y \times_P Z + \dim X - \dim X \times_P X \\ &= \dim Y \times_P Z + \dim P - \dim X \,, \end{aligned}$$

since $\dim X \times_P X = 2 \dim X - \dim P$, which in turn follows from the smoothness of $X \to P$ and the irreducibility of X, P.

(2): This follows directly from [Kal20, Remark 1.8]. □

Proposition 2.9 ([Kal20, Proposition 1.7])**.** *Let $\kappa \colon X \to P$ be a smooth morphism of smooth irreducible varieties and let a subgroup $G \subset \mathrm{Aut}_P(X)$ be generated by a family $\mathcal{G}$ of connected algebraic subgroups of $\mathrm{Aut}_P(X)$*

which is closed under conjugation by G. Moreover, assume that G acts transitively on each fiber of κ.

Then there exist $H_1, \dots, H_s \in \mathcal{G}$ such that $(H_1, \dots, H_s)$ is big enough for smoothness. □

2.2. **Sufficiently transitive group actions.** For a variety X we denote by $\mathrm{SAut}(X)$ the subgroup of $\mathrm{Aut}(X)$ that is generated by all unipotent algebraic subgroups; in particular, $\mathrm{SAut}(X) \subseteq \mathrm{Aut}^{\mathrm{alg}}(G)$. Transitivity of the natural action of $\mathrm{SAut}(X)$ on X implies m-transitivity for all m and that one can prescribe the tangent map of an automorphism of X at a finite number of fixed points:

Theorem 2.10 ([AFK$^+$13, Theorem 0.1, Theorem 4.14 and Remark 4.16])**.** *Let X be an irreducible smooth affine variety of dimension at least 2. If $\mathrm{SAut}(X)$ acts transitively on X, then:*

(1) $\mathrm{SAut}(X)$ acts m-transitively on X for each $m \geq 1$;
(2) for every finite subset $Z \subset X$ and every collection $\beta_z \in \mathrm{SL}(T_z X)$, $z \in Z$, there is an automorphism $\varphi \in \mathrm{SAut}(X)$ that fixes Z pointwise such that the differential satisfies $d_z\varphi = \beta_z$ for all $z \in Z$. □

Example 2.11. Let F be an irreducible smooth affine variety of dimension ≥ 2 such that $\mathrm{SAut}(F)$ acts transitively on it. Then, Theorem 2.10 implies that $\mathrm{SAut}(F)$ acts sufficiently transitively on F; see Definition 2.1.

Example 2.12. If G is a connected characterless algebraic group, then the group $\mathrm{Aut}^{\mathrm{alg}}(G)$ acts sufficiently transitively on G. Indeed, such a G is generated by its unipotent subgroups (see e.g. [Pop11, Lemma 1.1]) and thus $G \subseteq \mathrm{SAut}(G)$. In particular, $\mathrm{SAut}(G)$ acts transitively on G. Now, if $\dim G = 0$, then the statement is trivial. If $\dim G = 1$, then G is isomorphic to $\mathbb{G}_a$ and thus $\mathrm{Aut}^{\mathrm{alg}}(G) = \mathrm{Aut}(\mathbb{A}^1)$; hence, the statement is also clear. If $\dim G \geq 2$, then the statement follows from Example 2.11.

Incidentally, the above example characterizes algebraic groups G such that $\mathrm{Aut}^{\mathrm{alg}}(G)$ acts sufficiently transitively on G:

Proposition 2.13. *Let G be an algebraic group. Then $\mathrm{Aut}^{\mathrm{alg}}(G)$ acts sufficiently transitively on G if and only if G is connected and characterless.*

Proof. According to Example 2.12 we only have to show the 'only if'-part.

Let G be an algebraic group such that $\mathrm{Aut}^{\mathrm{alg}}(G)$ acts sufficiently transitively on it.

Let $g, g' \in G$. Since $\mathrm{Aut}^{\mathrm{alg}}(G)$ acts transitively on G, there exist connected algebraic subgroups $H_1, \dots, H_r$ in $\mathrm{Aut}(G)$ such that g' lies in the image of the morphism

$$H_1 \times \cdots \times H_r \to G\,, \quad (h_1, \dots, h_r) \mapsto (h_1 \circ \ldots \circ h_r)(g)\,.$$

Hence, $g, g' \in G$ lie in an irreducible closed subset of G. Since g, g' were arbitrary elements of G, it follows that G is connected.

Denote by G^u the algebraic subgroup of G that is generated by all unipotent elements in G. Then G^u is closed and normal in G and each invertible function on G^u is constant. There exists an algebraic torus $T \subseteq G$ (i.e. T is a product of finitely many copies of the underlying multiplicative group of the ground field) such that $G = G^u \rtimes T$; see e.g. [FvS19, Lemma 8.2].

Let $\pi\colon G \to T$ be the canonical projection. Take an arbitrary algebraic action $\rho\colon H \times G \to G$ of an arbitrary connected algebraic group H. Since each invertible function on each fiber of π is constant, the morphism

$$H \times G \xrightarrow{\rho} G \xrightarrow{\pi} T$$

is invariant under the algebraic action $N \times (H \times G) \to H \times G$ that is given by $n \cdot (h, g) = (h, ng)$. Hence, the morphism $\pi \circ \rho$ factors through $\mathrm{id}_H \times \pi$, i.e. there is a commutative diagram

$$\begin{tikzcd} H \times G \arrow[r, "\rho"] \arrow[d, "\mathrm{id}_H \times \pi"'] & G \arrow[d, "\pi"] \\ H \times T \arrow[r, "\rho_T"] & T \end{tikzcd} \tag{2}$$

for a unique morphism $\rho_T\colon H \times T \to T$. As ρ is an action, ρ_T is an action as well. Since $\mathrm{Aut}^{\mathrm{alg}}(G)$ acts 2-transitively on G and since each action ρ of a connected algebraic group on G induces an action ρ_T on T such that (2) commutes, we get that $\mathrm{Aut}^{\mathrm{alg}}(T)$ acts 2-transitively on T. By Lemma 2.14 below, we find that T is trivial, and thus $G = G^u$ is characterless. ☐

The following lemma is certainly well-known to the specialists. However, for lack of a reference we give a proof of it.

Lemma 2.14. *Let T be an algebraic torus. Then*

$$\mathrm{Aut}^{\mathrm{alg}}(T) = \{\, T \to T\,,\ t \mapsto st \mid s \in T \,\}\,.$$

Proof. Let $H \subset \mathrm{Aut}(T)$ be an algebraic subgroup. Hence there exists a faithful algebraic H-action $\rho\colon H \times T \to T$ such that the image of the induced homomorphism in $\mathrm{Aut}(T)$ is H (see Remark 2.3). By [Ros61, Theorem 2] there exist morphisms $\mu\colon H \to T$ and $\lambda\colon T \to T$ such that

$$\rho(h, t) = \mu(h)\lambda(t) \quad \text{for each } h \in H,\, t \in T\,.$$

After replacing μ and λ by $t_0\mu$ and $t_0^{-1}\lambda$, respectively, for some $t_0 \in T$, we may assume that $\mu(e_H) = e_T$, where e_H and e_T denote the neutral elements of H and T, respectively. Hence, $t = \rho(e_H, t) = \lambda(t)$ for each $t \in T$, and thus

$$\rho(h, t) = \mu(h)t \quad \text{for each } h \in H,\, t \in T\,.$$

This implies that H lies inside $\{\, T \to T\,,\ t \mapsto st \mid s \in T \,\}$, and thus the lemma follows. ☐

2.3. **Sufficiently transitive group actions on fibers.** In the next proposition, we provide a class of smooth morphisms $\pi\colon X \to P$ such that $\mathrm{Aut}_P^{\mathrm{alg}}(X)$ acts sufficiently transitively on each fiber of π.

Proposition 2.15. *Let G be a connected algebraic group and $H \subseteq G$ be a connected characterless algebraic subgroup of dimension ≥ 2. Then, the algebraic quotient $\pi\colon G \to G/H =: P$ is a smooth morphism such that $\mathrm{Aut}_P^{\mathrm{alg}}(G)$ acts sufficiently transitively on each fiber of π.*

We note that the dimension condition $\dim H \geq 2$ in Proposition 2.15 is necessary, as the following example shows.

Example 2.16. Denote by $\pi\colon \mathrm{SL}_2 \to P := \mathrm{SL}_2/H$ the algebraic quotient, where $H \subset \mathrm{SL}_2$ denotes the subgroup of unipotent upper triangular matrices. In this case each automorphism φ in $\mathrm{Aut}_P(\mathrm{SL}_2)$ acts as a translation on $\pi^{-1}(p) \simeq \mathbb{A}^1$ for each $p \in P$. In particular, for each $p \in P$ we have that $\mathrm{Aut}_P^{\mathrm{alg}}(\mathrm{SL}_2)$ does not act sufficiently transitively on $\pi^{-1}(p)$ (while the group $\mathrm{Aut}^{\mathrm{alg}}(\pi^{-1}(p))$ acts sufficiently transitively on $\pi^{-1}(p)$ by Example 2.12).

That φ acts as a translation on each fiber of π can be checked explicitly by writing φ with respect to the following parametrizations

$$\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \times \mathbb{A}^1 \to \mathrm{SL}_2\,, \quad (x, z, y) \mapsto \begin{pmatrix} x & y \\ z & \frac{yz+1}{x} \end{pmatrix}$$

and

$$\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \to \mathrm{SL}_2\,, \quad (x, z, w) \mapsto \begin{pmatrix} x & \frac{xw-1}{z} \\ z & w \end{pmatrix}\,.$$

For the proof of Proposition 2.15, we need some preparation. First, we recall a more general version of Theorem 2.10 stated in terms of the following definition.

Definition 2.17 ([AFK$^+$13, Definition 2.1])**.** Let X be an affine variety and let $\mathcal{N}$ be a set of locally nilpotent derivations on the coordinate ring $\mathcal{O}(X)$ and let $G(\mathcal{N})$ be the subgroup of $\mathrm{SAut}(X)$ that is generated by all automorphisms of X that are induced by the locally nilpotent derivations in $\mathcal{N}$. Then $\mathcal{N}$ is called saturated, if

(i) $\mathcal{N}$ is closed under conjugation by elements from $G(\mathcal{N})$ and
(ii) for each $D \in \mathcal{N}$ and each $f \in \ker(D)$ we have $fD \in \mathcal{N}$.

Remark 2.18. If X is an affine variety and if $\mathcal{N}$ is a set of locally nilpotent derivations on $\mathcal{O}(X)$ that satisfies (ii) from Definition 2.17, then there exists a bigger set $\mathcal{N}'$ of locally nilpotent derivations on $\mathcal{O}(X)$ that is saturated and satisfies $G(\mathcal{N}') = G(\mathcal{N})$; see [FKZ17, Lemma 4.6].

We come now to the promised generalization of Theorem 2.10.

Theorem 2.19 ([AFK$^+$13, Theorem 2.2, Theorem 4.14 and Remark 4.16])**.** *Let X be an irreducible smooth affine variety of dimension at least 2 and let $\mathcal{N}$ be a saturated set of locally nilpotent derivations on $\mathcal{O}(X)$. If the subgroup $G(\mathcal{N})$ of $\mathrm{SAut}(X)$ acts transitively on X, then:*

(1) $G(\mathcal{N})$ acts m-transitively on X for each $m \geq 1$;
(2) for every finite subset $Z \subset X$ and every collection $\beta_z \in \mathrm{SL}(T_z X)$, $z \in Z$, there is an automorphism $\varphi \in G(\mathcal{N})$ that fixes Z pointwise such that the differential satisfies $d_z\varphi = \beta_z$ for all $z \in Z$. $\square$

Lemma 2.20. *Let G be an algebraic groups and let $U \subseteq H \subseteq G$ be closed subgroups such that H is characterless and U is unipotent. Then, the restriction map $\mathcal{O}(G) \to \mathcal{O}(H)$, $q \mapsto q|_H$ induces a surjection on the U-invariant rings $\mathcal{O}(G)^U \to \mathcal{O}(H)^U$ where the U-actions are induced by right multiplication.*

The following example shows, that the assumption that H is characterless is necessary:

Example 2.21. Let $G = \mathrm{SL}_2$, H the subgroup of upper triangular matrices and let $U \subseteq H$ be the subgroup with 1 on the diagonal. Denote the coordinates on SL_2 by

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

Then $\mathcal{O}(G) \to \mathcal{O}(H)$ identifies with the homomorphism

$$\mathbf{k}[x,y,z,w]/(xw - yz - 1) \xrightarrow{x \mapsto x\,,\ y \mapsto y\,,\ z \mapsto 0\,,\ w \mapsto w} \mathbf{k}[x,y,w]/(xw - 1)$$

and thus $\mathcal{O}(G)^U \to \mathcal{O}(H)^U$ identifies with the non-surjective homomorphism

$$\mathbf{k}[x,z] \xrightarrow{x \mapsto x\,,\ z \mapsto 0} \mathbf{k}[x,w]/(xw - 1) \simeq \mathbf{k}[x, x^{-1}]\,.$$

We first provide the proof of Proposition 2.15 using Lemma 2.20. Afterwards, we provide the setup and the proof of Lemma 2.20.

Proof of Proposition 2.15. We have to show that $\mathrm{Aut}^{\mathrm{alg}}_{G/H}(G)$ acts sufficiently transitively on each fiber of $\pi\colon G \to G/H$. Since π is G-equivariant with respect to left multiplication by G, it is enough to show that $\mathrm{Aut}^{\mathrm{alg}}_{G/H}(G)$ acts sufficiently transitively on the closed subset H of G. Let

$$\mathcal{N} = \left\{ fD \;\middle|\; \begin{array}{l} D \text{ is a locally nilpotent derivation of } \mathcal{O}(H) \text{ induced} \\ \text{by right multiplication of a one-dimensional} \\ \text{unipotent subgroup of } H \text{ and } f \in \ker(D) \end{array} \right\}.$$

Since H is connected and characterless, H is spanned by all its one-dimensional unipotent subgroups; see [Pop11, Lemma 1.1] Hence, the subgroup $G(\mathcal{N})$ of $\mathrm{SAut}(H)$ generated by $\mathcal{N}$ acts transitively on H. By Remark 2.18, there exists a saturated set of locally nilpotent derivations $\mathcal{N}'$ with $G(\mathcal{N}) = G(\mathcal{N}')$; hence, Theorem 2.19 implies that $G(\mathcal{N})$ acts sufficiently transitively on H. Therefore, it suffices to show that every element of $G(\mathcal{N})$ can be extended to an automorphism in $\mathrm{Aut}^{\mathrm{alg}}_{G/H}(G)$.

Let $D \in \mathcal{N}$, $f \in \ker(D)$ and denote by $U \subseteq H$ the corresponding one-dimensional unipotent subgroup. Note that $\ker(D)$ is equal to the invariant ring $\mathcal{O}(H)^U$. By Lemma 2.20, there exists $q \in \mathcal{O}(G)^U$ such that $q|_H = f$.

Denote by E the locally nilpotent derivation of $\mathcal{O}(G)$ induced by right multiplication of U on G. Since $q \in \mathcal{O}(G)^U = \ker(E)$, the derivation qE of $\mathcal{O}(G)$ is locally nilpotent. Since U is a subgroup of H, the fibers gH, $g \in G$ of π are stable under the induced $\mathbb{G}_a$-action of qE and thus this $\mathbb{G}_a$-action gives an algebraic subgroup of $\mathrm{Aut}^{\mathrm{alg}}_{G/H}(G)$. Note that we have a commutative diagram of the following form

$$\begin{tikzcd} \mathcal{O}(G) \arrow[r, "qE"] \arrow[d] & \mathcal{O}(G) \arrow[d] \\ \mathcal{O}(H) \arrow[r, "fD"] & \mathcal{O}(H) \end{tikzcd}$$

where the vertical arrows are induced by the embedding $H \subseteq G$. Therefore we found our desired extension of the $\mathbb{G}_a$-action induced by fD. $\square$

Proof of Lemma 2.20. Since H and U are characterless, the quotients G/U, H/U and G/H are quasi-affine; see [Tim11, Example 3.10]. Hence, the canonical morphisms

- $\iota_{G/U} \colon G/U \to (G/U)_{\mathrm{aff}} \coloneqq \mathrm{Spec}(\mathcal{O}(G)^U)$
- $\iota_{H/U} \colon H/U \to (H/U)_{\mathrm{aff}} \coloneqq \mathrm{Spec}(\mathcal{O}(H)^U)$
- $\iota_{G/H} \colon G/H \to (G/H)_{\mathrm{aff}} \coloneqq \mathrm{Spec}(\mathcal{O}(G)^H)$

are dominant open immersions; see [Gro61, §5, Proposition 5.1.2]. The targets of these open immersions are affine schemes that are, in general, not of finite type over $\mathbf{k}$. There are unique G-actions on $(G/U)_{\mathrm{aff}}$ and $(G/H)_{\mathrm{aff}}$ such that $\iota_{G/U}$ and $\iota_{G/H}$ are G-equivariant and a unique H-action on $(H/U)_{\mathrm{aff}}$ such that $\iota_{H/U}$ is H-equivariant; see [KRvS19, Lemma 5]. Moreover, the canonical G-equivariant morphism $\rho \colon G/U \to G/H$ induces a unique G-equivariant morphism

$$\rho_{\mathrm{aff}} \colon (G/U)_{\mathrm{aff}} \to (G/H)_{\mathrm{aff}}$$

such that the following diagram commutes

$$\begin{tikzcd} G/U \arrow[r, "\iota_{G/U}"] \arrow[d, "\rho"] & (G/U)_{\mathrm{aff}} \arrow[d, "\rho_{\mathrm{aff}}"] \\ G/H \arrow[r, "\iota_{G/H}"] & (G/H)_{\mathrm{aff}} . \end{tikzcd}$$

Let $V \subseteq G/H$ be an open affine neighbourhood of $q \coloneqq H \in G/H$. We may assume that there is an $s \in \mathcal{O}(G/H) = \mathcal{O}(G)^H$ such that $V = (G/H)_s$, i.e. V consists of all points in G/H where s does not vanish. Further we may assume that the extension $s_{\mathrm{aff}} \colon (G/H)_{\mathrm{aff}} \to \mathbb{A}^1$ of $s \colon G/H \to \mathbb{A}^1$ via $\iota_{G/H}$ vanishes on the complement of $\iota_{G/H}(G/H)$ in $(G/H)_{\mathrm{aff}}$. Hence, $\iota_{G/H}(V) = ((G/H)_{\mathrm{aff}})_{s_{\mathrm{aff}}}$ and therefore

$$\iota_{G/U}(\rho^{-1}(V)) \subseteq \rho_{\mathrm{aff}}^{-1}(\iota_{G/H}(V)) = \rho_{\mathrm{aff}}^{-1}(((G/H)_{\mathrm{aff}})_{s_{\mathrm{aff}}}) = ((G/U)_{\mathrm{aff}})_{s_{\mathrm{aff}} \circ \rho_{\mathrm{aff}}} .$$

By [Sta21, Lemma 01P7] we have $(\mathcal{O}(G/U))_{s\circ\rho} = \mathcal{O}((G/U)_{s\circ\rho})$ and thus

$$((G/U)_{\mathrm{aff}})_{s_{\mathrm{aff}}\circ\rho_{\mathrm{aff}}} = ((G/U)_{s\circ\rho})_{\mathrm{aff}} = (\rho^{-1}(V))_{\mathrm{aff}}\,.$$

Hence, we have the following commutative diagram

$$\begin{tikzcd}
\rho^{-1}(V) \arrow[r, equal] & (G/U)_{s\circ\rho} \arrow[r] \arrow[d, "\text{open}"'] & ((G/U)_{\mathrm{aff}})_{s_{\mathrm{aff}}\circ\rho_{\mathrm{aff}}} \arrow[r, equal] \arrow[d, "\text{open}"] & (\rho^{-1}(V))_{\mathrm{aff}} \\
& G/U \arrow[r, "\iota_{G/U}"] & (G/U)_{\mathrm{aff}}\,. &
\end{tikzcd} \tag{$\triangle$}$$

Furthermore, we may shrink V such that there exists a finite Galois covering $\tau\colon V' \to V$ for some finite group Γ (i.e. τ is a geometric quotient for a free Γ-action on V') such that the pull-back map ρ' in the following pull-back diagram

$$\begin{tikzcd}
V' \times_V \rho^{-1}(V) \arrow[r, "\tau'"] \arrow[d, "\rho'"] & \rho^{-1}(V) \arrow[d, "\rho|_{\rho^{-1}(V)}"] \\
V' \arrow[r, "\tau"] & V
\end{tikzcd}$$

is a trivial H/U-bundle; see [Ser58, §1.5 and Proposition 3]. In particular, there exists an isomorphism $\varphi\colon V' \times (H/U) \to V' \times_V \rho^{-1}(V)$ such that $\rho'\circ\varphi\colon V'\times(H/U) \to V'$ is the projection onto the first factor. As $\tau\colon V' \to V$ is finite and V is affine, V' is affine as well. Note further, that the Γ-action on V' induces a natural free Γ-action on $V' \times_V \rho^{-1}(V)$ such that ρ' is Γ-equivariant and τ' is a geometric quotient for this Γ-action. Choose $q' \in V'$ such that $\tau(q') = q$.

Let $f \in \mathcal{O}(H/U) = \mathcal{O}(H)^U$. The goal is to extend f to an element in $\mathcal{O}(G)^U$. Consider the morphism

$$f'\colon \Gamma q' \times (H/U) \xrightarrow{\varphi|_{\Gamma q'\times(H/U)}} (\rho')^{-1}(\Gamma q') \xrightarrow{\tau'|_{(\rho')^{-1}(\Gamma q')}} \rho^{-1}(q) = H/U \xrightarrow{f} \mathbb{A}^1\,.$$

Then the extension $f'_{\mathrm{aff}}\colon \Gamma q' \times (H/U)_{\mathrm{aff}} = (\Gamma q' \times (H/U))_{\mathrm{aff}} \to \mathbb{A}^1$ of f' can be extended to a morphism

$$V' \times (H/U)_{\mathrm{aff}} \to \mathbb{A}^1\,, \tag{$*$}$$

as $\Gamma q' \times (H/U)_{\mathrm{aff}}$ is a closed subscheme in the affine scheme $V' \times (H/U)_{\mathrm{aff}}$. Let $F'\colon V' \times (H/U) \to \mathbb{A}^1$ be the composition of $\mathrm{id}_{V'} \times \iota_{H/U}$ with the morphism $(*)$. By construction we have that $F'|_{\Gamma q'\times(H/U)} = f'$. Now, let

$$G' \coloneqq \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma\cdot(F'\circ\varphi^{-1})\colon V' \times_V \rho^{-1}(V) \to \mathbb{A}^1$$

be the average of $F'\circ\varphi^{-1}$ over Γ. Since $f'\circ(\varphi|_{(\rho')^{-1}(\Gamma q')})^{-1} = f\circ\tau'|_{(\rho')^{-1}(\Gamma q')}$ is Γ-invariant, it follows that

$$f\circ\tau'|_{(\rho')^{-1}(\Gamma q')} = G'|_{(\rho')^{-1}(\Gamma q')}\,. \tag{$**$}$$

Since G' is Γ-invariant and since τ' is a geometric quotient for the Γ-action on $V' \times_V \rho^{-1}(V)$, there exists a morphism $F\colon \rho^{-1}(V) \to \mathbb{A}^1$ such that $G' =$

$F \circ \tau'$. Using $(**)$, we find $F|_{\rho^{-1}(q)} = f$. The commutative diagramm $(\triangle)$ implies that F extends to a morphism $F_{\mathrm{aff}} \colon (\rho^{-1}(V))_{\mathrm{aff}} = \rho_{\mathrm{aff}}^{-1}(\iota_{G/H}(V)) \to \mathbb{A}^1$ via $\iota_{G/U}$, i.e.

$$F_{\mathrm{aff}}(\iota_{G/U}(gU)) = F(gU) \quad \text{for all } gU \in \rho^{-1}(V)\,.$$

Hence the restriction $F_{\mathrm{aff}}|_{\rho_{\mathrm{aff}}^{-1}(H)} \colon \rho_{\mathrm{aff}}^{-1}(H) \to \mathbb{A}^1$ satisfies

$$F_{\mathrm{aff}}(\iota_{G/U}(hU)) = F(hU) = f(hU) \quad \text{for all } h \in H\,.$$

Since $\rho_{\mathrm{aff}}^{-1}(H)$ is a closed subscheme of the affine scheme $(G/U)_{\mathrm{aff}}$, there exists an extension of $F_{\mathrm{aff}}|_{\rho_{\mathrm{aff}}^{-1}(H)}$ to a morphism $(G/U)_{\mathrm{aff}} \to \mathbb{A}^1$. This is our desired element in $\mathcal{O}(G)^U$. □

2.4. **The proof of Theorem 2.5.** Throughout this subsection we use the following notation.

Notation. Let $f \colon X \to Z$ be a morphism of varieties, then we denote by $X_Z^{(2)}$ the complement of the diagonal in the fiber product $X \times_Z X$ and we denote by $(\ker \mathrm{d}f)^\circ$ the complement of the zero section in the kernel of the differential $\mathrm{d}f \colon TX \to TZ$.

We start with the following rather technical result that will turn out to be the key.

Proposition 2.22. *Let $\pi \colon X \to P$, $\rho \colon X \to Q$ be smooth morphisms of smooth irreducible varieties such that there exists a morphism $\eta \colon Q \to P$ with $\pi = \eta \circ \rho$. Assume that $\mathrm{Aut}_P^{\mathrm{alg}}(X)$ acts sufficiently transitively on each fiber of π.*

If Z is a smooth variety and $f \colon Z \to X$ is a morphism such that each non-empty fiber of $\pi \circ f \colon Z \to P$ has the same dimension $k \geq 0$, then there exists a $\varphi \in \mathrm{Aut}_P^{\mathrm{alg}}(X)$ with

$$\dim((\varphi \circ f) \times (\varphi \circ f))^{-1}(X_Q^{(2)}) \leq \dim Z + \dim P - \dim Q + k \tag{A}$$

$$\dim(d(\varphi \circ f))^{-1}(\ker d\rho)^\circ \leq \dim Z + \dim P - \dim Q + k\,. \tag{B}$$

For the proof of the estimate (B) in this key proposition, we need the following estimate:

Lemma 2.23. *Let $f \colon X \to Y$ be a morphism of varieties such that X is smooth and denote by k the maximal dimension among the fibers of f.*

Then the kernel of the differential $df \colon TX \to TY$, i.e. the closed subvariety

$$\ker(df) := \bigcup_{x \in X} \ker(d_x f) \subseteq TX\,,$$

satisfies $\dim \ker(df) \leq \dim X + k$.

Proof of Lemma 2.23. Let $X = \bigcup_{i=1}^n X_i$ be a partition into smooth, irreducible, locally closed subvarieties $X_1, \ldots, X_n$ in X such that

$$f_i \coloneqq f|_{X_i} \colon X_i \to \overline{f(X_i)}$$

is smooth for each $i = 1, \ldots, n$ (see [Har77, Lemma 10.5, Ch. III]). Note that $f(X_i)$ is an open subvariety of $\overline{f(X_i)}$ that is smooth, see [GR03, Proposition 3.1, Exposé II]. Let $x \in X_i$. Thus the differential $\mathrm{d}_x f_i \colon T_x X_i \to T_{f(x)}\overline{f(X_i)}$ is surjective and since $\dim f_i^{-1}(x) \leq k$, we get $\dim \ker(\mathrm{d}_x f_i) \leq k$. Then the kernel of

$$T_x X_i \hookrightarrow T_x X \xrightarrow{\mathrm{d}_x f} T_{f(x)} Y$$

has dimension $\leq k$, which implies $\dim \ker(\mathrm{d}_x f) \leq \dim T_x X - \dim T_x X_i + k$. Since X is smooth, we have $\dim T_x X \leq \dim X$ (we did not assumed that X is equidimensional, hence we do not necessarily have an equality) and since X_i is smooth and irreducible, we have $\dim T_x X_i = \dim X_i$. Thus we get

$$\begin{aligned}\dim \ker(\mathrm{d}f)|_{X_i} &\leq \dim X_i + \max_{x \in X_i} \dim \ker(\mathrm{d}_x f) \\ &\leq \dim X_i + \dim X - \dim X_i + k = \dim X + k\,.\end{aligned}$$

Hence, $\dim \ker(\mathrm{d}f) \leq \max_{1 \leq i \leq n} \dim(\ker \mathrm{d}_x f) \cap TX|_{X_i} \leq \dim X + k$. $\square$

Proof of Proposition 2.22. Let $G \coloneqq \mathrm{Aut}_P^{\mathrm{alg}}(X)$ and let $\mathcal{G}$ be the family of all connected algebraic subgroups of $\mathrm{Aut}(X)$ that lie in $\mathrm{Aut}_P(X)$. By definition G is generated by the subgroups inside $\mathcal{G}$ and $\mathcal{G}$ is closed under conjugation by elements of G.

Since $\pi \colon X \to P$ is smooth and G acts sufficiently transitively on each fiber of π, the morphisms

$$\kappa \colon X_P^{(2)} \to P\,, \quad (x, x') \mapsto \pi(x)$$

and

$$\kappa' \colon (\ker \mathrm{d}\pi)^\circ \to X \xrightarrow{\pi} P$$

are smooth and G acts transitively on each fiber of κ and κ'.

Applying Proposition 2.9 to κ and the image of G in $\mathrm{Aut}_P(X_P^{(2)})$ under $\varphi \mapsto \varphi \times_P \varphi$ gives $H_1, \ldots, H_s \in \mathcal{G}$ such that $\mathcal{H} = (H_1, \ldots, H_s)$ is big enough for smoothness with respect to κ. Likewise one gets $H'_1, \ldots, H'_{s'} \in \mathcal{G}$ such that $\mathcal{H}' = (H'_1, \ldots, H'_{s'})$ is big enough for smoothness with respect to κ'. Using Proposition 2.8(2), $\mathcal{M} = (H_1, \ldots, H_s, H'_1, \ldots, H'_{s'})$ is big enough for smoothness with respect to κ and κ'. By Proposition 2.8(1), $\mathcal{M}$ is also big enough for proper intersection with respect to κ and κ'. Hence, there is an open dense subset $U \subset H_1 \times \cdots \times H_s \times H'_1 \times \cdots \times H'_{s'}$ such that for each element in U the estimate (PI) in Definition 2.7 is satisfied with respect to

- the smooth morphism $\kappa \colon X_P^{(2)} \to P$,
- the morphism $(f \times f)|_{(f\times f)^{-1}(X_P^{(2)})} \colon (f \times f)^{-1}(X_P^{(2)}) \to X_P^{(2)}$ and
- the closed subset $X_Q^{(2)}$ in $X_P^{(2)}$

and

- the smooth morphism $\kappa'\colon (\ker \mathrm{d}\pi)^\circ \to P$,
- the morphism $\mathrm{d}f|_{(\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ}\colon (\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ \to (\ker \mathrm{d}\pi)^\circ$ and
- the closed subset $(\ker \mathrm{d}\rho)^\circ$ in $(\ker \mathrm{d}\pi)^\circ$.

That means that, if we choose an element $(h_1, \dots, h_s, h'_1, \dots, h'_{s'}) \in U$, then the automorphism $\varphi = (h_1 \cdots h_s \cdot h'_1 \cdots h'_{s'})^{-1} \in G$ satisfies the following estimates:

$$\begin{aligned}
&\dim((\varphi \circ f) \times (\varphi \circ f))^{-1}(X_Q^{(2)}) \\
&= \dim(f \times f)^{-1}(X_P^{(2)}) \times_{X_P^{(2)}} (\varphi \times \varphi)^{-1}(X_Q^{(2)}) \\
&\overset{\text{(PI)}}{\leq} \dim(f \times f)^{-1}(X_P^{(2)}) \times_P X_Q^{(2)} + \dim P - \dim X_P^{(2)}
\end{aligned}$$

and

$$\begin{aligned}
&\dim(\mathrm{d}(\varphi \circ f))^{-1}(\ker \mathrm{d}\rho)^\circ \\
&= \dim(\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ \times_{(\ker \mathrm{d}\pi)^\circ} (\mathrm{d}\varphi)^{-1}(\ker \mathrm{d}\rho)^\circ \\
&\overset{\text{(PI)}}{\leq} \dim(\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ \times_P (\ker \mathrm{d}\rho)^\circ + \dim P - \dim(\ker \mathrm{d}\pi)^\circ\,.
\end{aligned}$$

Since $\pi\colon X \to P$ and $\kappa\colon X \to Q$ are both smooth morphisms of smooth irreducible varieties, we get

- $\dim X_P^{(2)} = 2\dim X - \dim P$
- $\dim X_Q^{(2)} = 2\dim X - \dim Q$
- $\dim(\ker \mathrm{d}\pi)^\circ = 2\dim X - \dim P$.

Hence, it is enough to show the following estimates:

(1) $\dim(f \times f)^{-1}(X_P^{(2)}) \times_P X_Q^{(2)} \leq 2\dim X + \dim Z - \dim Q - \dim P + k$
(2) $\dim(\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ \times_P (\ker \mathrm{d}\rho)^\circ \leq 2\dim X + \dim Z - \dim Q - \dim P + k$

We establish (1): Consider the following pull-back diagram

$$\begin{tikzcd}
(f \times f)^{-1}(X_P^{(2)}) \times_P X_Q^{(2)} \arrow[r] \arrow[d] & X_Q^{(2)} \arrow[d, "\varepsilon"] \\
(f \times f)^{-1}(X_P^{(2)}) \arrow[r] & P
\end{tikzcd} \tag{3}$$

Let $Q_0 \subset Q$ be the image of $\rho\colon X \to Q$. Since ρ is smooth, Q_0 is an open dense subset of Q. Hence $\eta|_{Q_0}\colon Q_0 \to P$ is a morphism of smooth irreducible varieties. Since $\pi = \eta|_{Q_0} \circ \rho\colon X \to P$ is smooth, it follows that $\eta|_{Q_0}$ is smooth. Thus

$$\varepsilon\colon X_Q^{(2)} = X_{Q_0}^{(2)} \to Q_0 \xrightarrow{\eta|_{Q_0}} P$$

is smooth as well of relative dimension $2\dim X - \dim Q - \dim P$. Since each non-empty fiber of $\pi \circ f\colon Z \to P$ has dimension k, the image of $Z \times_P Z \to P$

is contained in $\pi(f(Z))$ and each non-empty fiber of it has dimension $\leq 2k$. Thus the same holds for

$$(f \times f)^{-1}(X_P^{(2)}) \to P\,.$$

Hence $\dim(f \times f)^{-1}(X_P^{(2)}) \leq \dim \overline{\pi(f(Z))} + 2k = \dim Z + k$ and the estimate (1) follows from the pull-back diagram (3).

We establish (2): Consider the following fiber product:

$$\begin{array}{ccc} (\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ \times_P (\ker \mathrm{d}\rho)^\circ & \to & (\ker \mathrm{d}\rho)^\circ \\ \downarrow & & \downarrow \\ (\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ & \longrightarrow & P \end{array} \tag{4}$$

Since $\rho\colon X \to Q$ is smooth, we get $\dim(\ker \mathrm{d}\rho)^\circ = 2\dim X - \dim Q$. Hence $(\ker \mathrm{d}\rho)^\circ \to P$ is smooth of relative dimension $2\dim X - \dim Q - \dim P$ (since $(\ker \mathrm{d}\rho)^\circ \to X$ and $\pi\colon X \to P$ are smooth). Moreover,

$$\dim(\mathrm{d}f)^{-1}(\ker \mathrm{d}\pi)^\circ \leq \dim \ker \mathrm{d}(\pi \circ f) \leq \dim Z + k$$

where the second inequality follows from Lemma 2.23, since each non-empty fiber of $\pi \circ f\colon Z \to P$ has dimension k and Z is smooth. Thus the desired estimate (2) follows from the pull-back diagram (4). ☐

Lemma 2.24. *Let $f\colon Z \to X$ and $\rho\colon X \to Q$ be morphisms of varieties. Then we have the following:*

$$\dim Z_Q^{(2)} = \max\left\{\dim(f \times f)^{-1}(X_Q^{(2)}), \dim Z_X^{(2)}\right\}$$
$$\dim \ker d(\rho \circ f)^\circ = \max\left\{\dim(df)^{-1}(\ker d\rho)^\circ, \dim(\ker df)^\circ\right\}\,.$$

Proof. The first equality follows, since the underlying set of $Z_Q^{(2)}$ is the disjoint union of

$$\{\,(z_1, z_2) \in Z \times Z \mid \rho(f(z_1)) = \rho(f(z_2))\,,\ f(z_1) \neq f(z_2)\,\} = (f \times f)^{-1}(X_Q^{(2)})$$

and the underlying subset of $Z_X^{(2)}$ in $Z \times Z$. The second equality follows, since the underlying set of $\ker \mathrm{d}(\rho \circ f)^\circ$ is the disjoint union of

$$\{\, v \in TZ \mid \mathrm{d}(\rho \circ f)(v) = 0\,,\ (\mathrm{d}f)(v) \neq 0\,\} = (\mathrm{d}f)^{-1}(\ker \mathrm{d}\rho)^\circ$$

and the underlying subset of $(\ker \mathrm{d}f)^\circ$ in TZ. ☐

In order to construct embeddings, we use the following characterization of them:

Proposition 2.25. *A morphism $f\colon Z \to X$ of varieties is an embedding if and only if the following conditions are satisfied*

- *f is proper*
- *f is injective*
- *for each $z \in Z$, the differential $d_z f\colon T_z Z \to T_{f(z)} X$ is injective.*

We prove this proposition in the Appendix B for the lack of a reference to an elementary proof; see Proposition B.1. From Proposition 2.22 and Lemma 2.24 we get now immediately the following consequence:

Corollary 2.26. *Let X be a smooth irreducible variety such that* $\mathrm{Aut}^{\mathrm{alg}}(X)$ *acts sufficiently transitively on X. If $\rho\colon X \to Q$ is a finite étale surjection and $Z \subset X$ is a smooth closed subvariety with* $\dim X \geq 2\dim Z + 1$, *then there exists $\varphi \in$* $\mathrm{Aut}^{\mathrm{alg}}(X)$ *such that $\rho \circ \varphi\colon X \to Q$ restricts to an isomorphism $Z \to \rho(\varphi(Z))$.*

Proof. We apply Proposition 2.22 to $\pi\colon X \to P \coloneqq \{\mathrm{pt}\}$, $\rho\colon X \to Q$ (note that Q is irreducible and smooth by [GR03, Proposition 3.1, Exposé II]), and the inclusion $f\colon Z \hookrightarrow X$ in order to get a $\varphi \in \mathrm{Aut}^{\mathrm{alg}}(X)$ such that

$$\dim((\varphi \circ f) \times (\varphi \circ f))^{-1}(X_Q^{(2)}) \leq 2\dim Z - \dim Q \leq \dim X - 1 - \dim Q < 0$$
$$\dim(\mathrm{d}(\varphi \circ f))^{-1}(\ker \mathrm{d}\rho)^{\circ} \leq 2\dim Z - \dim Q \leq \dim X - 1 - \dim Q < 0$$

where we used the assumption $\dim X \geq 2\dim Z + 1$. Applying Lemma 2.24 to $\varphi \circ f\colon Z \to X$ and $\rho\colon X \to Q$ yields, that the composition

$$Z \overset{f}{\hookrightarrow} X \xrightarrow{\varphi} X \xrightarrow{\rho} Q$$

is injective and the differential $\mathrm{d}_z(\rho \circ \varphi \circ f)\colon T_zZ \to T_{\rho(\varphi(z))}Q$ is injective for each $z \in Z$. As the composition $\rho \circ \varphi \circ f\colon Z \to Q$ is also proper, the statement follows from Proposition 2.25. □

The following number associated to each morphism will be crucial for the proof of Theorem 2.5:

Definition 2.27. For each morphism $f\colon Z \to X$ of varieties we define the *θ-invariant* by

$$\theta_f \coloneqq \max\{\, \dim Z_X^{(2)} \,,\ \dim(\ker \mathrm{d}f)^{\circ} \,\}\,.$$

In case $W \subseteq Z$ is locally closed, we define the *restricted θ-invariant* by

$$\theta_f|_W \coloneqq \max\{\, \dim W_X^{(2)} \,,\ \dim(\ker \mathrm{d}f)^{\circ}|_W \,\}\,.$$

Note that θ_f stays the same if we replace f with $\varphi \circ f$ for an automorphism $\varphi \in \mathrm{Aut}(X)$. Moreover, the following remarks hold.

Remark 2.28. If $f\colon Z \to X$ is a proper morphism, then f is an embedding if and only if $\theta_f < 0$. This follows directly from Proposition 2.25.

Remark 2.29. If $f\colon Z \to X$ is a morphism and if $X_1, \ldots, X_r \subseteq X$ are locally closed subsets with $\bigcup_i X_i = X$, then we have

$$\theta_f = \max_i \theta_f|_{f^{-1}(X_i)}\,.$$

The next result will enable us to inductively lower the θ-invariant in the proof of Theorem 2.5. We formulate it first in a general version suitable for the applications, and we formulate it afterwards in the special case needed for the proof of Theorem 2.5.

Proposition 2.30. *Let $\rho\colon X \to Q$ be a principal $\mathbb{G}_a$-bundle, Z an affine variety and $r\colon Z \to Q$ a finite morphism. Moreover, let $A \subseteq Z$ be a closed subset, let $g_A\colon A \to X$ be a morphism with $\rho \circ g_A = r|_A$ and let $Z_1, \ldots, Z_s \subseteq Z \setminus A$ be locally closed subsets.*

Then there is a morphism $g\colon Z \to X$ with $\rho \circ g = r$, $g|_A = g_A$ and such that the restricted θ-invariants satisfy $\theta_g|_{Z_i} \leq \theta_r|_{Z_i} - 1$ for all i.

Part of Proposition 2.30 can be illustrated by the following commutative diagram with filler g:

$$\begin{tikzcd} A \arrow[r, "g_A"] \arrow[d, phantom, "\cap"] & X \arrow[d, "\rho"] \\ Z \arrow[r, "r"] \arrow[ur, dotted, "\exists g"] & Q \end{tikzcd} .$$

Proof. Let $W \coloneqq r(Z) \subset Q$. Since $r\colon Z \to Q$ is finite and Z is affine, W is a closed affine subvariety of Q by Chevalley's Theorem, [GW10, Theorem 12.39]. The restriction $\rho^{-1}(W) \to W$ of ρ is locally trivial with respect to the Zariski topology (see [Ser58, Example, §2.3]) and since W is affine, it is a trivial principal $\mathbb{G}_a$-bundle (see e.g. [Gro58, Proposition 1, §1]); this means, there exists a W-isomorphism $\iota\colon W \times \mathbb{G}_a \to \rho^{-1}(W)$.

For $i \in \{1, \ldots, s\}$, we choose finite subsets

$$R_i \subseteq (Z_i)^{(2)}_Q \quad \text{and} \quad S_i \subseteq (\ker \mathrm{d}r)^\circ\,|_{Z_i}$$

such that each irreducible component of $(Z_i)^{(2)}_Q$ and of $(\ker \mathrm{d}r)^\circ\,|_{Z_i}$ contains a point of R_i and of S_i, respectively. Let $\mathrm{pr}_1, \mathrm{pr}_2\colon Z \times Z \to Z$ be the projection onto the first and second factor, respectively. As Z is affine and $Z_i \subset Z \setminus A$ for all i, there exists a morphism $q\colon Z \to \mathbb{G}_a$ such that

- q restricted to A is equal to $\mathrm{pr}_{\mathbb{G}_a} \circ \iota^{-1} \circ g_A$ where $\mathrm{pr}_{\mathbb{G}_a}\colon W \times \mathbb{G}_a \to \mathbb{G}_a$ denotes the natural projection onto $\mathbb{G}_a$,
- q restricted to $\mathrm{pr}_1(R_i) \cup \mathrm{pr}_2(R_i)$ is injective for all i and
- $\mathrm{d}q\colon TZ \to T\mathbb{G}_a$ restricted to S_i never vanishes for all i.

Now, we define

$$\begin{aligned} g\colon\quad Z &\longrightarrow W \times \mathbb{G}_a \xrightarrow[\simeq]{\iota} \rho^{-1}(W) \subset X\,. \\ z &\longmapsto (r(z), q(z)) \end{aligned}$$

Since $\iota\colon W \times \mathbb{G}_a \to \rho^{-1}(W)$ is a W-isomorphism, $\rho \circ g = r$. Moreover, by construction we have $g|_A = g_A$. Now, we claim that

$$\dim(Z_i)^{(2)}_X \leq \dim(Z_i)^{(2)}_Q - 1 \quad \text{for all } i\,, \tag{5}$$

$$\dim(\ker \mathrm{d}g)^\circ|_{Z_i} \leq \dim(\ker \mathrm{d}r)^\circ|_{Z_i} - 1 \quad \text{for all } i\,. \tag{6}$$

For proving (5), take an irreducible component V of $(Z_i)^{(2)}_Q$. Then

$$V^\circ \coloneqq \{\, (v_1, v_2) \in V \mid g(v_1) \neq g(v_2) \,\}$$

is an open subset of V. By construction, there exists $(z_1, z_2) \in R_i \cap V$ with $g(z_1) \neq g(z_2)$. Hence, V° is non-empty. This implies that $V \cap (Z_i)_X^{(2)}$ is properly contained in V. Since $(Z_i)_X^{(2)}$ is a closed subset of $(Z_i)_Q^{(2)}$, we get (5). Similarly, we get (6) by using that $\mathrm{d}g$ restricted to S_i never vanishes. Together, the estimates (5) and (6) imply that $\theta_g|_{Z_i} \leq \theta_r|_{Z_i} - 1$ for all i. $\square$

By choosing A as the empty set, $s = 1$, and Z_1 equal to Z, Proposition 2.30 becomes the following.

Corollary 2.31. *Let $\rho\colon X \to Q$ be a principal $\mathbb{G}_a$-bundle, Z an affine variety and $r\colon Z \to Q$ a finite morphism. Then there exists a morphism $g\colon Z \to X$ such that $\rho \circ g = r$ and $\theta_g \leq \theta_r - 1$.* $\square$

We prove Theorem 2.5 by inductively applying Corollary 2.31.

Proof of Theorem 2.5. Let Z be a smooth affine variety such that $\dim X \geq 2\dim Z + 1$ and such that condition d) is satisfied. Let $n := \dim P = \dim Z$.

The following claim will enable us to lower the θ-invariant.

Claim: $\exists\, f\colon Z \to X$ such that $\pi \circ f\colon Z \to P$ is finite and $\theta_f \geq 0$
$\implies \exists\, g\colon Z \to X$ such that $\pi \circ g\colon Z \to P$ is finite and $\theta_g < \theta_f$

Proof of Claim. Let $f\colon Z \to X$ be a morphism such that $\pi \circ f\colon Z \to P$ is finite and $\theta_f \geq 0$. By condition d), $\eta\colon Q \to P$ is surjective and since $\rho\colon X \to Q$ is surjective, we get that $\pi\colon X \to P$ is surjective as well. Since ρ and π are smooth surjections and since X is smooth and irreducible, it follows that P and Q are smooth and irreducible; see [GR03, Proposition 3.1, Exposé II]. By condition b), $\mathrm{Aut}_P^{\mathrm{alg}}(X)$ acts sufficiently transitively on each fiber of π. Thus we may apply Proposition 2.22 to $f\colon Z \to X$ and may choose a $\varphi \in \mathrm{Aut}_P^{\mathrm{alg}}(X)$ such that $f' := \varphi \circ f$ satisfies

$$\begin{aligned}&\max\{\, \dim(f' \times f')^{-1}(X_Q^{(2)})\,,\ \dim(\mathrm{d}f')^{-1}(\ker \mathrm{d}\rho)^\circ \,\}\\ &\leq \dim Z + \dim P - \dim Q\,,\end{aligned}$$

since $\pi \circ f\colon Z \to P$ is finite (see condition d)). Note that

$$\dim Z + \dim P - \dim Q = 2n - \dim Q \leq \dim X - 1 - \dim Q = 0\,,$$

since $\rho\colon X \to Q$ is a principal $\mathbb{G}_a$-bundle. Thus by Lemma 2.24:

$$\begin{aligned}\dim Z_{Q,\rho\circ f'}^{(2)} &\leq \max\{\, 0\,,\ \dim Z_{X,f'}^{(2)} \,\}\\ \dim \ker \mathrm{d}(\rho \circ f')^\circ &\leq \max\{\, 0\,,\ \dim \ker(\mathrm{d}f')^\circ \,\}\end{aligned}$$

where we compute $Z_{Q,\rho\circ f'}^{(2)}$ and $Z_{X,f'}^{(2)}$ with respect to $\rho \circ f'$ and f', respectively. Thus $\theta_{f'} \leq \theta_{\rho\circ f'} \leq \max\{0, \theta_{f'}\}$, which implies (as $\theta_f = \theta_{f'} \geq 0$)

$$\theta_f = \theta_{f'} = \theta_{\rho\circ f'}\,. \tag{7}$$

Note that $\rho \circ f'\colon Z \to Q$ is finite, since $\pi \circ f' = \pi \circ f$ is finite. Hence, applying Corollary 2.31 to $\rho \circ f'\colon Z \to Q$ yields a morphism $g\colon Z \to X$ such that $\rho \circ g = \rho \circ f'$ and $\theta_g < \theta_{\rho \circ f'}$. Thus, we get $\theta_g < \theta_f$ by (7). Since $\pi \circ g = \pi \circ f'$ is finite, this completes the proof of the claim. □

By condition d), the composition $\eta \circ r \colon Z \to P$ is finite. In particular, $r \colon Z \to Q$ is finite and since Z is affine, there exists a morphism $f \colon Z \to X$ such that $\rho \circ f = r$; see Corollary 2.31. By the finiteness of $\pi \circ f = \eta \circ r$, we can iteratively apply the claim in order to get a morphism $g \colon Z \to X$ such that $\pi \circ g \colon Z \to P$ is finite and $\theta_g < 0$. In particular, $g \colon Z \to X$ is proper, and, thus, $g \colon Z \to X$ is an embedding by Remark 2.28. □

3. Applications: Embeddings into algebraic groups

In this section we apply the results from Section 2 in order to construct embeddings of smooth affine varieties into characterless algebraic groups.

In the entire section, we use the language of and results about algebraic groups, with more notions showing up in later subsections. For the basic results on algebraic groups we refer to [Hum75] and for the basic results about Lie algebras and root systems we refer to [Hum78].

3.1. Embeddings into a product of the form $\mathbb{A}^m \times H$.

In this subsection, we study embeddings of smooth affine varieties into varieties of the from $\mathbb{A}^m \times H$ where H is a characterless algebraic group. While this is of independent interest, for us it is also a preparation to establish Theorem A; compare with the outline of the proof in the introduction.

Corollary 3.1. *Let H be a characterless algebraic group and let Z be a smooth affine variety with*

$$2 \dim Z + 1 \leq m + \dim H \,. \tag{$*$}$$

If $\dim Z \leq m$, then Z admits an embedding into $\mathbb{A}^m \times H$.

Proof. We may and do assume that H is connected. We set $d \coloneqq \dim Z \leq m$ and $G \coloneqq \mathbb{A}^{m-d} \times H$. Since G is a connected characterless algebraic group, $\mathrm{Aut}^{\mathrm{alg}}(G)$ acts sufficiently transitively on G by Example 2.12.

Let $X = \mathbb{A}^d \times G \simeq \mathbb{A}^m \times H$. Since $\dim G = m + \dim H - d \geq d + 1 \geq 1$ due to $(*)$ and since G is characterless, we may and do choose a one-dimensional unipotent subgroup $U \subseteq G$. Let $Q = \mathbb{A}^d \times G/U$. We apply Theorem 2.5 and Remark 2.6 to the natural projections

$$\pi \colon X \to \mathbb{A}^d \,, \quad \rho \colon X \to Q \quad \text{and} \quad \eta \colon Q \to \mathbb{A}^d$$

and get our desired embedding $Z \to X$. □

Remark 3.2. Corollary 3.1 gives us back the Holme-Kaliman-Srinivas embedding theorem, when we take for H the trivial group.

3.2. **Embeddings into a product of the form $\mathbb{A}^m \times (\mathrm{SL}_2)^s$.** In this subsection we study the special case $\mathbb{A}^m \times (\mathrm{SL}_2)^s$. The main result of the subsection is Proposition 3.3, which is an analog of Corollary 3.1 with a weaker dimension condition. This result will be used in order to get optimal dimension conditions for embeddings into characterless algebraic groups of low dimension in Subsection 3.4.

Proposition 3.3. *Let $s, m \geq 0$ be integers and let Z be a smooth affine variety with*

$$2 \dim Z + 1 \leq m + \dim\left((\mathrm{SL}_2)^s\right) . \tag{$**$}$$

If $\dim Z \leq m + s$, then Z admits an embedding into $\mathbb{A}^m \times (\mathrm{SL}_2)^s$.

Remark 3.4. In Proposition 3.3 we may replace the condition $\dim Z \leq m$ by $s-1 \leq m$ in case $m+3s$ is odd and by $s-2 \leq m$ in case $m+3s$ is even. Indeed, if $m+3s$ is odd, then $s-1 \leq m$ implies that

$$\dim Z \overset{(**)}{\leq} \frac{m+3s-1}{2} = \frac{m+(s-1)+2s}{2} \leq \frac{m+m+2s}{2} = m+s$$

and if $m+3s$ is even, then $s-2 \leq m$ implies that

$$\dim Z \overset{(**)}{\leq} \frac{m+3s-2}{2} = \frac{m+(s-2)+2s}{2} \leq \frac{m+m+2s}{2} = m+s\,.$$

Proof of Proposition 3.3. We may and do assume that $\dim Z \geq 0$. If $\dim Z < s$, we may replace Z with $Z \times \mathbb{A}^{s-\dim Z}$ and the assumed dimension estimates are still satisfied; thus, we may and do assume that $\dim Z \geq s$. We set $d \coloneqq \dim Z$.

Choose a finite morphism $r \colon Z \to \mathbb{A}^d$ (which exists due to Noether Normalization). For any subset $I \subseteq \{1, \ldots, s\}$, let

$$H_I \coloneqq \{\, (x_1, \ldots, x_d) \in \mathbb{A}^d \mid x_i = 0 \text{ for each } i \in I \text{ and } x_i \neq 0 \text{ for each } i \notin I \,\}\,.$$

Moreover, we denote for $k \in \{0, \ldots, d\}$

$$Z_k \coloneqq \{\, z \in Z \mid \operatorname{rank} \mathrm{d}_z r = k \,\}\,,$$

which is a locally closed subset of Z. Note that $\dim Z_k \leq k$. (Indeed, since $r|_{Z_k} \colon Z_k \to r(Z_k)$ is a finite morphism, there exists $z \in Z_k$ with $\dim Z_k = \operatorname{rank} \mathrm{d}_z(r|_{Z_k}) \leq \operatorname{rank} \mathrm{d}_z r = k$.) Using Kleiman's Transversality Theorem [Kle74, 2. Theorem] there exists an affine linear automorphism φ of $\mathbb{A}^d$ such that

$$\dim Z_k \cap r^{-1}(\varphi^{-1}(H_I)) \leq \dim Z_k + \dim H_I - d \leq k - |I|\,.$$

Hence, after replacing r by $\varphi \circ r$, we may assume that the dimension of the locally closed subset

$$Z_{k,I} \coloneqq Z_k \cap r^{-1}(H_I) \subseteq Z$$

is less than or equal to $k - |I|$. Since $\operatorname{rank} \mathrm{d}_z r = k$ for each $z \in Z_{k,I}$ we get

$$\dim(\ker \mathrm{d}r)^\circ|_{Z_{k,I}} \leq \dim(\ker \mathrm{d}r)|_{Z_{k,I}} = \dim Z_{k,I} + (d-k) \leq d - |I|\,. \tag{8}$$

Now, for $I \subseteq \{1, \ldots, s\}$, let

$$Z_I \coloneqq r^{-1}(H_I) = \bigcup_{k=0}^{d} Z_{k,I} \subset Z\,.$$

Since $\dim Z_{k,I} \leq k - |I|$ for all k, we get $\dim Z_I \leq d - |I|$. Since $r|_{Z_I} \colon Z_I \to \mathbb{A}^d$ is finite, the projection $\dim(Z_I)^{(2)}_{\mathbb{A}^d} \to Z_I$ to one of the factors is quasi-finite. Hence, $\dim(Z_I)^{(2)}_{\mathbb{A}^d} \leq \dim Z_I \leq d - |I|$, and by the estimate (8) we get $\dim(\ker \mathrm{d}r)^\circ|_{Z_I} \leq d - |I|$. In total the restricted θ-invariants of r satisfy

$$\theta_r|_{Z_I} \leq d - |I| \quad \text{for all } I \subseteq \{1, \ldots, s\}\,. \tag{9}$$

We set

$$X_l \coloneqq (\mathbb{A}^2 \setminus \{(0,0)\})^{l-1} \times \mathbb{A}^2 \times \mathbb{A}^{d-l}\,, \quad Q_l \coloneqq (\mathbb{A}^2 \setminus \{(0,0)\})^{l} \times \mathbb{A}^{d-l}\,,$$

and

$$\rho_l \coloneqq \mathrm{id}_{(\mathbb{A}^2 \setminus \{(0,0)\})^{l-1}} \times \mathrm{pr}_1 \times \mathrm{id}_{\mathbb{A}^{d-l}} \colon X_l \to Q_{l-1}\,,$$

where $\mathrm{pr}_1 \colon \mathbb{A}^2 \to \mathbb{A}^1$ denotes the projection onto the first factor. Next, for $l \in \{0, \ldots, s\}$, we construct inductively finite morphisms $g_l \colon Z \to Q_l$ such that we have $\rho_l \circ g_l = g_{l-1}$, $\theta_{g_l}|_{Z_I} \leq \theta_{g_{l-1}}|_{Z_I}$ for all $I \subseteq \{1, \ldots, s\}$, and $\theta_{g_l}|_{Z_I} \leq \theta_{g_{l-1}}|_{Z_I} - 1$ for $I \subset \{1, \ldots, s\}$ with $l \notin I$.

Let $g_0 \colon Z \to Q_0$ be the finite morphism $r \colon Z \to \mathbb{A}^d$. By induction, we assume that the finite morphism

$$g_{l-1} = \left(g_{l-1}^{(1)}, \ldots, g_{l-1}^{(l+d-1)}\right) \colon Z \to Q_{l-1}$$

is already constructed for some $1 \leq l \leq s$. We apply Proposition 2.30 to the trivial $\mathbb{G}_a$-bundle $\rho_l \colon X_l \to Q_{l-1}$, the closed subset

$$A \coloneqq r^{-1}\left(\{\, (x_1, \ldots, x_d) \in \mathbb{A}^d \mid x_l = 0 \,\}\right) = \bigcup_{I \subseteq \{1, \ldots, s\} \colon l \in I} Z_I \subseteq Z\,,$$

and the morphism

$$g_A \colon A \to Q_l\,, \quad a \mapsto \left(g_{l-1}^{(1)}(a), \ldots, g_{l-1}^{(2l-1)}(a), 1, g_{l-1}^{(2l)}(a), \ldots, g_{l-1}^{(l+d-1)}(a)\right)$$

in order to get a morphism $g_l \colon Z \to X_l$ with $\rho_l \circ g_l = g_{l-1}$, $g_l|_A = g_A$ and

$$\theta_{g_l}|_{Z_I} \leq \theta_{g_{l-1}}|_{Z_I} - 1 \quad \text{for all } I \subseteq \{1, \ldots, s\} \text{ with } l \notin I \tag{10}$$

(here we used that $Z_I \subseteq Z \setminus A$ for each I with $l \notin I$). Since $\rho_l \circ g_l = g_{l-1}$, we get $\theta_{g_l}|_{Z_I} \leq \theta_{g_{l-1}}|_{Z_I}$ for all $I \subseteq \{1, \ldots, s\}$. Since $g_l|_A = g_A$ and since $g_{l-1}^{(2l-1)}$ is equal to the l-th coordinate function of r, we get that the image of g_l is contained in Q_l. Thus, we may consider g_l as a morphism $Z \to Q_l$.

Now,

$$\theta_{g_s}|_{Z_I} \overset{(10)}{\leq} \theta_r|_{Z_I} - (s - |I|) \overset{(9)}{\leq} d - s \quad \text{for all } I \subseteq \{1, \ldots, s\}\,.$$

Since $r\colon Z \to \mathbb{A}^d$ factorizes through $g_s\colon Z \to Q_s$, $\mathbb{A}^d = \bigcup_I H_I$, and $Z_I = r^{-1}(H_I)$, Remark 2.29 implies that

$$\theta_{g_s} = \max_{I \subseteq \{1,\dots,s\}} \theta_{g_s}|_{Z_I} \leq d - s\,. \tag{11}$$

Finally, let $\rho \coloneqq \eta^s \times \mathrm{pr}\colon (\mathrm{SL}_2)^s \times \mathbb{A}^m \to Q_s = (\mathbb{A}^2 \setminus \{(0,0)\})^s \times \mathbb{A}^{d-s}$, where $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^2 \setminus \{(0,0)\}$ denotes the projection to the first column and $\mathrm{pr}\colon \mathbb{A}^m \to \mathbb{A}^{d-s}$ is a surjective linear map (such a map exists, since $d \leq s+m$). Since η is a $\mathbb{G}_a$-bundle, ρ is the composition of $2s+m-d$ many $\mathbb{G}_a$-bundles. Thus, Corollary 2.31 gives us a morphism $g\colon Z \to (\mathrm{SL}_2)^s \times \mathbb{A}^m$ such that $\rho \circ g = g_s$ and $\theta_g \leq \theta_{g_s} - (2s+m-d)$. Using the estimate (11) gives us $\theta_g \leq 2d - (3s+m)$. By $(**)$, we have $2d-(3s+m) < 0$. Since g is proper, Remark 2.28 implies that g is an embedding. $\square$

3.3. **Embeddings into (semi)simple algebraic groups.** In this subsection, we consider arbitrary (semi)simple algebraic groups G as targets of embeddings of smooth affine varieties Z. However, while doing so the price we have to pay is to relax the dimension condition $2\dim Z + 1 \leq \dim G$ in order to get an embedding of Z into G.

From the point of view of the outline of the proof of Theorem A in the introduction, the content of this subsection can be summarized as follows. Fixing a semisimple algebraic group G, we start with two lemmas (Lemma 3.5 and Lemma 3.6) that yield closed subvarieties $X_P \subseteq G$ with $X_P \simeq \mathbb{A}^m \times H$ based on a choice of a parabolic subgroup $P \subseteq G$. We then formulate a version of Theorem A for semisimple algebraic groups where the dimension assumption on Z depends on dimension estimates for a chosen parabolic subgroup P and its subgroups P^u and $R_u(P)$ defined below. Finally, we provide dimension estimates for P^u and $R_u(P)$ for good choices of $P \subset G$ for simple algebraic groups based on the classification of simple Lie algebras (Proposition 3.9). This suffices to yield Theorem A by applying Corollary 3.1 to X_P for a good choice of P (Theorem 3.7).

We recall a few notions. If G is an algebraic group, we denote by $R(G)$ the radical, by $R_u(G)$ its unipotent radical, and by G^u the closed subgroup of G that is generated by all unipotent elements of G. Recall that a connected algebraic group G is called semisimple if G is non-trivial and $R(G)$ is trivial, and it is called simple if G is non-commutative and contains no non-trivial proper closed connected normal subgroup. Moreover, a non-trivial algebraic group G is called reductive if $R_u(G)$ is trivial.

For lack of a reference, we insert a proof of the following classical facts:

Lemma 3.5. *Let G be a semisimple algebraic group and let $P \subset G$ be a parabolic subgroup. Then the following holds:*

(1) If L is a Levi factor of P, then $R_u(P) \rtimes L^u = P^u$.

(2) If $P^- \subset G$ is an opposite parabolic subgroup to P, then we have $\dim G = \dim R_u(P) + \dim P$ *and the product morphism*

$$R_u(P^-) \times R_u(P) \times (P \cap P^-)^u \to G$$

is an embedding[1].

(3) *If P is a maximal parabolic subgroup of G (i.e. P is a maximal proper subgroup of G that contains a Borel subgroup), then* $\dim P^u = \dim P - 1$.

For the proof of Lemma 3.5 we need the following Lemma.

Lemma 3.6. *Let G be a connected reductive algebraic group. Then*

a) $G^u = [G, G]$,

b) $G = G^u \cdot R(G)$, *$G^u \cap R(G)$ is finite and G^u is trivial or semisimple.*

Proof of Lemma 3.6. Note that G/G^u is an algebraic torus, as it is connected and contains only semisimple elements; see [Hum75, Proposition 21.4B and Theorem 19.3]. In particular, G^u contains the commutator subgroup $[G, G]$.

On the other hand, for every non-trivial character α of a maximal algebraic torus $T \subset G$, $[G, G]$ contains the root subgroup $U_\alpha \subset G$ with respect to T, since for each isomorphism $\lambda \colon \mathbb{G}_a \to U_\alpha$ we have

$$\lambda(\alpha(t) - 1) = \lambda(\alpha(t))\lambda(1)^{-1} = t\lambda(1)t^{-1}\lambda(1)^{-1} \in [G, G] \quad \text{for every } t \in T\,.$$

Hence $[G, G]$ contains G^u and thus we get the first statement.

The second statement follows from the first statement and from [Bor91, Proposition 14.2, Ch. IV]. $\square$

Proof of Lemma 3.5. (1): By definition we have $R_u(P) \ltimes L = P$. Hence, we get an inclusion $R_u(P) \rtimes L^u \subset P^u$. On the other hand, the inclusion $P^u \subset P$ induces an inclusion $P^u/R_u(P) \subset P/R_u(P)$ and $\pi \colon P \to P/R_u(P)$ restricts to an isomorphism $\pi|_L \colon L \to P/R_u(P)$. Hence,

$$L^u \xrightarrow[\simeq]{\pi|_{L^u}} (P/R_u(P))^u = P^u/R_u(P)\,,$$

which implies (1).

(2): By [Tim11, Example 3.10], the algebraic quotient G/P^u is quasi-affine. Let P^- be an opposite parabolic subgroup to P. The orbit in G/P^u through the class of the neutral element under the natural action of the unipotent radical $R_u(P^-)$ is therefore closed in G/P^u. This implies that $R_u(P^-)P^u$ is closed in G.

By definition, $L \coloneqq P \cap P^-$ is a Levi factor of P (and also of P^-). The product morphism induces an isomorphism of varieties

$$R_u(P^-) \times R_u(P) \times L \xrightarrow{\simeq} R_u(P^-) \times P \xrightarrow{\simeq} R_u(P^-)P$$

and $R_u(P^-)P$ is an open dense subset of G (see [Bor91, Proposition 14.21] or [FvS19, Appendix B.2]). This gives the first statement. Due to (1), we have $P^u = R_u(P) \rtimes L^u$. Hence, the above isomorphism restricts to an isomorphism:

$$R_u(P^-) \times R_u(P) \times L^u \xrightarrow{\simeq} R_u(P^-)P^u\,.$$

[1]By convention, for us embeddings are closed. In contrast, the product morphism $G \times G \times G \to G$ restricts to an isomorphism $R_u(P^-) \times R_u(P) \times (P \cap P^-) \to W$, where W is *open* in G.

(3): By construction $G/R_u(G)$ is a reductive or trivial algebraic group. In the second case, G contains no maximal parabolic subgroup and thus we may assume that $G/R_u(G)$ is reductive. Since $R_u(G)$ is contained in every Borel subgroup of G, it follows that $R_u(G)$ is contained in P. Thus $P/R_u(G)$ is a maximal parabolic subgroup of $G/R_u(G)$. Since $R_u(G) \subset P^u$, we get an isomorphism

$$P/P^u \simeq (P/R_u(G))/(P^u/R_u(G)).$$

Thus, it is enough to show (3) in case G is reductive (and by definition it is connected).

Let $B \subset G$ be a Borel subgroup, $T \subset B$ a maximal algebraic torus, $r = \dim T$, r is the rank of G, and let $\mathfrak{X}(T)$ be the group of characters of T. We may choose simple roots $\alpha_1, \dots, \alpha_r \in \mathfrak{X}(T)$ such that P is the parabolic subgroup with respect to $\alpha_1, \dots, \alpha_{r-1}$; see [Hum75, Theorem in §29.3]. Let

$$Z_i = \left(\bigcap_{j=1}^{i} \ker(\alpha_j) \right)^{\circ} \subset T \quad \text{for each } i = 1, \dots, r$$

where H° denotes the identity component of a closed subgroup $H \subset G$. Since by definition $\alpha_1, \dots, \alpha_r$ form a basis of $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{R}$, it follows that over $\mathbb{Z}$ the elements $\alpha_1, \dots, \alpha_r$ are linearly independent. Hence, the dimension of Z_i is $r - i$. From [Hum75, §30.2], it follows that

$$R(P) = R_u(P) \rtimes Z_{r-1}.$$

Now, let $Q \coloneqq P/R_u(P)$. Thus Q is a connected reductive algebraic group. Since P^u is the preimage of Q^u under the canonical projection $\pi\colon P \to Q$, we get

$$P/P^u \simeq Q/Q^u.$$

Note that $\pi(R(P))$ is a normal solvable connected subgroup of Q and thus $\pi(R(P)) \subset R(Q)$. On the other hand, $\pi^{-1}(R(Q))$ is a normal, connected subgroup and it is solvable, as $R_u(P) = \ker(\pi)$ and $R(Q)$ are solvable. The latter two statements together imply that $\pi^{-1}(R(Q)) = R(P)$ and thus

$$R(Q) \simeq Z_{r-1}.$$

By Lemma 3.6, $Q = Q^u \cdot R(Q)$ and $R(Q) \cap Q^u$ is finite. Thus, the canonical projection $Q \to Q/R(Q)$ restricts to an isogeny $Q^u \to Q/R(Q)$. In total:

$$1 = \dim Z_{r-1} = \dim R(Q) = \dim Q - \dim Q^u = \dim Q/Q^u = \dim P/P^u. \quad \square$$

Theorem 3.7. *Let G be a simple algebraic group, let $k \geq 0$ be an integer, and let Z be a smooth affine variety. If $\dim G + k > 2 \dim Z + 1$, then Z admits an embedding into $G \times \mathbb{A}^k$.*

For the proof of Theorem 3.7 we will use the two next propositions.

Proposition 3.8. *Let G be a semisimple algebraic group and let $k \geq 0$ be an integer. If there exists a parabolic subgroup $P \subset G$ with $\dim P^u - 1 \leq 3 \dim R_u(P)$, then for every smooth affine variety Z with*

$$2 \dim Z + \dim P - \dim P^u < \dim G + k$$

there exists an embedding of Z into $G \times \mathbb{A}^k$.

Proposition 3.9. *Let G be a simple algebraic group. Then there exists a maximal parabolic subgroup $P \subset G$ such that $\dim P^u \leq 3 \dim R_u(P)$.*

Proof of Theorem 3.7. Let $P \subset G$ be a maximal parabolic subgroup as in Proposition 3.9. By Lemma 3.5(3) we have $\dim P - \dim P^u = 1$. Thus the theorem follows from Proposition 3.8. □

Proof of Proposition 3.8. By Lemma 3.5(2) there exists an embedding of $\mathbb{A}^m \times H$ into $G \times \mathbb{A}^k$, where $m = 2 \dim R_u(P) + k$ and $H = (P \cap P^-)^u$ for an opposite parabolic subgroup $P^- \subset G$ of P. By Lemma 3.5(1) we have $\dim H = \dim P^u - \dim R_u(P)$. Now, we get

$$\dim H - 1 = \dim P^u - \dim R_u(P) - 1 \leq 2 \dim R_u(P) \leq m\,. \tag{12}$$

By Lemma 3.5(2) we get $\dim G = \dim R_u(P) + \dim P$. Hence

$$\begin{aligned} 2 \dim Z + 1 &\leq \dim G - \dim P + \dim P^u + k \\ &= \dim P^u + \dim R_u(P) + k \\ &= \dim H + m\,. \end{aligned}$$

Thus, we get $\dim Z \leq \frac{\dim H - 1 + m}{2} \leq m$ by (12). Hence, the proposition follows from Corollary 3.1. □

Proof of Proposition 3.9. Let $P \subset G$ be a maximal parabolic subgroup. By Lemma 3.5(2),(3) we get $\dim R_u(P) + \dim P^u + 1 = \dim G$. Let L be a Levi factor of P. Then, by Lemma 3.5(1) $\dim P^u = \dim L^u + \dim R_u(P)$. Now, if we find a maximal parabolic subgroup P in G such that

$$\dim G \geq 2 \dim L^u + 1\,, \tag{13}$$

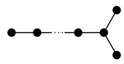
then we are done, as in this case we would get

$$\begin{aligned} 3 \dim R_u(P) &= \dim R_u(P) + \dim P^u + 1 - 1 + 2 \dim R_u(P) - \dim P^u \\ &= \dim G - 1 + 2 \dim R_u(P) - \dim P^u \\ &\geq 2 \dim L^u + 2 \dim R_u(P) - \dim P^u \\ &= \dim P^u\,. \end{aligned}$$

We treat first the case, when G is one of the classical Lie-types A_n, B_n, C_n or D_n. For $n \geq 1$, we denote by a_n, b_n, c_n, d_n the dimension of the Lie algebra of type A_n, B_n, C_n and D_n, respectively. By [Hum78, §1.2], we get

$$a_n = n^2 + 2n\,, \quad b_n = c_n = 2n^2 + n\,, \quad d_n = 2n^2 - n\,.$$

Now we choose $s \in \mathbb{N}_0$ according to the Lie-type as follows

Lie-type	Dynkin diagram	s
A_n, $n \geq 1$		$\lfloor (n+1)/2 \rfloor$
B_n, $n \geq 2$		$\lfloor (4n+1)/6 \rfloor$
C_n, $n \geq 3$		$\lfloor (4n+1)/6 \rfloor$
D_n, $n \geq 4$		$\lfloor (4n-1)/6 \rfloor$

where $\lfloor x \rfloor$ means the largest integer that is smaller or equal than x. In order to specify the maximal parabolic subgroup P of G, let I be the set of all simple roots in the Dynkin diagram of G, except the simple root at position s, when we count from the left in the Dynkin diagram. We let P be the standard parabolic subgroup with respect to I and some fixed chosen Borel subgroup of G and we let (as above) $L \subset P$ be a Levi factor. Then L^u is semisimple or trivial (by Lemma 3.6) and the corresponding Dynkin diagram is the Dynkin diagram of G with the vertex s (counted from the left) deleted; see [Hum75, §30.2]. For example, if the Lie type of G is B_4, then $s = \lfloor 17/6 \rfloor = 2$ and we have the following Dynkin diagrams (the cross $\times$ means to delete the corresponding simple root):

G: P: $\Longrightarrow$ $\dim L^u = a_1 + b_2 = 13\,.$

By considering the Dynkin diagrams for the classical types A_n, B_n, C_n and D_n and by using that $a_1 = b_1 = c_1$, $d_2 = 2a_1$ and $d_3 = a_3$, we get

Lie-type	s	$\dim L^u$	
A_n, $n \geq 1$	$\lfloor \frac{n+1}{2} \rfloor \geq 1$	$a_{s-1} + a_{n-s} =$	$2s^2 - (2n+2)s + n^2 + 2n - 1$
B_n, $n \geq 2$	$\lfloor \frac{4n+1}{6} \rfloor \geq 1$	$a_{s-1} + b_{n-s} =$	$3s^2 - (4n+1)s + 2n^2 + n - 1$
C_n, $n \geq 3$	$\lfloor \frac{4n+1}{6} \rfloor \geq 2$	$a_{s-1} + c_{n-s} =$	$3s^2 - (4n+1)s + 2n^2 + n - 1$
D_n, $n \geq 4$	$\lfloor \frac{4n-1}{6} \rfloor \geq 2$	$a_{s-1} + d_{n-s} =$	$3(s+1)^2 - (4n+5)(s+1)$ $+2n^2 + 3n + 1.$

From this table we conclude $\dim G - 2 \dim L^u \geq 0$ as desired. We provide the detailed calculation. For A_n with $n \geq 1$, we note

$$\begin{aligned}
\dim G - 2\dim L^u - 1 &= -n^2 + 4(n+1)\left\lfloor \frac{n+1}{2} \right\rfloor - 4\left\lfloor \frac{n+1}{2} \right\rfloor^2 - 2n + 1 \\
&= \begin{cases} -n^2 + 4(n+1)\frac{n+1}{2} - 4\left(\frac{n+1}{2}\right)^2 - 2n + 1 \\ \quad \text{if } n \text{ is odd} \\ -n^2 + 4(n+1)\frac{n}{2} - 4\left(\frac{n}{2}\right)^2 - 2n + 1 \\ \quad \text{if } n \text{ is even} \end{cases} \\
&= \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} \\
&\geq 0\,.
\end{aligned}$$

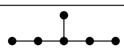
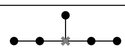
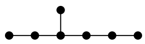
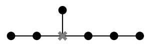
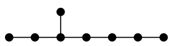
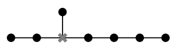
For B_n and C_n with $n \geq 2$ and $n \geq 3$, respectively, and $x \in \{0, -2, -4\}$ such that 6 divides $4n + x$, we calculate

$$\begin{aligned}\dim G - 2\dim L^u - 1 = & -2n^2 + 2(4n+1)\left\lfloor \frac{4n+1}{6} \right\rfloor - 6\left\lfloor \frac{4n+1}{6} \right\rfloor^2 - n + 1 \\ = & -2n^2 + 2(4n+1)\frac{4n+x}{6} - \frac{(4n+x)^2}{6} - n + 1 \\ = & \frac{2n^2+n}{3} + 1 + \frac{2x - x^2}{6} \geq \frac{2n^2+n}{3} + 1 - 4 \\ \geq & \, 0\,.\end{aligned}$$

For D_n with $n \geq 4$ and $x \in \{0, 2, 4\}$ such that 6 divides $4n + x$, we calculate

$$\begin{aligned}\dim G - 2\dim L^u - 1 = & -2n^2 + 2(4n+5)\left\lfloor \frac{4n+5}{6} \right\rfloor - 6\left\lfloor \frac{4n+5}{6} \right\rfloor^2 \\ & - 7n - 3 \\ = & -2n^2 + 2(4n+5)\frac{4n+x}{6} - \frac{(4n+x)^2}{6} \\ & - 7n - 3 \\ = & \frac{2n^2-n}{3} + \frac{10x - x^2}{6} - 3 \\ \geq & \frac{2n^2-n}{3} - 3 \\ \geq & \, 0\,.\end{aligned}$$

Now, for the exceptional Lie-types, we choose P as in the table below and the estimate (13) follows from the same table (again the cross × in the dynkin diagram of P means, to remove the corresponding simple root):

Lie-type	Dynkin diagram of G	$\dim G$	Dynkin diagram of P	$\dim L^u$
E_6		78		$a_1 + a_2 + a_2 = 19$
E_7		133		$a_1 + a_2 + a_3 = 26$
E_8		248		$a_1 + a_2 + a_4 = 35$
F_4		52		$b_3 = 21$
G_2		14		$a_1 = 3$. □

Having settled the case for simple algebraic groups, we go on to semisimple algebraic groups. The following result generalizes Theorem 3.7.

Theorem 3.10. *Let G be a semisimple algebraic group and let $k \geq 0$ be an integer. Let $r \geq 1$ be the number of minimal normal closed connected subgroups of G. If Z is a smooth affine variety with $\dim G + k > 2\dim Z + r$, then there exists an embedding of Z into $G \times \mathbb{A}^k$.*

Proof. Let $G_1, \dots, G_r$ be the minimal normal closed connected subgroups of G. By [Hum75, Theorem in §27.5], the product morphism $G_1 \times \cdots \times G_r \to G$ is a finite étale surjection. In the light of Corollary 2.26 we may thus assume $G = G_1 \times \cdots \times G_r$. Since G_i is a simple algebraic group, there exists a maximal parabolic subgroup $P_i \subset G_i$ such that $3 \dim R_u(P_i) \geq \dim P_i^u$, by Proposition 3.9. Let

$$P \coloneqq P_1 \times P_2 \times \cdots \times P_r \subset G_1 \times G_2 \times \cdots \times G_r \,.$$

Then we get $P^u = P_1^u \times \cdots \times P_r^u$ and $R_u(P) = R_u(P_1) \times \cdots \times R_u(P_r)$ and therefore $3 \dim R_u(P) \geq \dim P^u$. Since $\dim P_i - \dim P_i^u = 1$ for each $i \in \{1, \dots, r\}$ (Lemma 3.5(3)), we get $\dim P - \dim P^u = r$. Thus, the theorem follows from Proposition 3.8. □

3.4. **Embeddings into algebraic groups of low dimension.** Our main result concerning characterless algebraic groups of low dimension is the following.

Proposition 3.11. *Let G be a characterless algebraic group with $\dim G \leq 10$ and let Z be a smooth affine variety with $2 \dim Z + 1 \leq \dim G$. If the Lie algebra of G is non-isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ and non-isomorphic to $\mathfrak{sl}_3 \times \boldsymbol{k}$, then Z admits an embedding into G.*

Before giving the proof, let us shortly comment on the above result. Proposition 3.11 implies that for any characterless algebraic group G with $\dim G \leq 8$ the condition $2 \dim Z + 1 \leq \dim G$ suffices to get an embedding of Z into G.

Question. *Does every 4-dimensional smooth affine variety embed into the algebraic group $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ or into $\mathrm{SL}_3 \times \mathbb{G}_a$?*

Proof of Proposition 3.11. Let G be a characterless algebraic group of dimension ≤ 10 such that its Lie algebra is neither isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ nor to $\mathfrak{sl}_3 \times \mathbf{k}$. We may and will assume that G is connected. Using a Levi decomposition [OV90, Theorem 4, Ch. 6], G is isomorphic as a variety to $\mathbb{A}^m \times H$ where H is a connected reductive characterless algebraic group. In particular, H is semisimple or trivial; see [FvS19, Remark 8.3] and Lemma 3.6. In case H is trivial, the result follows from the Holme-Kaliman-Srinivas embedding theorem. Thus we may assume that H is semisimple. Since every semisimple algebraic group is the target of a finite homomorphism of a product of simple algebraic groups (see [Hum75, Theorem in §27.5]), we may assume that H is the product of simple algebraic groups by Corollary 2.26. From the classification of simple Lie algebras it follows that a simple algebraic group of dimension ≤ 10 has Lie algebra equal to $\mathfrak{sl}_2$, $\mathfrak{sl}_3$ or $\mathfrak{so}_5 = \mathfrak{sp}_4$. Again using Corollary 2.26, we may assume that the factors of H are simple algebraic groups that are not targets of non-trivial finite homomorphisms. Hence, H is a product of the groups

$$\mathrm{SL}_2 \,, \quad \mathrm{SL}_3 \quad \text{and} \quad \mathrm{Sp}_4 \,.$$

If H has a factor equal to SL_3 or Sp_4, then the statement follows from Theorem 3.7 (note we excluded the case $\mathbb{A}^1 \times \mathrm{SL}_3$). Hence, we are left with the case

$$G \simeq \mathbb{A}^m \times (\mathrm{SL}_2)^s \,.$$

for some $s \geq 1$. We distinguish two cases:

- $m + 3s$ is odd: In case $s - 1 \leq m$, the statement follows from Remark 3.4 and Proposition 3.3. Thus we assume that $s - 1 > m$. Since $m + 3s \leq 10$ by assumption, we get $0 \leq m \leq \min\{10 - 3s, s - 2\}$. Since $m + 3s$ is odd, this implies that $(s, m) = (3, 0)$, which contradicts the assumption that the Lie algebra of G is non-isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$.
- $m + 3s$ is even: Again using Remark 3.4 and Proposition 3.3 we may assume that $s - 2 > m$. Similarly as above we get $0 \leq m \leq \min\{10 - 3s, s - 3\}$. Hence, $(s, m) = (3, 0)$, and since $m + 3s$ is even, we arrive at a contradiction. □

4. Non-embedability results for algebraic groups

Recall from the last section that, for all simple algebraic groups G and smooth affine varieties Z such that $\dim G \geq 2 \dim Z + 2$, there exists an embedding of Z into G (see Theorem 3.7). In this section, for every algebraic group G and every integer d such that $\dim G \leq 2d$, we construct a smooth affine variety Z of dimension d such that Z does not allow an embedding into G (see Corollary 4.4 below). Thus, for a simple algebraic group G this gives optimality of our embedding result (Theorem 3.7) in case $\dim G$ is even, and optimality up to one dimension in case $\dim G$ is odd. We will focus more on this last case in Section 5.

We recall some facts of the Segre- and Chern class operations. For this we use the excellent book of Fulton [Ful98] as a reference. For a smooth irreducible variety X of dimension d we denote by $\mathrm{CH}_i(X)$ its i-th Chow group, i.e. the group of i-cycles modulo linear equivalence for each $0 \leq i \leq d$. For $i > d$ and $i < 0$ we set $\mathrm{CH}_i(X) = 0$. For each vector bundle $E \to X$ and each $i \geq 0$, we get the so-called *Segre class operations*

$$s_i(E) \colon \mathrm{CH}_k(X) \to \mathrm{CH}_{k-i}(X) \,, \quad \alpha \mapsto s_i(E) \cap \alpha$$

and thus endomorphisms $s_i(E)$ on $\mathrm{CH}(X) = \bigoplus_{i=0}^{k} \mathrm{CH}_i(X)$ (see [Ful98, §3.1]). By [Ful98, Propsition 3.1(a)] we have that $s_0(E) = 1$ is the identity in $\mathrm{End}(\mathrm{CH}(X))$. Following [Ful98, §3.2] we consider the formal power series $s_t(E) = \sum_{i=0}^{\infty} s_i(E) t^i$ and define $c_t(E) = \sum_{i=0}^{\infty} c_i(E) t^i$ as the inverse of $s_t(E)$ inside the formal power series ring $\mathrm{End}(\mathrm{CH}(X))[[t]]$. This makes sense since the endomorphisms $s_i(E)$, $i \geq 0$ commute pairwise [Ful98, Proposition 3.1(b)]. It follows that $c_i(E)$ maps $\mathrm{CH}_k(X)$ into $\mathrm{CH}_{k-i}(X)$ and we denote the image of $\alpha \in \mathrm{CH}_k(X)$ under $c_i(E)$ by $c_i(E) \cap \alpha \in \mathrm{CH}_{k-i}(X)$. The operations $c_i(E)$, $i \geq 0$ are called *Chern class operations*. Moreover, by [Ful98, Example 8.1.6] we have

$$c_i(E) \cap (c_j(E) \cap [X]) = (c_i(E) \cap [X]) \cdot (c_j(E) \cap [X]) \quad \text{for all } i, j \,,$$

where '$\cdot$' denotes the intersection product; see [Ful98, §8.1]. In the sequel we denote by $T^*X \to X$ the cotangent bundle of X.

Proposition 4.1. *For $d \geq 1$, there exists an irreducible smooth affine variety Z of dimension d such that $s_d(T^*Z) \neq 0$.*

Proof. By the proof of [BMS89, Theorem 5.8], there exists a smooth irreducible affine variety Z of dimension d such that the component in $\mathrm{CH}_0(Z)$ of the total Segre class of $T^*Z \to Z$ is non-vanishing, $s_d(T^*Z) \cap [Z] \neq 0$ in $\mathrm{CH}_0(Z)$. This implies that $s_d(T^*Z) \neq 0$ inside $\mathrm{End}(\mathrm{CH}(Z))$. □

From a Theorem of Grothendieck, [Gro58, Remarque p.21] or [Bri11, Proposition 2.8] we get the following result:

Proposition 4.2. *Let G be a connected algebraic group of dimension n. Then $\mathrm{CH}_i(G)$ is a torsion group for $0 \leq i \leq n-1$ and $\mathrm{CH}_n(G) = \mathbb{Z}$.* □

Lemma 4.3. *Let Z be an irreducible smooth affine variety of dimension $d \geq 1$. If there is a connected algebraic group G of dimension $2d$ such that there is an embedding $\iota\colon Z \to G$, then $s_d(T^*Z) = 0$.*

Proof. Since $d \geq 1$, by Proposition 4.2, we get that $\iota_*([Z]) \in \mathrm{CH}_d(G)$ is a torsion element where $[Z] \in \mathrm{CH}_d(Z)$ denotes the class associated to Z. By [Ful98, Corollary 6.3] we have

$$\iota^*(\iota_*([Z])) = c_d(N^*) \cap [Z] \in \mathrm{CH}_0(Z)$$

where N^* denotes the conormal bundle of Z in G. Hence $\iota^*(\iota_*([Z]))$ is a torsion element in $\mathrm{CH}_0(Z)$. In case $d = 1$, we have $\dim G = 2$ and thus G is solvable. In particular $\mathrm{CH}_1(G) = 0$. In case $d \geq 2$, it follows from [BMS89, Proposition 2.1] that $\mathrm{CH}_0(Z)$ is torsion free. Thus in both cases $\iota^*(\iota_*([Z]))$ is zero. Moreover $c_d(N^*) \cap \alpha = 0$ for each $\alpha \in \mathrm{CH}_k(Z)$ if $k < d$. This implies that $c_d(N^*) = 0$, it is the zero endomorphism of $\mathrm{CH}(Z)$.

Since G is an algebraic group, the cotangent bundle $T^*G \to G$ is trivial. Moreover, we have a short exact sequence of vector bundles over Z:

$$0 \to N^* \to \iota^*(T^*G) \to T^*Z \to 0\,.$$

Then we get

$$1 = c_t(\iota^*(T^*G)) = c_t(N^*)c_t(T^*Z) \quad \text{inside } \mathrm{End}(\mathrm{CH}(Z))[[\mathrm{t}]]$$

by [Ful98, Theorem 3.2(e)]. By definition we get $s_t(T^*Z) = c_t(N^*)$ and thus $s_d(T^*Z) = c_d(N^*) = 0$. □

Now, we apply the above results in order to get irreducible smooth affine varieties that do not admit an embedding into algebraic groups for appropriate dimensions.

Corollary 4.4. *Let G be an algebraic group of dimension $n > 0$. Then, for each integer $d \geq \frac{n}{2}$ there exists a smooth irreducible affine variety Z of dimension d that does not admit an embedding into G.*

Proof. By assumption $2d \geq n$. Let $k := 2d - n \geq 0$. By Proposition 4.1 there exists a smooth irreducible affine variety Z of dimension d such that $s_d(T^*Z) \neq 0$. Towards a contradiction, assume that Z allows an embedding into G. As Z is irreducible, there exists an embedding of Z into the identity component G° of G and hence also into $G^\circ \times (\mathbb{G}_a)^k$. Since $\dim G^\circ + k = n + 2d - n = 2d$, by Lemma 4.3 we get $s_d(T^*Z) = 0$, contradiction. □

5. Limits of our methods for odd dimensional simple groups

In Section 4 we proved that Theorem 3.7 is optimal for even dimensional simple algebraic groups G. Moreover, by Proposition 3.11 we also get optimality in case $\dim G \leq 8$. In this section we will explain, why we are not able to apply our method to an odd dimensional simple algebraic group G and smooth affine varieties Z with $\dim G = 2\dim Z + 1$ and $\dim Z > 1$.

Concretely, let G be an odd dimensional simple algebraic group. In order to apply our method (Theorem 2.5) to a smooth affine variety Z with $\dim G = 2\dim Z + 1$ we need at least the following: a smooth morphism

$$\pi\colon G \to P \quad \text{with} \quad \dim P = \dim Z$$

that factors through a principal $\mathbb{G}_a$-bundle, $\mathrm{Aut}_P^{\mathrm{alg}}(G)$ acts sufficiently transitively on each fiber of π, and a finite surjective morphism $Z \to P$.

The only way to construct such a $\pi\colon G \to P$ seems to be forming the algebraic quotient by some proper connected characterless algebraic subgroup $H \subset G$ of the right dimension; see Proposition 2.13 and Proposition 2.15. However, in this section we prove Proposition D which yields an obstruction to the existence of proper surjective morphisms $Z \to G/H$; see also the discussion in the introduction.

Since the obstruction comes from algebraic topology, in this section we work with varieties over the complex numbers, i.e. our ground field will be $\mathbb{C}$. However, using an appropriate Lefschetz principle, we promote a version of Proposition D back to every algebraically closed field of characteristic zero; see Appendix C.

In order to avoid confusion with the category of complex manifolds, below we write *algebraic morphism* instead of just *morphism*. We restate Proposition D:

Proposition 5.1. *Let Z be a simply-connected complex smooth algebraic variety with the rational homology of a point. If G/H is a $\dim Z$-dimensional complex homogeneous space of a complex simple algebraic group G, then there is no proper surjective algebraic morphism from Z to G/H.*

In Appendix C we prove that Proposition 5.1 holds for $Z = \mathbb{A}^{\dim G - \dim H}$ over any algebraically closed field of characteristic zero; see Proposition C.1.

Proof of Proposition 5.1. Let $\tilde{G}$ be the universal cover of G. Then $p\colon \tilde{G} \to G$ is a homomorphism of simple complex algebraic groups [GR03, Théorème

5.1, Exposé XII]. Since $\tilde{G}/p^{-1}(H)$ and G/H are isomorphic as algebraic varieties, we may assume that G is simply connected.

Assume that there exists a proper surjective algebraic morphism $Z \to G/H$. Let H° be the identity component of H. Denote by $p\colon G/H^\circ \to G/H$ the canonical projection, which is a finite algebraic étale surjection. As Z is simply connected, there exists a holomorphic map $f\colon Z \to G/H^\circ$ such that $p \circ f\colon Z \to G/H$ is the original proper surjective algebraic morphism. By [Ser58, Proposition 20], it follows that $f\colon Z \to G/H^\circ$ is an algebraic morphism, and it is also proper and surjective. Thus, by replacing H by H°, we may assume without loss of generality that H is connected.

Since G is simply connected and H is connected, the long exact homotopy sequence assocaited to $H \hookrightarrow G \twoheadrightarrow G/H$ yields the exact sequence

$$1 = \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) = 1\,.$$

Thus, since G is connected, we get that G/H is simply connected. Let

$$i_0 \coloneqq \inf\{\, i \geq 1 \mid \ \pi_i(G/H) \otimes_{\mathbb{Z}} \mathbb{Q} \text{ is non-vanishing}\,\}\,.$$

By Proposition 5.2 below, it follows that $1 < i_0 < \infty$. As G/H is simply connected, we may apply a rational version of the Hurewicz Theorem [KK04, Theorem 1.1] and get

$$0 \neq \pi_{i_0}(G/H) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H_{i_0}(G/H; \mathbb{Q})$$

where $H_*(\cdot; \mathbb{Q})$ denotes singular homology with rational coefficients. Since $f\colon Z \to G/H$ is a proper surjective algebraic morphism, Theorem C applies, and we get that $f_*\colon H_{i_0}(Z;\mathbb{Q}) \to H_{i_0}(G/H;\mathbb{Q})$ is surjective. However, this contradicts $H_{i_0}(Z;\mathbb{Q}) = 0$. □

Proposition 5.2. *Let G be a simple complex algebraic group. Then, for each proper closed complex subgroup $H \subset G$, there exists $i > 1$ such that*

$$\pi_i(G/H) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0\,.$$

For the proof of this proposition, we use facts about the rational homotopy groups of all simply connected simple complex algebraic groups. We recall those facts next.

Denote by G a simply connected semisimple complex algebraic group. Recall that there exists a maximal compact connected real Lie subgroup $K \subset G$ such that G and K are homotopy equivalent [Hel78, Theorem 2.2, Ch. VI]. In particular, K is simply connected, and thus we may apply [MT91, Theorem 6.27, Ch. IV] to get a continuous map of a product of odd dimensional spheres into K

$$f\colon S^{2n_1-1} \times \cdots \times S^{2n_l-1} \to K$$

that induces an isomorphism between the singular cohomology rings with rational coefficients

$$H^*(K;\mathbb{Q}) \simeq H^*(S^{2n_1-1} \times \cdots \times S^{2n_l-1}; \mathbb{Q})\,.$$

By the universal coefficient theorem for cohomology, f induces an isomorphism between singular homology groups with rational coefficients. Since K is simply connected, we get by Künneth's formula

$$H_1(S^{2n_1-1};\mathbb{Q}) \oplus \cdots \oplus H_1(S^{2n_l-1};\mathbb{Q}) \simeq H_1(K;\mathbb{Q}) = 0\,.$$

This implies $n_i \geq 2$ for each $i \in \{1,\ldots,l\}$. In particular, the product of spheres $S^{2n_1-1} \times \cdots \times S^{2n_l-1}$ is simply connected as well. Now, by the Whitehead-Serre Theorem [FHT01, Theorem 8.6], f induces for each $i \geq 0$ an isomorphism of rational homotopy groups

$$\pi_i(S^{2n_1-1}) \otimes_{\mathbb{Z}} \mathbb{Q} \times \cdots \times \pi_i(S^{2n_l-1}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \pi_i(K) \otimes_{\mathbb{Z}} \mathbb{Q} = \pi_i(G) \otimes_{\mathbb{Z}} \mathbb{Q}\,. \quad (14)$$

Note that by a Theorem of Serre ([FHT01, Example 1 in §15(d)] or [KK04, Theorem 1.3], for odd positive integers k, the group $\pi_i(S^k)\otimes_{\mathbb{Z}}\mathbb{Q}$ is isomorphic to $\mathbb{Q}$ if $i = k$ and otherwise it vanishes.

Definition 5.3. For a simply connected semisimple complex algebraic group G, we call the above constructed unordered l-tuple $\{2n_1 - 1, \ldots, 2n_l - 1\}$ the *rational homotopy type* of G.

In the following table we list the complex dimension and rational homotopy type for each Lie type (the statements follow from [MT91, Theorem 6.5, Ch. III and Theorem 5.10, Ch. VI]):

TABLE 1. Rational homotopy types

Lie-Type	Complex dimension	Rational homotopy type of the simply connected simple complex algebraic group
A_m, $m \geq 1$	$m^2 + 2m$	$\{3, 5, \ldots, 2m + 1\}$
B_m, $m \geq 2$	$2m^2 + m$	$\{3, 7, \ldots, 4m - 1\}$
C_m, $m \geq 3$	$2m^2 + m$	$\{3, 7, \ldots, 4m - 1\}$
D_m, $m \geq 4$	$2m^2 - m$	$\{3, 7, \ldots, 4m - 5\} \cup \{2m - 1\}$
E_6	78	$\{3, 9, 11, 15, 17, 23\}$
E_7	133	$\{3, 11, 15, 19, 23, 27, 35\}$
E_8	248	$\{3, 15, 23, 27, 35, 39, 47, 59\}$
F_4	52	$\{3, 11, 15, 23\}$
G_2	14	$\{3, 11\}$

Proof of Proposition 5.2. With the same argument as in the beginning of the proof of Proposition 5.1, we may assume that G is simply connected. Let $H^\circ \subset H$ be the identity component of H. Since $G/H^\circ \to G/H$ is a finite étale surjection, we get for each $i > 1$ an isomorphism $\pi_i(G/H^\circ) \simeq \pi_i(G/H)$. Hence, in addition we may assume that H is connected.

Let $R(H)$ be the radical of H. By definition $H/R(H)$ is a semisimple complex algebraic group. Let $S \to H/R(H)$ be the universal covering. As before, S is a simply connected semisimple complex algebraic group. Since $R(H)$ is the product of an algebraic torus and a unipotent algebraic group, it follows from the long exact homotopy sequence that

$$\pi_i(H) = \pi_i(H/R(H)) = \pi_i(S) \quad \text{for each } i > 2$$

and

$$\pi_2(H) \hookrightarrow \pi_2(H/R(H)) = \pi_2(S)$$

is injective. From (14) it follows that $\pi_2(S) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Hence we get

$$\pi_i(H) \otimes_{\mathbb{Z}} \mathbb{Q} = \pi_i(S) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{for each } i > 1\,.$$

Let $S_1, \ldots, S_l$ be the connected normal minimal closed complex subgroups of S. Then each S_i is a simple complex algebraic group and the product morphism $S_1 \times \cdots \times S_l \to S$ is a finite étale surjection (see [Hum75, Theorem in §27.5]). Hence, we get

$$\pi_i(S) = \pi_i(S_1) \times \cdots \times \pi_i(S_l) \quad \text{for each } i > 1\,.$$

Now, assume towards a contradiction that $\pi_i(G/H) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for each $i > 1$. By tensoring the long exact homotopy sequence associated to $H \hookrightarrow G \twoheadrightarrow G/H$ with $\mathbb{Q}$, we get isomorphisms

$$\pi_i(H) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \pi_i(G) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{for each } i > 1\,.$$

In particular,

$$\pi_3(S_1) \otimes_{\mathbb{Z}} \mathbb{Q} \times \cdots \times \pi_3(S_l) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \pi_3(G) \otimes_{\mathbb{Z}} \mathbb{Q}\,. \tag{15}$$

According to Table 1, we have $\pi_3(S_i) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \pi_3(G) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$ for each $i \in \{1, \ldots, l\}$. Hence, due to (15), we get $l = 1$, S is already simple, and

$$\pi_i(S) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \pi_i(G) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{for each } i > 1\,.$$

Since S and G are both simply connected we get even

$$\pi_i(S) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \pi_i(G) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{for each } i \geq 0\,.$$

This implies that S and G have the same rational homotopy type. However, according to Table 1 this can only happen if the Lie types of S and G coincide or the Lie types of S and G are B_m and C_m, respectively, for some $m \geq 3$ (note that the 5^{th} rational homotopy group is non-vanishing only for A_m with $m \geq 2$ and the 7^{th} rational homotopy group is non-vanishing only for B_m, C_m and D_m). In both cases the complex dimension of S and G coincide, which contradicts the fact that H is a proper closed complex subgroup of G. □

Appendix A. Hopf's Umkehrungshomomorphismus theorem

In this chapter we use a version of Hopf's Umkehrungshomomorphismus theorem to prove Theorem C. Apart from the proof of Theorem C, in this section we consider the Euclidean topology on (topological, smooth and complex) manifolds and subsets thereof. For lack of reference, we provide a proof of the following version for (in general) non-closed manifolds of a result going back to the work of Hopf in the case of closed (smooth) manifolds [Hop30]. While we will apply the result only for smooth maps, we take the opportunity to formulate the statements for topological manifolds and proper continuous maps between them. Aspects of our proof are written with smooth concepts in mind (definition of degree, exhaustion of manifolds by full-dimensional compact manifolds with boundary), even if the proficient topologist might have worked differently, e.g. to avoid topological transversality in Lemma A.5. An advantage is that this proof works very naturally in the smooth setup as well, and it seems like the fastest path from citable literature to the theorem.

The reader may read what follows for the ring R being $\mathbb{Z}$ or $\mathbb{Q}$ without loss for the application in this paper. Recall that an orientation on a manifold is a $\mathbb{Z}$-orientation. The notions used in the result will be explained afterwards.

Theorem A.1. *Let R be a commutative unital ring, M and N be R-oriented non-empty topological manifolds of the same dimension where N is connected, and let $f\colon M \to N$ be a proper continuous map. Denote by $d \in R$ the degree of f, by $f_k\colon H_k(M;R) \to H_k(N;R)$ the induced map in k-th homology, and by $f_{!,k} : H_k(N;R) \to H_k(M;R)$ the Umkehrungshomomorphismus. Then, for all non-negative integers k and all $c \in H_k(N;R)$, we have $f_k \circ f_{!,k}(c) = dc$.*

We use Theorem A.1 to prove Theorem C. In fact we prove the following.

Theorem A.2. *Let $f\colon X \to Y$ be a proper surjective holomorphic map between complex n-dimensional manifolds. Assume that Y is connected and let the integer $d \geq 1$ be the number of preimages of a regular value of f. Then the following hold.*

(a) The image of the induced map on k-th homology $H_k(X;R) \to H_k(Y;R)$ contains $dH_k(Y;R)$ for all integers $k \geq 0$.
(b) Assume that X is connected. Then for all x in X, the image of the induced homomorphism on the fundamental groups $f_*\colon \pi_1(X,x) \to \pi_1(Y,f(x))$ has finite index in $\pi_1(Y,f(x))$ and this index divides d. In case X has the rational homology of a point, then f_* is a surjection.

Proof of Theorem C. Since f is proper (in the sense of algebraic geometry), it is proper as a map when X and Y are endowed with the Euclidean topology (i.e. their topology as (complex) differentiable manifolds); see [GR03, Proposition 3.2, Exp. XII], [Bou71, Proposition 6, §10]. From here on we consider X and Y with their Euclidean topology. W.l.o.G. X and Y are connected. Since $d \neq 0$ is a unit in $\mathbb{Q}$, the statement follows by applying Theorem A.2(a) for $R = \mathbb{Q}$. □

Proof of Theorem A.2. Since X and Y are complex manifolds, they are canonically oriented, and since f is holomorphic, for every regular point $x \in X$, f maps a neighborhood orientation-preservingly to Y. Consequently, the local degree of f at a regular value $y \in Y$ (see Remark A.6 for a topological definition) equals the non-negative integer d of the number of elements of $f^{-1}(y)$.

The degree of f is well-defined as the local degree $d \geq 0$ of any regular value $y \in Y$, and, since preimages of regular values are non-empty, the degree of f is non-zero.

(a): We apply Theorem A.1 to find that

$$f_k(H_k(X;R)) \supseteq f_k(f_{!,k}(H_k(Y;R))) = dH_k(Y;R)\,.$$

(b): W.l.o.g., we fix $x \in X$ such that $y := f(x) \in Y$ is a regular value of f. Let $p\colon \widetilde{Y} \to Y$ be the covering of Y corresponding to the subgroup $f_*(\pi_1(X,x)) \subseteq \pi_1(Y,y)$. We show that this is a finite cover, whence $f_*(\pi_1(X,x))$ has finite index in $\pi_1(Y,y)$.

We pick $\widetilde{y} \in p^{-1}(y)$ and denote by $\widetilde{f}\colon X \to \widetilde{Y}$ a lift of f with $\widetilde{f}(x) = \widetilde{y}$.

First we show that $p^{-1}(y)$ is contained in the image of $\widetilde{f}$. Indeed, take $\widetilde{z} \in p^{-1}(y)$ and let $\widetilde{\beta}\colon [0,1] \to \widetilde{Y}$ be a path connecting $\widetilde{y}$ and $\widetilde{z}$. We arrange for $\widetilde{\beta}$ to lie in $p^{-1}(Y^{\mathrm{reg}})$, where $Y^{\mathrm{reg}} \subseteq Y$ denotes the subset of regular values of f. This can, for example, be achieved by composing $\tilde{\beta}$ with p, homotoping the resulting path rel endpoints into Y^{reg} (here we invoke that f is holomorphic[2]), and lifting the resulting path. The loop $\beta := p \circ \widetilde{\beta}$ can be lifted to a path $\alpha\colon [0,1] \to f^{-1}(Y^{\mathrm{reg}})$ starting at x since f restricts to a covering $f^{-1}(Y^{\mathrm{reg}}) \to Y^{\mathrm{reg}}$ (recall that proper local homeomorphisms are coverings). By construction, $\widetilde{f} \circ \alpha$ and $\widetilde{\beta}$ are lifts of β starting at $\widetilde{y}$; in particular, $\widetilde{f}(\alpha(1)) = \widetilde{z}$ as desired.

We conclude that $p^{-1}(y)$, which is the index of $f_*(\pi_1(X,x))$ in $\pi_1(Y,y)$, must be finite. In fact,

$$\left|p^{-1}(y)\right| \left|\widetilde{f}^{-1}(\widetilde{y})\right| = \left|\left(p \circ \widetilde{f}\right)^{-1}(y)\right| = \left|f^{-1}(y)\right| = d \in \mathbb{N},$$

where the first equality follows since $\tilde{f}$ restricts to a covering $f^{-1}(Y^{\mathrm{reg}}) \to p^{-1}(Y^{\mathrm{reg}})$ and, thus, $|\tilde{f}^{-1}(y')| = |\tilde{f}^{-1}(\widetilde{y})|$ for all $y' \in p^{-1}(y)$.

Assume now, that X has the rational homology of a point. Note that $\widetilde{f}$ is a holomorphic map between complex n-dimensional manifolds; hence, its degree $\widetilde{d}$ equals $\widetilde{f}^{-1}(\widetilde{y})$ by the same argument we used above to find

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[2]We provide an argument for the claim that any smooth path $\beta\colon [0,1] \to Y$ with endpoints in Y^{reg} can be homotoped relative endpoints to a smooth path in Y^{reg}.

Note that $Y^{\mathrm{sing}} := Y \setminus Y^{\mathrm{reg}}$ is the image $f(X^{\mathrm{sing}})$ of the closed analytic subset $X^{\mathrm{sing}} = \{\, x \in X \mid f \text{ is singular at } x \,\} \subseteq X$, and thus Y^{sing} is a closed analytic subset of Y by Remmert's Proper Mapping Theorem [Rem57, Satz 23]. As such, Y^{sing} can be stratified as a finite union $M_1 \dot\cup \cdots \dot\cup M_k$ of complex submanifolds $M_i \subseteq Y$ of complex codimesion at least 1; see e.g. [Chi89, §5.5. Stratifications]. Let $G\colon [0,1] \times \mathbb{R}^N \to Y$ be a smooth map for some $N \in \mathbb{N}$ with $G|_{[0,1]\times\{0\}} = \beta$ such that for each $t \in [0,1]$, the map $\mathbb{R}^N \to Y$, $v \mapsto G(t,v)$ is submersive (see e.g. [GP74, Corollary to the ε-Neighbourhood Theorem]). Let $F\colon [0,1] \times \mathbb{R}^N \to Y$ be given by $F(t,v) := G(t, t(1-t)v)$. Then, F and $\partial F\colon \partial[0,1] \times \mathbb{R}^N \to Y$ are both transversal to each submanifold M_i of Y. By Thom's Transversality Theorem, $\beta_v\colon [0,1] \to Y$, $t \mapsto F(t,v)$ is transversal to Y^{sing} for each $v \in \mathbb{R}^N$ away from a nullset. As the M_i have real codimension at least 2 in Y, this means that the image of β_v is disjoint from each M_i. We set $\beta' = \beta_v$ for some v not in that nullset.

$d = f^{-1}(y)$. Hence, since f and $\widetilde{f}$ have non-zero-degree, Theorem A.1 implies that they both induce surjections on rational homology. Hence, both Y and $\widetilde{Y}$ have the rational homology of a point and, in particular, they both have Euler characteristic 1. However, for a finite covering $p\colon \widetilde{Y} \to Y$ of index k, the Euler characteristic of $\widetilde{Y}$ is k-times that of Y, thus $k = 1$. $\square$

Remark A.3. An n-dimensional manifold M is said to *dominate* an n-dimensional connected manifold N, if there exists a proper continuous map $f\colon M \to N$ of non-zero degree. Using this term, the above proof of Theorem A.2(a) amounts to observing that a proper surjective holomorphic map between complex n-dimensional manifolds is a map that establishes that the domain dominates the target and then applying Theorem A.1.

Remark A.4. Only after a preprint of this article appeared on the arXiv, the authors became aware of Gurjar's result [Gur80]. This was the motivation to add (b) to Theorem A.2, so that Theorem A.2 specializes to Gurjar's result by setting $X = \mathbb{C}^n$. Our proof of part (b) can also be understood as the natural generalization of the argument from [Gur80].

Before providing the proof of Theorem A.1, we recall orientations, dualities, the Umkehrungshomomorphismus, and the degree. We do this somewhat detailed and in a for us suitable way since we need all notions to work for non-compact manifolds. We take [Hat02] as our reference for algebraic topology.

For readability, we will drop the coefficients from the notation of homology and cohomology.

Manifold. A topological manifold, short manifold, of dimension n is a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$. In particular, manifolds have no boundary unless otherwise stated. A manifold is said to be closed if it is compact.

Orientation. An R-orientation is a map $o\colon M \to \bigcup_{x\in M} H_n(M, M\setminus\{x\})$ such that $o(x) \in H_n(M, M\setminus\{x\}) \simeq R$ is a generator (i.e. $Ro(x) = H_n(M, M\setminus\{x\})$) and o is continuous. Here, $\bigcup_{x\in M} H_n(M, M\setminus\{x\})$ is endowed with the following topology, which turns the canonical projection to M into a covering map and o into a section of this covering map. The topology is the inductive limit topology with respect to the maps

$$\mathbb{R}^n \times R \xrightarrow{(x,r)\mapsto r\varepsilon_x(\mu)} \bigcup_{y\in\mathbb{R}^n} H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{y\}) \xrightarrow{\phi_*} \bigcup_{x\in M} H_n(M, M\setminus\{x\})$$

for all local charts $\phi\colon \mathbb{R}^n \to U$ where R carries the discrete topology, $\mu \in H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$ is a fixed generator and $\varepsilon_x\colon H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\})$ is induced by the translation $\mathbb{R}^n \to \mathbb{R}^n$, $y \mapsto y + x$; see [Hat02, R-orientation]).

For every compact $K \subset M$, we denote by $o_K \in H_n(M, M\setminus K)$ the unique element in $H_n(M, M\setminus K)$ that maps to $o(x)$ under the map induced by inclusion of pairs $(M, M \setminus K) \subset (M, M \setminus \{x\})$ for all $x \in K$; see [Hat02, Lemma 3.27].

For context, recall that for a closed oriented manifold M, o_M is the fundamental class in $H_n(M)$.

Cohomology groups with compact supports as a limit. For any topological space X, we denote by $H^k_{\mathrm{comp}}(X)$ *cohomology groups with compact supports*; that is, the limit group of the directed system of groups given by the groups and maps

$$\Big\{H^k(X, X \setminus K)\Big\}_{K \subset X,\ K \text{ compact}} \quad \text{and} \quad H^k(X, X \setminus K) \xrightarrow{\iota^*} H^k(X, X \setminus L),$$

where $K \subseteq L \subseteq X$ are compacts and $i : (X, X \setminus L) \to (X, X \setminus K)$ denotes the inclusions of pairs, respectively; see [Hat02, Paragraph after Prop 3.33].

This yields a functor from the category of topological space with morphisms given by *proper* continuous maps to the category of R-modules for each non-negative integer k: to a proper continuous map $f\colon X \to Y$ we associate

$$f^k_{\mathrm{comp}}\colon H^k_{\mathrm{comp}}(Y) \to H^k_{\mathrm{comp}}(X)\,, \quad [\phi] \mapsto [f^k(\phi) \in H^k(X, X \setminus f^{-1}(J))]\,,$$

where $\phi \in H^k(Y, Y \setminus J)$ for some compact subset $J \subseteq Y$, and we denote by $f^k\colon H^k(Y, Y \setminus J) \to H^k(X, X \setminus f^{-1}(J))$ the homomorphism induced by $f\colon (X, X \setminus f^{-1}(J)) \to (Y, Y \setminus J)$.

Poincaré duality and the Umkehrungshomomorphismus. We recall that for an R-oriented topological manifolds M we have the Poincare duality isomorphism. One can write the *Poincaré duality* map

$$PD_k(M)\colon H^{n-k}_{\mathrm{comp}}(M) \to H_k(M)$$

as the homomorphism induced by

$$H^{n-k}(M, M \setminus K) \to H_k(M)\,, \quad \psi \mapsto o_K \cap \psi$$

for all compact subsets K in X; see [Hat02, Theorem 3.35].

Correspondingly, for all non-negative integers k, one defines the *Umkehrungshomomorphismus* in homology of a proper continuous map $f\colon M \to N$ between R-oriented n-manifolds as

$$f_{!,k} \coloneqq PD_k(M) \circ f^{n-k}_{\mathrm{comp}} \circ (PD_k(N))^{-1}\colon H_k(N) \to H_k(M).$$

Alexander duality. For an R-orientable manifold M and a locally contractible compact path-connected subset $K \subset M$, one has the following

$$H_l(M, M \setminus K) \simeq H^{n-l}(K) \text{ for all } l \in \{0, \dots, n\}, \tag{16}$$

which we only use for $l = n$ and K path-connected, so that $H^{n-l}(K) \simeq R$.

Proof of (16). Let K be a compact in an R-orientable n-dimensional manifold M. If M is closed, see [Hat02, Theorem 3.44] for a proof (the proof given there works as stated for every R).

If instead M is not closed (i.e. M is not compact), we find a compact n-dimensional submanifold $M_0 \subset M$ with boundary such that K is contained in the interior of M_0° of M_0; see Lemma A.5 below. Now (16) follows from the case above since by excision

$$H_l(M, M\backslash K) \simeq H_l(M_0^\circ, M_0^\circ\backslash K) \simeq H_l\left(M_0 \cup \mathrm{id}_{\partial M_0} M_0, M_0 \cup \mathrm{id}_{\partial M_0}\overline{M_0} \setminus K\right),$$

where $M_0 \cup \mathrm{id}_{\partial M_0} M_0$ denotes the doubling of M_0, i.e. the closed R-orientable n-manifold obtained by gluing M_0 to a copy of itself along their boundary via the identity. □

Every compact sits in a compact submanifold. The following lemma was used above to assure that Alexander duality holds for non-compact manifolds. We will also use it below for degree calculations.

Lemma A.5. *Let M be an n-dimensional manifold. If $K \subset M$ is a compact subset, then there exists a compact n-dimensional manifold M_0 (with boundary if K has non-empty intersection with at least one non-compact component of M) such that the interior of M_0 contains K. If M is connected, then M_0 can be chosen to be path-connected.*

Proof. If K has empty intersection with all non-compact components of M, set M_0 to be the union of connected components of M that have non-empty intersection with K. Hence, we consider the case that K has non-empty intersection with at least one non-compact component of M (in particular, M is non-compact).

Pick a proper continuous map $f\colon M \to \mathbb{R}$. (For example, exhaust M by a countable union of compacts $K_1 \subset K_2 \subset \cdots$ with $K_i \subseteq K_{i+1}^\circ$ (possible by second countability), define f to be i on the compacts $K_i \setminus K_i^\circ$, and extend it to map $K_{i+1} \setminus K_i^\circ$ to $[i, i+1]$ by the Tietze extension theorem.)

Let $a, b \in \mathbb{R}$ be such that $a + 1 < f(x) < b - 1$ for all x in K. Up to changing f by a homotopy that is constant outside of the compact $f^{-1}([a-1, a+1] \cup [b-1, b+1]$ (in particular, the resulting f stays proper), we may and do assume that f is transversal to a and b, which in particular implies that $M_0 \coloneqq f^{-1}([a, b])$ is a compact manifold with boundary $f^{-1}(a) \cup f^{-1}(b)$; see [FNOP19, Definition 10.7 and Theorem 10.8] for necessary definitions and statements.[3]

In case M is connected, one can easily arrange for M_0 to be connected. Indeed, let L be the union of M_0 with the image of (finitely many) paths in M between components of M_0. Thus L is a path-connected compact subset of M that contains the original K. Find a compact submanifold (with boundary) of dimension n of M that contains L (as done in the previous paragraph) and take its connected component that contains L. □

[3]We abstain from providing the details of topological transversality (details and further references can be found in [FNOP19]). We note that in the rest of the paper we use this appendix only for smooth manifolds, and the proof is written such that replacing f by a smooth map the argument works with the notion of transversality and the corresponding transversality theorems in smooth manifold theory.

Degree. Let $f\colon M \to N$ be a proper continuous map, where M and N are R-oriented n-manifolds.

For $y \in N$, we set $K \coloneqq f^{-1}(y)$ and consider the induced map

$$f_n\colon H_n(M, M\backslash f^{-1}(y)) \to H_n(N, N\backslash\{y\}) = Ro(y) \simeq R\,.$$

We define the *local degree* d_y of f at a point $y \in N$ as the unique $d_y \in R$ such that $f_n(o_K) = d_y o(y)$.

Remark A.6 (Local degree for y with finite preimage). In the special case that K is finite, say given by pairwise distinct points $x_1, \cdots x_l$, we have that

$$d_y = \sum_{i=0}^{l} r(x_i),$$

where $r(x_i) \in R$ is such that for an open neighborhood U of x_i with $U \cap K = \{x_i\}$ the induced map of pairs $f_n\colon H_n(M, M \setminus \{x_i\}) \simeq H_n(U, U \setminus \{x_i\}) \to H_n(N, N \setminus \{y\})$ satisfies $f_n(o(x_i)) = r(x_i)o(y)$.

If $y_1 \neq y_2$ are in the same connected component of N, then $d_{y_1} = d_{y_2}$. This follows from the following lemma, which is immediate from naturality of induced maps in homology of pairs.

Lemma A.7. *Let $f\colon M \to N$ be a proper continuous map, where M and N are R-oriented n-manifolds.*

If J is a compact subset of N such that $H_n(N, N \setminus J) \simeq R$, e.g. J is path connected and locally contractible (Alexander duality; see (16)*), then the unique $d \in R$ such that $f_n(o_{f^{-1}(J)}) = do_J$ satisfies $d = d_y$ for all $y \in J$.* □

And, indeed, it follows that $d_{y_1} = d_{y_2}$: let J be a closed arc embedded in N with endpoints y_1 and y_2 (such an arc exists since connected components of manifolds are arc-connected), hence $d_{y_1} = d_{y_2}$ by Lemma A.7.

Hence, if N is connected, the *degree* d of f is defined to be the local degree of f at a $y \in N$.

The proof.

Proof of Theorem A.1. Let $f\colon M \to N$ be a proper continuous map between R-oriented manifolds M and N, where N is connected. Fix a non-negative integer k and $c_Y \in H_k(N)$. We choose a compact $J \subset N$ such that $PD_k(N)([\psi]) = c_Y$ for some $\psi \in H^{n-k}(Y, Y \setminus J)$. In fact, by increasing J if necessary (and changing ψ to the corresponding class given by the inclusion map), we may and do choose J to be connected and locally contractible (indeed, we may choose it as a connected submanifold with boundary by Lemma A.5). We set $K \coloneqq f^{-1}(J) \subset M$, which is compact since f is proper.

With this setup we calculate

$$\begin{aligned}
f_k\left(f_{!,k}(c_Y)\right) &= f_k\left(PD_k(M)\circ f^{n-k}_{\mathrm{comp}}\circ(PD_k(N))^{-1}(c_Y)\right) && (17)\\
&= f_k\left(PD_k(M)(f^{n-k}_{\mathrm{comp}}([\psi]))\right) && (18)\\
&= f_k\left(PD_k(M)([f^{n-k}(\psi)])\right) && (19)\\
&= f_k\left(o_K\cap f^{n-k}(\psi)\right) && (20)\\
&= f_n(o_K)\cap\psi && (21)\\
&= do_J\cap\psi && (22)\\
&= dPD_k(N)([\psi]) = dc_Y \qquad , && (23)
\end{aligned}$$

where we use the following. (17) holds by the definition of the Umkehrungshomomorphism. (18) holds by our choice of ψ. (19) follows by the definition of the induced map on cohomology with compact support. (20) holds by the definition of $PD_k(M)$. (21) is an application of the naturality of the cap product

$$\cap\colon H_n(M, M\setminus K)\times H^{n-k}(M, M\setminus K)\to H_k(M);$$

see [Hat02, more general relative cap product, The Duality Theorem, p. 240]. For (22), note that $do_J = f_n(o_K)$ by Lemma A.7 since $K = f^{-1}(J)$ and by Alexander duality (see (16)) we have $H_n(N, N\setminus J)\simeq R$ by our choice of J. Finally, (23) holds by the definition of $PD_k(N)$ and since $PD_k(N)([\psi]) = c_Y$. □

Appendix B. A characterization of embeddings

For the lack of a reference to an elementary proof of the following characterization of embeddings, we provide a proof here.

Proposition B.1. *Let $f\colon X\to Y$ be a morphism of varieties. Then the following are equivalent:*

a) f is an embedding;
b) f is proper, injective and for each $x\in X$ the differential $d_xf\colon T_xX\to T_{f(x)}Y$ is injective.

For the proof, we use the following two lemmas from commutative algebra.

Lemma B.2. *Let B be a ring and $S\subset B$ be a multiplicative set such that the localization $R := S^{-1}B$ is a local ring. Denote by $\mathfrak{n}$ the maximal ideal of R, by $\varphi\colon B\to R$ the canonical homomorphism and set $\mathfrak{m} := \varphi^{-1}(\mathfrak{n})$.*

Then there exists an isomorphism $\psi\colon R\to B_{\mathfrak{m}}$ such that $\psi\circ\varphi\colon B\to B_{\mathfrak{m}}$ is the canonical homomorphism of the localization.

Proof of Lemma B.2. As $\varphi(S)$ consists of units in R, we get $\varphi(S)\subset R\setminus\mathfrak{n}$, i.e. $S\subset B\setminus\mathfrak{m}$. By the universal property of localizations there exists a homomorphism $\psi\colon R\to B_{\mathfrak{m}}$ such that $\psi\circ\varphi$ is equal to the canonical homomorphism $\iota\colon B\to B_{\mathfrak{m}}$. Thus it is enough to show that ψ is an isomorphism.

By definition $\varphi(B \setminus \mathfrak{m}) \subset R \setminus \mathfrak{n}$, i.e. $\varphi(B \setminus \mathfrak{m})$ consists of units in R. Hence $\varphi\colon B \to R$ factors through $\iota\colon B \to B_{\mathfrak{m}}$, there exists $\theta\colon B_{\mathfrak{m}} \to R$ such that $\theta \circ \iota = \varphi$. Thus the following commutative diagram exists:

$$\begin{tikzcd} & B \arrow[dl, "\varphi"'] \arrow[d, "\iota"] \arrow[dr, "\varphi"] & \\ R \arrow[r, "\psi"] & B_{\mathfrak{m}} \arrow[r, "\theta"] & R\,. \end{tikzcd}$$

For $r \in R$ there exist $b \in B$ and $s \in S$ with $r = \frac{b}{s}$ in R and we get

$$(\theta \circ \psi)(r) = (\theta \circ \psi)(\varphi(b)\varphi(s)^{-1}) = \varphi(b)\varphi(s)^{-1} = r\,.$$

Hence $\theta \circ \psi$ is the identity on R and in particular, ψ is injective. On the other hand, let $\frac{b}{t} \in B_{\mathfrak{m}}$ where $b \in B$ and $t \in B \setminus \mathfrak{m}$. Since $\varphi(B \setminus \mathfrak{m})$ consists of units in R, we get $\varphi(b)\varphi(t)^{-1} \in R$ and thus $\psi(\varphi(b)\varphi(t)^{-1}) = \iota(b)\iota(t)^{-1} = \frac{b}{t}$. This shows that ψ is surjective. □

Lemma B.3. *Let $A \subset B$ be a ring extension of Noetherian local rings where $\mathfrak{m}_A$ and $\mathfrak{m}_B$ denote the maximal ideals of A and B, respectively. If*

a) $\mathfrak{m}_A \subset \mathfrak{m}_B$,
b) the induced field extension $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is trivial,
c) the induced homomorphism $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective,
d) B is a finite A-module,

then $A = B$.

Proof of Lemma B.3. We claim that $\mathfrak{m}_A B = \mathfrak{m}_B$. Indeed, by a) we know that $\mathfrak{m}_A B \subset \mathfrak{m}_B$. Since $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, we get $\mathfrak{m}_B = \mathfrak{m}_A + \mathfrak{m}_B^2$ and inductively

$$\mathfrak{m}_B = \mathfrak{m}_A + \mathfrak{m}_B^n \quad \text{for each } n \geq 2\,. \tag{24}$$

Let $\pi\colon B \to B/\mathfrak{m}_A B$ be the canonical projection. Since $B/\mathfrak{m}_A B$ is a Noetherian local ring and $\pi(\mathfrak{m}_B)$ is a proper ideal of $B/\mathfrak{m}_A B$, Krull's intersection theorem implies the second equality below:

$$\pi(\mathfrak{m}_B) \overset{(24)}{=} \bigcap_{n \geq 1} \pi(\mathfrak{m}_B)^n = (0)\,.$$

This implies $\mathfrak{m}_B \subset \mathfrak{m}_A B$ and proves the claim.

Since by b), we have that the field extension $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is trivial, the claim implies now that

$$B = A + \mathfrak{m}_B = A + \mathfrak{m}_A B\,.$$

This in turn gives us $M = \mathfrak{m}_A M$ for $M = B/A$. Since B is a finite A-module, M is a finite A-module as well. Since A is a local ring with maximal ideal $\mathfrak{m}_A$, we conclude by Nakayama's lemma that $M = 0$, $A = B$. □

Proof of Proposition B.1. Clearly, a) implies b), hence we are left with the proof of the reverse implication and thus we assume b) holds.

Note that $f(X)$ is closed in Y, since f is proper. We may therefore replace Y with $f(X)$ and assume in addition that f is surjective. Now, we have to show that f is locally an isomorphism. Since f is proper and injective, it is finite; see [GW10, Corollary 12.89]. Thus, for each $x \in X$ there exists an open affine neighbourhood $U \subset Y$ of $f(x)$ such that $f^{-1}(U)$ is affine and $\mathcal{O}_X(f^{-1}(U))$ is a finite $\mathcal{O}_Y(U)$-module via the induced homomorphism $f_U^*\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. As f is surjective, f_U^* is injective.

Let $A := \mathcal{O}_Y(U)$, $B := \mathcal{O}_X(f^{-1}(U))$ and denote by $\mathfrak{m}_A$, $\mathfrak{m}_B$ the maximal ideals corresponding to the points $f(x)$, x, respectively. We identify A with a subring of B and then $\mathfrak{m}_A = \mathfrak{m}_B \cap A$. By the flatness of $A \to A_{\mathfrak{m}_A}$ the homomorphism $A_{\mathfrak{m}_A} \to A_{\mathfrak{m}_A} \otimes_A B$ is injective and it is finite, since $A \subset B$ is finite. Let $R := A_{\mathfrak{m}_A} \otimes_A B$. Then R is the localization of B at the multiplicative set $A \setminus \mathfrak{m}_A$. Hence, we have a commutative push-out diagram

$$\begin{array}{ccc} A_{\mathfrak{m}_A} & \subset & R \\ {\scriptstyle \iota_A}\uparrow & & \uparrow{\scriptstyle \varphi} \\ A & \subset & B \end{array} \tag{25}$$

where ι_A and φ denote the canonical homomorphisms into the corresponding localizations.

Let $\mathfrak{n}$ be a maximal ideal in R. We claim that $\mathfrak{n} = \varphi(\mathfrak{m}_B)R$. Indeed, $\mathfrak{n} \cap A_{\mathfrak{m}_A}$ is a maximal ideal of $A_{\mathfrak{m}_A}$, since $A_{\mathfrak{m}_A} \subset R$ is finite; see [Mat86, Lemma 2, §9]. This implies the first equality below and the second one follows from the commutativity of (25):

$$\mathfrak{m}_A = \iota_A^{-1}(\mathfrak{n} \cap A_{\mathfrak{m}_A}) = \varphi^{-1}(\mathfrak{n}) \cap A\,. \tag{26}$$

Since $\varphi^{-1}(\mathfrak{n})$ is a prime ideal of B, $\varphi^{-1}(\mathfrak{n}) \cap A = \mathfrak{m}_A$ is a maximal ideal of A and since $A \subset B$ is finite, it follows from [Mat86, Lemma 2, §9] that $\varphi^{-1}(\mathfrak{n})$ is a maximal ideal of B. Since $f\colon X \to Y$ is injective, $\mathfrak{m}_B$ is the only maximal ideal in B with $\mathfrak{m}_B \cap A = \mathfrak{m}_A$. By (26), we get now $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}_B$, which implies the claim.

Using the claim, $\varphi(\mathfrak{m}_B)R$ is the unique maximal ideal in R. In particular R is a local ring and $\mathfrak{m}_B = \varphi^{-1}(\varphi(\mathfrak{m}_B)R)$. By Lemma B.2 there is an isomorphism $\psi\colon R \to B_{\mathfrak{m}_B}$ such that $\psi \circ \varphi$ is equal to the canonical homomorphism $\iota_B\colon B \to B_{\mathfrak{m}_B}$ of the localization. Hence we may identify R with $B_{\mathfrak{m}_B}$ and φ with ι_B and we have to show now that $A_{\mathfrak{m}_A} = B_{\mathfrak{m}_B}$. However, this follows from Lemma B.3 applied to the ring extension $A_{\mathfrak{m}_A} \subset B_{\mathfrak{m}_B}$ (condition c) in Lemma B.3 follows from the injectivity of $\mathrm{d}_x f\colon T_x X \to T_{f(x)} Y$ and condition b) follows from the fact that $A/\mathfrak{m}_A = A_{\mathfrak{m}_A}/\mathfrak{m}_A A_{\mathfrak{m}_A}$, $B/\mathfrak{m}_B = B_{\mathfrak{m}_B}/\mathfrak{m}_B B_{\mathfrak{m}_B}$ and from the assumption that the ground field is algebraically closed). $\square$

Appendix C. Non-existence of proper surjective morphisms of affine spaces into homogeneous spaces

In this last appendix, we prove a version of Proposition D that works over an arbitrary algebraically closed field $\mathbf{k}$ of characteristic zero:

Proposition C.1. *Let $d \geq 1$. If G/H is a d-dimensional homogeneous space of a simple algebraic group G, then there is no proper surjective morphism from $\mathbb{A}^d$ to G/H.*

The idea is simply to reduce the situation to the case of complex numbers and then to use Proposition D. In other words, we check that the Lefschetz principle holds for the specific statement we need.

For the proof we make the following convention. If X is a variety over $\mathbf{k}$ and if $\mathbf{k} \subset K$ is a field extension such that K is algebraically closed as well, we denote by X_K the fiber product $X \times_{\operatorname{Spec} \mathbf{k}} \operatorname{Spec} K$. In case X is affine, we will denote the coordinate ring of X by $\mathbf{k}[X]$; in particular we then have $K[X_K] = K \otimes_{\mathbf{k}} \mathbf{k}[X]$. In the proof we will use the following properties of G_K for an algebraic group G over $\mathbf{k}$:

Lemma C.2. *Let $\mathbf{k} \subset K$ be a field extension such that K is algebraically closed and let G be an algebraic group over $\mathbf{k}$. Then the following holds:*

(1) The algebraic group G is connected if and only if G_K is connected.
(2) The group of $\mathbf{k}$-rational points $G(\mathbf{k})$ is dense in G_K.
(3) Let H be a closed subgroup over $\mathbf{k}$ of G. Then $G_K/H_K = (G/H)_K$.
(4) If G° denotes the identity component of G, then $(G^\circ)_K = (G_K)^\circ$.
(5) Assume that G is connected. Then, G is simple (semisimple, reductive) if and only if G_K is simple (semisimple, reductive).

Remark C.3. Let G be a non-trivial algebraic group G over $\mathbf{k}$. Then G is reductive if and only if the identity component G° is reductive or trivial. Hence, for any field extension $\mathbf{k} \subset K$ where K is algebraically closed, the algebraic group G is reductive if and only if G_K is (see Lemma C.2).

Proof of Lemma C.2. (1): If G is connected, then $\mathbf{k}[G]$ is an integral domain. There is a canonical inclusion $K[G_K] = K \otimes_{\mathbf{k}} \mathbf{k}[G] \subset K \otimes_{\mathbf{k}} \mathbf{k}(G)$ where $\mathbf{k}(G)$ denotes the field of rational functions on G. Since $\mathbf{k}$ is algebraically closed, by [ZS58, Corollary 1, §15, Ch. III], we get that $K \otimes_{\mathbf{k}} \mathbf{k}(G)$ is an integral domain and thus G_K is connected.

If G_K is connected, then $K[G_K] = K \otimes_{\mathbf{k}} \mathbf{k}[G]$ is an integral domain. As $\mathbf{k} \subset K$ is an inclusion, it follows that $\mathbf{k}[G] \to K \otimes_{\mathbf{k}} \mathbf{k}[G]$ is an inclusion and thus $\mathbf{k}[G]$ is an integral domain as well. This shows that G is connected.

(2): Note that a $\mathbf{k}$-rational point of G corresponds to a $\mathbf{k}$-algebra homomorphism $\mathbf{k}[G] \to \mathbf{k}$ which in turn induces a K-algebra homomorphism $K[G_K] = K \otimes_{\mathbf{k}} \mathbf{k}[G] \to K \otimes_{\mathbf{k}} \mathbf{k} = K$ and thus gives a (closed) point in G_K. In this way we see $G(\mathbf{k})$ as a subgroup of G_K.

Denote by G° the identity component. Hence there exists a finite set $E \subset G(\mathbf{k})$ such that $G = \coprod_{e \in E} e \cdot G^\circ$. Since $(G^\circ)_K$ is connected (see (1)), it follows from [Bor91, 18.3 Corollary] that $G^\circ(\mathbf{k})$ is dense in $(G^\circ)_K$. Hence

$$G(\mathbf{k}) = \coprod_{e \in E} e \cdot G^\circ(\mathbf{k}) \quad \text{is dense in} \quad G_K = \coprod_{e \in E} e \cdot (G^\circ)_K \,.$$

(3): Denote by $\pi_K \colon G_K \to (G/H)_K$ the pull-back of the natural projection $\pi \colon G \to G/H$. Let $\mathrm{pr} \colon H \times G \to G$ be the projection onto the second factor. Since π is H-invariant, we get the commutativity of

$$\begin{tikzcd} G \times H \arrow[r, "(g,h)\mapsto g\cdot h"] \arrow[d, "\mathrm{pr}"'] & G \arrow[d, "\pi"] \\ G \arrow[r, "\pi"] & G/H \end{tikzcd} \quad \text{and thus of} \quad \begin{tikzcd} G_K \times H_K \arrow[r, "(g,h)\mapsto g\cdot h"] \arrow[d, "\mathrm{pr}_K"'] & G_K \arrow[d, "\pi_K"] \\ G_K \arrow[r, "\pi_K"] & (G/H)_K \,. \end{tikzcd}$$

This shows that π_K is H_K-invariant. In particular, there exists a morphism $\theta \colon G_K/H_K \to (G/H)_K$ such that π_K factors as

$$G_K \to G_K/H_K \xrightarrow{\theta} (G/H)_K \tag{27}$$

where the first morphism denotes the canonical projection.

Let $U \subset G/H$ be an open affine subvariety and let $V \to U$ be a finite étale morphism such that $V \times_U \pi^{-1}(U) \to V$ is a trivial principal H-bundle. In particular, $V \times_U \pi^{-1}(U) \simeq U \times H$ is affine and since $V \times_U G \to \pi^{-1}(U)$ is finite, it follows that $\pi^{-1}(U)$ is affine by Chevalley's Theorem [GW10, Theorem 12.39]. Using (27), we get that the restriction $\pi_K|_{\pi^{-1}(U)_K} \colon \pi^{-1}(U)_K \to U_K$ factorizes as

$$\pi^{-1}(U)_K \to \pi^{-1}(U)_K/H_K \xrightarrow{\theta_{U_K}} U_K \,,$$

where θ_{U_K} denotes the restriction of θ to $\pi^{-1}(U)_K/H_K$. Since $\pi^{-1}(U)$ is affine, we get $K[\pi^{-1}(U)_K] = K \otimes_{\mathbf{k}} \mathbf{k}[\pi^{-1}(U)]$.

We claim that θ_{U_K} is an isomorphism. To achieve this it is enough to show that the induced map of θ_{U_K} on global sections of the structure sheaves is a K-algebra isomorphism (since U_K is affine). Since $U_K = (\pi^{-1}(U)/H)_K$, this amounts to showing that the invariant rings satisfy

$$(K \otimes_{\mathbf{k}} \mathbf{k}[\pi^{-1}(U)])^{H_K} = K \otimes_{\mathbf{k}} \mathbf{k}[\pi^{-1}(U)]^H \quad \text{inside } K \otimes_{\mathbf{k}} \mathbf{k}[\pi^{-1}(U)] \,.$$

The inclusion '$\supseteq$' follows from the existence of θ_{U_K}. For the reverse inclusion let $(e_i)_i$ be a $\mathbf{k}$-basis of the $\mathbf{k}$-vector space K and let $\sum_i e_i \otimes_{\mathbf{k}} f_i \in K \otimes_{\mathbf{k}} \mathbf{k}[\pi^{-1}(U)]$ be H_K-invariant (almost all $f_i \in \mathbf{k}[\pi^{-1}(U)]$ are zero). In particular, we get for all $h \in H(\mathbf{k})$ and $g \in \pi^{-1}(U)$ that

$$\sum_i e_i f_i(h \cdot g) = \sum_i e_i f_i(g) \quad \text{inside } K \,.$$

As $(e_i)_i$ is a $\mathbf{k}$-basis for K, we get $f_i(h \cdot g) = f_i(g)$ for each $h \in H(\mathbf{k})$, each $g \in \pi^{-1}(U)$ and each i. This implies $f_i \in \mathbf{k}[\pi^{-1}(U)]^H$ for each i and shows '$\subseteq$'. Hence θ_{U_K} is an isomorphism.

As we may cover G/H by open affine subvarieties U such that there is a finite étale morphism $V \to U$ that trivializes π over U, it follows that θ is an isomorphism.

(4): The connectedness of $(G^\circ)_K$ follows from the connectedness of G°; see (1). Since $(G/G^\circ)_K = G_K/(G^\circ)_K$ is finite (see (3)), we get that $(G^\circ)_K$ is the identity component in G_K.

(5): Let T be a maximal algebraic torus of G, denote by $\mathfrak{X}(T)$ the character lattice of T and denote by $\mathfrak{g}$ the Lie algebra of G. Moreover, let $R \subset \mathfrak{X}(T)$ be the roots of $\mathfrak{g}$ with respect to T and for each $\alpha \in R$, let $\mathfrak{g}^\alpha$ denote the corresponding eigenspace. Hence we get

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha \, .$$

Note that $K \otimes_{\mathbf{k}} \mathfrak{g}$ is the Lie algebra of G_K, that we may naturally identify $\mathfrak{X}(T)$ with $\mathfrak{X}(T_K)$ and that the natural T-action on $\mathfrak{g}$ induces naturally a T_K-action on $K \otimes_{\mathbf{k}} \mathfrak{g}$. Since $(K \otimes_{\mathbf{k}} \mathfrak{g})^\alpha \supset K \otimes_{\mathbf{k}} \mathfrak{g}^\alpha$ for each $\alpha \in R$ and $(K \otimes_{\mathbf{k}} \mathfrak{g})^0 \supset K \otimes_{\mathbf{k}} \mathfrak{g}^0$, we get

$$(K \otimes_{\mathbf{k}} \mathfrak{g})^0 \oplus \bigoplus_{\alpha \in R} (K \otimes_{\mathbf{k}} \mathfrak{g})^\alpha = K \otimes_{\mathbf{k}} \mathfrak{g} = K \otimes_{\mathbf{k}} \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in R} K \otimes_{\mathbf{k}} \mathfrak{g}^\alpha$$

and

$$(K \otimes_{\mathbf{k}} \mathfrak{g})^0 = K \otimes_{\mathbf{k}} \mathfrak{g}^0 \, , \quad (K \otimes_{\mathbf{k}} \mathfrak{g})^\alpha = K \otimes_{\mathbf{k}} \mathfrak{g}^\alpha \text{ for each } \alpha \in R \, .$$

We assume first that G is semisimple (reductive). Using that G_K is connected, it follows from [DG11, Proposition 1.12, Corollaire 1.13, Exp. XIX] that T_K is a maximal algebraic torus in G_K and G_K is semisimple (reductive). If G is simple, then G_K is semisimple. Moreover, the roots system of G with respect to T is irreducible, and thus the root system of G_K with respect to T_K is irreducible as well. Hence, if G is simple, then G_K is simple as well.

Assume now that G_K is simple. If there exists a proper connected normal subgroup N over $\mathbf{k}$ of G, then N_K is a proper connected normal subgroup of G_K, since $G(\mathbf{k})$ is dense in G_K and $N(\mathbf{k})$ is dense in N_K; see (2). Hence, N_K contains only the identity and thus N as well. Moreover, as G_K is non-commutative and $G(\mathbf{k})$ is dense in G_K, we get that G is non-commutative. Analogously one shows that G is semisimple (reductive) in case G_K is semisimple (reductive). □

Proof of Proposition C.1. We assume towards a contradiction that there exists a proper surjective morphism $\varphi\colon \mathbb{A}^n \to G/H$. In particular, φ is quasi-finite and since φ is proper, we conclude that φ is finite; see [GW10, Corollary 12.89]. By Chevalley's Theorem [GW10, Theorem 12.39], G/H is affine. In particular, we get a finite ring extension

$$\mathbf{k}[G/H] \subseteq \mathbf{k}[x_1, \ldots, x_n] \, ,$$

where $x_1, \ldots, x_n$ are variables. By the assumption there exist monic polynomials $f_1, \ldots, f_n \in \mathbf{k}[G/H][T]$ such that $f_1(x_1) = \ldots = f_n(x_n) = 0$.

There exists an algebraically closed subfield $\mathbf{k}' \subset \mathbf{k}$ of finite transcendence degree over $\mathbb{Q}$, an algebraic group G' over $\mathbf{k}'$, and a proper subgroup H' over $\mathbf{k}'$ of G' such that $G = G'_{\mathbf{k}}$ and $H = H'_{\mathbf{k}}$.

Since G/H is affine and G is reductive, H is reductive or trivial (see [Tim11, Theorem 3.8]). By Remark C.3, H' is reductive or trivial, and thus G'/H' is affine. Hence, $\mathbf{k}'[G'/H']$ is a finitely generated $\mathbf{k}'$-algebra, and thus there exists a surjective $\mathbf{k}'$-algebra homomorphism $\eta' \colon \mathbf{k}'[y_1, \ldots, y_m] \to \mathbf{k}'[G'/H']$, where $y_1, \ldots, y_m$ are new variables. By Lemma C.2(3)

$$\eta \coloneqq \mathbf{k} \otimes_{\mathbf{k}'} \eta \colon \mathbf{k}[y_1, \ldots, y_m] \to \mathbf{k} \otimes_{\mathbf{k}'} \mathbf{k}'[G'/H'] = \mathbf{k}[G/H]$$

is a surjective $\mathbf{k}$-algebra homomorphism. For each $i = 1, \ldots, n$, let $d_i \coloneqq \deg(f_i) > 0$ and let $p_{ij} \in \mathbf{k}[y_1, \ldots, y_m]$, where $j = 0, \ldots, d_i - 1$, such that

$$f_i = T^{d_i} + \sum_{j=0}^{d_i-1} \eta(p_{ij}) T^j \,.$$

By enlarging $\mathbf{k}'$ we may assume in addition that the coefficients of all the $p_{ij} \in \mathbf{k}[y_1, \ldots, y_m]$ and all the $\eta(y_i) \in \mathbf{k}[x_1, \ldots, x_n]$ are contained in $\mathbf{k}'$. In particular, the polynomial f_i has coefficients in $\mathbf{k}'[G'/H']$ for each i and

$$\mathbf{k}'[G'/H'] \subseteq \mathbf{k}'[x_1, \ldots, x_n] \,. \tag{28}$$

As $f_i(x_i) = 0$ for each i, we get that (28) is a finite ring extension.

Since the field extension $\mathbb{Q} \subset \mathbf{k}'$ has finite transcendence degree, there exists an embedding of $\mathbf{k}'$ into the field of complex numbers $\mathbb{C}$. Hence,

$$\mathbb{C}[(G'/H')_{\mathbb{C}}] = \mathbb{C} \otimes_{\mathbf{k}'} \mathbf{k}'[G'/H'] \subset \mathbb{C}[x_1, \ldots, x_n]$$

is a finite ring extension and $G'_{\mathbb{C}}/H'_{\mathbb{C}} = (G'/H')_{\mathbb{C}}$ is affine. Thus, we get a finite surjective morphism $\mathbb{A}^n_{\mathbb{C}} \to G'_{\mathbb{C}}/H'_{\mathbb{C}}$. Since G simple, we get that G' is simple, and thus also $G'_{\mathbb{C}}$ is simple; see Lemma C.2(5). This contradicts Proposition D. $\square$

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Peter Feller, ETH Zürich, Department of Mathematics, Rämistrasse 101, CH-8092 Zürich, Switzerland
Email address: `peter.feller@math.ch`

Immanuel van Santen, University of Basel, Department of Mathematics and Computer Science, Spiegelgasse 1, CH-4051 Basel, Switzerland
Email address: `immanuel.van.santen@math.ch`

DYNAMICAL DEGREES OF AFFINE-TRIANGULAR AUTOMORPHISMS OF AFFINE SPACES

JÉRÉMY BLANC AND IMMANUEL VAN SANTEN

Abstract. We study the possible dynamical degrees of automorphisms of the affine space $\mathbb{A}^n$. In dimension $n = 3$, we determine all dynamical degrees arising from the composition of an affine automorphism with a triangular one. This generalises the easier case of shift-like automorphisms which can be studied in any dimension. We also prove that each weak Perron number is the dynamical degree of an affine-triangular automorphism of the affine space $\mathbb{A}^n$ for some n, and we give the best possible n for quadratic integers, which is either 3 or 4.

Contents

1. Introduction

1.1. Dynamical degrees of polynomial endomorphisms. In this text, we work over an arbitrary field $\mathbf{k}$. For each $n \geq 1$, recall that an endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$ of $\mathbb{A}^n = \mathbb{A}^n_{\mathbf{k}}$ is given by

$$f\colon (x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$$

where $f_1, \ldots, f_n \in \mathbf{k}[x_1, \ldots, x_n]$. To simplify the notation, we often write $f = (f_1, \ldots, f_n)$ and thus identify $\mathrm{End}(\mathbb{A}^n)$ with $(\mathbf{k}[x_1, \ldots, x_n])^n$.

The degree of an endomorphism $f = (f_1, \ldots, f_n)$, denoted by $\deg(f)$, is defined to be $\deg(f) = \max(\deg(f_1), \ldots, \deg(f_n))$. The set $\mathrm{End}(\mathbb{A}^n)$ of endomorphisms of $\mathbb{A}^n$ is a monoid, for the composition law, and the subset of invertible elements is the group $\mathrm{Aut}(\mathbb{A}^n)$ of automorphisms of $\mathbb{A}^n$.

The dynamics of endomorphisms of $\mathbb{A}^n$, specially in the case of the ground field $\mathbf{k} = \mathbb{C}$, was studied intensively in the last decades, see for instance [FsW98, Sib99, Mae00, BFs00, Mae01a, Mae01b, Gue02, GS02, Gue04, Ued04, FJ11, JW12, Xie17,

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DL18]. For each dominant endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$, the (*first*) *dynamical degree* is defined as the real number

$$\lambda(f) = \lim_{r\to\infty} \deg(f^r)^{\frac{1}{r}} \in \mathbb{R}_{\geq 1}$$

(the limit exists by Fekete's subadditivity Lemma, see Lemma 2.2.1). If $f \in \mathrm{End}(\mathbb{A}^1)$ or $f \in \mathrm{Aut}(\mathbb{A}^2)$, then $\lambda(f)$ is an integer, but in higher dimensions, it can be quite complicated to understand the possible dynamical degrees. In [DF, Corollary 3], the authors conjecture that $\lambda(f)$ is an algebraic integer of degree $\leq n$, and of degree $\leq n-1$ if $f \in \mathrm{Aut}(\mathbb{A}^n)$, a conjecture proven until now only for $n \leq 2$.

In this article, we study some particular family of automorphisms of $\mathbb{A}^n$, that we call *affine-triangular*. These are compositions consisting of one affine automorphism and one triangular automorphism (see Definition 2.1.1) below. Our two main results are Theorem 1 and Theorem 2 below:

Theorem 1. *For each field* $\boldsymbol{k}$ *and each integer* $d \geq 2$*, the set of dynamical degrees of all affine-triangular automorphisms of* $\mathbb{A}^3$ *of degree* $\leq d$ *is equal to*

$$\left\{ \frac{a+\sqrt{a^2+4bc}}{2} \,\middle|\, (a,b,c) \in \mathbb{N}^3, a+b \leq d, c \leq d \right\} \setminus \{0\}.$$

Moreover, for $a, b, c \in \mathbb{N}$ *such that* $\lambda = \frac{a+\sqrt{a^2+4bc}}{2} \neq 0$*, the dynamical degre* λ *is achieved by either of the automorphisms*

$$(x_3 + x_1^a x_2^b, x_2 + x_1^c, x_1) \text{ and } (x_3 + x_1^a x_2^{bc}, x_1, x_2).$$

Using Theorem 1, we prove in [BvS, Theorem 2] that the set of dynamical degrees of all automorphisms of degree 3 of $\mathbb{A}^3$ is equal to

$$\left\{ 1, \sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \sqrt{3}, 2,\ \frac{1+\sqrt{13}}{2},\ 1+\sqrt{2},\ \sqrt{6},\ \frac{1+\sqrt{17}}{2},\ \frac{3+\sqrt{5}}{2},\ 1+\sqrt{3},\ 3 \right\}.$$

Note that $\frac{3+\sqrt{5}}{2}$ is the only number that does not belong to the list in Theorem 1 and thus it is the dynamical degree of an automorphism of degree 3 of $\mathbb{A}^3$ that is not conjugate to an affine-triangular automorphism of any degree.

For the next theorem, we recall the definition of (weak)-Perron numbers (see Theorem 3.2.4 for some equivalent characterisations).

Definition 1.1.1. A *Perron number* (respectively *weak Perron number*) is a real number $\lambda \geq 1$ that is an algebraic integer such that all other Galois conjugates $\mu \in \mathbb{C}$ satisfy $|\mu| < \lambda$ (respectively $|\mu| \leq \lambda$).

Theorem 2. *Each weak-Perron number* λ *is the dynamical degree of an affine-triangular automorphism of* $\mathbb{A}^n$ *for some integer* n*. Moreover:*

(1) *If* $\lambda > 1$ *is an integer, the least* n *possible is* 2.
(2) *If* λ *is a quadratic integer and its conjugate is negative, the least possible* n *is* 3.
(3) *If* λ *is a quadratic integer and its conjugate is positive, the least possible* n *is* 4.

Note that Statement (1) in Theorem 2 is well-known, as $\{\lambda(f) \mid f \in \mathrm{Aut}(\mathbb{A}^2)\} = \mathbb{Z}_{\geq 1}$. We include it to emphasise the relation between the degree of the weak-Perron numbers and the possible n. In view of the above theorems and of the techniques developped in this text, it is natural to ask the following

Question 1.1.2. *Is every dynamical degree of any element of* $\mathrm{End}(\mathbb{A}^n)$ (*respectively* $\mathrm{Aut}(\mathbb{A}^n)$) *equal to a weak Perron number of degree* $\leq n$ (*respectively of degree* $\leq n-1$)*?*

As already mentioned above, a positive answer to this question, where "weak Perron number" is replaced by "algebraic integer", was conjectured in the recent preprint [DF, Corollary 3] (that appeared after we asked the above question in a first version of this text). In [DF], it is also proven that the dynamical degree of any element in $\mathrm{Aut}(\mathbb{A}^3)$ is an algebraic number of degree at most six. More generally they prove that the dynamical degree of any element of $\mathrm{End}(\mathbb{A}^n)$ is an algebraic number of degree at most n in case the square of the first dynamical degree is bigger than the second dynamical degree of f [DF, Theorem 2].

Theorem 1 shows in particular that the dynamical degree of every affine-triangular automorphism of $\mathbb{A}^3$ is equal to the dynamical degree of a shift-like automorphism. However, for each $d \geq 3$ the set of dynamical degrees of all affine-triangular automorphisms of $\mathbb{A}^3$ of degree d strictly contains the set of dynamical degrees of all shift-like automorphisms of $\mathbb{A}^3$ of degree d. Indeed, the latter set of dynamical degrees consists of the numbers $(a+\sqrt{a^2+4d-4a})/2$ where $0 \leq a \leq d$ and does not contain $(1+\sqrt{1+4d})/2$, which is the dynamical degree of the affine-triangular automorphism $(x_3+x_1x_2, x_2+x_1^d, x_1)$, see Corollary 4.3.7.

d	dynamical degrees of shift-like automorphisms of $\mathbb{A}^3$ of degree d not appearing in degree $<d$	dynamical degrees of affine-triangular automorphisms of $\mathbb{A}^3$ of degree d not appearing in degree $<d$
1	$\{1\}$	$\{1\}$
2	$\{\sqrt{2}, \frac{1+\sqrt{5}}{2}, 2\}$	$\{\sqrt{2}, \frac{1+\sqrt{5}}{2}, 2\}$
3	$\{\sqrt{3}, 1+\sqrt{2}, 3\}$	$\{\sqrt{3}, \frac{1+\sqrt{13}}{2}, 1+\sqrt{2}, \sqrt{6}, \frac{1+\sqrt{17}}{2}, 1+\sqrt{3}, 3\}$
4	$\{\frac{1+\sqrt{13}}{2}, 1+\sqrt{3}, \frac{3+\sqrt{13}}{2}, 4\}$	$\{2\sqrt{2}, 1+\sqrt{5}, \frac{3+\sqrt{13}}{2}, \frac{1+\sqrt{33}}{2}, 2\sqrt{3}, \frac{1+\sqrt{37}}{2}, \frac{3+\sqrt{17}}{2}, 1+\sqrt{7}, \frac{3+\sqrt{21}}{2}, 4\}$

Note that $2\sqrt{2}$ and $\sqrt{3}$ appear as dynamical degrees of affine-triangular automorphisms in degree 4 and 3, respectively (and not smaller), even if $2\sqrt{2} < 3$ and $\sqrt{3} < 2$. Similarly, for each prime p, the number $\sqrt{p}$ is the dynamical degree of a shift-like automorphism of degree p, but it is not the dynamical degree of an affine-triangular automorphism of degree $< p$.

1.2. **Dynamical degrees of affine-triangular automorphisms in higher dimensions.** In dimension $n \geq 4$, we are not able to compute all dynamical degrees of all affine-triangular automorphisms, but can get some large families. The case of shift-like automorphisms is covered by our method, and we retrieve a proof of the result of Mattias Jonsson (Proposition 4.2.5), but we can also study wider classes. We give the dynamical degrees of all permutation-elementary automorphisms (a family that strictly includes the shift-like automorphisms) in §4.2 (especially Proposition 4.2.3) and also give the dynamical degrees of other affine-triangular automorphisms. In particular, we show that in any dimension $n \geq 4$, there are affine-triangular automorphisms of $\mathbb{A}^n$ whose dynamical degrees are not those of a shift-like automorphisms or more generally of a permutation-elementary automorphisms, contrary to the case of dimension $n \leq 3$. The reason is that dynamical degrees of shift-like automorphisms are special kinds of weak Perron numbers.

Indeed, they are positive real numbers that are roots of a monic integral polynomial where all coefficients (except the first one) are non-positive. These numbers are called *Handelman numbers* in [Bas97] (see especially [Bas97, Lemma 10]) and they have no other positive real Galois conjugates (Lemma 3.2.7). This implies that Handelman numbers are weak Perron numbers (see Corollary 3.2.8). Theorem 1 implies that the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ is a Handelman number (and the same holds for all automorphisms of $\mathbb{A}^1$ and $\mathbb{A}^2$), but for any $n \geq 4$, there are affine-triangular automorphisms of $\mathbb{A}^n$ whose dynamical degrees are not Handelman numbers. This follows in particular from Theorem 2, applied to any weak Perron quadratic integer with a positive conjugate, for instance to $(3+\sqrt{5})/2$. We can also apply Theorem 2 to weak Perron numbers of arbitrary large degree.

1.3. **Results in the literature on dynamical degrees of endomorphisms of $\mathbb{A}^n$.** Let us recall what is known on the dynamical degrees of elements of $\mathrm{End}(\mathbb{A}^n)$.

(1) The case where $n = 1$ is obvious: in this case we have $\lambda(f) = \deg(f)$, so each dynamical degree is an integer, which is moreover equal to 1 in the case of automorphisms.

(2) When $n = 2$, the case of automorphisms follows from the Jung-van der Kulk Theorem [Jun42, vdK53]: every dynamical degree is an integer, as $\deg(f^r) = \deg(f)^r$ for each r, when f is taken to be cyclically reduced (this is explained in Corollary 2.4.3 below, or in [Fur99, Proposition 3]). The set of all dynamical degrees of quadratic endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$ is equal to $\{1, \sqrt{2}, (1+\sqrt{5})/2, 2\}$ by [Gue04, Theorem 2.1]. Moreover, the dynamical degree of every element of $\mathrm{End}(\mathbb{A}^2_{\mathbb{C}})$ is a quadratic integer, by [FJ07, Theorem A'].

(3) The case of dimension $n \geq 3$ is open in general: there is for the moment no hope of classifying all dynamical degrees, even when studying only automorphisms.

The set of dynamical degrees of all automorphisms of $\mathbb{A}^3_{\mathbb{C}}$ of degree 2 is equal to $\{1, \sqrt{2}, (1+\sqrt{5})/2, 2\}$ by [Mae01a, Theorem 3.1] (and the same holds over any field [BvS, Theorem 2]).

Apart from the above classification results, two natural families are also known: the *monomial endomorphisms* and the *shift-like automorphisms*.

(A) A *monomial endomorphism* of $\mathbb{A}^n$ is an endomorphism of the form $f = (f_1, \ldots, f_n)$, where each f_i is a monomial. When we write $f_i = \alpha_i x_1^{m_{i,1}} \cdots x_n^{m_{i,n}}$ with $\alpha_i \in \mathbf{k}^*$ and $m_{i,1}, \ldots, m_{i,n} \in \mathbb{N}$ and assume that f is dominant, then the dynamical degree of f is the spectral radius of the corresponding matrix $M = (m_{i,j})_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{N})$. This classical result is proven again in Corollary 3.2.5 below. The numbers arising this way are the weak Perron numbers (see Theorem 3.2.4).

(B) For each $n \geq 1$, a *shift-like automorphism* of $\mathbb{A}^{n+1}$ is an automorphism of the form $(x_{n+1}+p(x_1, \ldots, x_n), x_1, \ldots, x_n)$ for some polynomial $p \in \mathbf{k}[x_1, \ldots, x_n]$. These are particular examples of affine-triangular automorphisms. The dynamics of such automorphisms have been studied in various texts (see for instance [BP98, Mae00, Mae01b, Ued04, BV18]). The dynamical degrees of shift-like automorphisms are known, by a result of Mattias Jonsson (see Proposition 4.2.5 below). For a proof of this result, together with a generalisation, see §4.2.

1.4. **Description of the techniques associated to degrees.** In the rest of this introduction, we describe the main technique that we introduce in order to compute dynamical degrees of endomorphisms of $\mathbb{A}^n$. This is related to degree functions (or monomial valuations), and may be applied to endomorphisms of $\mathbb{A}^n$, not only affine-triangular automorphisms. We also give an outline of the whole article.

Definition 1.4.1. For each $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$, we define a *degree function* $\deg_\mu \colon \mathbf{k}[x_1, \dots, x_n] \to \mathbb{R}_{\geq 0} \cup \{-\infty\}$ by $\deg_\mu(0) = -\infty$ and

$$\deg_\mu\Big(\sum_{(a_1,\dots,a_n)\in\mathbb{N}^n} \underbrace{c_{(a_1,\dots,a_n)}}_{\in\mathbf{k}} \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\Big) = \max\left\{ \sum_{i=1}^n a_i\mu_i \,\middle|\, c_{(a_1,\dots,a_n)} \neq 0 \right\}.$$

We say that a polynomial $p \in \mathbf{k}[x_1, \dots, x_n]$ is *μ-homogeneous of degree $\theta \in \mathbb{R}$* if p is a finite sum of monomials p_i with $\deg_\mu(p_i) = \theta$ for each i (where the zero polynomial is μ-homogeneous of degree θ for each θ).

We can then write every element $q \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$ uniquely as

$$q = \sum_{\theta\in\mathbb{R}_{\geq 0}} q_\theta \,,$$

where each $q_\theta \in \mathbf{k}[x_1, \dots, x_n]$ is μ-homogeneous of degree θ (and only finitely many q_θ are non-zero). We then say that q_θ is the *μ-homogeneous part of q of degree θ*. The *μ-leading part* of q is the μ-homogeneous part of q of degree $\deg_\mu(q)$.

Remark 1.4.2. Note that if $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$, then

$$\mathbf{k}(x_1, \dots, x_n) \to \mathbb{R} \cup \{\infty\}\,, \quad f/g \mapsto \deg_\mu(g) - \deg_\mu(f)$$

is a valuation in the sense of [Mat89, p.75] where $\mathbf{k}(x_1, \dots, x_n)$ denotes the field of rational functions in $x_1, \dots, x_n$ over $\mathbf{k}$. Such valuations are often called “monomial valuations” in the literature.

Definition 1.4.3. Let $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. For each $f = (f_1, \dots, f_n) \in \mathrm{End}(\mathbb{A}^n) \setminus \{0\}$ we denote the *μ-degree of f* by

$$\deg_\mu(f) = \inf\{\theta \in \mathbb{R}_{\geq 0} \mid \deg_\mu(f_i) \leq \theta\mu_i \text{ for each } i \in \{1, \dots, n\}\}$$

and we say that $\deg_\mu(f) = \infty$ if the above set is empty.

We moreover say that f is *μ-algebraically stable* if $\deg_\mu(f) < \infty$ and $\deg_\mu(f^r) = \deg_\mu(f)^r$ for each $r \geq 1$.

Remark 1.4.4. If $\mu = (1, \dots, 1)$, then $\deg_\mu(f) = \deg(f)$ is the standard degree and the notion of being μ-algebraically stable is the standard notion of “algebraically stable”, studied for instance in [GS02, Bis08, Bla16]. The fact of being algebraically stable can be interpreted geometrically by looking at the behaviour of the endomorphism at infinity: [Bla16, Corollary 2.16].

In order to compute the dynamical degree of an endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$, the following endomorphism associated to f will be of great importance for us:

Definition 1.4.5. Let $f = (f_1, \dots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism, let $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ be such that $\deg_\mu(f) = \theta < \infty$. We define the *$\mu$-leading part of f* to be the endomorphism $g = (g_1, \dots, g_n) \in \mathrm{End}(\mathbb{A}^n)$, where $g_j \in \mathbf{k}[x_1, \dots, x_n]$ is the μ-homogeneous part of f_j of degree $\theta\mu_j$ for each $j \in \{1, \dots, n\}$.

The degree functions are studied in §2. Basic properties are given in §2.3, and the relation with μ-homogeneous endomorphisms is given in §2.5 (we explain in particular when $\deg_\mu(f) = \infty$ in Lemma 2.5.6). In §2.6, we explain how degree functions allow us to give an estimate on the dynamical degrees, and sometimes to compute it exactly. In particular, we prove the following result (at the end of §2.6).

Proposition A. *Let* $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ *be a dominant endomorphism. For each* $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{>0})^n$ *the following hold:*

(1) $\theta := \deg_\mu(f) < \infty$,
(2) *The dynamical degree of* f *satisfies* $1 \leq \lambda(f) \leq \theta$.
(3) *Let* $g \in \mathrm{End}(\mathbb{A}^n)$ *be the* μ*-leading part of* f. *If* $\theta > 1$, *then*

$$\lambda(f) = \theta \Leftrightarrow f \textit{ is } \mu\textit{-algebraically stable} \Leftrightarrow g^r \neq 0 \textit{ for each } r \geq 1.$$

Remark 1.4.6. Let $\mu = (1, \ldots, 1)$. In this case, the μ-degree is the classical degree and Proposition A(2) is the classical inequality $\lambda(f) \leq \deg(f)$.

Remark 1.4.7. Proposition A is false when we apply it to $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. For instance, if $f = (x_1, x_2^2)$, $\mu = (1, 0)$, then $\deg_\mu(f) = 1$ but $1 < \lambda(f) = 2$.

To apply Proposition A to compute the dynamical degree, we need to find some eigenvectors and eigenvalues. This is done here by looking at monomial maps associated to endomorphisms in $\mathrm{End}(\mathbb{A}^n)$. These behave quite well with respect to degree functions (see Corollary 3.2.5).

Definition 1.4.8. Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be an endomorphism such that $f_i \neq 0$ for each i. We will say that a square matrix $M = (m_{i,j})_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{N})$ is *contained in* f if for each $i \in \{1, \ldots, n\}$, the coefficient of the monomial $\prod_{j=1}^n x_j^{m_{i,j}}$ in f_i is nonzero. The set of matrices that are contained in f is then finite and non-empty.

The *maximal eigenvalue of* f is defined to be

$$\theta = \max \{ |\xi| \in \mathbb{R} \mid \xi \text{ is an eigenvalue of a matrix that is contained in } f \} .$$

An element $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ is a *maximal eigenvector of* f if $\deg_\mu(f_i) = \theta\mu_i$ for each $i \in \{1, \ldots, n\}$. In particular, we then get $\deg_\mu(f) = \theta < \infty$.

It often happens that we cannot apply Proposition A to compute the dynamical degree, but that we can do it by allowing μ to have some coordinates, but not all, to be equal to zero. In fact, the following generalization of Proposition A is our main tool to compute dynamical degrees:

Proposition B. *Let* $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ *be a dominant endomorphism with maximal eigenvalue* θ. *Then the following holds:*

(1) *There exists a maximal eigenvector* $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ *of* f.
(2) *We have* $1 \leq \lambda(f) \leq \theta \leq \deg(f)$.
(3) *For each maximal eigenvector* μ *of* f, *we have* $\theta = \deg_\mu(f)$, *and the following hold:*
    (i) *If* f *is* μ*-algebraically stable, then* $\lambda(f) = \theta$.
    (ii) *If* $\lambda(f) = \theta$, $\theta > 1$ *and* $\mu \in (\mathbb{R}_{>0})^n$, *then* f *is* μ*-algebraically stable.*
    (iii) *Let* $g \in \mathrm{End}(\mathbb{A}^n)$ *be the* μ*-leading part of* f. *If* $\theta > 1$, *then* f *is* μ*-algebraically stable if and only if for each* $r \geq 1$ *there is* $i \in \{1, \ldots, n\}$ *with* $\mu_i > 0$ *and such that the* i*-th component of* g^r *is non-zero.*

Remark 1.4.9. In Proposition B(1), there are examples with no possibility for μ to be in $(\mathbb{R}_{>0})^n$, as the examples $f = (x_1, x_2^2) \in \mathrm{End}(\mathbb{A}^2)$ or $f = (x_1, x_3, x_2 + x_3^2) \in$

$\mathrm{Aut}(\mathbb{A}^3)$ show. Hence, Proposition A cannot be directly applied in order to prove Proposition B. However, if some coordinates of μ are zero, then a linear projection is preserved (this follows from Lemma 2.5.6, see also Corollary 2.6.2). To prove Proposition B, we will use Lemma 2.6.1, that is a version of Proposition A that also works for $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$.

Remark 1.4.10. The implication of Proposition B(3)(i) is not an equivalence, as we show in Example 3.4.2 below.

The proof of Proposition B is given in Section 3. For each dominant endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$, Proposition B(1) gives the existence of a maximal eigenvector μ. Moreover, Proposition B(3) shows that if f is μ-algebraically stable then $\lambda(f)$ is equal to the maximal eigenvalue θ of f. We will use this to compute the dynamical degree of many endomorphisms of $\mathbb{A}^n$.

The following result allows to compute all dynamical degrees of permutation-elementary endomorphism of $\mathbb{A}^n$, and generalises in particular Proposition 4.2.5. Its proof is given in §4.2:

Proposition C. *Let $f \in \mathrm{Aut}(\mathbb{A}^n)$ be a permutation-elementary automorphism. If the maximal eigenvalue θ of f is bigger than 1, there exists a maximal eigenvector μ of f such that f is μ-algebraically stable. In particular, the dynamical degree $\lambda(f)$ is equal to the maximal eigenvalue θ of f, which is a Handelman number.*

Proposition C is false if we replace "permutation-elementary" by "permutation-triangular" (see Example 4.3.4 for examples in dimension 3). We can however obtain the following result, which is proven in §4.3:

Proposition D. *Every affine-triangular automorphism $f \in \mathrm{Aut}(\mathbb{A}^3)$ is conjugate to a permutation-triangular automorphism $f' \in \mathrm{Aut}(\mathbb{A}^3)$ such that $\deg(f') \leq \deg(f)$ and such that f' has the following property: either the maximal eigenvalue θ of f' is equal to 1, or f' is μ-algebraically stable for each maximal eigenvector μ. In particular, the dynamical degrees $\lambda(f)$ and $\lambda(f')$ are equal to the maximal eigenvalue θ of f', which is a Handelman number.*

The proof of Theorem 1 is given at the end of §4.3, directly after proving Proposition D, as it follows almost directly from this result. We use these results in §4.4, to prove Theorem 2.

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2. Inequalities associated to degree functions and the proof of Proposition A

2.1. **Definitions of elementary, affine and triangular automorphisms.** Let us recall the following classical definitions (even if our definition of elementary is slightly more restrictive than what is used in the literature):

Definition 2.1.1. An endomorphism $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ is said to be

- *triangular* if $f_i \in \mathbf{k}[x_1, \ldots, x_i]$ for each $i \in \{1, \ldots, n\}$,

- *elementary* if $f_i = x_i$ for for each $i \in \{1, \dots, n-1\}$.
- *an affine automorphism* if $f \in \mathrm{Aut}(\mathbb{A}^n)$ and if $\deg(f) = 1$,
- *a permutation of the coordinates* if $\{f_1, \dots, f_n\} = \{x_1, \dots, x_n\}$,
- *affine-triangular* if $f = \alpha \circ \tau$ where α is an affine automorphism and τ is a triangular endomorphism,
- *affine-elementary* if $f = \alpha \circ e$ where α is an affine automorphism and e is an elementary endomorphism,
- *permutation-triangular* if $f = \alpha \circ \tau$ where α is a permutation of the coordinates and τ is a triangular endomorphism.
- *permutation-elementary* if $f = \alpha \circ e$ where α is a permutation of the coordinates and e is an elementary endomorphism.

For each $n \leq 4$, if $\mathrm{char}(\mathbf{k}) \neq 2$, every automorphism of $\mathbb{A}^n$ of degree 2 is conjugate, by an affine automorphism, to an affine-triangular automorphism, see [MO91]. This result is false in dimension $n = 5$ [Sun14], as for example

$$f = (x_1 + x_2x_4, x_2 + x_1x_5 + x_3x_4, x_3 - x_2x_5, x_4, x_5) \in \mathrm{Aut}(\mathbb{A}^5)$$

shows: the Jacobian of the homogeneous part of degree 2 of an affine-triangular automorphism of degree ≤ 2 contains a zero-column, but the Jacobian of the homogeneous part of degree 2 of f contains linearly independent columns (see also [Sun14, Theorem 3.2]).

There are quite a few automorphisms of $\mathbb{A}^3$ of degree 3 that are not conjugate, by an affine automorphism, to affine-triangular automorphisms. More precisely, when $\mathbf{k}$ is algebraically closed, then each automorphism of $\mathbb{A}^3 = \mathrm{Spec}(\mathbf{k}[x, y, z])$ of degree 3 is conjugate, by an affine automorphism, either to an affine-triangular automorphism or to an automorphism of the form

$$(*) \qquad \alpha(x + yz + za(x, z), y + a(x, z) + r(z), z) \in \mathrm{Aut}(\mathbb{A}^3)$$

where $a \in \mathbf{k}[x, z] \setminus \mathbf{k}[z]$ is homogeneous of degree 2, $r \in \mathbf{k}[z]$ is of degree ≤ 3 and α is an affine automorphism, see [BvS, Theorem 3]. In fact, non of the automorphisms in $(*)$ is conjugated, by an affine automorphism, to an affine-triangular automorphism, see [BvS, Proposition 3.9.4].

For $\mathbf{k} = \mathbb{C}$ various (dynamical) properties of the affine-elementary automorphisms $(x_0 + x_1 + x_0^q x_2^d, x_0, \alpha x_2) \in \mathrm{Aut}(\mathbb{A}^3)$ with $\alpha \in \mathbb{C}$, $0 < |\alpha| \leq 1$, $q \geq 2$, $d \geq 1$ are studied in [DL18] and in particular their dynamical degree is computed, which is equal to the integer q.

2.2. **Existence of dynamical degrees.** We recall the following folklore result, which implies that the dynamical degree is well-defined.

Lemma 2.2.1. *Let $(a_r)_{r\geq 1}$ be a sequence of real numbers in $\mathbb{R}_{\geq 1}$ such that $a_{r+s} \leq a_r \cdot a_s$ for each $r, s \geq 1$. Then, $((a_r)^{1/r})_{r\geq 1}$ is a sequence that converges towards* $\inf_{r\geq 1}((a_r)^{1/r}) \in \mathbb{R}_{\geq 1}$.

Proof. As $(\log(a_r))_{r\geq 1}$ is subbadditive, $(\frac{\log(a_r)}{r})_{r\geq 1}$ converges to $\inf_{r\geq 1}(\frac{\log(a_r)}{r}) \geq 0$ by Fekete's subadditivity Lemma (see [Fek23, Satz II] or [Ste97, Lemma 1.2.1]). $\square$

In case $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ is of the from $\mu_1 = \dots = \mu_m = 0$ and $\mu_{m+1} = \dots = \mu_n = 1$ for some $0 \leq m < n$ we denote for any polynomial

$p \in \mathbf{k}[x_1, \dots, x_n]$ its μ-degree $\deg_\mu(p)$ by $\deg_{x_{m+1},\dots,x_n}(p)$. Moreover, we denote for an endomorphism $f = (f_1, \dots, f_n) \in \mathrm{End}(\mathbb{A}^n)$

$$\deg_{x_{m+1},\dots,x_n}(f) = \max_{j \in \{1,\dots,n\}} \deg_{x_{m+1},\dots,x_n}(f_j)\,.$$

If $m = 0$, then $\deg_\mu(f)$ is simply the classical degree that we denote by $\deg(f)$. If $m > 0$, then $\deg_\mu(f)$ is in general not equal to $\deg_{x_{m+1},\dots,x_n}(f)$. In fact, $\deg_\mu(f)$ is equal to $\deg_{x_{m+1},\dots,x_n}(f)$ in case $\deg_{x_1,\dots,x_m}(f_i) = 0$ for all $i \in \{1, \dots, m\}$ and otherwise it is equal to ∞.

Corollary 2.2.2. *Let $f \in \mathrm{End}(\mathbb{A}^n)$ be an endomorphism. For each integer $m \in \{0, \dots, n-1\}$, the sequence*

$$\deg_{x_{m+1},\dots,x_n}(f^r)^{1/r}$$

converges to a real number $\mu_m \geq 1$. This gives in particular the dynamical degree $\lambda(f) = \mu_0$, which satisfies $\lambda(f^d) = \lambda(f)^d$ for each $d \geq 1$.

Proof. This follows from Lemma 2.2.1, as

$$\deg_{x_{m+1},\dots,x_n}(f^{r+s}) \leq \deg_{x_{m+1},\dots,x_n}(f^r) \cdot \deg_{x_{m+1},\dots,x_n}(f^s),$$

for all $r, s \geq 1$. □

2.3. **Basic properties of degree functions.** Below we list several properties of degree functions (see Definition 1.4.1). Apart from the easy observations $\deg_\mu|_{\mathbf{k}^*} = 0$, $\deg_\mu(f \cdot g) = \deg_\mu(f) + \deg_\mu(g)$ and $\deg_\mu(f+g) \leq \max(\deg_\mu(f), \deg_\mu(g))$, which correspond to say that $-\deg_\mu$ is a valuation (see Remark 1.4.2), we have:

Remark 2.3.1. We fix $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ and get:

(1) As explained in Definition 1.4.1, each polynomial $p \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$ can be written uniquely as a finite sum

$$p = \sum_{\theta \in \mathbb{R}_{\geq 0}} p_\theta$$

where each $p_\theta \in \mathbf{k}[x_1, \dots, x_n]$ is μ-homogeneous of degree θ. We then obtain $\deg_\mu(p) = \max\{\theta \mid p_\theta \neq 0\}$.

(2) Let $m \in \{0, \dots, n-1\}$ and assume that $\mu_i = 0$ for $i \leq m$, but $\mu_i > 0$ for $i > m$. Then we have for each polynomial $p \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$

$$\mu_{\min} \cdot \deg_{x_{m+1},\dots,x_n}(p) \leq \deg_\mu(p) \leq \mu_{\max} \cdot \deg_{x_{m+1},\dots,x_n}(p)$$

where $\mu_{\min} = \min_{m+1 \leq i \leq n} \mu_i$ and $\mu_{\max} = \max_{m+1 \leq i \leq n} \mu_i$. In particular, for each dominant endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$ we have

$$\lim_{r\to\infty} \deg_{x_{m+1},\dots,x_n}(f^r)^{\frac{1}{r}} = \lim_{r\to\infty} \max_{i \in \{1,\dots,n\}} \deg_\mu((f^r)_i)^{\frac{1}{r}}$$

where $(f^r)_i$ denotes the i-th coordinate function of f^r. Note that the left hand side is the dynamical degree $\lambda(f)$ in case $m = 0$, i.e. when $\mu \in (\mathbb{R}_{>0})^n$.

2.4. **Endomorphisms that preserve a linear projection.** The following is an algebraic analogue of the application of [DN11, Theorem 1.1] to endomorphisms of $\mathbb{A}^n$ that preserve a linear projection:

Lemma 2.4.1. *Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. For each $r \geq 1$, we write*

$$f^r = ((f^r)_1, \ldots, (f^r)_n)\,.$$

Let $m \in \{0, \ldots, n-1\}$ be such that $f_1, \ldots, f_m \in \boldsymbol{k}[x_1, \ldots, x_m]$. Then, the dynamical degree of f is given by $\lambda(f) = \max\{\lambda_1, \lambda_2\}$, where

$$\begin{array}{rcl} \lambda_1 & = & \lim\limits_{r\to\infty} \max\{\deg((f^r)_1), \ldots, \deg((f^r)_m)\}^{1/r} = \lambda((f_1, \ldots, f_m)) \\ \lambda_2 & = & \lim\limits_{r\to\infty} \max\{\deg_{x_{m+1},\ldots,x_n}((f^r)_{m+1}), \ldots, \deg_{x_{m+1},\ldots,x_n}((f^r)_n)\}^{1/r} \\ & = & \lim\limits_{r\to\infty} \deg_{x_{m+1},\ldots,x_n}(f^r)^{1/r} \end{array}$$

are two limits which exist. (If $m = 0$, by convention we set $\lambda_1 = 1$.)

Proof. For each $r \geq 1$, we write

$$\begin{array}{rcl} a_r & = & \max\{\deg((f^r)_1), \ldots, \deg((f^r)_m)\} \\ b_r & = & \max\{\deg((f^r)_{m+1}), \ldots, \deg((f^r)_n)\} \\ c_r & = & \max\{\deg_{x_{m+1},\ldots,x_n}((f^r)_{m+1}), \ldots, \deg_{x_{m+1},\ldots,x_n}((f^r)_n)\} \\ & = & \deg_{x_{m+1},\ldots,x_n}(f^r). \end{array}$$

As $b_r \geq c_r$, we obtain for each $r \geq 1$

$$\deg(f^r) = \max\{a_r, b_r\} \geq \max\{a_r, c_r\}.$$

It follows from Corollary 2.2.2 that the limits

$$\lambda_1 = \lim_{r\to\infty} a_r^{1/r}\,,\ \lambda_2 = \lim_{r\to\infty} c_r^{1/r} \quad \text{and} \quad \lambda(f) = \lim_{r\to\infty} \deg(f^r)^{1/r}$$

exist (and all belong to $\mathbb{R}_{\geq 1}$). We obtain

$$\lambda(f) = \lim_{r\to\infty} \max\{a_r^{1/r}, b_r^{1/r}\} \geq \lim_{r\to\infty} \max\{a_r^{1/r}, c_r^{1/r}\} = \max\{\lambda_1, \lambda_2\}\,.$$

We may thus assume that $\lambda(f) > \lambda_1$, which implies that $\lim_{r\to\infty} b_r^{1/r}$ exists, and is equal to $\lambda(f)$. It remains to see that in this case $\lambda(f) \leq \max\{\lambda_1, \lambda_2\}$.

For all $r, s \geq 1$ and each $i \in \{m+1, \ldots, n\}$, the polynomial $(f^{r+s})_i$ is obtained by replacing $x_1, \ldots, x_n$ with $(f^r)_1, \ldots, (f^r)_n$ in $(f^s)_i$, so the degree of $(f^{r+s})_i$ is at most

$$\begin{array}{l} \deg_{x_1,\ldots,x_m}((f^s)_i) \cdot \deg((f^r)_1, \ldots, (f^r)_m) \\ + \deg_{x_{m+1},\ldots,x_n}((f^s)_i) \cdot \deg((f^r)_{m+1}, \ldots, (f^r)_n)\,. \end{array}$$

This gives $b_{r+s} \leq b_s \cdot a_r + c_s \cdot b_r$. When we choose then $s = r$, we obtain

$$b_{2r} \leq b_r \cdot (a_r + c_r)\,.$$

As $\lambda(f) = \lim\limits_{r\to\infty} b_{2r}^{1/2r}$, we have $\lambda(f)^2 = \lim\limits_{r\to\infty} b_{2r}^{1/r}$. The above inequality gives

$$\begin{array}{rcl} \lambda(f)^2 & = & \lim\limits_{r\to\infty} b_{2r}^{1/r} \\ & \leq & \lim\limits_{r\to\infty} b_r^{1/r} \cdot \limsup\limits_{r\to\infty} (a_r + c_r)^{1/r} \\ & \leq & \lambda(f) \cdot \limsup\limits_{r\to\infty} (2\max\{a_r, c_r\})^{1/r} \\ & = & \lambda(f) \cdot \max\{\lambda_1, \lambda_2\}, \end{array}$$

so $\lambda(f) \leq \max\{\lambda_1, \lambda_2\}$. $\square$

Corollary 2.4.2. *Let $n \geq 2$ and let $f \in \mathrm{Aut}(\mathbb{A}^n)$ be an automorphism such that $f_1, \ldots, f_{n-2} \in \boldsymbol{k}[x_1, \ldots, x_{n-2}]$ and such that the dynamical degree of $g = (f_1, \ldots, f_{n-2}) \in \mathrm{Aut}(\mathbb{A}^{n-2})$ is an integer. Then, the dynamical degree of f is an integer.*

Proof. By Lemma 2.4.1, one has $\lambda(f) = \max\{\lambda(g), \lambda_2\}$, where

$$\lambda_2 = \lim_{r\to\infty} \max\{\deg_{x_{n-1},x_n}((f^r)_{n-1}), \deg_{x_{n-1},x_n}((f^r)_n)\}^{1/r} .$$

It remains to see that λ_2 is an integer. As $\mathbf{k}[x_1, \ldots, x_{n-2}, f_{n-1}, f_n] = \mathbf{k}[x_1, \ldots, x_n]$, one has $K[f_{n-1}, f_n] = K[x_{n-1}, x_n]$, where $K = \mathbf{k}(x_1, \ldots, x_{n-2})$. Hence, one can see the automorphism $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, f_{n-1}, f_n)$ of $\mathbb{A}^n$ as an automorphism $F \in \mathrm{Aut}_K(\mathbb{A}^2)$ of $\mathbb{A}^2$ defined over K. For each $i \geq 0$, the automorphism $g^{-i} \circ (x_1, \ldots, x_{n-1}, f_{n-1}, f_n) \circ g^i$ of $\mathbb{A}^n$ can be seen as an element of $\mathrm{Aut}_K(\mathbb{A}^2)$ that we denote by F^{g^i} where we identify g with the automorphism $(f_1, \ldots, f_{n-2}, x_{n-1}, x_n) \in \mathrm{Aut}(\mathbb{A}^n)$. This gives

$$\max\{\deg_{x_{n-1},x_n}((f^r)_{n-1}), \deg_{x_{n-1},x_n}((f^r)_n)\} = \deg(G_r)$$

where $G_r = F^{g^{r-1}} \circ \cdots \circ F^{g^2} \circ F^g \circ F \in \mathrm{Aut}_K(\mathbb{A}^2)$, since $G_r = g^{-r} \circ f^r$ when we consider G_r, g and f as automorphisms of $\mathbb{A}^n$.

According to the Jung-van der Kulk Theorem [Jun42, vdK53], one can write $F = F_1 \circ \cdots \circ F_s$ where each $F_i \in \mathrm{Aut}_K(\mathbb{A}^2)$ is either triangular or affine. One can moreover assume that two consecutive F_i are not both affine or both triangular (as otherwise one may reduce the description), and get then $\deg(F) = \prod_{i=1}^{s} \deg(F_i)$ (follows by looking at what happens at infinity or by [vdE00, Lemma 5.1.2]). We prove that λ_2 is an integer by induction on s. If $s = 1$, then F is either affine or triangular; this implies that the set $\{\deg(G_r) \mid r \geq 1\}$ is bounded, so $\lambda_2 = 1$. If $s > 1$ and F_1, F_s are both affine or both triangular, we replace F with $(F_1)^g \circ F \circ F_1^{-1}$. This replaces $G_r = F^{g^{r-1}} \circ \cdots \circ F^g \circ F$ with $\tilde{G}_r = (F_1)^{g^r} \circ G_r \circ F_1^{-1}$. As $\deg((F_1)^{g^r}) = \deg(F_1)$ for each $r \geq 1$, one has

$$\frac{1}{\deg(F_1)^2} \deg(G_r) \leq \deg(\tilde{G}_r) \leq \deg(G_r) \cdot \deg(F_1)^2 ,$$

so this replacement does not change the value of λ_2. As this decreases the value of s, we may assume that F_1 and F_s are not both triangular or affine. Hence, for each $r \geq 1$, G_r is a product of rs elements that are affine or triangular, with no two consecutive in the same group. This gives $\deg(G_r) = \prod_{i=0}^{r-1} \prod_{j=1}^{s} \deg(F_j^{g^i}) = \prod_{i=0}^{r-1} \prod_{j=1}^{s} \deg(F_j) = \deg(F)^r$. Hence, $\lambda_2 = \deg(F)$ is an integer. □

Corollary 2.4.3. *The dynamical degree of any element of $\mathrm{Aut}(\mathbb{A}^2)$ is an integer. Similarly, the dynamical degree of any element of $\mathrm{Aut}(\mathbb{A}^3)$ (respectively $\mathrm{Aut}(\mathbb{A}^4)$) which preserves the set of fibres of a linear projection $\mathbb{A}^3 \to \mathbb{A}^1$ or $\mathbb{A}^3 \to \mathbb{A}^2$ (respectively $\mathbb{A}^4 \to \mathbb{A}^2$) is an integer.*

Proof. The fact that the dynamical degree of any element of $\mathrm{Aut}(\mathbb{A}^2)$ is an integer follows from Corollary 2.4.2 applied to $n = 2$. If $f \in \mathrm{Aut}(\mathbb{A}^3)$ is an automorphism that preserves the set of fibres of a linear projection $\mathbb{A}^3 \to \mathbb{A}^1$ or $\mathbb{A}^3 \to \mathbb{A}^2$, then one may conjugate by an element of GL_3 and obtain $f = (f_1, f_2, f_3)$ with either $f_1 \in \mathbf{k}[x_1]$ or $f_1, f_2 \in \mathbf{k}[x_1, x_2]$. The fact that $\lambda(f)$ is an integer follows then from Corollary 2.4.2 and Lemma 2.4.1, respectively (in the second case, one uses the fact that the dynamical degree of $(f_1, f_2) \in \mathrm{Aut}(\mathbb{A}^2)$ is an integer). Similarly, in the case of an automorphism of $\mathbb{A}^4$ preserving a linear projection $\mathbb{A}^4 \to \mathbb{A}^2$, one restricts to the case $f = (f_1, \ldots, f_4) \in \mathrm{Aut}(\mathbb{A}^4)$ with $f_1, f_2 \in \mathbf{k}[x_1, x_2]$, and applies Corollary 2.4.2. $\square$

2.5. **Homogeneous endomorphisms.**

Lemma 2.5.1. *Let $h = (h_1, \ldots, h_n) \in \mathrm{End}(\mathbb{A}^n)$, let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ and let $\theta \in \mathbb{R}_{\geq 0}$. The following conditions are equivalent:*

(1) *The polynomial h_i is μ-homogeneous of degree $\theta\mu_i$ for each $i \in \{1, \ldots, n\}$.*
(2) *For each μ-homogeneous polynomial p of degree ξ and each integer $r \geq 1$, the polynomial $p \circ h^r$ is μ-homogeneous of degree $\theta^r\xi$.*

If additionally $h_i \neq 0$ for each $i \in \{1, \ldots, n\}$, then (1) and (2) are equivalent to

(3) *For each Matrix M contained in h, μ is an eigenvector to the eigenvalue θ.*

Proof. The implication (2) $\Rightarrow$ (1) is given by choosing $p = x_i$ for $i = 1, \ldots, n$, so we may assume (1) and prove (2). It suffices to prove (2) for $r = 1$, as the general result follows by induction.

If $p = 0$, then $h(p) = 0$ is μ-homogeneous of any degree. It then suffices to do the case where p is a monomial: we write $p = \zeta x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $\zeta \in \mathbf{k}^*$, $a_1, \ldots, a_n \geq 0$, which is μ-homogeneous of degree $\deg_\mu(p) = \sum_{i=1}^n a_i\mu_i$. As h_i is μ-homogeneous of degree $\theta\mu_i$, the polynomial $p \circ h = \zeta h_1^{a_1} h_2^{a_2} \cdots h_n^{a_n}$ is μ-homogeneous of degree $\sum_{i=1}^n a_i\theta\mu_i = \theta \deg_\mu(p)$.

Now, we assume additionally that $h_i \neq 0$ for each $i \in \{1, \ldots, n\}$. The equivalence between (1) and (3) follows immediately from the definition of the μ-degree. $\square$

Definition 2.5.2. Let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ and let $\theta \in \mathbb{R}_{\geq 0}$. We say that $h \in \mathrm{End}(\mathbb{A}^n)$ is *μ-homogeneous of degree θ* if the conditions of Lemma 2.5.1 are satisfied.

Lemma 2.5.3. *Let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. For each $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ and each $\theta \in \mathbb{R}_{\geq 0}$, the following are equivalent:*

(1) *We can write f as a finite sum $f = \sum_{0 \leq \xi \leq \theta} g_\xi$, where each $g_\xi \in \mathrm{End}(\mathbb{A}^n)$ is μ-homogeneous of degree ξ.*
(2) $\deg_\mu(f) \leq \theta$.

Proof. (1) $\Rightarrow$ (2): For each $i \in \{1, \ldots, n\}$, the polynomial f_i is the sum of the i-th components of the endomorphisms g_ξ. As each of these polynomials has degree $\xi\mu_i \leq \theta\mu_i$, the polynomial f_i is of μ-degree $\deg_\mu(f_i) \leq \theta\mu_i$.

(2) $\Rightarrow$ (1): As in Remark 2.3.1(1), we write each f_i, $i \in \{1, \ldots, n\}$ as $f_i = \sum_{0 \leq \kappa \leq \theta\mu_i} p_{i,\kappa}$ where each $p_{i,\kappa}$ is μ-homogeneous of degree κ.

We define $g_0 = (p_{1,0}, \ldots, p_{n,0}) \in \mathrm{End}(\mathbb{A}^n)$, which is μ-homogeneous of degree 0.

For each $\xi \in \mathbb{R}$ with $0 \leq \xi \leq \theta$, we define the i-th component $(g_\xi)_i$ of g_ξ as follows: if $\mu_i = 0$ and $\xi > 0$, then $(g_\xi)_i = 0$ and otherwise, we choose $(g_\xi)_i = p_{i,\xi\mu_i}$. By construction, g_ξ is μ-homogeneous of degree ξ.

Moreover, $f_i = \sum_{0 \leq \kappa \leq \theta\mu_i} p_{i,\kappa} = \sum_{0 \leq \xi \leq \theta} (g_\xi)_i$ for each $i \in \{1, \ldots, n\}$ with $\mu_i > 0$. If $\mu_i = 0$, then $f_i = \sum_{0 \leq \kappa \leq \theta\mu_i} p_{i,\kappa} = p_{i,0} = \sum_{0 \leq \xi \leq \theta} (g_\xi)_i$. This yields $f = \sum_{0 \leq \xi \leq \theta} g_\xi$. $\square$

Remark 2.5.4. In the decomposition of Lemma 2.5.3(1), the i-th component of each g_ξ is unique, if $\mu_i > 0$, but is not unique if $\mu_i = 0$.

Example 2.5.5. We have $\deg_{(1,\dots,1)}(f) = \deg(f)$ and $\deg_\mu(\mathrm{id}_{\mathbb{A}^n}) = 1$ for each $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. However, $\deg_{(2,3,0)}(x_1, x_2 + x_1^2 x_3, x_3) = \frac{4}{3}$.

Lemma 2.5.6. *Let $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. For each $f = (f_1, \dots, f_n) \in \mathrm{End}(\mathbb{A}^n)$, the following are equivalent:*

(1) $\deg_\mu(f) < \infty$.
(2) *For each $i \in \{1, \dots, n\}$ such that $\mu_i = 0$, the element f_i is a polynomial in the variables $\{x_j \mid j \in \{1, \dots, n\}, \mu_j = 0\}$.*

In particular, if $\mu \in (\mathbb{R}_{>0})^n$ then the above conditions hold.

Proof. (1) $\Rightarrow$ (2): Suppose that $\theta = \deg_\mu(f) < \infty$. For each $i \in \{1, \dots, n\}$, we get $\deg_\mu(f_i) \leq \theta\mu_i$ (Definition 1.4.3). If $\mu_i = 0$, then $\deg_\mu(f_i) = 0$, which means that f_i is a polynomial in the variables $\{x_j \mid j \in \{1, \dots, n\}, \mu_j = 0\}$.

(2) $\Rightarrow$ (1): it follows from (2) that $\deg_\mu(f_i) \leq 0$ for each $i \in \{1, \dots, n\}$ such that $\mu_i = 0$. This gives $\deg_\mu(f) = \max\left\{\deg_\mu(f_i)/\mu_i \;\middle|\; \mu_i > 0\right\} < \infty$. □

Lemma 2.5.7. *Let $f = (f_1, \dots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. For each maximal eigenvector μ of f, the μ-leading part $g = (g_1, \dots, g_n) \in \mathrm{End}(\mathbb{A}^n)$ of f has the following properties:*

(1) *The maximal eigenvalue θ of f is such that $\deg_\mu(g) = \deg_\mu(f) = \theta < \infty$;*
(2) *For each $i \in \{1, \dots, n\}$, the polynomial g_i is non-constant.*

Proof. As $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ is a maximal eigenvector of f, we have $\deg_\mu(f_i) = \theta\mu_i$ for each $i \in \{1, \dots, n\}$, where θ is the maximal eigenvalue of f. This gives $\deg_\mu(f) = \theta < \infty$ and therefore $\deg_\mu(g_i) = \theta\mu_i = \deg_\mu(f_i)$ for each $i \in \{1, \dots, n\}$. Hence, we get (1). In case $\mu_i > 0$, we have $\deg_\mu(g_i) = \theta\mu_i > 0$ and thus g_i is non-constant. In case $\mu_i = 0$, we have $\deg_\mu(f_i) = \theta\mu_i = 0$ and thus $g_i = f_i$. As f is dominant, the latter polynomial is non-constant. This shows (2). □

2.6. Inequalities obtained by iterations.

Lemma 2.6.1. *Let $f = (f_1, \dots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. Suppose that $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ and that $\theta = \deg_\mu(f) \in \mathbb{R}_{\geq 0}$. Let $g = (g_1, \dots, g_n) \in \mathrm{End}(\mathbb{A}^n)$ be the μ-leading part of f. Then the following hold:*

(1) *We can write f as a finite sum $f = g + \sum_{0 \leq \xi < \theta} g_\xi$, where each $g_\xi \in \mathrm{End}(\mathbb{A}^n)$ is μ-homogeneous of degree ξ.*
(2) *The i-the coordinate function $(g^r)_i$ of g^r is the μ-homogeneous part of degree $\theta^r\mu_i$ of $(f^r)_i$ for each $i \in \{1, \dots, n\}$ and each $r \geq 1$.*
(3) $\deg_\mu(f^r) \leq \theta^r$ *for each $r \geq 1$.*
(4) *We have*
$$1 \leq \lim_{r\to\infty} \max_{i \in \{1,\dots,n\}} \deg_\mu((f^r)_i)^{1/r} = \lim_{r\to\infty} (\deg_\mu(f^r))^{1/r} \leq \theta\,.$$
(5) *If $\theta > 1$, the following are equivalent:*
   (i) $\lim_{r\to\infty}(\deg_\mu(f^r))^{1/r} = \theta$.
   (ii) *f is μ-algebraically stable.*
   (iii) *For each $r \geq 1$ there is $i \in \{1, \dots, n\}$ with $\mu_i > 0$ and $(g^r)_i \neq 0$.*

Proof. As $\deg_\mu(f) = \theta$, we have $\deg_\mu(f_i) \leq \theta\mu_i$ for each $i \in \{1, \dots, n\}$. Moreover, as f is dominant and $\mu \neq 0$, there are $i, j \in \{1, \dots, n\}$ such that $\mu_i > 0$ and $\deg_{x_i}(f_j) \geq 1$. This implies that $\deg_\mu(f_j) \geq \mu_i > 0$ and thus
$$0 < \deg_\mu(f) = \theta\,.$$

We now observe that $\deg_\mu(f-g)<\theta$. Indeed, for each $j\in\{1,\ldots,n\}$, the j-th component g_j of g is the μ-homogeneous part of f_j of degree $\theta\mu_j\geq\deg_\mu(f_j)$. If $\mu_j=0$, then $f_j=g_j$, and if $\mu_j>0$, then $\deg_\mu(f_j-g_j)<\theta\mu_j$.

By Lemma 2.5.3, we can write $f-g$ as a finite sum $f-g=\sum_{0\leq\xi<\theta}g_\xi$, where each $g_\xi\in\mathrm{End}(\mathbb{A}^n)$ is μ-homogeneous of degree ξ. This gives (1).

We now prove (2)-(3) by induction on $r\geq 1$. For $r=1$, (2) follows from the definition of g. Moreover, (3) is given by hypothesis.

We now assume (2)-(3) for some integer $r\geq 1$ and prove them for $r+1$. For each $i\in\{1,\ldots,r\}$, we write $(f^r)_i=(g^r)_i+s_i$, where $(g^r)_i$ is μ-homogeneous of degree $\theta^r\mu_i$ and $\deg_\mu(s_i)<\theta^r\mu_i$. This gives

$$\begin{aligned}(f^{r+1})_i &= ((g^r)_i+s_i)\circ f\\ &\overset{(1)}{=} (g^{r+1})_i+s_i\circ g+\sum_{0\leq\xi<\theta}((g^r)_i+s_i)\circ g_\xi\end{aligned}$$

As g is μ-homogeneous of degree θ, the polynomial $(g^{r+1})_i$ is μ-homogeneous of degree $\theta^{r+1}\mu_i$ (Lemma 2.5.1). As s_i is a sum of μ-homogeneous polynomials of degree $<\theta^r\mu_i$ and g_ξ is μ-homogeneous of degree $\xi<\theta$, we have

$$\deg_\mu(s_i\circ g+\sum_{0\leq\xi<\theta}((g^r)_i+s_i)\circ g_\xi)<\theta^{r+1}\mu_i$$

(by using Lemma 2.5.1 again). This yields (2)-(3) for $r+1$.

We now prove (4). We choose $i\in\{1,\ldots,n\}$ such that $\mu_i=\max\{\mu_1,\ldots,\mu_n\}$, and observe that for each $r\geq 1$, there is $j\in\{1,\ldots,n\}$ such that $\deg_{x_i}((f^r)_j)>0$ (as f is dominant), so $\deg_\mu((f^r)_j)\geq\mu_i=\max\{\mu_1,\ldots,\mu_n\}>0$. This implies that

$$1\leq\lim_{r\to\infty}\max_{i\in\{1,\ldots,n\}}\deg_\mu((f^r)_i)^{1/r}$$

(the limit exists by Remark 2.3.1(2)). Let us write $I_0=\{i\in\{1,\ldots,n\}\mid\mu_i=0\}$. For each $i\in I_0$, we have $\deg_\mu(f_i)\leq\theta\mu_i=0$, so f_i is a polynomial in the variables $\{x_j\mid j\in I_0\}$. This implies that the same holds for $(f^r)_i$, for each integer $r\geq 1$. Hence, $\deg_\mu((f^r)_i)=0$ for each $i\in I_0$. Writing $I_{>0}=\{i\in\{1,\ldots,n\}\mid\mu_i>0\}$, we get for each $r\geq 1$,

$$\deg_\mu(f^r)=\max\left\{\frac{\deg_\mu((f^r)_i)}{\mu_i}\Big| i\in I_{>0}\right\}.$$

As $\deg_\mu(f^r)\leq\theta^r$ (Assertion (3)), we obtain

$$\lim_{r\to\infty}\left(\max_{i\in\{1,\ldots,n\}}\deg_\mu(f^r)_i\right)^{1/r}=\lim_{r\to\infty}\left(\deg_\mu(f^r)\right)^{1/r}\leq\theta.$$

It remains to prove (5); for this, we assume that $\theta>1$. For each $r\geq 1$, Assertion (3) gives $\deg_\mu(f^r)\leq\theta^r$, or equivalently $\deg_\mu((f^r)_i)\leq\theta^r\mu_i$ for each $i\in\{1,\ldots,n\}$. The equality $\deg_\mu(f^r)=\theta^r$ holds if and only if there exists $i\in\{1,\ldots,n\}$ such that $\mu_i>0$ and $\deg_\mu((f^r)_i)=\theta^r\mu_i$. Since $(g^r)_i$ is the μ-homogeneous part of $(f^r)_i$ of degree $\theta^r\mu_i$ (follows from (2)), this gives the equivalence between *(ii)* and *(iii)*. It remains then to prove *(i)* $\Leftrightarrow$ *(iii)*.

"*(iii)* $\Rightarrow$ *(i)*": Suppose that for each $r\geq 1$ there is $i\in\{1,\ldots,n\}$ such that $\mu_i>0$ and $(g^r)_i\neq 0$. There is then $j\in\{1,\ldots,n\}$ and an infinite set $I\subset\mathbb{N}$ such that

$\mu_j > 0$ and $(g^r)_j \neq 0$ for each $r \in I$. Assertion (2) implies that $\deg_\mu((f^r)_j) \geq \theta^r \mu_j$, for each $r \in I$, which implies that

$$\lim_{r\to\infty} \left(\max_{i\in\{1,\dots,n\}} \deg_\mu(f^r)_i \right)^{1/r} \geq \theta\,.$$

This, together with (4), gives $\lim_{r\to\infty}(\deg_\mu(f^r))^{1/r} = \theta$.

"$(i) \Rightarrow (iii)$": Conversely, suppose that there exists $s \geq 1$ such that $(g^s)_i = 0$ for each $i \in \{1,\dots,n\}$ with $\mu_i > 0$. For all such i we obtain $\deg_\mu((f^s)_i) < \theta^s \mu_i$ (by (2) and (3)). As $\theta > 1$, there exists then $\theta' \in \mathbb{R}$ with $1 < \theta' < \theta$ such that

$$\deg_\mu((f^s)_i) \leq \theta'^s \mu_i$$

for each $i \in \{1,\dots,n\}$. Applying the inequality of (4) for f^s, we obtain

$$\lim_{r\to\infty} \left(\max_{i\in\{1,\dots,n\}} (\deg_\mu(f^{sr})_i)^{1/r} \right) \leq \theta'^s$$

which gives, by taking the s-th root,

$$\lim_{r\to\infty} \left(\max_{i\in\{1,\dots,n\}} (\deg_\mu(f^r)_i)^{1/r} \right) \leq \theta' < \theta.$$

□

Now we can give a short proof of Proposition A.

Proof of Proposition A. (1): As $\mu \in (\mathbb{R}_{>0})^n$, we have $\theta := \deg_\mu(f) < \infty$ (Lemma 2.5.6). Using Remark 2.3.1(2) we get

$$\lambda(f) = \lim_{r\to\infty} \max_{i\in\{1,\dots,n\}} (\deg_\mu(f^r)_i)^{1/r}\,.$$

By definition, g is the μ-leading part of f. Now, Lemma 2.6.1(4) implies that $1 \leq \lambda(f) \leq \theta$. If $\theta > 1$, we moreover obtain

$$\lambda(f) = \theta \Leftrightarrow \deg_\mu(f^r) = \theta^r \text{ for each } r \geq 1 \Leftrightarrow g^r \neq 0 \text{ for each } r \geq 1$$

(by Lemma 2.6.1(4) and Lemma 2.6.1(5)). □

Another consequence of Lemma 2.6.1 is the following result, that generalises Proposition A to the case where some coordinates of μ are zero.

Corollary 2.6.2. *Let $f = (f_1,\dots,f_n) \in \operatorname{End}(\mathbb{A}^n)$ be a dominant endomorphism and let $\mu = (\mu_1,\dots,\mu_n) \in (\mathbb{R}_{\geq 0})^n$ be such that $\theta = \deg_\mu(f) < \infty$, and assume that $m \in \{0,\dots,n\}$ exists, such that $\mu_i = 0$ for $i \in \{1,\dots,m\}$ and $\mu_i > 0$ for $i \in \{m+1,\dots,n\}$ (which can always be obtained by conjugating with a permutation). Then, the following hold:*

1. *For each $i \in \{1,\dots,m\}$, we have $f_i \in \boldsymbol{k}[x_1,\dots,x_m]$. Hence, the element $\hat{f} = (f_1,\dots,f_m)$ belongs to $\operatorname{End}(\mathbb{A}^m)$.*
2. *If $\lambda(\hat{f}) = \theta$, then $\lambda(f) = \theta$.*
3. *If $\lambda(\hat{f}) < \theta$, then $\lambda(f) = \theta \Leftrightarrow f$ is μ-algebraically stable.*

Proof. Assertion (1) follows from the fact that $\deg_\mu(f) < \infty$ and the choice of m (Lemma 2.5.6(2)).

Lemma 2.4.1 then gives $\lambda(f) = \max\{\lambda(\hat{f}), \lim_{r\to\infty} \deg_{x_{m+1},\ldots,x_n}(f^r)^{1/r}\}$. By using the equality $\lim_{r\to\infty} \deg_{x_{m+1},\ldots,x_n}(f^r)^{1/r} = \lim_{r\to\infty} \deg_\mu(f^r)^{1/r}$ (see Remark 2.3.1(2) and Lemma 2.6.1(4)), we obtain

$$\lambda(f) = \max\{\lambda(\hat{f}), \lim_{r\to\infty} \deg_\mu(f^r)^{1/r}\}.$$

Moreover, Lemma 2.6.1(4) implies that $\lim_{r\to\infty} \deg_\mu(f^r)^{1/r} \leq \deg_\mu(f) = \theta$. This provides (2). To show (3), we assume that $\lambda(\hat{f}) < \theta$ and obtain $\lambda(f) = \theta \Leftrightarrow \lim_{r\to\infty} \deg_\mu(f^r)^{1/r} = \theta$. This is equivalent to ask that f is μ-algebraically stable, by Lemma 2.6.1(5) (note that $1 \leq \lambda(\hat{f})$, since f and thus $\hat{f}$ is dominant). $\square$

We finish this section by the following simple observation:

Lemma 2.6.3. *Let $f \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. For each $\mu \in (\mathbb{R}_{>0})^n$ such that $\theta = \deg_\mu(f) \in \mathbb{R}_{>1}$ and each translation $\tau = (x_1 + c_1, \ldots, x_n + c_n) \in \mathrm{Aut}(\mathbb{A}^n)$ where $c_1, \ldots, c_n \in \boldsymbol{k}$, the following hold:*

$$f \text{ is } \mu\text{-algebraically stable} \Leftrightarrow \tau \circ f \text{ is } \mu\text{-algebraically stable}.$$

Proof. Denote by g the μ-leading part of f. As $\mu \in (\mathbb{R}_{>0})^n$, no component of g contains any constant. Hence, g is also the μ-leading part of $\tau \circ f$. By Lemma 2.6.1(5), f (respectively $\tau \circ f$) is μ-algebraically stable if and only if for each $r \geq 1$ there is $i \in \{1, \ldots, n\}$ such that $(g^r)_i \neq 0$. $\square$

3. Matrices associated to endomorphisms and the proof of Proposition B

3.1. **Spectral radii of N-uples of matrices.** In the sequel, we fix the usual Euclidean norm on $\mathbb{R}^n$, and on $n \times n$-matrices:

Definition 3.1.1. Let $n \geq 1$.

(1) We endow $\mathbb{R}^n$ will the usual norm:

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2}, \text{ for each } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(2) This endows the ring $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$-real matrices with the norm

$$\|M\| = \sup\left\{ \frac{\|Mv\|}{\|v\|} \,\middle|\, v \in \mathbb{R}^n \setminus \{0\} \right\}, \text{ for each } M \in \mathrm{Mat}_n(\mathbb{R}).$$

(3) The *spectrum* of $M \in \mathrm{Mat}_n(\mathbb{R})$ is the finite subset $\sigma(M) \subset \mathbb{C}$ of eigenvalues of M.

(4) The *spectral radius* of $M \in \mathrm{Mat}_n(\mathbb{R})$ is defined by

$$\rho(M) = \max_{\lambda\in\sigma(M)} |\lambda|$$

and satisfies

$$\rho(M) = \lim_{n\to\infty} \|M^n\|^{1/n}.$$

If $M = (m_{i,j})_{i,j=1}^n$ and $N = (n_{i,j})_{i,j=1}^n$ are matrices in $\mathrm{Mat}_n(\mathbb{R})$ such that for each (i, j) we have $0 \leq m_{i,j} \leq n_{i,j}$, then $\rho(M) \leq \rho(N)$.

(5) We have a partial order on $\mathbb{R}^n$ given by

$$x \leq y \quad \text{iff} \quad x_i \leq y_i \text{ for all } i = 1, \dots, n$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Note that for $0 \leq x \leq y$ we have $\|x\| \leq \|y\|$.

(6) For $M \in \mathrm{Mat}_n(\mathbb{R})$ we denote by χ_M the characteristic polynomial of M.

3.2. **The Perron-Frobenius Theorem and its applications.** The Perron-Frobenius theory was first established for matrices with positive coefficients, then generalised to irreducible matrices with non-negative coefficients and then to any matrices with non-negative coefficients. There are three equivalent definitions of reducible matrices (see [Gan59, Vol. 2, Chap. XIII, §1, Definitions 2,2',2"]). Let us recall one of them:

Definition 3.2.1. [Gan59, Vol. 2, Chap. XIII, §1, Definition 2'] For each $n \geq 1$, a matrix $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$ is called *reducible* if there is a permutation matrix $S \in \mathrm{GL}_n(\mathbb{Z})$ such that the matrix $SMS^{-1} \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$ is block-triangular, i.e.

$$SMS^{-1} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

where A, D are square matrices, and where the zero matrix has positive dimensions.

A matrix $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$ is called *irreducible* if it is not reducible.

Lemma 3.2.2. [Gan59, Vol. 2, Chap. XIII, §4] *For each reducible matrix $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$, there is a permutation matrix $S \in \mathrm{GL}_n(\mathbb{Z})$ such that SMS^{-1} is a lower triangular block-matrix*

$$\begin{pmatrix} A_{1,1} & 0 & \cdots & 0 \\ A_{2,1} & A_{2,2} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A_{m,1} & \cdots & A_{m,m-1} & A_{m,m} \end{pmatrix}$$

where $A_{1,1}, \dots, A_{m,m}$ are irreducible matrices.

Theorem 3.2.3 (Perron-Frobenius Theorem)**.** [Gan59, Vol. 2, Chap. XIII, §2 and §3, Theorems 2 and 3] *For each $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$, there exists an eigenvector $v \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ to the eigenvalue $\rho(M)$. If M is moreover irreducible, we can choose v in $(\mathbb{R}_{>0})^n$.*

Theorem 3.2.4 (Theorem of Lind on weak-Perron numbers)**.** *For each $\lambda \in \mathbb{R}$, the following conditions are equivalent:*

(1) *λ is a weak Perron number (see Definition 1.1.1);*

(2) *λ is the spectral radius of a non-zero square matrix with non-negative integral coefficients;*

(3) *λ is the spectral radius of an irreducible square matrix with non-negative integral coefficients;*

(4) *$\lambda > 0$ and λ^m is a Perron number for some $m \geq 1$.*

Proof. The equivalence between (1) and (3) follows from [Lin84, Theorem 3, page 291], and the equivalence between (2) and (3) follows from Lemma 3.2.2. The equivalence between (1) and (4) can be found for instance in [Sch97, Lemma 4] or [Bru13, Theorem 2]. □

As a consequence of Corollary 2.6.2 and of the Perron-Frobenius theorem, we obtain the following result (which is classical, see for instance [FW12, Lin12]):

Corollary 3.2.5. *For each matrix* $M = (m_{i,j})_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{N})$ *and for each* $(\alpha_1, \dots, \alpha_n) \in (\boldsymbol{k}^*)^n$*, the monomial endomorphism*

$$f_M = (\alpha_1 x_1^{m_{1,1}} \cdots x_n^{m_{1,n}}, \cdots, \alpha_n x_1^{m_{n,1}} \cdots x_n^{m_{n,n}}) \in \mathrm{End}(\mathbb{A}^n)$$

is dominant if and only if $\det(M) \neq 0$*. In this case, the dynamical degree of* f_M *is equal to the spectral radius of* M*:*

$$\lambda(f_M) = \rho(M) \in \mathbb{R}_{\geq 1}.$$

Proof. Note that the endomorphism $f_M \in \mathrm{End}(\mathbb{A}^n)$ restricts to an endomorphism $h_M \in \mathrm{End}((\mathbb{A}^1 \setminus \{0\})^n)$.

If $\det(M) = 0$, any non-zero element of the kernel of the transpose of M gives rise to a non-constant element p in the Laurent polynomial ring $\mathbf{k}[x_1^{\pm}, \dots, x_n^{\pm}]$ such that $p \circ h_M$ is constant, so h_M and thus f_M is not dominant. We then assume that $\det(M) \neq 0$. This implies that $h_M \in \mathrm{End}((\mathbb{A}^1 \setminus \{0\})^n)$ is surjective on $\overline{\mathbf{k}}$-points and thus f_M is dominant. In particular, $\lambda(f_M) \geq 1$. Thus we only have to show that $\lambda(f_M) = \rho(M)$. By the Perron-Frobenius-Theorem (Theorem 3.2.3), there exists an eigenvector $\mu \in (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ of M to the eigenvalue $\rho(M)$. Since the spectral radius of M and the dynamical degree of f_M do not change if we conjugate M with a permutation matrix, we may assume that there is $m < n$ such that $\mu_1 = \ldots = \mu_m = 0$ and $\mu_i > 0$ for each $i \geq m+1$. Since $(f_M)^r = f_{M^r}$ we get for each $r \geq 1$ and each $i \in \{1, \dots, n\}$ that $\deg_\mu(((f_M)^r)_i) = (M^r \mu)_i = \rho(M)^r \mu_i$. This implies that $\deg_\mu((f_M)^r) = \rho(M)^r$ for each $r \geq 1$. Thus f_M is μ-algebraically stable and $\deg_\mu(f_M) = \rho(M) < \infty$. By Corollary 2.6.2(1), we may write

$$M = \left(\begin{array}{c|c} \hat{M} & 0 \\ \hline * & * \end{array}\right)$$

where $\hat{M} \in \mathrm{Mat}_m(\mathbb{N})$ with $\det(\hat{M}) \neq 0$. By induction, the endomorphism $f_{\hat{M}} \in \mathrm{End}(\mathbb{A}^m)$ satisfies $\lambda(f_{\hat{M}}) = \rho(\hat{M}) \leq \rho(M)$. By Corollary 2.6.2(2),(3) we get then $\lambda(f_M) = \deg_\mu(f_M) = \rho(M)$. □

Corollary 3.2.6. *For each endomorphism* $f \in \mathrm{End}(\mathbb{A}^n)$ *and each matrix* $M \in \mathrm{Mat}_n(\mathbb{N})$ *that is contained in* f*, we have* $\rho(M) \leq \deg(f)$*.*

Proof. By the Perron-Frobenius-Theorem (Theorem 3.2.3), there exists an eigenvector $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ of M to the eigenvalue $\rho(M)$. Hence, $\sum_{j=1}^n m_{i,j}\mu_j = \rho(M)\mu_i$ for each $i \in \{1, \dots, n\}$. By choosing an integer $r \in \{1, \dots, n\}$ such that $\mu_r = \max\{\mu_1, \dots, \mu_n\}$, we obtain

$$\rho(M)\mu_r = \sum_{j=1}^n m_{r,j}\mu_j \leq \mu_r \sum_{j=1}^n m_{r,j}.$$

The coefficient of the monomial $\prod_{j=1}^n x_j^{m_{r,j}}$ in f_r is nonzero (as M is contained in f, see Definition 1.4.8). This monomial has degree $\sum_{j=1}^n m_{r,j}$, so $\deg(f) \geq \sum_{j=1}^n m_{r,j}$. As $\mu_r > 0$, this gives $\rho(M) \leq \deg(f)$. □

In the following we will use the next basic property of Handelman numbers. It is a straightforward application of Descarte's Rule of Signs, see e.g. [Str86, p.91]:

Lemma 3.2.7 (Basic property of Handelman numbers)**.** *Let $n \geq 1$. For each $(a_0, \ldots, a_{n-1}) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$, the polynomial $x^n - \sum_{i=0}^{n-1} a_i x^i \in \mathbb{R}[x]$ has a unique positive real root. In particular, a Handelman number has no other positive real Galois conjugate.*

Corollary 3.2.8. *Each Handelman number is a weak Perron number.*

Proof. Let $\lambda \in \mathbb{R}_{>0}$ be a Handelman number. There exists $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n \setminus \{0\}$ such that λ is a root of $P(x) = x^n - \sum_{i=0}^{n-1} a_i x^i \in \mathbb{Z}[x]$. By Lemma 3.2.7, all roots of P, except λ, are either non-real or real and non-positive. Since P is the characteristic polynomial of the matrix

$$A = \begin{pmatrix} a_{n-1} & \cdots & a_1 & a_0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0}),$$

it follows by the Perron-Frobenius-Theorem (Theorem 3.2.3) that the spectral radius of A is equal to λ. This implies that λ is a weak Perron number (Theorem 3.2.4). $\square$

3.3. **Sequences of matrices.** To study endomorphisms of $\mathbb{A}^n$, we will need to consider finite sets of elements of $\mathrm{Mat}_n(\mathbb{R})$ that have the property that we can exchange rows. In order to take the norm on such sets, we will have to see them ordered, and thus see these in $\mathrm{Mat}_n(\mathbb{R})^N$ for some $N \geq 1$.

Notation 3.3.1. Let $n, N \geq 1$. We denote by $\widehat{\mathcal{M}}_{n,N} \subset \mathrm{Mat}_n(\mathbb{R})^N$ the $\mathbb{R}$-vector subspace of N-tuples $(M_1, \ldots, M_N)$ that have the following property:

For each $i, j \in \{1, \ldots, N\}$ and each $l \in \{1, \ldots, n\}$, the replacement of the l-th row of M_i with the l-th row of M_j gives a matrix which lies in $\{M_1, \ldots, M_N\}$.

We then denote by $\mathcal{M}_{n,N} \subset \widehat{\mathcal{M}}_{n,N}$ the subset that consists of the N-tuples $(M_1, \ldots, M_N)$ where $M_1, \ldots, M_N$ are N distinct matrices with non-negative coefficients.

Remark 3.3.2. If $f \in \mathrm{End}(\mathbb{A}^n)$ is an endomorphism, then there exists some integer $N \geq 1$ and some N-tuple $(S_1, \ldots, S_N) \in \mathcal{M}_{n,N}$ such that $\{S_1, \ldots, S_N\}$ is the set of matrices that are contained in f (as in Definition 1.4.8).

The following two lemmas build the key ingredients for proving the existence of maximal eigenvectors of endomorphisms of $\mathbb{A}^n$ in the next subsection (see Proposition 3.4.1). This eventually leads then to a proof of Proposition B.

Lemma 3.3.3. *Let $n, N \geq 1$. For each $M = (M_1, \ldots, M_N) \in \mathcal{M}_{n,N}$, there exists a sequence $(D_t)_{t \in \mathbb{N}}$ of elements $D_t = (D_{t,1}, \ldots, D_{t,N}) \in \mathcal{M}_{n,N}$ that converges towards M (with respect to the topology of $\mathrm{Mat}_n(\mathbb{R})^N$ that is given by the norm as in Definition 3.1.1) and such that for each $t \in \mathbb{N}$, there is no complex number which is an eigenvalue of two elements of $D_{t,1}, \ldots, D_{t,N}$.*

Proof. The result being trivially true for $N = 1$, we will assume $N \geq 2$. For each $i \in \{1, \ldots, n\}$, we denote by $\Gamma_i \subset \mathbb{R}^n$ the finite set of i-th rows of the matrices $M_1, \ldots, M_N$:

$$\Gamma_i = \{r \in \mathbb{R}^n \mid r \text{ is the } i\text{-th row of one of the matrices } M_1, \ldots, M_N\}.$$

We then write $\Gamma_i = \{r_{i,1}, \dots, r_{i,s_i}\}$, where $s_i \geq 1$ is the cardinality of Γ_i.

As all matrices $M_1, \dots, M_N$ are pairwise distinct and as one can "exchange rows" (see Notation 3.3.1), we have $N = s_1 \cdots s_n$, and obtain a unique $\mathbb{R}$-linear map

$$\varphi \colon \prod_{i=1}^{n} (\mathbb{R}^n)^{s_i} \to \widehat{\mathcal{M}}_{n,N}$$

with the following properties:

(1) For each $k \in \{1, \dots, N\}$, the composition of φ with the projection map $\pi_k \colon \mathrm{Mat}_n(\mathbb{R})^N \to \mathrm{Mat}_n(\mathbb{R})$ onto the k-th factor is of the form

$$\begin{array}{rccc} \pi_k \circ \varphi \colon & \prod_{i=1}^{n} (\mathbb{R}^n)^{s_i} & \to & \mathrm{Mat}_n(\mathbb{R}) \\ & (v_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i} & \mapsto & \begin{pmatrix} v_{1,j_1} \\ \vdots \\ v_{n,j_n} \end{pmatrix} \end{array}$$

where $j_i \in \{1, \dots, s_i\}$ for each $i \in \{1, \dots, n\}$.

(2) $(M_1, \dots, M_N) = \varphi((r_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i})$.

Indeed, the possibilities for maps $\pi_k \circ \varphi$ as in (1) are parametrised by the N possible choices of $j_i \in \{1, \dots, s_i\}$ for each $i \in \{1, \dots, n\}$, and by (2) the image of $(r_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i}$ by the maps $\pi_1 \circ \varphi, \dots, \pi_N \circ \varphi$ give the matrices $M_1, \dots, M_N$; this gives the existence and the unicity of φ.

We now identify $\prod_{i=1}^{n} (\mathbb{R}^n)^{s_i}$ with the real locus $X(\mathbb{R})$ of the affine space $X = \mathbb{A}^{n \sum s_i}$.

For any two matrices $A, B \in \mathrm{Mat}_n(\mathbb{R})$, the resultant of the characteristic polynomials χ_A and χ_B is denoted by $r(A, B)$. Recall that $r(A, B) = 0$ if and only if A and B have a common eigenvalue. Hence, for any distinct $a, b \in \{1, \dots, N\}$, the set

$$Z_{a,b} = \left\{ x \in \prod_{i=1}^{n} (\mathbb{R}^n)^{s_i} \;\middle|\; \begin{array}{l} \text{the matrices } \pi_a(\varphi(x)) \text{ and } \pi_b(\varphi(x)) \\ \text{have a common eigenvalue} \end{array} \right\}$$

corresponds to the elements of $X(\mathbb{R})$ that satisfy one polynomial equation $P_{a,b} \in \mathbb{R}[X]$.

We now prove that $P_{a,b} \neq 0$, or equivalently that $Z_{a,b} \neq X(\mathbb{R}) = \prod_{i=1}^{n} (\mathbb{R}^n)^{s_i}$, by showing that $\pi_a(\varphi(x))$ and $\pi_b(\varphi(x))$ have no common eigenvalue for at least one $x \in X(\mathbb{R})$. We consider $j_1, \dots, j_n$ and $j'_1, \dots, j'_n$ so that $\pi_a \circ \varphi$ and $\pi_b \circ \varphi$ are respectively given by

$$\begin{array}{ccc} \prod_{i=1}^{n} (\mathbb{R}^n)^{s_i} & \to & \mathrm{Mat}_n(\mathbb{R}) \\ (v_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq s_i}} & \mapsto & \begin{pmatrix} v_{1,j_1} \\ \vdots \\ v_{n,j_n} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} \prod_{i=1}^{n} (\mathbb{R}^n)^{s_i} & \to & \mathrm{Mat}_n(\mathbb{R}) \\ (v_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq s_i}} & \mapsto & \begin{pmatrix} v_{1,j'_1} \\ \vdots \\ v_{n,j'_n} \end{pmatrix} \end{array}.$$

Since the matrices M_a and M_b are distinct, the linear maps $\pi_a \circ \varphi$ and $\pi_b \circ \varphi$ are also distinct. There is thus $l \in \{1, \dots, n\}$ such that $j_l \neq j'_l$. Suppose first that $l = 1$, i.e. $j_1 \neq j'_1$. We may choose $x \in X(\mathbb{R})$ such that

$$\pi_a(\varphi(x)) = \left(\begin{array}{c|c} 0 & 1 \\ \hline I_{n-1} & 0 \end{array} \right) \quad \text{and} \quad \pi_b(\varphi(x)) = \left(\begin{array}{c|c} 0 & 0 \\ \hline I_{n-1} & 0 \end{array} \right).$$

These matrices have characteristic polynomials $t^n - 1$ and t^n, respectively. If $l > 1$, we simply consider conjugation of the above matrices by permutations. In all cases, we find an $x \in X(\mathbb{R})$ such that $\pi_a(\varphi(x))$ and $\pi_b(\varphi(x))$ are matrices without common eigenvalue in $\mathbb{C}$. This shows that $Z_{a,b} \neq X(\mathbb{R})$, i.e. $P_{a,b} \neq 0$.

The product of all polynomials $P_{a,b}$ with distinct $a, b \in \{1, \dots, n\}$ gives a non-zero polynomial $P \in \mathbb{R}[X]$. We can thus take a real affine linear map $\ell \colon \mathbb{A}^1 \to X = \mathbb{A}^{n \sum s_i}$ such that $\ell(0) = (r_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i}$, such that the coordinates of $\ell(\mathbb{R}_{\geq 0})$ are non-negative and such that the restriction of P to $\ell(\mathbb{R})$ is non-zero. We obtain that $P(\ell(\frac{1}{n})) \neq 0$ for any sufficiently large positive integer n. It suffices then to fix a sufficiently large $c \geq 1$ and to define $D_t = \varphi(\ell(\frac{1}{t+c}))$ for each integer $t \geq 0$. □

Lemma 3.3.4. *Let $S = (S_1, \dots, S_N) \in \mathcal{M}_{n,N}$ and let $v \geq 0$ be an eigenvector of S_1 to the eigenvalue $\lambda \geq 0$. Suppose moreover that $\lambda > \rho(S_i)$ for each $i \in \{2, \dots, N\}$. Then $S_i v \leq \lambda v$ for each $i \in \{1, \dots, N\}$.*

Proof. Assume for contradiction that there is $i \in \{2, \dots, N\}$ such that $S_i v \not\leq \lambda v$. Denote by v_j the j-th component of v for each $j \in \{1, \dots, n\}$. Since we may replace each row R_j in S_i such that $R_j v < \lambda v_j$ with the j-th row from S_1 and still get an element in $\{S_1, \dots, S_N\}$, we may assume that $S_i v \geq \lambda v \geq 0$. As the coefficients of v and S_i are non-negative, we obtain by induction that $(S_i)^r v \geq \lambda^r v \geq 0$ for each $r \geq 1$. In particular,

$$\|(S_i)^r\| \geq \frac{\|(S_i)^r v\|}{\|v\|} \geq \lambda^r$$

and we obtain $\rho(S_i) = \lim_{r \to \infty} \|(S_i)^r\|^{1/r} \geq \lambda$. This contradicts the assumption that $\lambda > \rho(S_i)$. □

3.4. Existence of maximal eigenvectors of endomorphisms of $\mathbb{A}^n$.

Proposition 3.4.1. *For each $n, N \geq 1$ and each $S = (S_1, \dots, S_N) \in \mathcal{M}_{n,N}$, there exists $j \in \{1, \dots, N\}$ and an eigenvector $v \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ of S_j to the eigenvalue $\lambda = \max\{\rho(S_1), \dots, \rho(S_N)\}$ such that for each $i \in \{1, \dots, N\}$ we have*

$$S_i v \leq S_j v = \lambda v\,.$$

Proof. Let $S = (S_1, \dots, S_N) \in \mathcal{M}_{n,N}$. By Lemma 3.3.3, there exists a sequence $(D_t)_{t \in \mathbb{N}}$ of elements $D_t = (D_{t,1}, \dots, D_{t,N}) \in \mathcal{M}_{n,N}$ that converges towards S and such that for each $t \in \mathbb{N}$, there is no complex number which is an eigenvalue of two elements of $D_{t,1}, \dots, D_{t,N}$. In particular, $\rho(D_{t,i}) \neq \rho(D_{t,j})$ for distinct i, j by the Perron-Frobenius-Theorem (Theorem 3.2.3).

By possibly replacing this sequence with a subsequence, we may assume that there is a $j \in \{1, \dots, N\}$ such that $\rho(D_{t,j}) > \rho(D_{t,i})$ for all $i \in \{1, \dots, N\} \setminus \{j\}$ and each $t \in \mathbb{N}$. After exchanging the ordering of $S_1, \dots, S_N$, we may assume that $j = 1$. For each $i \in \{1, \dots, N\}$, the sequence $(D_{t,i})_{t \in \mathbb{N}}$ converges towards S_i, so $(\rho(D_{t,i}))_{t \in \mathbb{N}}$ converges towards $\rho(S_i)$ [Ost73, Theorem in Appendix A]. In particular, $\rho(S_1) = \lambda = \max\{\rho(S_1), \dots, \rho(S_n)\}$. By the Perron-Frobenius-Theorem (Theorem 3.2.3), there is for each $t \in \mathbb{N}$ an eigenvector $v_t \geq 0$ of $D_{t,1}$ to the eigenvalue $\rho(D_{t,1})$. Lemma 3.3.4 then gives for each $i \in \{1, \dots, N\}$ and each $t \in \mathbb{N}$

$$D_{t,i} v_t \leq \rho(D_{t,1}) v_t\,.$$

Now, we may assume that $\|v_t\| = 1$ for all t (after normalizing v_t). Let

$$\mathbb{S}^{n-1} = \{\, w \in \mathbb{R}^n \mid \|w\| = 1 \,\}\,.$$

Since $\mathbb{S}^{n-1}$ is compact (with respect to the Euclidean topology), we may take a subsequence and assume that $(v_t)_{t\in\mathbb{N}}$ converges to a $v \geq 0$ in $\mathbb{S}^{n-1}$. Thus we get

$$\lambda v = \rho(S_1)v = \lim_{t\to\infty} \rho(D_{t,1})v_t = \lim_{t\to\infty} D_{t,1}v_t = S_1 v$$

and for each $i \in \{1, \dots, N\}$

$$S_i v = \lim_{t\to\infty} D_{t,i}v_t \leq \lim_{t\to\infty} \rho(D_{t,1})v_t = \rho(S_1)v = \lambda v \quad .$$

This finishes the proof of the proposition. □

Proof of Proposition B. By Remark 3.3.2, there exists $(S_1, \dots, S_N) \in \mathcal{M}_{n,N}$ such that $\{S_1, \dots, S_N\}$ is the set of matrices contained in f. By Proposition 3.4.1 there exists $j \in \{1, \dots, N\}$ and an eigenvector $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ of S_j to the eigenvalue $\theta = \max\{\rho(S_1), \dots, \rho(S_N)\}$ such that $S_i\mu \leq S_j\mu = \theta\mu$ for each $i \in \{1, \dots, N\}$. We now prove that this implies that $\deg_\mu(f_l) = \theta\mu_l$ for each $l \in \{1, \dots, n\}$, which shows that $\mu = (\mu_1, \dots, \mu_n)$ is a maximal eigenvector of f, and thus proves (1). For each monomial $m = \chi x_1^{r_1} \cdots x_n^{r_n}$ of f_l with $\chi \in \mathbf{k}^*$ there is a matrix S_i with its l-th line equal to $(r_1 \; r_2 \; \cdots \; r_n)$. The l-th component of $S_i\mu$ is equal to $r_1\mu_1 + \cdots + r_n\mu_n = \deg_\mu(m)$. The inequality $S_i\mu \leq \theta\mu$ then yields $\deg_\mu(m) \leq \theta\mu_l$. As this holds for each monomial of f_l, we obtain $\deg_\mu(f_l) \leq \theta\mu_l$. The equality follows from $S_j\mu = \theta\mu$, since the monomial m that corresponds to the l-th row of S_j has μ-degree equal to $\theta\mu_l$.

We now prove (2). The dominance of f implies that $1 \leq \deg(f^r)$ for each r and this in turn gives $1 \leq \lambda(f)$. The inequality $\theta \leq \deg(f)$ follows from Corollary 3.2.6, so we only need to prove $\lambda(f) \leq \theta$. This is done by induction on n. If $n = 1$, then $\mu \in (\mathbb{R}_{>0})^1$ and the statement follows from Proposition A(2). Now, let $n > 1$. We may assume (after a permutation of the coordinates) that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$. Now, let $m \in \{0, \dots, n-1\}$ with $\mu_i = 0$ for $i \leq m$ and $\mu_i > 0$ for $i > m$. From Remark 2.3.1(2) we get

$$\lambda_2 := \lim_{r\to\infty} \deg_{x_{m+1},\dots,x_n}(f^r)^{\frac{1}{r}} = \lim_{r\to\infty} \max_{i\in\{1,\dots,n\}} \deg_\mu((f^r)_i)^{\frac{1}{r}} .$$

From Lemma 2.5.6 we get that for each $i \in \{1, \dots, m\}$, the element f_i is a polynomial in the variables $\{x_1, \dots, x_m\}$. Thus we get from Lemma 2.4.1 that $\lambda(f) = \max\{\lambda_1, \lambda_2\}$ where

$$\lambda_1 = \lambda(\hat{f}) = \lim_{r\to\infty} \deg(\hat{f}^r)^{\frac{1}{r}} \quad \text{and} \quad \hat{f} := (f_1, \dots, f_m) \in \operatorname{End}(\mathbb{A}^m) .$$

Since $m \leq n - 1$, by induction hypothesis we have

$$\lambda_1 \leq \theta_1 := \max\left\{ |\xi| \in \mathbb{R} \mid \xi \text{ is an eigenvalue of a matrix that is contained in } \hat{f} \right\} .$$

Note that each eigenvalue of a matrix that is contained in $\hat{f}$ is an eigenvalue of a matrix that is contained in f. Thus we get $\theta_1 \leq \theta$. From Lemma 2.6.1(4), it follows that $\lambda_2 \leq \theta$. In summary we proved that $\lambda(f) = \max\{\lambda_1, \lambda_2\} \leq \theta$, i.e. (2) holds for n.

We now prove (3). We take a maximal eigenvector μ of f. As $\deg_\mu(f_i) = \theta\mu_i$ for each $i \in \{1, \dots, n\}$, we have $\deg_\mu(f) = \theta$. If $\theta = 1$, *(i)* follows from (2) and *(ii)* is trivially true, so we may assume that $\theta > 1$. If f is μ-algebraically stable, then Lemma 2.6.1(5) gives $\lambda_2 = \theta$ and thus $\lambda(f) = \theta$, so *(i)* is proven. Conversely, if $\mu \in (\mathbb{R}_{>0})^n$ and $\lambda(f) = \theta > 1$, then f is μ-algebraically stable by Proposition A(3).

This achieves the proof of *(ii)*. As $\theta = \deg_\mu(f) \in \mathbb{R}_{\geq 0}$ (i.e. is not equal to $+\infty$), *(iii)* is a direct consequence of Lemma 2.6.1(5). □

We now give an example that shows that the implication of Proposition B(3)(*i*) is not an equivalence.

Example 3.4.2. We consider the automorphism

$$f = (f_1, f_2, f_3, f_4) = ((x_1)^2 + x_2, x_1, x_3 + (x_3 + x_4)^2, x_4 - (x_3 + x_4)^2) \in \mathrm{Aut}(\mathbb{A}^4)\,.$$

As $\deg(f) = 2$, the maximal eigenvalue θ of f (see Definition 1.4.8) satisfies $\theta \leq 2$ (Corollary 3.2.6). Moreover, $\theta = 2$, as the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is contained in f. When we choose $\mu = (0, 0, 1, 1)$, we get $\deg_\mu(f) = 2$, and we see that f is not μ-algebraically stable, as $\deg_\mu(f^2) = 2 < 4$. Moreover, $\deg_\mu(f_i) = 0$ for $i \in \{1, 2\}$ and $\deg_\mu(f_i) = 2$ for $i \in \{3, 4\}$. Thus μ is a maximal eigenvector of f (see Definition 1.4.8). However, $\lambda(f) = \theta$. Indeed, $\lambda(f) \leq \deg(f) = 2$, and $((x_1)^2 + x_2, x_1)$ is algebraically stable for the standard degree, as its homogeneous part of degree 2 is $((x_1)^2, 0)$, which satisfies $((x_1)^2, 0)^r = ((x_1)^{2^r}, 0)$ for each $r \geq 1$ (see Proposition A).

4. Explicit calculation of dynamical degrees of affine-triangular automorphisms

In this section, we apply Proposition B to compute the dynamical degrees of affine-triangular dominant endomorphisms of $\mathbb{A}^n$. We prove Proposition 4.2.3, which implies Propositions 4.2.5 and C.

Notation 4.0.1. We denote by $\mathrm{TEnd}(\mathbb{A}^n)$ and $\mathrm{TAut}(\mathbb{A}^n)$ (respectively $\mathrm{EEnd}(\mathbb{A}^n)$ and $\mathrm{EAut}(\mathbb{A}^n)$) the monoid and group of triangular (respectively elementary) endomorphisms and automorphisms of $\mathbb{A}^n$. We denote by $\mathrm{Aff}(\mathbb{A}^n)$ the group of affine automorphisms of $\mathbb{A}^n$ and by $\mathrm{Sym}(\mathbb{A}^n) \subset \mathrm{Aff}(\mathbb{A}^n)$ the group of permutations of the coordinates.

4.1. **From affine-triangular to permutation-triangular endomorphisms.**

We can restrict ourselves to permutation-triangular endomorphisms, as the next simple result shows.

Proposition 4.1.1. *Each affine-triangular endomorphism of $\mathbb{A}^n$ is conjugate by an element of $\mathrm{Aff}(\mathbb{A}^n)$ to a permutation-triangular endomorphism.*

Proof. We take $\alpha \in \mathrm{Aff}(\mathbb{A}^n)$ and $\tau \in \mathrm{TEnd}(\mathbb{A}^n)$ and show that we can conjugate $f = \alpha \circ \tau$ to a permutation-triangular endomorphism by an element of $\mathrm{Aff}(\mathbb{A}^n)$.

Let $p = (p_1, \ldots, p_n) \in \mathbb{A}^n$ be the point such that $\alpha(p) = 0$ and consider the translation $\tau_p = (x_1 + p_1, \ldots, x_n + p_n) \in \mathrm{Aff}(\mathbb{A}^n) \cap \mathrm{TAut}(\mathbb{A}^n)$. Then $\alpha' = \alpha \circ \tau_p \in \mathrm{Aff}(\mathbb{A}^n)$ fixes the origin $(0, \ldots, 0) \in \mathbb{A}^n$. We then replace α with α' and τ with $\tau_p^{-1} \circ \tau$, and may assume that α belongs to the subgroup $\mathrm{GL}_n = \mathrm{GL}_n(\mathbf{k}) \subset \mathrm{Aff}(\mathbb{A}^n)$ of elements that fix the origin.

The group $B = \mathrm{TAut}(\mathbb{A}^n) \cap \mathrm{GL}_n$ is a Borel subgroup of GL_n. It consists of all lower triangular matrices. The so-called Bruhat decomposition of GL_n:

$$\mathrm{GL}_n = B\,\mathrm{Sym}(\mathbb{A}^n)B$$

yields $\beta, \gamma \in B$ and $\sigma \in \mathrm{Sym}(\mathbb{A}^n)$ such that $\alpha = \beta \circ \sigma \circ \gamma$. This gives

$$\beta^{-1} \circ f \circ \beta = \beta^{-1} \circ \alpha \circ \tau \circ \beta = \sigma \circ \gamma \circ \tau \circ \beta$$

where $\gamma \circ \tau \circ \beta \in \mathrm{TEnd}(\mathbb{A}^n)$. This achieves the proof. □

4.2. **Permutation-elementary automorphisms.** Up to conjugation, each permutation-elementary automorphism has a particular form. This shows the following easy observation.

Lemma 4.2.1. *Let $n \geq 1$ and let $h \in \mathrm{End}(\mathbb{A}^{n+1})$ be a permutation-elementary automorphism. There is a permutation of the coordinates $\alpha \in \mathrm{Sym}(\mathbb{A}^{n+1})$ such that*

$$f = \alpha \circ h \circ \alpha^{-1} = (f_1, \ldots, f_m, \xi x_{n+1} + p(x_1, \ldots, x_n), x_{m+1}, \ldots, x_n),$$

where $0 \leq m \leq n$, $\{x_1, \ldots, x_m\} = \{f_1, \ldots, f_m\}$, $\xi \in \boldsymbol{k}^*$ and $p \in \boldsymbol{k}[x_1, \ldots, x_n]$.

Proof. We write $h = \sigma \circ \tau$ where $\sigma \in \mathrm{Sym}(\mathbb{A}^{n+1})$ and $\tau \in \mathrm{EAut}(\mathbb{A}^{n+1})$. We may choose $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathrm{Sym}(\mathbb{A}^{n+1})$ such that $\alpha_{n+1} = x_{n+1}$ and $\alpha \circ \sigma \circ \alpha^{-1}$ induces the following cyclic permutation on the last coordinates

$$(\alpha \circ \sigma \circ \alpha^{-1})_{m+1} = x_{n+1}\,,\ (\alpha \circ \sigma \circ \alpha^{-1})_{m+2} = x_{m+1}\,,\ \ldots,\ (\alpha \circ \sigma \circ \alpha^{-1})_{n+1} = x_n\,,$$

for some integer m with $0 \leq m \leq n$. This gives

$$\alpha \circ \sigma \circ \alpha^{-1} = (f_1, \ldots, f_m, x_{n+1}, x_{m+1}, \ldots, x_n)$$

where $\{x_1, \ldots, x_m\} = \{f_1, \ldots, f_m\}$. As $\alpha_{n+1} = x_{n+1}$, we obtain

$$\alpha \circ \tau \circ \alpha^{-1} = (x_1, \ldots, x_n, \xi x_{n+1} + p(x_1, \ldots, x_n))$$

for some $\xi \in \mathbf{k}^*$ and $p \in \mathbf{k}[x_1, \ldots, x_n]$. This implies that $\alpha \circ h \circ \alpha^{-1}$ is equal to

$$(\alpha \circ \sigma \circ \alpha^{-1}) \circ (\alpha \circ \tau \circ \alpha^{-1}) = (f_1, \ldots, f_m, \xi x_{n+1} + p(x_1, \ldots, x_n), x_{m+1}, \ldots, x_n)\,.$$

□

We will need the following result to obtain Proposition 4.2.3 below. Proposition 4.2.3 will be the key ingredient to show Proposition 4.2.5 and Proposition *C*.

Lemma 4.2.2. *Let $0 \leq m \leq n$, let $\hat{f} = (f_1, \ldots, f_m) \in \mathrm{Aut}(\mathbb{A}^m)$ and let $q \in \boldsymbol{k}[x_1, \ldots, x_{n+1}] \setminus \{0\}$. For each $r \geq 1$, every component of g^r is non-zero where*

$$g = (f_1, \ldots, f_m, q, x_{m+1}, \ldots, x_n) \in \mathrm{End}(\mathbb{A}^{n+1})\,.$$

Proof. For each $r \geq 1$, we write $g^r = ((g^r)_1, \ldots, (g^r)_{n+1})$. The result is true by assumption when $r = 1$. For each $r \geq 1$ and $1 \leq i \leq m$ we have $(g^r)_i = (\hat{f}^r)_i \neq 0$.

As $(f_1, \ldots, f_m) \in \mathrm{Aut}(\mathbb{A}^m)$, we also have $(f_1, \ldots, f_m, x_{m+1}, \ldots, x_n) \in \mathrm{Aut}(\mathbb{A}^n)$. In particular, g is dominant if $q \notin \mathbf{k}[x_1, \ldots, x_n]$, i.e. if $\deg_{x_{n+1}}(q) \geq 1$. Thus we assume that $q \in \mathbf{k}[x_1, \ldots, x_n] \setminus \{0\}$.

Suppose first that $m = n$, in which case $g = (f_1, \ldots, f_m, q)$. For each $r \geq 2$, we get $g^r = ((g^r)_1, \ldots, (g^r)_m, q((g^{r-1})_1, \ldots, (g^{r-1})_m))$. As $\hat{f}$ is dominant and q is not the zero polynomial, every component of g^r is not zero.

We then assume that $n > m$ and prove the result by induction on $n - m$. As $(f_1, \ldots, f_m, x_{m+1}, \ldots, x_n) \in \mathrm{Aut}(\mathbb{A}^n)$, there is a polynomial $h \in \mathbf{k}[x_1, \ldots, x_n]$

such that $h(f_1,\dots,f_m,x_{m+1},\dots,x_n)=q$, since $q\in\mathbf{k}[x_1,\dots,x_n]$. We denote by $\phi\colon\mathbb{A}^n\hookrightarrow\mathbb{A}^{n+1}$ the closed embedding that is given by

$$(x_1,\dots,x_n)\mapsto(x_1,\dots,x_m,h(x_1,\dots,x_n),x_{m+1},\dots,x_n)$$

and we write $\tau=(f_1,\dots,f_m,h,x_{m+1},\dots,x_{n-1})\in\mathrm{End}(\mathbb{A}^n)$. We now prove that $g\circ\phi=\phi\circ\tau$:

$$\begin{aligned}&g\circ\phi(x_1,\dots,x_n)\\&=(f_1,\dots,f_m,q(x_1,\dots,x_m,h,x_{m+1},\dots,x_{n-1}),h,x_{m+1},\dots,x_{n-1})\\&=(f_1,\dots,f_m,h(f_1,\dots,f_m,h,x_{m+1},\dots,x_{n-1}),h,x_{m+1},\dots,x_{n-1})\\&=\phi\circ\tau(x_1,\dots,x_n).\end{aligned}$$

Hence, $g^r\circ\phi=\phi\circ\tau^r$ for each $r\geq 1$. By induction, every component of τ^r is non-zero, so every component of g^r is non-zero, except maybe the $(m+1)$-th one. But if the $(m+1)$-th component of g^r were zero, then the $(m+2)$-th of g^{r+1} would be zero, impossible as the $(m+2)$-th component of $g^{r+1}\circ\phi=\phi\circ\tau^{r+1}$ is not equal to zero. □

Proposition 4.2.3. *Let $0\leq m<n$, let $\hat{f}=(f_1,\dots,f_m)\in\mathrm{Aut}(\mathbb{A}^m)$, $\xi\in\boldsymbol{k}^*$ and $p\in\boldsymbol{k}[x_1,\dots,x_n]$. Denote by $I\subset\mathbb{N}^n$ the finite subset of indices of the monomials of p, and define*

$$\theta=\max\left\{\lambda\in\mathbb{R}\;\middle|\;\lambda^{n-m}=\sum_{j=m+1}^{n}i_j\lambda^{n-j}\text{ for some }(i_1,\dots,i_n)\in I\right\}.$$

Then,

$$f=(f_1,\dots,f_m,\xi x_{n+1}+p(x_1,\dots,x_n),x_{m+1},\dots,x_n)\in\mathrm{Aut}(\mathbb{A}^{n+1})$$

has the following properties:

(1) *If $\deg_{x_{m+1},\dots,x_n}(p)\leq 1$, then $\lambda(f)=\lambda(\hat{f})$.*
(2) *If $\deg_{x_{m+1},\dots,x_n}(p)\geq 2$, define*

$$\mu=(\mu_1,\dots,\mu_{n+1})=(0,\dots,0,\theta^{n-m},\theta^{n-m-1},\dots,\theta,1),$$

i.e. $\mu_j=0$ for $j\leq m$ and $\mu_j=\theta^{n+1-j}$ for $j\geq m+1$. Then we have $\theta>1$, $\deg_\mu(f_j)=\theta\mu_j$ for each j (in particular $\deg_\mu(f)=\theta$) and f is μ-algebraically stable. If moreover $\lambda(\hat{f})\leq\theta$ (in particular, if $m=0$), then $\lambda(f)=\theta$.
(3) *Assume $\{f_1,\dots,f_m\}=\{x_1,\dots,x_m\}$. If $\deg_{x_{m+1},\dots,x_n}(p)\leq 1$, then the maximal eigenvalue of f is equal to 1 and otherwise it is equal to θ.*

Remark 4.2.4. The case $m=n$, not treated in Proposition 4.2.3, is rather trivial. We have $f=(f_1,\dots,f_n,\xi x_{n+1}+p(x_1,\dots,x_n))$ where $\{f_1,\dots,f_n\}=\{x_1,\dots,x_n\}$. Every matrix contained in f is then a block-matrix with a $(n\times n)$-permutation-matrix and a (1×1)-matrix with a 0 or a 1 on the diagonal, so every eigenvalue is either 0 or a root of unity. This implies that $\theta=1$ is the only possible maximal eigenvalue.

Proof of Proposition 4.2.3. (1) Since $\deg_{x_{m+1},\dots,x_n}(p)\leq 1$, Lemma 2.4.1 implies that

$$\lambda(f)=\max\{\lambda(\hat{f}),\lim_{r\to\infty}\deg_{x_{m+1},\dots,x_{n+1}}(f^r)^{1/r}\}=\lambda(\hat{f})$$

where by convention $\lambda(\hat{f}) = 1$ in case $m = 0$.

(2): For each $i = (i_1, \dots, i_n) \in I$, we set

$$p_i = \sum_{j=m+1}^{n} i_j x^{n-j} \in \mathbb{Z}[x]$$

and $q_i = x^{n-m} - p_i \in \mathbb{Z}[x]$. Then θ is the biggest real root of one of the polynomials in $\{q_i \mid i \in I\}$. Note that q_i is monic and of degree $n-m > 0$. As $\deg_{x_{m+1},\dots,x_n}(p) \geq 2$, there is $i = (i_1, \dots, i_n) \in I$ such that $p_i(1) \geq 2$. This implies that $q_i(1) = 1 - p_i(1) < 0$, so q_i has a real root that is bigger than 1. This proves that $\theta > 1$. For each $i \in I$, we moreover have $q_i(\theta) \geq 0$, since q_i has no real root bigger than θ. This gives $\theta^{n-m} \geq p_i(\theta)$, with equality for at least one $i \in I$.

We now prove that $\deg_\mu(f_j) = \theta\mu_j$ for each $j \in \{1, \dots, n+1\}$ where f_j denotes the j-th component of f: For each $j \in \{1, \dots, m\}$ we have $\deg_\mu(f_j) = 0 = \theta\mu_j$ and for each $j \in \{m+2, \dots, n+1\}$, we have $\deg_\mu(f_j) = \deg_\mu(x_{j-1}) = \mu_{j-1} = \theta\mu_j$. We moreover have

$$\begin{aligned}
\deg_\mu(f_{m+1}) &= \max\left(\{\deg_\mu(x_{n+1})\} \cup \left\{ \sum_{j=m+1}^{n} i_j\mu_j \;\middle|\; (i_1, \dots, i_n) \in \mathbb{N}^n \right\}\right) \\
&= \max\left(\{1\} \cup \{\, \theta \cdot p_i(\theta) \mid i = (i_1, \dots, i_n) \in \mathbb{N}^n \,\}\right) = \theta^{n-m+1} = \theta\mu_{m+1}.
\end{aligned}$$

This gives in particular $\theta = \deg_\mu(f)$.

It remains to prove that f is μ-algebraically stable, i.e. that $\deg_\mu(f^r) = \theta^r$ for each $r \geq 1$; this will then give the result by Corollary 2.6.2.

By Lemma 2.6.1(5), this corresponds to ask that for each $r \geq 1$, there exists $j \in \{m+1, \dots, n\}$ such that $(g^r)_j \neq 0$, where $g = (g_1, \dots, g_{n+1}) \in \mathrm{End}(\mathbb{A}^{n+1})$ is the μ-leading part of f and $(g^r)_j$ denotes the j-th component of g^r. We observe that

$$g = (f_1, \dots, f_m, g_{m+1}, x_{m+1}, \dots, x_n)$$

where $g_{m+1} \in \mathbf{k}[x_1, \dots, x_{n+1}] \setminus \{0\}$. The result then follows from Lemma 4.2.2

(3): The maximal eigenvalue of f is the biggest real number that is an eigenvalue of one of the matrices contained in f. Each such matrix is either contained in $(f_1, \dots, f_m, \xi x_{n+1}, x_{m+1}, \dots, x_n)$, but then has spectral radius equal to 1, or is contained in $(f_1, \dots, f_m, \prod_{j=1}^{n} x_j^{i_j}, x_{m+1}, \dots, x_n)$ for some $(i_1, \dots, i_n) \in I$. In this latter case, the spectral radius is the one of the matrix

$$\begin{pmatrix} i_{m+1} & \cdots & i_n & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

and thus equal to the biggest real root of the polynomial $x^{n-m} - \sum_{j=m+1}^{n} i_j x^{n-j}$. If $\deg_{x_{m+1},\dots,x_n}(p) \leq 1$, the maximal eigenvalue is again equal to 1, and if $\deg_{x_{m+1},\dots,x_n}(p) \geq 2$, we get that θ is the maximal eigenvalue of f. □

As mentioned in the introduction, the following result is due to Mattias Jonsson (unpublished).

Proposition 4.2.5. *For each $n \geq 1$ and each polynomial $p \in \boldsymbol{k}[x_1, \dots, x_n]$ of degree ≥ 2, let $e_p \in \mathrm{Aut}(\mathbb{A}^{n+1})$ be the automorphism*

$$e_p = (x_{n+1} + p(x_1, \dots, x_n), x_1, \dots, x_n) \in \mathrm{Aut}(\mathbb{A}^{n+1}).$$

Let $I \subset \mathbb{N}^n$ be the finite subset of indices of the monomials of p. We get

$$\lambda(e_p) = \max \left\{ \lambda \in \mathbb{R} \;\middle|\; \lambda^n = \sum_{j=1}^{n} i_j \lambda^{n-j} \quad \text{for some } (i_1, \dots, i_n) \in I \right\}$$

Proof. Apply Proposition 4.2.3(2) with $m = 0$ and $\xi = 1$. □

Proof of Proposition C. Let $h \in \mathrm{End}(\mathbb{A}^{n+1})$ be a permutation-elementary automorphism. By Lemma 4.2.1 there is a permutation of the coordinates $\alpha \in \mathrm{Sym}(\mathbb{A}^{n+1})$ such that

$$f = \alpha \circ h \circ \alpha^{-1} = (f_1, \dots, f_m, \xi x_{n+1} + p(x_1, \dots, x_n), x_{m+1}, \dots, x_n),$$

where $0 \leq m \leq n$, $\{x_1, \dots, x_m\} = \{f_1, \dots, f_m\}$, $\xi \in \mathbf{k}^*$ and $p \in \mathbf{k}[x_1, \dots, x_n]$. In particular $\lambda(\hat{f}) = 1$ where $\hat{f} = (f_1, \dots, f_m) \in \mathrm{Sym}(\mathbb{A}^m)$.

As the maximal eigenvalue θ of f is bigger than 1, we have $m < n$ (see Remark 4.2.4). Moreover, Proposition 4.2.3(3) yields that $\deg_{x_{m+1},\dots,x_n}(p) \geq 2$. Then, Proposition 4.2.3(2),(3) give the existence of a maximal eigenvector μ such that f is μ-algebraically stable and prove that the dynamical degree $\lambda(f)$ is equal to the maximal eigenvalue θ of f (this latter fact also follows from Proposition B). Since $\alpha \in \mathrm{Sym}(\mathbb{A}^{n+1})$ we get that $\alpha^{-1}(\mu)$ is a maximal eigenvector of $h = \alpha^{-1} \circ f \circ \alpha$, h is $\alpha^{-1}(\mu)$-algebraically stable and θ is the maximal eigenvalue of h. Moreover, $\lambda(h) = \lambda(f)$. Proposition 4.2.3(2) shows that θ is the root of a monic integral polynomial where all coefficients (except the first one) are non-positive, so it is a Handelman number by definition. □

4.3. **Affine-triangular automorphisms of $\mathbb{A}^3$.** In this section, we apply Proposition B to affine-triangular automorphisms $f \in \mathrm{Aut}(\mathbb{A}^3)$ and prove Proposition D and Theorem 1. By Proposition 4.1.1, we can reduce to the case of permutation-triangular automorphisms. If the maximal eigenvalue θ of f is equal to 1, then Proposition B gives $\lambda(f) = \theta$. If $\theta > 1$, there is a maximal eigenvector $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ of f, and if f is μ-algebraically stable, we obtain $\lambda(f) = \theta$ (Proposition B(3)). We will then study the cases where f is not μ-algebraically stable. This implies that the μ-leading part g of f is such that one component of g^r is equal to zero for some $r \geq 1$. The possibilities for such endomorphisms g are studied in Lemma 4.3.2 below. The following result is a simple observation, whose proof is left as an exercise.

Lemma 4.3.1. *Let $n \geq 1$ and let $f = (f_1, \dots, f_n) \in \mathrm{TEnd}(\mathbb{A}^n)$ be a triangular endomorphism. Then,*

1. *f is dominant if and only if $\deg_{x_i}(f_i) \geq 1$ for each $i \in \{1, \dots, n\}$;*
2. *f is an automorphism if and only if $\deg_{x_i}(f_i) = 1$ for each $i \in \{1, \dots, n\}$.*

Lemma 4.3.2. *Let $g = (g_1, g_2, g_3) = \sigma \circ \tau \in \mathrm{End}(\mathbb{A}^3)$ where $\tau \in \mathrm{TEnd}(\mathbb{A}^3)$ is a triangular endomorphism, $\sigma \in \mathrm{Sym}(\mathbb{A}^3)$ is a permutation of the coordinates, where all g_i are non-constant and such that one of the components of g^r is a constant for some $r \geq 2$. Then, one of the following holds:*

(1) *$g_2, g_3 \in \boldsymbol{k}[x_1]$, $g_1 \in \boldsymbol{k}[x_1, x_2, x_3] \setminus (\boldsymbol{k}[x_1, x_2] \cup \boldsymbol{k}[x_1, x_3])$ and there exists $\zeta \in \boldsymbol{k}$ such that $g_1(t, g_2, g_3) = \zeta$ for each $t \in \boldsymbol{k}$;*
(2) *$g_1, g_3 \in \boldsymbol{k}[x_1]$, $g_2 \in \boldsymbol{k}[x_1, x_3] \setminus \boldsymbol{k}[x_1]$;*
(3) *$g_1, g_2 \in \boldsymbol{k}[x_1]$, $g_3 \in \boldsymbol{k}[x_1, x_2] \setminus \boldsymbol{k}[x_1]$;*
(4) *$g_1, g_2 \in \boldsymbol{k}[x_1, x_2] \setminus \boldsymbol{k}[x_1]$, $g_3 \in \boldsymbol{k}[x_1]$ and $g_1(g_1, g_2) = \zeta_1$, $g_2(g_1, g_2) = \zeta_2$ for some $\zeta_1, \zeta_2 \in \boldsymbol{k}$.*

Proof. We distinguish some cases, depending on which of the polynomials g_1, g_2, g_3 belong to $\mathbf{k}[x_1]$.

We first observe that $g_1, g_2, g_3 \in \mathbf{k}[x_1]$ is impossible, as each component of g^r, for each $r \geq 1$, would then be obtained by composing dominant endomorphisms of $\mathbb{A}^1$ and thus would not be constant.

• Suppose that $g_1, g_3 \in \mathbf{k}[x_1]$. By induction, we obtain $(g^r)_1, (g^r)_3 \in \mathbf{k}[x_1] \setminus \mathbf{k}$ for each $r \geq 1$, so $(g^r)_2 \in \mathbf{k}$ for some $r \geq 2$. If $g_2 \in \mathbf{k}[x_1, x_3]$, we obtain (2). Otherwise, $\deg_{x_2}(g_2) = d \geq 1$ and proceeding by induction we obtain $\deg_{x_2}((g^r)_2) = d^r \geq 1$ for each $r \geq 1$, impossible.

• If $g_1, g_2 \in \mathbf{k}[x_1]$ we do the same argument as before (by exchanging the roles of x_2 and x_3) and obtain (3).

• Suppose now that $g_2, g_3 \in \mathbf{k}[x_1]$. As $g_1 \in \mathbf{k}[x_1, x_2, x_3] \setminus \mathbf{k}[x_1]$, the closure of the image of $g \in \mathrm{End}(\mathbb{A}^3)$ is then equal to $\mathbb{A}^1 \times \Gamma$, where $\Gamma \subset \mathbb{A}^2$ is the irreducible curve that is the closure of the image of $\mathbb{A}^1 \to \mathbb{A}^2$, $x_1 \mapsto (g_2(x_1), g_3(x_1))$. The restriction of g gives an endomorphism $h = g|_{\mathbb{A}^1 \times \Gamma} \in \mathrm{End}(\mathbb{A}^1 \times \Gamma)$.

We now prove that h is not dominant. For each $r \geq 1$ and each $i \in \{1, 2, 3\}$, the restriction of $(g^r)_i$ to $\mathbb{A}^1 \times \Gamma$ is equal to $\pi_i \circ h^r$, where $\pi_i \colon \mathbb{A}^1 \times \Gamma \to \mathbb{A}^1$ is the i-th projection. Choosing i and r such that $(g^r)_i$ is constant, we find that $\pi_i \circ h^r$ is constant, so h^r is not dominant, as π_i is dominant. This proves that h is not dominant.

Denote by $\Gamma' \subset \mathbb{A}^1 \times \Gamma$ the closure of $h(\mathbb{A}^1 \times \Gamma)$, which is an irreducible curve, that contains $\left\{ (g_1(x, g_2(y), g_3(y)), g_2(x), g_3(x)) \mid (x, y) \in \mathbb{A}^2 \right\}$. This implies that the polynomial $s = g_1(x, g_2(y), g_3(y)) \in \mathbf{k}[x, y]$ is contained in $\mathbf{k}[x]$. We moreover observe that s is a constant. Indeed, otherwise the restriction of h to Γ' would be a dominant map $\Gamma' \to \Gamma'$ and since $\pi_i|_{\Gamma'} \colon \Gamma' \to \mathbb{A}^1$ is non-constant for each $i \in \{1, 2, 3\}$, the restriction of $(g^r)_i$ to Γ' would be non-constant for each $r \geq 1$ and each $i \in \{1, 2, 3\}$, contradiction. Hence, $\pi_1 \circ h = g_1|_{\mathbb{A}^1 \times \Gamma} \colon \mathbb{A}^1 \times \Gamma \to \mathbb{A}^1$ is equal to a constant $\zeta \in \mathbf{k}$. This yields$g_1(t, g_2, g_3) = \zeta$ for each $t \in \mathbf{k}$ and implies that $g_1 \notin \mathbf{k}[x_1, x_2] \cup \mathbf{k}[x_1, x_3]$, since g_1, g_2, g_3 are non-constant, whence (1).

• It remains to assume that at most one of the g_i belongs to $\mathbf{k}[x_1]$. We write $\tau = (\tau_1, \tau_2, \tau_3)$, and observe that $\{g_1, g_2, g_3\} = \{\tau_1, \tau_2, \tau_3\}$. As $\tau_1 \in \mathbf{k}[x_1] \setminus \mathbf{k}$, we get that exactly one of the g_i belongs to $\mathbf{k}[x_1]$ and that $\tau_2 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$. As g is not dominant, neither is τ; Lemma 4.3.1 then implies that $\tau_3 \in \mathbf{k}[x_1, x_2]$. So $g_1, g_2, g_3 \in \mathbf{k}[x_1, x_2]$ and exactly one of the three belongs to $\mathbf{k}[x_1]$. Note that the endomorphism $h = (g_1, g_2) \in \mathrm{End}(\mathbb{A}^2)$ is not dominant. Indeed, otherwise no component of g^r is constant for each $r \geq 1$, as $g_3 \in \mathbf{k}[x_1, x_2]$ is non-constant. It is thus impossible that $g_1 \in \mathbf{k}[x_1]$ or $g_2 \in \mathbf{k}[x_1]$, as (g_1, g_2) (respectively (g_2, g_1)) would be a dominant triangular endomorphism of $\mathbb{A}^2$ (Lemma 4.3.1). Hence, $g_3 \in \mathbf{k}[x_1] \setminus \mathbf{k}$ and $g_1, g_2 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$. As h is not dominant, the closure of $h(\mathbb{A}^2)$ is an irreducible curve $\Gamma \subset \mathbb{A}^2$.

If $g_j(\Gamma)$ is not a point for $j = 1$ or $j = 2$, then the restriction $h|_\Gamma \colon \Gamma \to \Gamma$ would be dominant. As g_3 is not constant on Γ (because $g_3(g_1(x_1, x_2))$ is not constant), we get that $(g^r)_i$ is non-constant for each $r \geq 1$ and each $i \in \{1,2,3\}$, contradiction. Thus $g_i(\Gamma) = \{\zeta_i\}$ for $i = 1,2$ where $\zeta_i \in \mathbf{k}$. This gives (4). $\square$

Lemma 4.3.3. *Let $f = \sigma \circ \nu \in \mathrm{Aut}(\mathbb{A}^3)$ be a permutation-triangular automorphism, where $\sigma \in \mathrm{Sym}(\mathbb{A}^3)$ and $\nu \in \mathrm{TAut}(\mathbb{A}^3)$. Suppose that the maximal eigenvalue θ of f is bigger than 1 and let μ be a maximal eigenvector of f such that f is not μ-algebraically stable. Then, one of the following cases holds:*

- (i) *$f = (\xi_3 x_3 + p_3(x_1,x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1))$ where $\xi_2, \xi_3 \in \boldsymbol{k}^*$, $p_1, p_2 \in \boldsymbol{k}[x_1]$, $p_3 \in \boldsymbol{k}[x_1,x_2]$, $\deg(p_1) = 1$, and $\deg(p_2) = \theta^2 > 1$. Moreover, there exists $s \in \boldsymbol{k}[x_2]$ such that the conjugation of f by $(x_1, x_2, x_3 + s(x_2))$ does not increase the degree of p_3 and (strictly) decreases the degree of p_2.*
- (ii) *$f = (\xi_2 x_2 + p_2(x_1), \xi_3 x_3 + p_3(x_1,x_2), p_1(x_1))$ where $\xi_2, \xi_3 \in \boldsymbol{k}^*$, $p_1, p_2 \in \boldsymbol{k}[x_1]$, $p_3 \in \boldsymbol{k}[x_1,x_2]$, $\deg(p_1) = 1$, and $\deg(p_2) = \theta > 1$. Moreover, there exists $s \in \boldsymbol{k}[x_1]$ such that the conjugation of f by $(x_1, x_2 + s(x_1), x_3)$ (strictly) decreases the degrees of p_2 and p_3.*

Proof. Denote by $g = (g_1, g_2, g_3)$ the μ-leading part of f. As $\mu = (\mu_1, \mu_2, \mu_3) \in (\mathbb{R}_{\geq 0})^3 \setminus \{0\}$ is a maximal eigenvector of f, $g_i \notin \mathbf{k}$ for each $i \in \{1,2,3\}$ (Lemma 2.5.7). Moreover, as f is not μ-algebraically stable, there is some $r \geq 1$ such that $(g^r)_i = 0$ for all $i \in \{1,2,3\}$ with $\mu_i > 0$ (Lemma 2.6.1(5)). We write $g = \sigma \circ \tau$ where $\tau = (\tau_1, \tau_2, \tau_3) \in \mathrm{TEnd}(\mathbb{A}^3)$; Lemma 4.3.2 gives then four possibilities (1)-(2)-(3)-(4) for g, that we consider separately. We will show that (i) and (ii) occur in Cases (1) and (4), respectively and that (2)-(3) do not occur.

(2)-(3): Let us first observe that Case (2) (respectively (3)) of Lemma 4.3.2 does not occur. Indeed, otherwise the first and the last (respectively the first two) components of g^r belong to $\mathbf{k}[x_1] \setminus \mathbf{k}$ for each $r \geq 1$, so $\mu = (0, \mu_2, 0)$ (respectively $\mu = (0,0,\mu_3)$), since $(g^r)_i = 0$ for each $i \in \{1,2,3\}$ with $\mu_i > 0$. This gives $\deg_\mu(g_i) = 0$ for $i = 1,2,3$, as g_1, g_2, g_3 belong to $\mathbf{k}[x_1,x_3]$ (respectively $\mathbf{k}[x_1,x_2]$), impossible as $\deg_\mu(g) = \deg_\mu(f) = \theta > 1$ (Lemma 2.5.7).

(1): Suppose now that Case (1) of Lemma 4.3.2 occurs: As $g_1 \in \mathbf{k}[x_1,x_2,x_3] \setminus (\mathbf{k}[x_1,x_2] \cup \mathbf{k}[x_1,x_3])$ and since the monomials of g_1 are some of those of f_1, that is one of the coordinates of the triangular automorphism $\nu \in \mathrm{TAut}(\mathbb{A}^3)$, the polynomial f_1 is equal to the third coordinate of ν and g_1 is of the form $g_1 = \xi_3 x_3 + q(x_1,x_2)$ for some $\xi_3 \in \mathbf{k}^*$ and $q \in \mathbf{k}[x_1,x_2] \setminus \mathbf{k}[x_1]$. Since $g_1(t, g_2, g_3) = \zeta \in \mathbf{k}$ for each $t \in \mathbf{k}$, we obtain $\xi_3 g_3 + q(t, g_2) = \zeta$ for each $t \in \mathbf{k}$, so $q \in \mathbf{k}[x_2] \setminus \mathbf{k}$ and

$$g = (\xi_3 x_3 + q(x_2), g_2, (\zeta - q(g_2))/\xi_3),$$

where $g_2 \in \mathbf{k}[x_1]$. By definition (Definition 1.4.5), g_i is the μ-homogeneous part of f_i of degree $\theta\mu_i$, for each $i \in \{1,2,3\}$ so each monomial of g_i is of μ-degree $\theta\mu_i$. The explicit form of g_1, g_2, g_3 directly gives

$$\theta\mu_1 = \mu_3 = \deg(q)\mu_2\,,\ \theta\mu_2 = \deg(g_2)\mu_1 \ \text{ and } \ \theta\mu_3 = \deg(g_3)\mu_1 = \deg(q)\deg(g_2)\mu_1\,.$$

In particular, $\mu_1, \mu_2, \mu_3 \in \mathbb{R}_{>0}$ and $\deg(g_3) = \deg(q)\deg(g_2) = \theta^2 > 1$. Since two monomials in the same variables have distinct μ-degrees, we moreover find that q, g_2 and g_3 are monomials, so $\zeta = 0$.

One component of f (and of τ) belongs to $\mathbf{k}[x_1]$ and is of degree 1. As $g_1 \notin \mathbf{k}[x_1]$ and $\deg(g_3) > 1$, we find that $f_2 \in \mathbf{k}[x_1]$ is of degree 1. This yields $\sigma = (x_3, x_1, x_2)$ and $\deg(f_2) = \deg(g_2) = 1$, whence $\deg(q) = \deg(g_3) = \theta^2 > 1$. We obtain the form given in (i): the automorphism f is equal to

$$f = (\xi_3 x_3 + p_3(x_1,x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1))$$

where $\xi_2, \xi_3 \in \mathbf{k}^*$, $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$, $\deg(p_1) = 1$. Moreover, $g_3 = -q(g_2)/\xi_3 \in \mathbf{k}[x_1]$ is the μ-leading part of $f_3 = \xi_2 x_2 + p_2(x_1)$, so g_3 is only one monomial, of degree $\theta^2 = \deg(g_3) = \deg(p_2)$.

To prove that we are indeed in Case *(i)*, it remains to show that the conjugation by $h = (x_1, x_2, x_3 + \xi_3^{-1} q(x_2))$ does not increase the degree p_3 and strictly decreases the degree of p_2. We calculate

$$h \circ f \circ h^{-1} = (\xi_3 x_3 + p_3(x_1, x_2) - q(x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1) + q(p_1(x_1))/\xi_3).$$

As every monomial of $g_1 = \xi_3 x_3 + q(x_2)$ is contained in $f_1 = \xi_3 x_3 + p_3(x_1, x_2)$, the degree of $p_3(x_1, x_2) - q(x_2)$ is at most the one of $p_3(x_1, x_2)$. It remains to see that $\deg(p_2 + q(p_1)/\xi_3) < \deg(p_2)$ which follows from the fact that $g_3 = -q(g_2)/\xi_3 \in \mathbf{k}[x_1]$ is the μ-leading part of $f_3 = \xi_2 x_2 + p_2(x_1)$, and that g_2 is the leading monomial of p_1 (of degree 1.

(4): It remains to consider Case (4) of Lemma 4.3.2. As $g_1, g_2 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$, the only component of f which belongs to $\mathbf{k}[x_1]$ (and is of degree 1) is f_3, so $\sigma = (x_3, x_2, x_1)$ or $\sigma = (x_2, x_3, x_1)$. Let $j \in \{1, 2\}$ be such that $f_j = \nu_2$, where $\nu = (\nu_1, \nu_2, \nu_3)$. We then have $f_j = \xi_2 x_2 + p_2(x_1)$ for some $\xi_2 \in \mathbf{k}^*$ and some $p_2 \in \mathbf{k}[x_1]$. As $g_j \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$, we get $g_j = \xi_2 x_2 + q(x_1)$ for some $q \in \mathbf{k}[x_1]$, that consists of some monomials of p_2. Since $\zeta_j = g_j(g_1, g_2)$, we obtain $\zeta_j = \xi_2 g_2 + q(g_1)$.

We now show that $j = 2$ leads to a contradiction. It gives

$$g_2 = \xi_2 x_2 + q(x_1) = \xi_2^{-1}(\zeta_2 - q(g_1)).$$

Since $\xi_2 x_2 + q(x_1)$ is irreducible, the polynomial $\zeta_2 - q(g_1)$ is irreducible, and thus $\deg(q) = 1$, which in turn implies that g_2 and thus g_1 is of degree 1. Hence, g_1, g_2, g_3 are of degree 1, impossible, as $\theta > 1$ is the eigenvalue of a matrix that is contained in g (Lemma 2.5.1).

This contradiction proves that $j = 1$, so $\sigma = (x_2, x_3, x_1)$. This yields

$$f = (\xi_2 x_2 + p_2(x_1), \xi_3 x_3 + p_3(x_1, x_2), p_1(x_1))$$

where $\xi_2, \xi_3 \in \mathbf{k}^*$, $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$ and $\deg(p_1) = 1$, as in *(ii)*.

We also have $g_1 = \xi_2 x_2 + q(x_1)$ and $\zeta_1 = \xi_2 g_2 + q(g_1)$, which yields $g_2 = (\zeta_1 - q(g_1))/\xi_2 = (\zeta_1 - q(\xi_2 x_2 + q(x_1)))/\xi_2$. As g is the μ-leading part of f, the polynomial g_2 is not constant (Lemma 2.5.7), so $\deg(q) \geq 1$. Recall that g_i is the μ-homogeneous part of f_i of degree $\theta\mu_i$, for each $i \in \{1, 2, 3\}$ (Definition 1.4.5) so each monomial of g_i is of μ-degree $\theta\mu_i$. We thus obtain

$$\theta\mu_1 = \mu_2 = \deg(q)\mu_1 \quad \text{and} \quad \theta\mu_3 = \mu_1 .$$

This proves that $\mu_1, \mu_2, \mu_3 \in \mathbb{R}_{>0}$, that $\deg(q) = \theta > 1$ and that $\mu = (\theta\mu_3, \theta^2\mu_3, \mu_3)$.

Since two monomials in the same variables have distinct μ-degrees, we moreover find that q is a monomial, the leading monomial of p_2, so $\deg(p_2) = \deg(q) = \theta > 1$, as stated in *(ii)*.

To prove that we are indeed in Case *(ii)*, it remains to show that the conjugation by $h = (x_1, x_2 + q(x_1)/\xi_2, x_3)$ strictly decreases the degree of p_2 and p_3. We calculate

$$h \circ f \circ h^{-1} = (\xi_2 x_2 + p_2'(x_1), \xi_3 x_3 + p_3'(x_1, x_2), p_1(x_1)),$$

where

$$\begin{aligned} p_2'(x_1) &= p_2(x_1) - q(x_1), \\ p_3'(x_1, x_2) &= p_3(x_1, x_2 - q(x_1)/\xi_2) + q(\xi_2 x_2 + p_2'(x_1))/\xi_2 . \end{aligned}$$

As q is the leading monomial of p_2, this conjugation decreases the degree of p_2, i.e. $\deg(p_2') < \deg(p_2) = \theta$. It remains to see that $\deg(p_3') < \deg(p_3)$. To simplify the calculations, we replace μ by a multiple of itself (this is still a maximal eigenvector) and may assume that $\mu = (1, \theta, \theta^{-1})$. As $g_2 = (\zeta_1 - q(\xi_2 x_2 + q(x_1)))/\xi_2$ is the μ-homogeneous part of $f_2 = \xi_3 x_3 + p_3(x_1, x_2)$ of μ-degree $\theta\mu_2 = \theta^2$, the polynomial $\Delta = p_3 - g_2 \in \mathbf{k}[x_1, x_2]$ is equal to

$$\Delta = \sum_{i=0}^{\theta-1} x_2^i \Delta_i$$

where each $\Delta_i \in \mathbf{k}[x_1]$ is such that $\deg(\Delta_i) + i\theta < \theta^2$. As $\theta > 1$, this implies that $\deg(x_2^i \Delta_i) = i + \deg(\Delta_i) < \theta^2$ for each i, so $\deg(\Delta) < \theta^2$, which implies that the degree of $p_3 = \Delta + g_2$ is equal to θ^2, since $\deg(g_2) = \theta^2$. We then need to show that $\deg(p_3') < \theta^2$. Since $\deg(p_2') < \deg(q) = \theta$, we have $\deg(q(\xi_2 x_2 + p_2'(x_1))/\xi_2) < \theta^2$, so we only need to show that $\deg(p_3(x_1, x_2 - q(x_1)/\xi_2)) < \theta^2$. This is given by

$$\begin{aligned} p_3(x_1, x_2 - q(x_1)/\xi_2) &= \Delta(x_1, x_2 - q(x_1)/\xi_2) + g_2(x_1, x_2 - q(x_1)/\xi_2). \\ &= \sum_{i=0}^{\theta-1} (x_2 - q(x_1)/\xi_2)^i \Delta_i + (\zeta_1 - q(\xi_2 x_2))/\xi_2 \end{aligned}$$

and by the fact that $\deg(\Delta_i) + i\theta < \theta^2$ for each i. □

Example 4.3.4. We now give two distinct examples to show that Cases *(i)*-*(ii)* of Lemma 4.3.3 indeed occur.

(i) Let $n \geq 2$, and let $f = (x_3 - x_2^n, x_1, x_2 + x_1^n) \in \mathrm{Aut}(\mathbb{A}^3)$. Because of the matrix contained in (x_3, x_1, x_1^n), the maximal eigenvalue satisfies $\theta \geq \sqrt{n} > 1$ and as $f^2 = (x_2, x_3 - x_2^n, x_1 + (x_3 - x_2^n)^n)$ and $f^3 = (x_1, x_2, x_3)$, the map f is not μ-algebraically stable for any maximal eigenvector μ of f. It has then to satisfy Case *(i)* of Lemma 4.3.3, so $\theta = \sqrt{n}$.

(ii) Let $n \geq 2$, and let $f = (x_2 - x_1^n, x_3 + (x_2 - x_1^n)^n, x_1) \in \mathrm{Aut}(\mathbb{A}^3)$. Because of the matrix contained in $(-x_1^n, x_3, x_1)$, the maximal eigenvalue satisfies $\theta \geq n > 1$ and as $f^2 = (x_3, x_1 + x_3^n, x_2 - x_1^n)$ and $f^3 = (x_1, x_2, x_3)$, the element f is not μ-algebraically stable for each maximal eigenvector μ of f. It has then to satisfy Case *(ii)* of Lemma 4.3.3, so $\theta = n$.

We now give examples of permutation-triangular automorphisms of $\mathbb{A}^3$ which are μ-algebraically stable. These will be useful in the proof of Theorem 1.

Lemma 4.3.5. *For all $a, b, c \in \mathbb{N}$ such that $\lambda = \frac{a+\sqrt{a^2+4bc}}{2} \neq 0$, the maximal eigenvalue and the dynamical degree of the automorphisms*

$$f = (x_1^a x_2^b + x_3, x_2 + x_1^c, x_1) \text{ and } f' = (x_3 + x_1^a x_2^{bc}, x_1, x_2)$$

are equal to λ. Both automorphisms are μ-algebraic stable for each maximal eigenvector μ.

Proof. The matrices that are contained in f are

$$\begin{pmatrix} a & b & 0 \\ c & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ c & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

whose characteristic polynomials are $x(x^2 - ax - bc)$, $x(x-a)(x-1)$, $x(x^2-1)$ and $(x+1)(x^2-1)$, respectively. The corresponding spectral radii are respectively λ, a, 1 and 1. Hence, the maximal eigenvalue of f is λ.

Similarly, the matrices contained in $f'$are

$$\begin{pmatrix} a & bc & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

whose characteristic polynomials are $x(x^2 - ax - bc)$ and $x^3 - 1$. The maximal eigenvalue of f' is then also λ.

As neither f nor f' satisfies any of the two Cases *(i)*-*(ii)* of Lemma 4.3.3, both f and f' are μ-algebraically stable for each maximal eigenvector μ (of f and f', respectively). This gives then $\lambda(f) = \lambda(f') = \lambda$ (Proposition B) and achieves the proof. □

Lemma 4.3.6. *The maximal eigenvalue θ of a permutation-triangular automorphism $f \in \mathrm{Aut}(\mathbb{A}^3)$ of degree $d \geq 1$ is a non-zero number equal to $(a+\sqrt{a^2+4bc})/2$ for some $(a,b,c) \in \mathbb{N}^3$ where $a + b \leq d$ and $c \leq d$. It is thus a positive integer or a quadratic integer and a Handelman number.*

Proof. Each real number $\theta = \frac{a+\sqrt{a^2+4bc}}{2} \neq 0$, where $(a,b,c) \in \mathbb{N}^3$ is a root of the polynomial $P(x) = x^2 - ax - bc$, with $a, b, c \in \mathbb{N}^2 \setminus \{0\}$ so it is a Handelman number. If P is irreducible, then θ is a quadratic integer, and otherwise it is a positive integer. It remains to see that the maximal eigenvalue of every f is of the desired form.

We write $f = \sigma \circ \tau$, where $\sigma \in \mathrm{Sym}(\mathbb{A}^3)$ and $\tau \in \mathrm{TAut}(\mathbb{A}^3)$ is a triangular automorphism, that we write as $\tau = (\nu_1 x_1 + \epsilon, \nu_2 x_2 + p(x_1), \nu_3 x_3 + q(x_1, x_2))$ where $\nu_1, \nu_2, \nu_3 \in \mathbf{k}^*$, $\epsilon \in \mathbf{k}$, $p \in \mathbf{k}[x_1]$ and $q \in \mathbf{k}[x_1, x_2]$. The matrices contained in τ are all of the form

$$\begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} m & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ i & j & 0 \end{pmatrix}, \begin{pmatrix} m & 0 & 0 \\ k & 0 & 0 \\ i & j & 0 \end{pmatrix}$$

where m, k, i, j are non-negative integers and $0 \leq m \leq 1$, $k \leq \deg(p) \leq d$ and $i + j \leq \deg(q) \leq d$. Since the spectral radius is order-preserving on real square matrices with non-negative coefficients (see Definition 3.1.1(4)) and since $\nu_1 \neq 0$, the maximal eigenvalue is the spectral radius of a matrix where $m = 1$. The matrices contained in f are obtained from one of the above four types by permuting the rows. Permuting the rows of the identity matrix only gives a spectral radius equal to 1. In the second case, we conjugate by the permutation of the last two. In any case, we obtain that θ is either equal to 1 or is the spectral radius of a matrix $\sigma' M$, where σ' is a permutation matrix and M is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & j & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ k & 0 & 0 \\ i & j & 0 \end{pmatrix}$$

where $k \leq d$ and $i + j \leq d$. We obtain

$$\sigma' M = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & 0 \end{pmatrix}$$

for some $m_{ij} \in \mathbb{N}$, so θ is the spectral radius of the matrix

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

This last matrix is one of the following:

$$\begin{pmatrix} r & 0 \\ s & 0 \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} r & 0 \\ i & j \end{pmatrix}, \begin{pmatrix} i & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & j \\ r & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ i & j \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix},$$

where $r, s \in \{1, i, j, k\}$. In the first four cases, θ is an integer in $\{1, \ldots, d\}$, so has the desired form, with $a = \theta$, and $b = c = 0$. In the fifth case, the characteristic polynomial is $x^2 - ix - jr$. Choosing $a = i$, $b = j$ and $c = r$ we get $\theta = (a + \sqrt{a^2 + 4bc})/2$. In the sixth case, the characteristic polynomial is $x^2 - jx - i$. When we choose $a = j$, $b = i$ and $c = 1$, we get again $\theta = (a + \sqrt{a^2 + 4bc})/2$. In the last case, the characteristic polynomial is $x^2 - rs$. We then choose $a = 0$, $b = r$ and $c = s$. □

We can now give the proof of Proposition D.

Proof of Proposition D. We take an affine-triangular automorphism $f \in \mathrm{Aut}(\mathbb{A}^3)$. By Proposition 4.1.1, there exists $\alpha \in \mathrm{Aff}(\mathbb{A}^3)$ such that $f' = \alpha f \alpha^{-1}$ is a permutation-triangular automorphism. We then have $\deg(f') = \deg(f)$. Moreover, Proposition B shows that there exists a maximal eigenvector of f. We denote by θ the maximal eigenvalue of f'. If $\theta = 1$ or if f' is μ-algebraically stable for each maximal eigenvector μ, the dynamical degrees $\lambda(f)$ and $\lambda(f')$ are equal to the maximal eigenvalue θ of f' (Proposition B), which is a Handelman number (Lemma 4.3.6) so the result holds.

Suppose now that $\theta > 1$ and that f' is not μ-algebraically stable for some maximal eigenvector μ. Lemma 4.3.3 gives two possibilities for f':

$$\begin{aligned} f' &= (\xi_3 x_3 + p_3(x_1, x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1)) \quad \text{or} \\ f' &= (\xi_2 x_2 + p_2(x_1), \xi_3 x_3 + p_3(x_1, x_2), p_1(x_1)) \end{aligned}$$

where $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$, $\xi_2, \xi_3 \in \mathbf{k}^*$, $\deg(p_1) = 1$ and $\deg(p_2) > 1$. In both cases, Lemma 4.3.3 shows that one can replace f' by a conjugate, decrease the degree of p_2 and do not increase the degree of f'. After finitely many steps, we obtain the desired case where $\theta = 1$ or f' is μ-algebraically stable for each maximal eigenvector μ. Moreover, we still have $\deg(f') \leq \deg(f)$. □

Proof of Theorem 1. Let $f \in \mathrm{Aut}(\mathbb{A}^3)$ is an affine-triangular automorphism of $\mathbb{A}^3$ of degree d. Proposition D gives the existence of a permutation-triangular automorphism $f' \in \mathrm{Aut}(\mathbb{A}^3)$ such that $\deg(f') \leq \deg(f)$ and such that either the maximal eigenvalue θ of f' is equal to 1, or $\theta > 1$ and f' is μ-algebraically stable for each maximal eigenvector μ. In the first case, the dynamical degree $\lambda(f)$ is equal to $\lambda(f') = 1$, by Proposition B(2). In the second case, we obtain $\lambda(f) = \lambda(f') = \theta$, by Proposition B(3). Moreover, Lemma 4.3.6 proves that $\theta = \frac{a+\sqrt{a^2+4bc}}{2}$ for some $a, b, c \in \mathbb{N}$ with $a + b \leq d, c \leq d$ (and that $\theta \neq 0$).

Conversely, for all $a, b, c \in \mathbb{N}$ such that $\theta = \frac{a+\sqrt{a^2+4bc}}{2} \neq 0$, the element θ is the dynamical degree of $(x_1^a x_2^b + x_3, x_2 + x_1^c, x_1)$ and $(x_3 + x_1^a x_2^{bc}, x_1, x_2)$ (Lemma 4.3.5), and thus of a permutation-triangular automorphism of $\mathbb{A}^3$. This achieves the proof. □

Corollary 4.3.7. *For each $d \geq 3$ the set of all dynamical degrees of shift-like automorphisms of $\mathbb{A}^3$ of degree d is strictly contained in the set of all dynamical degrees of affine-triangular automorphisms of degree d.*

Proof. As each shift-like automorphism is also an affine-triangular automorphism, we have an inclusion, that we need to prove to be strict. From Proposition 4.2.5 it follows that the set of dynamical degrees of all shift-like automorphisms of $\mathbb{A}^3$ of degree d is equal to

$$\left\{ (a + \sqrt{a^2 + 4d - 4a})/2 \;\middle|\; 0 \leq a \leq d \right\}.$$

From Theorem 1 it follows that $\lambda_d = (1 + \sqrt{1+4d})/2$ is the dynamical degree of the affine-triangular automorphism $(x_3 + x_1x_2, x_2 + x_1^d, x_1)$. In order to show that λ_d is not the dynamical degree of any shift-like automorphism of $\mathbb{A}^3$ of degree d, for each $d \geq 3$, we only have to show that there exists no $d \geq 3$ and no $a \in \{0, \ldots, d\}$ such that

$$\sqrt{1 + 4d} = (a - 1) + \sqrt{a^2 + 4d - 4a}\,.$$

Indeed, if this would be the case, then $1 + 4d = (a-1)^2 + 2(a-1)\sqrt{a^2 + 4d - 4a} + a^2 + 4d - 4a$, which yields

$$a(3 - a) = (a - 1)\sqrt{a^2 + 4d - 4a}\,.$$

This implies that $a \leq 3$ and $a \notin \{0, 1\}$, i.e. $a = 2$. However, in this case $d = 2$. □

4.4. **Automorphisms of affine spaces associated to weak-Perron numbers.** In this section, we construct some affine-triangular automorphisms associated to weak-Perron numbers and prove Theorem 2.

Lemma 4.4.1. *Let $n \geq 1$ and let $A = (a_{i,j})_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{N})$ be an irreducible matrix with spectral radius $\rho(A) > 1$. The automorphism $f \in \mathrm{Aut}(\mathbb{A}^{2n})$ given by*

$$(*)\qquad \left(x_{n+1} + \prod_{i=1}^n x_i^{a_{1,i}}, x_{n+2} + \prod_{i=1}^n x_i^{a_{2,i}}, \ldots, x_{2n} + \prod_{i=1}^n x_i^{a_{n,i}}, x_1, \ldots, x_n \right)$$

has dynamical degree $\lambda(f) = \rho(A)$.

Proof. Let us write $\theta = \rho(A)$ and choose an eigenvector $v = (v_1, \ldots, v_n) \in (\mathbb{R}_{>0})^n$ of A to the eigenvalue θ (which exists by Theorem 3.2.3). We then choose $\mu = (\theta v_1, \ldots, \theta v_n, v_1, \ldots, v_n) \in (\mathbb{R}_{>0})^{2n}$. The matrix

$$M = \begin{pmatrix} A & 0 \\ I_n & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{N})$$

is contained in f, its spectral radius is θ and μ is an eigenvector of M to the eigenvalue θ. Writing $f = (f_1, \ldots, f_{2n})$, we now prove that $\deg_\mu(f_j) = \theta\mu_j$ for each $j \in \{1, \ldots, 2n\}$, and compute the μ-homogeneous part g_j of f_j of degree $\theta\mu_j$:

(1) For each $j \in \{1, \ldots, n\}$, we have $\deg_\mu(x_{n+j}) = v_j$ and $\deg_\mu(\prod_{i=1}^n x_i^{a_{j,i}}) = \sum_{i=1}^n \theta a_{j,i} v_i = \theta^2 v_j$, so $\deg_\mu(f_j) = \theta^2 v_j = \theta\mu_j$ and $g_j = \prod_{i=1}^n x_i^{a_{j,i}}$.
(2) For each $j \in \{n+1, \ldots, 2n\}$ we have $\deg_\mu(f_j) = \deg_\mu(x_{j-n}) = \theta v_{j-n} = \theta\mu_j$ and $g_j = f_j$.

This implies that $\deg_\mu(f) = \theta$. As the endomorphism $g = (g_1, \ldots, g_{2n}) \in \mathrm{End}(\mathbb{A}^{2n})$ is monomial, it satisfies $g^r \neq 0$ for each $r \geq 1$ (and moreover each component of g^r is not zero). This implies that f is μ-algebraically stable and that $\lambda(f) = \theta$ (see Proposition A). □

Proposition 4.4.2. *Let $\lambda \in \mathbb{R}$ be a weak Perron number that is a quadratic integer, and let $x^2 - ax - b$ be its minimal polynomial, with $a, b \in \mathbb{Z}$. We then have $a \geq 0$ and the following hold:*

(1) *If $b \geq 0$, then λ is the dynamical degree of the shift-like automorphism*
$$(x_3 + x_1^a x_2^b, x_1, x_2) \in \operatorname{Aut}(\mathbb{A}^3)\,.$$

(2) *If $b < 0$, then λ is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$, but is the dynamical degree of a permutation-triangular automorphism of $\mathbb{A}^4$ of the form $(*)$ in Lemma 4.4.1.*

Proof. Let us write $x^2 - ax - b = (x-\lambda)(x-\mu)$ for some $\mu \in \mathbb{R}$. Note that $\mu \neq \lambda$, as otherwise $\lambda^2 \in \mathbb{Z}$ and $2\lambda \in \mathbb{Z}$ would imply that $\lambda \in \mathbb{Z}$, impossible as λ is a quadratic integer. Since λ is a weak-Perron number, we have $\lambda \geq 1$ and $-\lambda \leq \mu < \lambda$. In particular, $a = \lambda + \mu \geq 0$. As $x^2 - ax - b$ is irreducible and has a real root by assumption, the discriminant is $a^2 + 4b \geq 1$.

If $b \geq 0$, Assertion (1) follows from Lemma 4.3.5 (and also from Proposition 4.2.3).

Suppose now that $b < 0$. As $\lambda\mu = -b$, this implies that $\mu > 0$, so λ is not a Handelman number (Lemma 3.2.7) and thus is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ (Proposition D). It is now enough to show that
$$f = (x_3 + x_1^{\alpha} x_2, x_4 + x_1^{\alpha(a-\alpha)+b} x_2^{a-\alpha}, x_1, x_2) \in \operatorname{Aut}(\mathbb{A}^4)$$
is a permutation-triangular automorphism with dynamical degree $\lambda(f) = \lambda$.

Firstly, we prove that f is a permutation-triangular automorphism of $\mathbb{A}^4$ by showing that the exponents are non-negative. As $a \geq 0$, the numbers $\alpha = \lfloor a/2 \rfloor$ and $a - \alpha$ are non-negative integers, so we only need to see that $\alpha(a-\alpha) + b \geq 0$. Since $a^2 + 4b \geq 1$ we get in case a is even, that $\alpha(a-\alpha)+b = \alpha^2 + b = (a^2+4b)/4 > 0$ and in case a is odd, that $\alpha = (a-1)/2$, so $\alpha(a-\alpha)+b = ((a-1)/2)\cdot((a+1)/2)+b = (a^2 + 4b - 1)/4 \geq 0$.

Secondly, the matrix
$$A = \begin{pmatrix} \alpha & 1 \\ \alpha(a-\alpha)+b & a-\alpha \end{pmatrix} \in \operatorname{Mat}_2(\mathbb{N})$$
has characteristic polynomial $x^2 - ax - b$ and thus spectral radius $\rho(A) = \lambda$. As $x^2 - ax - b$ is irreducible by assumption, it follows that A is an irreducible matrix. Moreover, as $b \leq -1$ and as $x^2 - ax - b$ has a real root, we get $a \neq 0$, hence $a \geq 1$. Since $a^2 + 4b \geq 1$, we get $\lambda = (a + \sqrt{a^2+4b})/2 \geq 1$. Now, if $\lambda = 1$, then $1 \leq a \leq 2$ and thus $a^2 + 4b \leq 0$ (as $b \leq -1$), contradiction. Thus $\lambda > 1$ and we can apply Lemma 4.4.1 and get that the dynamical degree of f is $\lambda(f) = \rho(A) = \lambda$. □

Proof of Theorem 2. Let $\lambda \geq 1$ be a weak-Perron number. By Theorem 3.2.4, λ is the spectral radius of an irreducible square matrix with non-negative integral coefficients. Lemma 4.4.1 then shows that λ is the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^n$ for some integer n. We denote by n_0 the least possible such n.

If $\lambda = 1$, then $n_0 = 1$, by taking the identity.

If $\lambda > 1$ is an integer, then $n_0 \geq 2$, as every automorphism of $\mathbb{A}^1$ is affine and thus has dynamical degree 1. Moreover, $n_0 = 2$ as $f = (x_1^{\lambda} + x_2, x_1)$ has dynamical degree equal to λ (f is μ-algebraic stable for $\mu = (1,0)$ and $\deg_{\mu}(f) = \lambda$).

If λ is not an integer, then $n_0 \geq 3$, as the dynamical degree of every automorphism of $\mathbb{A}^2$ is an integer (Corollary 2.4.3). If λ is a quadratic integer, the minimal polynomial of λ is equal to $x^2 - ax - b$ with $a \geq 0$ and $b \in \mathbb{Z}$ (Proposition 4.4.2). If the conjugate of λ is negative, we have $b > 0$, so $n_0 = 3$ by Proposition 4.4.2(1). If the conjugate of λ is positive, we have $b < 0$, so $n_0 = 4$ by Proposition 4.4.2(2). □

To complement Theorem 2, we now give a family of examples of quadratic integers that do not arise as dynamical degrees of affine-triangular automorphisms of $\mathrm{Aut}(\mathbb{A}^3)$ but which arise as dynamical degrees of some other automorphisms of $\mathbb{A}^3$.

Lemma 4.4.3. *For all integers $r, s, t \geq 1$, the dynamical degree of the automorphism*

$$f = (y + x^r z^t, z, x + z^s(y + x^r z^t)) \in \mathrm{Aut}(\mathbb{A}^3)$$

is the biggest root of $x^2 - ax + b \in \mathbb{R}[x]$, with $a = r + s + t$, $b = rs$ and satisfies $\lambda(f) > s + 1$. In particular, if $\lambda(f)$ is not an integer, it is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$, so f is not conjugate to an affine-triangular automorphism of $\mathbb{A}^3$.

Proof. Let θ be the biggest root of $P(x) = x^2 - ax + b = (x - r)(x - s) - tx \in \mathbb{R}[x]$ As $P(s+1) = (s+1-r) - t(s+1) = (s+1)(1-t) - r < 0$, we find that $\theta > s+1$. In particular, $\mu = (\theta - s, 1, \theta) \in \mathbb{R}_{\geq 1}$.

We compute $\deg_\mu(x^r z^{s+t}) = r(\theta - s) + (s+t)\theta = (r+s+t)\theta - rs = \theta^2$ and $\deg_\mu(x^r z^t) = \theta^2 - s\theta = \theta(\theta - s)$. This gives $\deg_\mu(f) = \theta$, with μ-leading part $g = (x^r z^t, z, x^r z^{s+t})$. Hence, $\lambda(f) = \theta$ by Proposition A.

If θ is not an integer, the other root of $P(x)$ is positive, so θ is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ (Theorem 2). This implies that f is not conjugate to an affine-triangular automorphism of $\mathbb{A}^3$. □

Example 4.4.4. We now apply Lemma 4.4.3 to small values of r, s, t, and find some examples of automorphisms $f = (y + x^r z^t, z, x + z^s(y + x^r z^t)) \in \mathrm{Aut}(\mathbb{A}^3)$ whose dynamical degree $\lambda(f)$ is not the one of an affine-triangular automorphism of $\mathbb{A}^3$. We give below all examples of $\lambda(f) \leq 5$ given by Lemma 4.4.3. Exchanging r and s does not change the value of $\lambda(f)$, so we will assume that $r \leq s \leq 3$.

r	s	t	f	$\lambda(f)$
1	1	1	$(y + xz, z, x + z(y + xz))$	$(3 + \sqrt{5})/2$
1	1	2	$(y + xz^2, z, x + z(y + xz^2))$	$2 + \sqrt{3}$
1	1	3	$(y + xz^3, z, x + z(y + xz^3))$	$(5 + \sqrt{21})/2$
1	2	1	$(y + xz, z, x + z^2(y + xz))$	$2 + \sqrt{2}$
1	2	2	$(y + xz^2, z, x + z^2(y + xz^2))$	$(5 + \sqrt{17})/2$
1	3	1	$(y + xz, z, x + z^3(y + xz))$	$(5 + \sqrt{13})/2$
2	3	1	$(y + x^2 z, z, x + z^3(y + x^2 z))$	$3 + \sqrt{3}$

Remark 4.4.5. Let λ be a weak-Perron number that is a quadratic integer.

By Theorem 2, λ is the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^4$ but is the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ if and only if its conjugate λ' is negative. If $\lambda' > 0$, then one can ask if λ is the dynamical degree of an automorphism of $\mathbb{A}^3$ (which would then necessarily be not conjugate to an affine-triangular automorphism). Writing $x^2 - ax + b$ the minimal polynomial of λ, with a, b positive integers, Lemma 4.4.3 shows that this is indeed true if one can write $b = rs$ with $r, s \geq 1$ and $a > r + s$. In particular, this holds if $b \leq 4$, as $a^2 - 4b > 0$, so $a > 2\sqrt{b}$. If $b = 5$, then $a \geq 5$ (as $a > 2\sqrt{b}$), and Lemma 4.4.3 applies as soon as $a \geq 6$. The case where $a = b = 5$ corresponds to $\lambda = (5+\sqrt{5})/2$, which is then the "simplest" weak-Perron quadratic integer that is not covered by Theorem 2 or Lemma 4.4.3.

According to Remark 4.4.5, it seems natural to ask if every quadric weak-Perron number is the dynamical degree of an automorphism of $\mathbb{A}^3$. A first intriguing case concerns the following question, which was in fact already asked to us by Jean-Philippe Furter and Pierre-Marie Poloni:

Question 4.4.6. *Is $(5 + \sqrt{5})/2$ the dynamical degree of an automorphism of $\mathbb{A}^3$?*

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Jérémy Blanc, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland

Email address: `jeremy.blanc@unibas.ch`

Immanuel van Santen, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland

Email address: `immanuel.van.santen@math.ch`

AUTOMORPHISMS OF THE AFFINE 3-SPACE OF DEGREE 3

JÉRÉMY BLANC AND IMMANUEL VAN SANTEN

Abstract. In this article we give two explicit families of automorphisms of degree ≤ 3 of the affine 3-space $\mathbb{A}^3$ such that each automorphism of degree ≤ 3 of $\mathbb{A}^3$ is a member of one of these families up to composition of affine automorphisms at the source and target; this shows in particular that all of them are tame. As an application, we give the list of all dynamical degrees of automorphisms of degree ≤ 3 of $\mathbb{A}^3$; this is a set of 3 integers and 9 quadratic integers. Moreover, we also describe up to compositions with affine automorphisms for $n \geq 1$ all morphisms $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3 with the property that the preimage of every affine hyperplane in $\mathbb{A}^n$ is isomorphic to $\mathbb{A}^2$.

Contents

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1. Introduction

1.1. **The results.** In this text, we fix an algebraically closed field $\mathbf{k}$ of any characteristic. We denote by $\mathbb{A}^n$ or sometimes $\mathbb{A}^n_{\mathbf{k}}$ the *affine n-space* $\mathrm{Spec}(\mathbf{k}[x_1, \ldots, x_n])$ over $\mathbf{k}$ for a specified choice of coordinates $x_1, \ldots, x_n$. Every morphism $f\colon \mathbb{A}^n \to \mathbb{A}^m$ is given by

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{f} & \mathbb{A}^m \\ (x_1, \ldots, x_m) & \longmapsto & (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \end{array}$$

for polynomials $f_1, \ldots, f_m \in \mathbf{k}[x_1, \ldots, x_n]$. If $n = 3$, we often use x, y, z instead of x_1, x_2, x_3 as coordinates. For simplicity we denote the above morphism sometimes by $f = (f_1, \ldots, f_m)$. For a morphism $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^m$ we denote by $\deg(f)$ its degree which is by definition equal to the maximum of the degrees $\deg(f_1), \ldots, \deg(f_n)$.

Let $\mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n)$ be the group of all automorphisms of $\mathbb{A}^n$ over $\mathbf{k}$. In the last decades, there has been done a lot of research on this group $\mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n)$, see e.g. the survey [vdE00]. There are two prominent subgroups of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n)$, namely the group of *affine automorphisms*

$$\mathrm{Aff}_{\mathbf{k}}(\mathbb{A}^n) = \left\{ (f_1, \ldots, f_n) \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n) \;\middle|\; \begin{array}{l} f_i \in \mathbf{k}[x_1, \ldots, x_n] \text{ and } \deg(f_i) = 1 \\ \text{for all } i = 1, \ldots, n \end{array} \right\}$$

and the group of *triangular automorphisms*

$$\mathrm{Triang}_{\mathbf{k}}(\mathbb{A}^n) = \left\{ (f_1, \ldots, f_n) \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n) \;\middle|\; \begin{array}{l} f_i \in \mathbf{k}[x_i, \ldots, x_n] \\ \text{for all } i = 1, \ldots, n \end{array} \right\}.$$

The subgroup generated by $\mathrm{Aff}_{\mathbf{k}}(\mathbb{A}^n)$ and $\mathrm{Triang}_{\mathbf{k}}(\mathbb{A}^n)$ inside $\mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n)$ is called the group of *tame automorphisms* and we denote it by $\mathrm{Tame}_{\mathbf{k}}(\mathbb{A}^n)$. In case $n = 1$, all automorphisms of $\mathbb{A}^1$ are tame (in fact they are affine) and for $n = 2$ it is proven by Jung and van der Kulk [Jun42, vdK53] that all automorphisms of $\mathbb{A}^2$ are tame. Since a long time it was conjectured that the famous Nagata-automorphism

$$(x - 2y(zx + y^2) - z(zx + y^2)^2, y + z(zx + y^2), z) \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^3)$$

is non-tame, until Shestakov and Umirbaev gave fifteen years ago an affirmative answer if $\mathrm{char}(\mathbf{k}) = 0$, see [SU04]. It is still an open problem whether $\mathrm{Tame}_{\mathbf{k}}(\mathbb{A}^n) \neq \mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n)$ for $n \geq 4$ and when $\mathrm{char}(\mathbf{k}) \neq 0$ also for $n = 3$.

It is conjectured by Rusek [Rus88] that all automorphisms of $\mathbb{A}^n$ of degree 2 are tame. If $n = 3$ and $\mathbf{k} = \mathbb{C}$, Fornaes and Wu [FsW98] classified all automorphisms of $\mathbb{A}^3_{\mathbb{C}}$ of degree 2 up to conjugation by affine automorphisms and it turned out that all of them are triangular up to composition of affine automorphisms at the source and target. For $n = 4$ and $\mathbf{k} = \mathbb{R}$, Meisters and Olech [MO91] and for $n = 5$ and $\mathbf{k} = \mathbb{C}$, Sun [Sun14] gave affirmative answers to Rusek's conjecture.

Motivated by these investigations of the tame automorphisms in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n)$, we study in this paper automorphisms of $\mathbb{A}^3$ of degree 3. For this let us introduce the following equivalence relation: $f, g \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n)$ are *equivalent* if there exist $\alpha, \beta \in \mathrm{Aff}_{\mathbf{k}}(\mathbb{A}^n)$ such that $f = \alpha \circ g \circ \beta$. The main theorem of this article is the following description of degree 3 automorphisms of $\mathbb{A}^3$:

Theorem 1 (see Theorem 3)**.** *Each automorphism of $\mathbb{A}^3$ of degree ≤ 3 is either equivalent to a triangular automorphism or to an automorphism of the form*

$$(*) \qquad (x + yz + za(x, z), y + a(x, z) + r(z), z) \in \mathrm{Aut}_{\boldsymbol{k}}(\mathbb{A}^3)$$

where $a \in \boldsymbol{k}[x,z] \setminus \boldsymbol{k}[z]$ is homogeneous of degree 2 and $r \in \boldsymbol{k}[z]$ is of degree ≤ 3.

In fact we prove that none of the automorphisms of $(*)$ is equivalent to a triangular automorphism, see Proposition 3.9.4.

Theorem 1 implies in particular that all automorphisms of degree ≤ 3 of $\mathbb{A}^3$ are tame, see Corollary 3.9.5.

As an other application of Theorem 1 we compute all dynamical degrees of automorphisms of degree ≤ 3. Recall, that the dynamical degree of an automorphism $f \in \mathrm{Aut}(\mathbb{A}^n)$ is defined by

$$\lambda(f) = \lim_{r\to\infty} \deg(f^r)^{\frac{1}{r}} \in \mathbb{R}_{\geq 1},$$

satisfies $1 \leq \lambda(f) \leq \deg(f)$ and is invariant under conjugation (in $\mathrm{Aut}(\mathbb{A}^n)$ but also in the bigger group $\mathrm{Bir}(\mathbb{A}^n)$ of birational transformations of $\mathbb{A}^n$). It gives information about the iteration of the automorphism f. The dynamical degree of an automorphism of $\mathbb{A}^2$ is always an integer, and all possible integers are possible, by simply taking $(x,y) \mapsto (y, x+y^d)$, for each $d \geq 1$. The set of dynamical degrees of automorphisms of $\mathbb{A}^3$ is still quite mysterious. In 2001, K. Maegawa proved that the set of dynamical degrees of all automorphisms of $\mathbb{A}^3_{\mathbb{C}}$ of degree 2 is equal to $\{1, \sqrt{2}, (1+\sqrt{5})/2, 2\}$ [Mae01, Theorem 3.1]. This also holds for each field (Theorem 2 below). Recently, we proved that for each $d \geq 1$ and each ground field k, the set of all dynamical degrees of automorphisms of $\mathbb{A}^3_{\mathrm{k}}$ of degree $\leq d$ that are equivalent to a triangular automorphism is

$$\left\{ \frac{a+\sqrt{a^2+4bc}}{2} \;\middle|\; (a,b,c) \in \mathbb{N}^3, a+b \leq d, c \leq d \right\} \setminus \{0\},$$

see [BvS19a, Theorem 1], reproduced below as Theorem 4.1.1. Using Theorem 1, we prove the following result:

Theorem 2. *For each $d \geq 1$ and each field k, let us denote by $\Lambda_{d,\mathrm{k}} \subset \mathbb{R}$ the set of dynamical degrees of all automorphisms of $\mathbb{A}^3_{\mathrm{k}}$ of degree d. We then have*

$$\begin{array}{lcl} \Lambda_{1,\mathrm{k}} &=& \{1\} \\ \Lambda_{2,\mathrm{k}} &=& \{1, \sqrt{2}, (1+\sqrt{5})/2, 2\} \\ \Lambda_{3,\mathrm{k}} &=& \{1, \sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \sqrt{3}, 2,\ \frac{1+\sqrt{13}}{2},\ 1+\sqrt{2},\ \sqrt{6},\ \frac{1+\sqrt{17}}{2},\ \frac{3+\sqrt{5}}{2},\ 1+\sqrt{3},\ 3\}. \end{array}$$

Note that the automorphisms in $(*)$ in Theorem 1 all fix a linear projection $\mathbb{A}^3 \to \mathbb{A}^1$ and thus the dynamical degree of these automorphisms are integers, see e.g. [BvS19a, Corollary 2.4.3]. Thus one has to permute the coordinate functions of these automorphisms in order to produce interesting dynamical degrees. The most interesting number in Theorem 2 is $(3+\sqrt{5})/2$. It is the dynamical degree of $f = (y+xz, z, x+z(y+xz)) \in \mathrm{Aut}(\mathbb{A}^3_{\mathrm{k}})$, for each field k. It follows from [BvS19a, Theorem 1] that $\lambda(f) = (3+\sqrt{5})/2$ is not the dynamical degree of any automorphism of $\mathbb{A}^3$ that is equivalent (over k or over its algebraic closure $\mathbf{k} = \overline{\mathrm{k}}$) to a triangular automorphism, of any degree, see [BvS19a, Example 4.4.6]. The fact that all dynamical degrees above arise essentially follows from [BvS19a], the main contribution of this text to Theorem 2 is to show that we cannot get more dynamical degrees. Theorem 2 implies that every dynamical degree of an element of $\mathrm{Aut}(\mathbb{A}^3)$ of degree 2 is also the dynamical degree of an element of $\mathrm{Aut}(\mathbb{A}^3)$ of degree 3, contrary to the case of dimension 2 (an element of $\mathrm{Aut}(\mathbb{A}^2)$ of degree 3 has dynamical degree equal to either 1 or 3).

1.2. **Outline of the article.** In order to classify all automorphisms of degree ≤ 3 up to equivalence we study first degree 3 polynomials in $\mathbf{k}[x,y,z]$ that define the affine plane $\mathbb{A}^2$ in $\mathbb{A}^3$ in Section 2. The closure in $\mathbb{P}^3$ of such a hypersurface in $\mathbb{A}^3$ is singular, so the polynomial has the form $xp+q$ for some $p,q \in \mathbf{k}[y,z]$ up to an affine automorphism, see Corollary 2.1.2. These polynomials were studied by Sathaye [Sat76] for fields with $\mathrm{char}(\mathbf{k})=0$ and by Russell [Rus76] for all fields and it turns out that all of them are variables of $\mathbf{k}[x,y,z]$, i.e. there are polynomials $g,h \in \mathbf{k}[x,y,z]$ with $\mathbf{k}[xp+q,g,h]=\mathbf{k}[x,y,z]$, see also Propositions 2.2.1, 2.2.2 and Corollary 2.2.3 for more detailed informations. We then give a description of all such hypersurfaces up to affine automorphisms (Proposition 2.3.5). As the polynomials of degree 3 of the form $xp+q$ correspond to cubic hypersurfaces of $\mathbb{A}^3$ whose closures in $\mathbb{P}^3$ are singular at $[0:1:0:0]$ (Lemma 2.1.1), it is also useful to classify them up to affine automorphisms that fix this point; this is done in Proposition 2.3.4, where a bigger list is given. Corollary 2.3.7 then corresponds to the case where we focus on a line at infinity instead of a point.

Then we investigate these hypersurfaces in families in Section 3. The best suited notion for us is the following: a morphism $f\colon \mathbb{A}^d \to \mathbb{A}^n$ is called an *affine linear system of affine spaces* if the preimage of each affine hyperplane of $\mathbb{A}^n$ is isomorphic to $\mathbb{A}^{d-1}$, see Definition 3.2.1. In case $d=3$, we say that f is *in standard form* if $f=(xp_1+q_1,\ldots,xp_n+q_n)$ for some polynomials $p_i,q_i \in \mathbf{k}[y,z]$. An affine linear system of affine spaces $g\colon \mathbb{A}^3 \to \mathbb{A}^n$ of degree 3 is equivalent to one in standard form if and only if for general affine hyperplanes $H \subset \mathbb{A}^3$ the closures of $g^{-1}(H)$ in $\mathbb{P}^3$ have a common singularity, see Lemma 2.1.1. Two affine linear systems of affine spaces $f,g\colon \mathbb{A}^d \to \mathbb{A}^n$ are called *equivalent* if there are $\alpha \in \mathrm{Aff}_{\mathbf{k}}(\mathbb{A}^n)$ and $\beta \in \mathrm{Aff}_{\mathbf{k}}(\mathbb{A}^3)$ such that $f=\alpha\circ g\circ\beta$. The key point in the proof of Theorems 1 and 3 is to show that each affine linear system of affine spaces $\mathbb{A}^3 \to \mathbb{A}^3$ of degree ≤ 3 is equivalent to one in standard form, see Proposition 3.6.1.

In Section 3.9, we give a description of all affine linear systems of affine spaces $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3 which implies Theorem 1. We call a morphism $f\colon Y \to X$ an $\mathbb{A}^1$-*fibration* if each closed fiber is (schematically) isomorphic to $\mathbb{A}^1$ and we call f a *trivial* $\mathbb{A}^1$-*fibration* if there exists an isomorphism $\varphi\colon X\times\mathbb{A}^1 \to Y$ such that the composition $f\circ\varphi\colon X\times\mathbb{A}^1 \to X$ is the projection onto the first factor. Note that the above definition of an $\mathbb{A}^1$-fibration differs from the notions of $\mathbb{A}^1$-fibrations in [GMM12] and [KM78]. In fact we show:

Theorem 3. *Every affine linear system of affine spaces $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3 is equivalent to an element of the following eleven families. Case* I) *corresponds to $n=1$, Cases* IIa) *and* IIb) *correspond to $n=2$ and Case* III) *corresponds to $n=3$.*

I) *variables of $\boldsymbol{k}[x,y,z]$:*
- (1) $x+r_2(y,z)+r_3(y,z)$ *where $r_i \in \boldsymbol{k}[y,z]$ is homogeneous of degree i;*
- (2) $xy+yr_2(y,z)+z$ *where $r_2 \in \boldsymbol{k}[y,z]\setminus\boldsymbol{k}[y]$ is homogeneous of degree 2;*
- (3) $xy^2+y(z^2+az+b)+z$ *where $a,b \in \boldsymbol{k}$.*

IIa) *trivial $\mathbb{A}^1$-fibrations:*
- (4) $(x+p_2(y,z)+p_3(y,z),\, y+q_2z^2+q_3z^3)$ *where $p_i \in \boldsymbol{k}[y,z]$ is homogeneous of degree i and $q_2,q_3 \in \boldsymbol{k}$;*
- (5) $(yz+za_2(x,z)+x,\, y+a_2(x,z)+r_1z+r_2z^2+r_3z^3)$ *where $a_2 \in \boldsymbol{k}[x,z]\setminus\boldsymbol{k}[z]$ is homogeneous of degree 2 and $r_i \in \boldsymbol{k}$;*
- (6) $(yz+za_2(x,z)+x,\, z)$ *where $a_2 \in \boldsymbol{k}[x,z]\setminus\boldsymbol{k}[z]$ is homogeneous of degree 2;*
- (7) $(xy^2+y(z^2+az+b)+z,\, y)$ *where $a,b \in \boldsymbol{k}$.*

IIb) *non-trivial* $\mathbb{A}^1$*-fibrations:*

(8) $(x + z^2 + y^3, y + x^2)$ *where* $\mathrm{char}(\boldsymbol{k}) = 2$*;*
(9) $(x + z^2 + y^3, z + x^3)$ *where* $\mathrm{char}(\boldsymbol{k}) = 3$.

III) *automorphisms of* $\mathbb{A}^3$*:*

(10) $(x + p_2(y, z) + p_3(y, z), y + q_2 z^2 + q_3 z^3, z)$ *where* $p_i \in \boldsymbol{k}[y, z]$ *is homogeneous of degree* i *and* $q_2, q_3 \in \boldsymbol{k}$*;*
(11) $(yz + za_2(x, z) + x, y + a_2(x, z) + r_2 z^2 + r_3 z^3, z)$ *where* $a_2 \in \boldsymbol{k}[x, z] \setminus \boldsymbol{k}[z]$ *is homogeneous of degree* 2 *and* $r_2, r_3 \in \boldsymbol{k}$.

The proof of Theorem 3 is given towards the end of Section 3.9. All the eleven cases in our list are in fact pairwise non-equivalent, see Proposition 3.9.4. For $n = 1$ and $\mathbf{k} = \mathbb{C}$, Ohta gave in [Oht99, Theorem 1] a list of all possibilities for affine linear systems of affine spaces $\mathbb{A}^3 \to \mathbb{A}^1$ of degree ≤ 3, together with a description of the curve at infinity. This corresponds then to a refined list of the items (1)-(2)-(3) of Theorem 3. Note that the fact that each affine linear system $\mathbb{A}^3 \to \mathbb{A}^1$ of affine spaces of degree ≤ 3 is equivalent to one of the items (1)-(2)-(3) is proven in Proposition 2.3.5 below, and is thus the very first part of our study. Moreover, Ohta gave in [Oht01, Theorem 2] and [Oht09, Theorem 2] lists of all possible affine linear systems $\mathbb{A}^3 \to \mathbb{A}^1$ of affine spaces of degree 4 in case the closure of the corresponding hypersurface in $\mathbb{P}^3$ is normal. In particular, he proves that all of them are variables of $\mathbb{A}^3$.

Let us give the connection of our results to the Jacobian conjecture. Recall that an endomorphism $f \in \mathrm{End}_{\mathbf{k}}(\mathbb{A}^n)$ has a constant non-zero Jacobian determinant $\det(\mathrm{Jac}(f)) \in \mathbf{k}^*$ if and only if for all affine hyperplanes $H \subset \mathbb{A}^n$ the preimage $f^{-1}(H)$ is a smooth hypersurface of $\mathbb{A}^n$, see Lemma 3.2.6. Thus for all $f \in \mathrm{End}_{\mathbf{k}}(\mathbb{A}^n)$ we have the following implications

$$f \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{A}^n) \implies \begin{array}{l} f \text{ is an affine linear system} \\ \text{of affine spaces} \end{array} \implies \det(\mathrm{Jac}(f)) \in \mathbf{k}^* .$$

For fields with $\mathrm{char}(\mathbf{k}) = 0$, the Jacobian conjecture asserts that the implications are equivalences. For $n = 3$, Vistoli proved the Jacobian conjecture in case $f \in \mathrm{End}_{\mathbf{k}}(\mathbb{A}^3)$ has degree 3, see [Vis99]. For fields with $\mathrm{char}(\mathbf{k}) = p > 0$, the last implication is certainly not an equivalence, take e.g. $(x_1 + x_1^p, x_2, \ldots, x_n) \in \mathrm{End}_{\mathbf{k}}(\mathbb{A}^n)$. However, Theorem 3 shows that in case $n = 3$ and $f \in \mathrm{End}_{\mathbf{k}}(\mathbb{A}^3)$ is of degree ≤ 3, the first implication is an equivalence.

It is also worth to mention that in case $n = 2$, there are affine linear systems of affine spaces $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3 that are $\mathbb{A}^{3-n}$-fibrations which are not trivial $\mathbb{A}^{3-n}$-fibrations, contrary to the cases $n = 1$ and $n = 3$. In fact, an affine linear system of affine spaces $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3 is a trivial $\mathbb{A}^{3-n}$-fibration if and only if it is equivalent to a linear system in standard form, see Corollary 3.9.2. Note that there are even non-trivial $\mathbb{A}^1$-fibrations $\mathbb{A}^2 \to \mathbb{A}^1$ in positive characteristic, see [KM78, Example on p.670].

In the last Section, we then compute the dynamical degree of all automorphisms of $\mathbb{A}^3$ of degree ≤ 3 by using the technique introduced in [BvS19a] and we prove Theorem 2 at the end of this section.

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1.3. **Conventions.** All schemes, varieties, rational maps and morphisms between them are defined over $\mathbf{k}$. Points of varieties refer to closed points of the associated scheme. If $f\colon X \to Y$ is a morphism of varieties, then the fibre over a point $y \in Y$ refers to the *schematic* fibre of f over y, i.e. $f^{-1}(y) = \mathrm{Spec}(\kappa(y)) \times_Y X$ where $\mathrm{Spec}(\kappa(y)) \to Y$ corresponds to the embedding of the point y in Y. More generally, the preimage of a closed subvariety Y' of Y corresponds to the *schematic* preimage of Y' under f, i.e. $f^{-1}(Y') = Y' \times_Y X$. If we speak of an n-dimensional scheme X, then we mean that every irreducible component of X has dimension n.

We denote for each $d \geq 0$ by $\mathbf{k}[x_1, \ldots, x_n]_d$ the vector space of homogeneous polynomials of degree d in the variables $x_1, \ldots, x_n$. By convention, the zero polynomial will be assumed to be homogeneous of any degree $d \geq 0$ (even if it has degree $-\infty$).

2. Hypersurfaces of $\mathbb{A}^3$ that are isomorphic to $\mathbb{A}^2$

2.1. **Existence of singularities at infinity.** In the sequel, we always see $\mathbb{A}^3$ as an open subvariety of $\mathbb{P}^3$ via the open embedding $\mathbb{A}^3 \hookrightarrow \mathbb{P}^3$, $(x, y, z) \mapsto [1 : x : y : z]$ and denote by $[w : x : y : z]$ the homogeneous coordiantes of $\mathbb{P}^3$.

Recall that the multiplicity m of a hypersurface $Y \subseteq \mathbb{P}^n$ at a given point $p \in Y$ is the multiplicity of the equation at this point, that can be computed locally, or is equivalently the multiplicity at p of the polynomial obtained by restriction of Y to a general line through p.

Lemma 2.1.1. *Let $F \in \boldsymbol{k}[w, x, y, z]$ be a homogeneous polynomial of degree d, let $f = F(1, x, y, z) \in \boldsymbol{k}[x, y, z]$ and let $X = \mathrm{Spec}(\boldsymbol{k}[x, y, z]/(f)) \subset \mathbb{A}^3$ be the corresponding hypersurface. The following conditions are equivalent:*

(1) *$f = xp + q$ for some polynomials $p, q \in \boldsymbol{k}[y, z]$.*
(2) *The closure $\overline{X}$ in $\mathbb{P}^3$ has multiplicity $\geq d - 1$ at the point $[0 : 1 : 0 : 0]$.*

Proof. We write $f = \sum_{i=0}^{d} x^{d-i} f_i(y, z)$ where $f_i \in \mathbf{k}[y, z]$ is of degree $\leq i$ for $i = 0, \ldots, d$. For each i, we denote by $F_i \in \mathbf{k}[w, y, z]$ the homogeneous polynomial of degree i such that $F_i(1, y, z) = f_i$. This implies that $F = \sum_{i=0} x^{d-i} F_i$. Note that $\deg(F) = d$ and that $\overline{X}$ is given by F in $\mathbb{P}^3$. Note that the multiplicity of $\overline{X}$, or equivalently of F, at the point $[0 : 1 : 0 : 0]$ is the smallest integer $m \geq 0$ such that F_m is not zero. Hence, this multiplicity m satisfies $m \geq d-1$ if and only $F = xF_{d-1} + F_d$, which corresponds to ask that $f = xf_{d-1} + f_d$. □

Corollary 2.1.2. *Let $X \subset \mathbb{A}^3$ be a hypersurface of degree $d \leq 3$ with $X \simeq \mathbb{A}^2$.*

(1) *If $d = 3$, then the closure $\overline{X}$ in $\mathbb{P}^3$ is singular.*
(2) *Up to an affine coordinate change, X is given by $xp + q = 0$ for polynomials $p, q \in \boldsymbol{k}[y, z]$ with $\max(\deg(p) + 1, \deg(q)) = d$.*

Proof. (1): If $\overline{X}$ is a smooth cubic hypersurface of $\mathbb{P}^3$, then $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}^7$, see [Har77, Chp. V, Proposition 4.8(a)]. However, since $\overline{X} \setminus X$ has at most 3 irreducible components $\mathrm{Pic}(X)$ is not trivial, so X cannot be isomorphic to $\mathbb{A}^2$.

(2): There exists a point in $\overline{X} \subset \mathbb{P}^3$ having multiplicity $\geq d - 1$: this is clear if $d \leq 2$ and follows from (1) if $d = 3$. Applying an affine automorphism of $\mathbb{A}^3$, we can assume that this point is $[0:1:0:0]$, and the result then follows from Lemma 2.1.1. $\square$

Remark 2.1.3. Corollary 2.1.2(1) is also true for $d \geq 4$: If $\bar{X}$ is smooth, then it is a K3-surface in case $d = 4$ and of general type in case $d > 4$. In both situations $\overline{X}$ is not rational.

Corollary 2.1.2(2) is false for $d \geq 4$: Consider the hypersurface X in $\mathbb{A}^3$ which is given by $f := z + (x + yz)^2 \cdot y^{d-4} = 0$. Note that X is isomorphic to $\mathbb{A}^2$, since f is the first component of the composition $\varphi_2 \circ \varphi_1$ of the automorphisms

$$\begin{array}{ccc} \mathbb{A}^3 & \xrightarrow{\varphi_1} & \mathbb{A}^3 \\ (x,y,z) & \longmapsto & (x+yz,y,z) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{A}^3 & \xrightarrow{\varphi_2} & \mathbb{A}^3 \\ (x,y,z) & \longmapsto & (x,y,z+x^2y^{d-4}) \,. \end{array}$$

Note that the closure $\overline{X}$ in $\mathbb{P}^3$ is singular only along the lines $w = y = 0$ and $w = z = 0$ and that the multiplicity at each of these points is $\leq d-2$. In particular, by Lemma 2.1.1 there is no affine coordinate change of $\mathbb{A}^3$ such that X is given by $xp + q = 0$ for $p, q \in \mathbf{k}[y,z]$.

2.2. **Hypersurfaces of $\mathbb{A}^3$ of degree 1 in one variable.** Motivated by Corollary 2.1.2, this section is devoted to the study of hypersurfaces $X \subset \mathbb{A}^3$ given by

$$xp(y,z) + q(y,z) = 0$$

for some polynomials $p, q \in \mathbf{k}[y,z]$ where $p \neq 0$. We start with the following result which is due to Russell [Rus76, Theorem 2.3]

Proposition 2.2.1. *Let $p, q \in \boldsymbol{k}[y,z]$ be such that*

$$X = \operatorname{Spec}(\boldsymbol{k}[x,y,z]/(xp+q))$$

is isomorphic to $\mathbb{A}^2$ and such that $p \notin \boldsymbol{k}$. Then there is an automorphism of $\boldsymbol{k}[y,z]$ that sends p onto an element of $\boldsymbol{k}[y]$. In particular, the irreducible components of the scheme $\operatorname{Spec}(\boldsymbol{k}[y,z]/(p))$ are disjoint and isomorphic to $\mathbb{A}^1$.

By Proposition 2.2.1 we are led to study the case of hypersurfaces in $\mathbb{A}^3$ of the form $xp(y) + q(y,z)$. This is done in the next result.

Proposition 2.2.2. *Let $p \in \boldsymbol{k}[y] \setminus \boldsymbol{k}$, $q \in \boldsymbol{k}[y,z]$ and consider the polynomial*

$$f = xp(y) + q(y,z) \in \boldsymbol{k}[x,y,z] \,.$$

Write $\tilde{p} = \prod_{i=1}^{r}(y - a_i)$ where $a_1, \ldots, a_r \in \boldsymbol{k}$ are the r distinct roots of p. Then the following statements are equivalent:

1. *$X = \operatorname{Spec}(\boldsymbol{k}[x,y,z]/(f))$ is isomorphic to $\mathbb{A}^2$;*
2. *There exists $\varphi \in \operatorname{Aut}_{\boldsymbol{k}}(\boldsymbol{k}[x,y,z])$ such that $\varphi(x) = f$ and $\varphi(y) = y$;*
3. *There exist $a \in \boldsymbol{k}[y,z], r_0, r_1 \in \boldsymbol{k}[y]$ with $\deg(r_i) < r$ for $i = 0, 1$ such that $r_1(a_i) \neq 0$ for each $i \in \{1, \ldots, r\}$ and*

$$q(y,z) = a\tilde{p} + zr_1 + r_0 \,.$$

Proof. (1) $\Rightarrow$ (2): This is done in [Rus76, Theorem 2.3], see also [Sat76] for the case $\operatorname{char}(\mathbf{k}) = 0$.

(2) $\Rightarrow$ (1): The automorphism φ corresponds to an automorphism of $\mathbb{A}^3$ that sends X onto $\operatorname{Spec}(\mathbf{k}[x,y,z]/(x)) \simeq \mathbb{A}^2$.

(1) $\Rightarrow$ (3): We consider the morphism $\pi\colon X \to \mathbb{A}^1$ given by $(x,y,z) \mapsto y$. Then, outside of $\{a_1, \ldots, a_r\}$, π is a trivial $\mathbb{A}^1$-bundle. If X is isomorphic to $\mathbb{A}^2$, then

each fibre of π needs to be isomorphic to $\mathbb{A}^1$ (this follows for instance from [Gan11, Theorem 4.12]). We write $q(y,z) = \sum_{j=0}^{d} z^j(q_j\tilde{p} + r_j)$, with $q_j, r_j \in \mathbf{k}[y]$ and $\deg(r_j) < r = \deg(\tilde{p})$ for each j.

For each $i \in \{1,\dots,r\}$, the fibre of π over a_i is $\operatorname{Spec}(\mathbf{k}[x,z]/(q(a_i,z)))$, so $q(a_i,z)$ is a polynomial of degree 1 in z (as each fibre of π is isomorphic to $\mathbb{A}^1$). This implies that $r_j(a_i) = 0$ for each $j \geq 2$ and that $r_1(a_i) \neq 0$. As $\deg(r_j) < r$, we obtain that $r_j = 0$ for $j \geq 2$. This gives (3) with $a = \sum_{j=0}^{d} z^j q_j$.

(3) $\Rightarrow$ (1): Let $R = \mathbf{k}[x,y,z]/(f)$ be the ring of regular functions on X. For each $i \in \{1,\dots,r\}$, Assertion (3) gives $f(x,a_i,z) = q(a_i,z) = zr_1(a_i) + r_0(a_i)$, so $R/(y-a_i) \simeq \mathbf{k}[\mathbb{A}^1]$, which implies that $(y-a_i)$ is a prime ideal of R and that $\pi^{-1}(a_i) = X \cap \{y = a_i\}$ is isomorphic to $\mathbb{A}^1$. Hence, every (closed) fibre of π is isomorphic to $\mathbb{A}^1$.

We consider $h_0 = z$ and construct inductively a finite sequence $h_0, h_1, \dots, h_{N_1}$ of regular functions on X such that $(\pi, h_i)\colon X \to \mathbb{A}^2$ restricts to an isomorphism $\pi^{-1}(U) \xrightarrow{\simeq} U \times \mathbb{A}^1$, where $U = \mathbb{A}^1 \setminus \{a_1,\dots,a_r\}$.

If h_i is constant on $\pi^{-1}(a_1)$, then there is a $c_i \in \mathbf{k}$ such that $h_i - c_i$ is a multiple of $y - a_1$. We then choose $h_{i+1} \in R$ such that $h_i - c_i = (y-a_1)\cdot h_{i+1}$. This sequence ends up at some point, i.e. that there exists $N_1 \geq 0$ such that h_{N_1} is not constant on $\pi^{-1}(a_1)$. Indeed, this is a direct application of [KW85, Lemma 1.1] where we use that R is a Noetherian integral domain.

Now, we start with $h_{N_1} \in R$. With the same argument as above, there exists now $h_{N_2} \in R$ that is not constant on $\pi^{-1}(a_1)$, not constant on $\pi^{-1}(a_2)$ and (π, h_{N_2}) restricts to an isomorphism $\pi^{-1}(U) \xrightarrow{\simeq} U \times \mathbb{A}^1$. Proceeding the same way with $i = 3,\dots,r$ we find $h \in R$ that is not constant on each $\pi^{-1}(a_j)$ for $j = 1,\dots,r$ and such that (π, h) restricts to an isomorphism $\pi^{-1}(U) \xrightarrow{\simeq} U \times \mathbb{A}^1$.

We observe that $(\pi,h)\colon X \to \mathbb{A}^2$ is birational, quasi-finite and surjective. By Zariski's Main Theorem [Gro61, Corollaire (4.4.9)] it is thus an isomorphism. □

Remark that the implication (3) $\Rightarrow$ (1) of Proposition 2.2.2 also follows from [BvS19b, Lemma 3.10] (the argument is essentially due to Asanuma [Asa87, Corollary 3.2]), but the argument given above is much simpler and goes back to [KW85].

Corollary 2.2.3. *Let $f \in \mathbf{k}[x,y,z]$ be a polynomial of degree ≤ 3. Then f is a variable of $\mathbf{k}[x,y,z]$ if and only if $\operatorname{Spec}(\mathbf{k}[x,y,z]/(f)) \simeq \mathbb{A}^2$. In particular, if this holds, then $\operatorname{Spec}(\mathbf{k}[x,y,z]/(f-\lambda)) \simeq \mathbb{A}^2$ for each $\lambda \in \mathbf{k}$.*

Proof. If f is a variable of $\mathbf{k}[x,y,z]$, then $\operatorname{Spec}(\mathbf{k}[x,y,z]/(f-\lambda)) \simeq \mathbb{A}^2$ for each $\lambda \in \mathbf{k}$, and thus in particular for $\lambda = 0$. Conversely, we suppose that $\operatorname{Spec}(\mathbf{k}[x,y,z]/(f))$ is isomorphic to $\mathbb{A}^2$, and prove that f is a variable.

After an affine coordinate change we may assume $f = xp(y) + q(y,z)$ with $p \in \mathbf{k}[y] \setminus \{0\}$ and $q \in \mathbf{k}[y,z]$ (Proposition 2.3.5). If $p \in \mathbf{k}^*$, then f is a variable as $(f,y,z) \in \operatorname{Aut}(\mathbb{A}^3)$. If $p \in \mathbf{k}[y] \setminus \mathbf{k}$, then Proposition 2.2.2(2) implies that f is a variable. □

2.3. Hypersurfaces of $\mathbb{A}^3$ of small degree that are isomorphic to $\mathbb{A}^2$.

Lemma 2.3.1. *Let $p, q \in \mathbf{k}[t]$ be two polynomials such that*

$$\mathbf{k}[t] = \mathbf{k}[p,q] \text{ and } \deg(p) < \deg(q).$$

Then, either $1 \in \{\deg(p), \deg(q)\}$ or $2 \leq \deg(p) \leq \deg(q) - 2$.

Proof. Suppose first that $\deg(p) \leq 0$, which is equivalent to $p \in \mathbf{k}$. We obtain $\mathbf{k}[t] = \mathbf{k}[q]$, which implies that $\deg(q) = 1$. Indeed, $\deg(q) \geq 1$ since $q \notin \mathbf{k}$ and $\deg(q) > 1$ is impossible, as the degree of any element of $\mathbf{k}[q]$ is a multiple of $\deg(q)$.

If $\deg(p) = 1$, the result holds, so we may assume that $\deg(p) \geq 2$. It remains to see that $\deg(p) < \deg(q) - 1$. We then consider the closed embedding $f\colon \mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ given by $t \mapsto (p(t), q(t))$, which extends to a morphism $\hat{f}\colon \mathbb{P}^1 \to \mathbb{P}^2$ given by $[t : u] \mapsto [u^d : P(t,u) : Q(t,u)]$, where $d = \deg(q)$ and where $P(t,u) = u^d \cdot p(\frac{t}{u})$, $Q(t,u) = u^d \cdot q(\frac{t}{u})$ are homogeneous polynomials of degree d. The image $\Gamma = \hat{f}(\mathbb{P}^1)$ is a closed curve of $\mathbb{P}^2$ that is rational and smooth outside of $[0 : 0 : 1] = \hat{f}([1 : 0])$. The degree of Γ is the intersection of Γ with a general line, which is then equal to $d = \deg(q) \geq 3$. The multiplicity m of Γ at the point $[0 : 0 : 1]$ satisfies then $m > 1$, as a smooth curve of degree $d \geq 3$ has genus $\frac{(d-1)(d-2)}{2} \geq 1$. It remains to observe that $m = \deg(q) - \deg(p)$. This can be checked in coordinates, or simply seen geometrically: a general line of $\mathbb{P}^2$ passing through $[0 : 0 : 1]$ intersects the curve $\Gamma \setminus \{[0 : 0 : 1]\}$ in $\deg(q) - m$ points and these points correspond to the roots of $p - \lambda$ for some general λ. □

Corollary 2.3.2. *Let $C \subset \mathbb{A}^2 = \operatorname{Spec}(\boldsymbol{k}[x,y])$ be a closed curve isomorphic to $\mathbb{A}^1$, of degree ≤ 3. Then, up to applying an element of $\operatorname{Aff}(\mathbb{A}^2)$, the curve C is given by $x + p(y) = 0$ for some $p \in \boldsymbol{k}[y]$ of degree ≤ 3 with no constant or linear term.*

Proof. Let $p, q \in \mathbf{k}[t]$ be such that $t \mapsto (p(t), q(t))$ is an isomorphism $\mathbb{A}^1 \to C$ defined over $\mathbf{k}$. The polynomials p, q satisfy then $\mathbf{k}[p,q] = \mathbf{k}[t]$. After applying an affine automorphism of $\mathbb{A}^2$, we may assume that $\deg(p) < \deg(q)$. By Lemma 2.3.1, we obtain $1 \in \{\deg(p), \deg(q)\}$.

We first assume that $\deg(q) = 1$, which implies that $\deg(p) < 1$, so $p \in \mathbf{k}$. After applying an affine automorphism of $\mathbb{A}^2$, we get $p = 0$ and $q = t$, so the curve C is given by $x = 0$.

We then assume that $\deg(p) = 1$. After applying an automorphism of $\mathbb{A}^1$, we may assume that $p = t$. Hence, C is given by $y - q(x) = 0$. After applying the automorphism $(x,y) \mapsto (y,x)$, the equation is $x - q(y) = 0$. By using an automorphism of the form $(x,y) \mapsto (x + ay + b, y)$ for some $a, b \in \mathbf{k}$, we may assume that q has no constant or linear term. □

Lemma 2.3.3. *Let $f \in \boldsymbol{k}[x,y,z]$ be a polynomial of the form*

$$f = xp(y,z) + q(y,z)\,,$$

for some $p, q \in \boldsymbol{k}[y,z]$ with $p \neq 0$ and $\deg(p) \leq 3$. If the surface $\operatorname{Spec}(\boldsymbol{k}[x,y,z]/(f))$ is isomorphic to $\mathbb{A}^2$, then after applying an affine automorphism on y and z, one of the following cases hold:

(1) *$p \in \boldsymbol{k}[y]$ has degree ≤ 3;*
(2) *$p = y + r(z)$ for some $r \in \boldsymbol{k}[z]$ of degree 2 or 3.*

Proof. If $p \in \mathbf{k}$, then we are in case (1). We may thus assume that $p \notin \mathbf{k}$. By Proposition 2.2.1, the irreducible components of $F_p = \operatorname{Spec}(\mathbf{k}[y,z]/(p))$ are disjoint and isomorphic to $\mathbb{A}^1$.

We use the embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$, $(y,z) \mapsto [1 : y : z]$ and denote by $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$ the line at infinity.

If the irreducible components of F_p are lines, then their closures in $\mathbb{P}^2$ have to pass through the same point in L_∞. After applying an affine automorphism, we may assume that the point is $[0:0:1]$, which implies that $p \in \mathbf{k}[y]$.

It remains to study the case where at least one irreducible component has degree ≥ 2. This component corresponds to an irreducible curve $C \subset \mathbb{A}^2$ of degree $d \in \{2,3\}$ whose closure $\overline{C}$ in $\mathbb{P}^2$ is again an irreducible curve of degree d.

By Corollary 2.3.2, we may apply an affine automorphism and assume that C is the zero locus of $y + r(z)$ for some polynomial r of degree d. If F_p is equal to C, then $p = y + r(z)$ (up to some constant which can be removed by an affine automorphism). Otherwise, as F_p has degree ≤ 3, we get that F_p is reduced, and it is the disjoint union of the degree 2 curve C with some line. But there is no such line in $\mathbb{A}^2$: by Bézout's theorem, the closure of the line in $\mathbb{P}^2$ would be tangent to the conic $\overline{C}$ at the point at infinity of $\overline{C}$, impossible as already L_∞ is tangent to $\overline{C}$ at that point. □

Proposition 2.3.4. *Let $f \in \boldsymbol{k}[x,y,z]$ be a polynomial of degree ≤ 3 of the form*

$$f = xp(y,z) + q(y,z)\,,$$

for some $p, q \in \boldsymbol{k}[y,z]$. If the surface $\operatorname{Spec}(\boldsymbol{k}[x,y,z]/(f))$ is isomorphic to $\mathbb{A}^2$, then after applying an affine automorphism that fixes the point $[0:1:0:0]$, one of the following cases occurs:

1. *$f = y + s(z)$ for some polynomial $s \in \boldsymbol{k}[z]$ of degree ≤ 3;*
2. *$f = x(y+z^2) + z$;*
3. *$f = x + r_2(y,z) + r_3(y,z)$ for some homogeneous $r_i \in \boldsymbol{k}[y,z]$ of degree i;*
4. *$f = xy + yr_2(y,z) + z$ for a homogeneous polynomial $r_2 \in \boldsymbol{k}[y,z]$ of degree 2;*
5. *$f = xy^2 + ys(z) + z$ for a polynomial $s \in \boldsymbol{k}[z]$ of degree ≤ 2;*
6. *$f = xy(y+1) + s(y)z + t(y)$ for some polynomials $s, t \in \boldsymbol{k}[y]$ of degree ≤ 1 with $s(0)s(-1) \neq 0$.*

Proof. If $p = 0$, then $f = q \in \mathbf{k}[y,z]$, so $\operatorname{Spec}(\mathbf{k}[x,y,z]/(f)) = \mathbb{A}^1 \times \operatorname{Spec}(\mathbf{k}[y,z]/(f))$, which implies that $\operatorname{Spec}(\mathbf{k}[y,z]/(f)) \simeq \mathbb{A}^1$. By Corollary 2.3.2, we may apply an affine automorphism on y and z in order to be in case (1). We may thus assume in the sequel that $p \neq 0$.

According to Lemma 2.3.3, we only need to consider the following two cases: either $p \in \mathbf{k}[y]$ or $p = y + r(z)$ for some $r \in \mathbf{k}[z]$ of degree 2.

Suppose first that $p = y + r(z)$ for some $r \in \mathbf{k}[z]$ of degree 2. By using the (non-affine) automorphism $(x,y,z) \mapsto (x, y - r(z), z)$ of $\mathbb{A}^3$, we get

$$\operatorname{Spec}(\mathbf{k}[x,y,z]/(f)) \simeq \operatorname{Spec}(\mathbf{k}[x,y,z]/(xy + q(y - r(z), z)).$$

Then, Proposition 2.2.2 shows that $q(y - r(z), z) = ay + \lambda z + \mu$ for some $a \in \mathbf{k}[y,z]$, $\lambda \in \mathbf{k}^*$ and $\mu \in \mathbf{k}$. This gives

$$f = xp + q = (x+s)(y+r) + \lambda z + \mu,$$

where $s = a(y+r, z) \in \mathbf{k}[y,z]$. As $\deg(r) = 2$, we obtain that $\deg(s) \leq 1$. Hence, after applying the affine automorphism $(x,y,z) \mapsto (x - s(y,z), y, z)$, we may assume that f is equal to $x(y + r(z)) + \lambda z + \mu$. Using the affine automorphism $(x,y,z) \mapsto (x, y, \lambda^{-1}(z - \mu))$, we obtain $x(y + r'(z)) + z$ for some $r' = \sum_{i=0}^{2} \mu_i z^2 \in \mathbf{k}[z]$ of degree 2. After replacing y with $y - \mu_0 - \mu_1 z$ we get $x(y + \mu_2 z^2) + z$. We then apply $(x,y,z) \mapsto (\mu_2^{-1}x, \mu_2 y, z)$ in order to be in case (2).

It remains to consider the case where $p \in \mathbf{k}[y]$. We distinguish the different cases:

If $p \in \mathbf{k}^*$, we may assume that $p = 1$ and after applying $(x, y, z) \mapsto (x - q_0 - q_1(y, z))$ we are in case (3), where $q_0, q_1 \in \mathbf{k}[y, z]$ are the constant and linear part of q, respectively.

If p has one single root, we may assume that $p = y^i$ for some $i \in \{1, 2\}$. Then, Proposition 2.2.2 shows that $q(y, z) = ay + \lambda z + \mu$ for some $a \in \mathbf{k}[y, z]$, $\lambda \in \mathbf{k}^*$ and $\mu \in \mathbf{k}$. After applying the affine automorphism $(x, y, z) \mapsto (x, y, \lambda^{-1}(z - \mu))$ we may assume that $\lambda = 1$ and $\mu = 0$.

If $i = 1$, then $f = xy + yr(y, z) + z$ for some polynomial r of degree ≤ 2. Let $r_1, r_0 \in \mathbf{k}[y, z]$ be the homogeneous parts of degree 1 and degree 0 of r, respectively. We may apply the affine automorphism $(x, y, z) \mapsto (x - r_1(y, z) - r_0, y, z)$ and thus we may assume that r is homogeneous of degree 2. Hence, we are in case (4).

If $i = 2$, then $f = xy^2 + yr(y, z) + z$ for some polynomial r of degree ≤ 2. Now, after applying a suitable affine automorphism of the form $(x, y, z) \mapsto (x - b(y, z), y, z)$ we may assume that $r \in \mathbf{k}[z]$ and thus we are in case (5).

We then assume that p has two distinct roots. We may assume that $p = y(y+1)$. Proposition 2.2.2 shows that $q(y, z) = ay(y + 1) + sz + t$ for some $a \in \mathbf{k}[y, z]$ of degree ≤ 1, and some $s, t \in \mathbf{k}[y]$ of degree ≤ 1 with $s(0) \neq 0$, $s(-1) \neq 0$. After applying $(x, y, z) \mapsto (x - a(y, z), y, z)$ we are in case (6). $\square$

Proposition 2.3.5 (Hypersurfaces isomorphic to $\mathbb{A}^2$ of degree ≤ 3)**.** *Let $f \in \boldsymbol{k}[x, y, z]$ be an irreducible polynomial of degree ≤ 3. If the surface $\mathrm{Spec}(\boldsymbol{k}[x, y, z]/(f))$ is isomorphic to $\mathbb{A}^2$, then there is $\alpha \in \mathrm{Aff}(\mathbb{A}^3)$, such that one of the following cases occur:*

A) $\alpha^*(f) = x + r_2(y, z) + r_3(y, z)$ *for some homogeneous* $r_i \in \boldsymbol{k}[y, z]$ *of degree* i;
B) $\alpha^*(f) = xy + yr_2(y, z) + z$ *for a homogeneous* $r_2 \in \boldsymbol{k}[y, z] \setminus \boldsymbol{k}[y]$ *of degree* 2;
C) $\alpha^*(f) = xy^2 + y(z^2 + az + b) + z$ *for some* $a, b \in \boldsymbol{k}$.

Moreover, if $f \in \boldsymbol{k}[x, y, z]$ is one of the polynomials from cases (3)-(6) of Proposition 2.3.4, then we may in addition assume that $\alpha^*(y) \in \boldsymbol{k}[y]$.

Proof. By Corollary 2.1.2 we may assume that

$$f = xp + q$$

for some $p, q \in \mathbf{k}[y, z]$ with $\deg(p) \leq 2$ and $\deg(q) \leq 3$. We go through the different cases of Proposition 2.3.4.

(1): We exchange x, y and get $f = x + s(z)$ and then we replace x with $x + a + bz$ for some $a, b \in \mathbf{k}$ in order to be in case A).

(2): We exchange x, y and get $f = y(x + z^2) + z = xy + yz^2 + z$ which is a subcase of B).

(3) and (4) directly give A) and B), except if we are in case (4) with $r_2 \in \mathbf{k}[y]$, in which case we exchange x, z in order to be in case A).

(5): We have $f = xy^2 + ys(z) + z$ for some polynomial s of degree ≤ 2. We distinguish three cases:

If $\deg(s) \leq 0$, we have $s \in \mathbf{k}$. After the coordinate change $(x, y, z) \mapsto (x, y, z - sy)$ and the exchange of x, z we are in case A).

If $\deg(s) = 1$, we have $f = xy^2 + y(az + b) + z$ for some $a \in \mathbf{k}^*$ and $b \in \mathbf{k}$. We replace x, y, z with $a(az + b), (y - 1)/a, x$ and obtain $xy + yr_2(y, z) + z$ where

$r_2 = yz + uy + vz + w$ for some $u, v, w \in \mathbf{k}$. After replacing x with $x - uy - vz - w$, we may assume that r_2 is homogeneous and still not in $\mathbf{k}[y]$; this gives B).

If $\deg(s) = 2$ we apply a homothety in x and y, and obtain C).

(6): We exchange x and z and get $f = xs(y) + y(y+1)z + t(y)$ for some polynomials $s, t \in \mathbf{k}[y]$ of degree ≤ 1 with $s(0)s(-1) \neq 0$. If $s \in \mathbf{k}$, then $s \neq 0$ and after applying $(x, y, z) \mapsto (s^{-1}(x - t(y)), y, z)$ we are in case A). Otherwise, we replace $s(y)$ with y and get $xy + u(y)z + v(y)$ where $u, v \in \mathbf{k}[y]$, $\deg(u) = 2$, $\deg(v) \leq 1$ and $u(0) \neq 0$. Hence, we get $xy + ya(y, z) + \lambda z + \mu$ with $a \in \mathbf{k}[y, z]$, $\lambda \in \mathbf{k}^*$ and $\mu \in \mathbf{k}$. After replacing $\lambda z + \mu$ with z we get $f = xy + yb(y, z) + z$ for some $b \in \mathbf{k}[y, z]$. When we write b as $b_0 + b_1 + b_2$, where each $b_i \in \mathbf{k}[y, z]$ is homogeneous of degree i, we may replace x with $x - b_0 - b_1$ and obtain B), except when $b_2 \in \mathbf{k}[y]$: then we exchange x and z in order to be in case A).

Moreover, in cases (3)-(6) we see that the constructed affine coordinate change maps $\mathbf{k}[y]$ onto itself. This shows the last statement. □

In the next corollary, we list several properties of the closure in $\mathbb{P}^3$ of a hypersurface in $\mathbb{A}^3$ of degree 3 which is isomorphic to $\mathbb{A}^2$.

Corollary 2.3.6. *Let $f \in \boldsymbol{k}[x, y, z]$ be a polynomial of degree 3 such that $X = \operatorname{Spec}(\boldsymbol{k}[x, y, z]/(f)) \simeq \mathbb{A}^2$ and write $f = f_0 + f_1 + f_2 + f_3$ where $f_i \in \boldsymbol{k}[x, y, z]$ is homogeneous of degree i.*

(1) *If f_3 defines a conic Γ and a tangent line L in $\mathbb{P}^2$, then the singular locus of $\overline{X} \subset \mathbb{P}^3$ equals the point $(\Gamma \cap L)_{red}$.*
(2) *If f_3 defines one line (with multiplicity 3) in $\mathbb{P}^2$, then f_2 is either zero or defines some lines in $\mathbb{P}^2$ and all the lines given by f_3 and f_2 have a point in $\mathbb{P}^2$ in common. Moreover, the singular locus of $\overline{X} \subset \mathbb{P}^3$ is given by $w = f_2 = f_3 = 0$.*
(3) *If f_3 neither defines a conic and a tangent line in $\mathbb{P}^2$, nor one line in $\mathbb{P}^2$, then f_3 defines several lines in $\mathbb{P}^2$ and all these lines pass through the same point $q \in \mathbb{P}^2$. Moreover, q lies in the singular locus of $\overline{X} \subset \mathbb{P}^3$.*

Proof. Applying an affine automorphism, we are in one of the three cases A)-B)-C) of Proposition 2.3.5. The affine automorphism induces an automorphisms of the plane at infinity and thus an isomorphism between the curve in $\mathbb{P}^2$ given by $f_3 = 0$ and respectively $r_3(y, z) = 0$, $yr_2(y, z) = 0$ and $y(xy + z^2) = 0$ where $r_i \in \mathbf{k}[y, z]$ is homogeneous of degree i for $i = 1, 2$. We thus obtain two cases for $f_3 = 0$, namely a conic and a tangent line (1), or a set of lines through the same point: (2)-(3). The distinction between (2) and (3) corresponds to ask whether the lines are all the same or not. We study the three cases separately.

(1): Here we are in Case C) of Proposition 2.3.5. There exist thus $\psi \in \operatorname{Aff}(\mathbb{A}^3)$ and $a, b \in \mathbf{k}$ with $f = \psi^*(g)$ where $g = xy^2 + y(z^2 + az + b) + z$. Let $G \in \mathbf{k}[w, x, y, z]$ be the homogeneous polynomial of degree 3 such that $G(1, x, y, z) = g$. The gradient of G

$$\begin{aligned}&\left(\frac{\partial G}{\partial w}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right)\\ &= \big(y(az + 2bw) + 2zw, y^2, 2xy + z^2 + azw + bw^2, y(2z + aw) + w^2\big)\end{aligned}$$

is equal to zero if and only if $w = y = z = 0$ and thus $[0 : 1 : 0 : 0]$ is the only singularity of the hypersurface $G = 0$ in $\mathbb{P}^3$.

(2): Here we are in Case A) of Proposition 2.3.5. There exist thus $\psi \in \mathrm{Aff}(\mathbb{A}^3)$ and a homogeneous $r_2 \in \mathbf{k}[y,z]$ of degree 2 such that $f = \psi^*(h)$ where $h = x + r_2(y,z) + y^3$. Let $\varphi \in \mathrm{GL}_3(\mathbf{k})$ be the linear part of ψ. Then $f_3 = \varphi^*(y)^3$ and $f_2 = r_2(\varphi^*(y), \varphi^*(z)) + 3\delta\varphi^*(y)^2$ where $\psi^*(y) = \varphi^*(y) + \delta$. Thus $f_2, f_3 \in \mathbf{k}[s,t]$ for $s = \varphi^*(y)$, $t = \varphi^*(z)$ and the first claim follows. Let $H \in \mathbf{k}[w,x,y,z]$ such that $H(1,x,y,z) = h$. The gradient of H

$$\left(\frac{\partial H}{\partial w}, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}\right) = \left(2xw + r_2(y,z), w^2, w\frac{\partial r_2}{\partial y}(y,z) + 3y^2, w\frac{\partial r_2}{\partial z}(y,z)\right)$$

is equal to zero if and only if

$$\begin{cases} w = y = r_2(y,z) = 0 & \text{if } \mathrm{char}(\mathbf{k}) \neq 3 \\ \quad w = r_2(y,z) = 0 & \text{if } \mathrm{char}(\mathbf{k}) = 3 \end{cases}.$$

Since the intersection of $H = 0$ with the plane $w = 0$ at infinity only consists of the line $w = y = 0$, the singular locus of $H = 0$ is equal to $w = y = r_2(y,z) = 0$ (where $\mathbf{k}$ has any characteristic). Note that this singular locus is mapped via ψ^{-1} onto $w = s = r_2(s,t) + 3\delta s^2 = 0$ and thus the second claim follows.

(3): The first claim directly follows from Proposition 2.3.5 and we may assume (after an affine automorphism) that f is as in case A) or in case B). In both cases the common intersection point of the lines defined by f_3 is $[0:1:0:0]$ which is a singularity of $\overline{X} \subset \mathbb{P}^3$ by Lemma 2.1.1. □

Corollary 2.3.7. *Let $f \in \mathbf{k}[x,y,z]$ be an irreducible polynomial of degree 3 such that the hypersurface $X = V_{\mathbb{A}^3}(f)$ is isomorphic to $\mathbb{A}^2$ and such that the closure of X in $\mathbb{P}^3$ contains the line $w = y = 0$. After applying an affine automorphism of $\mathbb{A}^3$ that preserves the line $w = y = 0$, we obtain one of the following cases:*

a) *$f = x + r_2(y,z) + ys_2(y,z)$ for some homogeneous $r_2, s_2 \in \mathbf{k}[y,z]$ of degree 2, with $s_2 \neq 0$;*
b) *$f = xy + yr_2(y,z) + z$ for a homogeneous $r_2 \in \mathbf{k}[y,z] \setminus \mathbf{k}[y]$ of degree 2;*
c) *$f = xz + yzr_1(y,z) + y + \delta z$ for some homogeneous $r_1 \in \mathbf{k}[y,z] \setminus \{0\}$ of degree 1 and $\delta \in \mathbf{k}$;*
d) *$f = xy^2 + y(z^2 + az + b) + z$ for some $a, b \in \mathbf{k}$;*

Proof. There exists an affine automorphism that sends f onto a $g \in \mathbf{k}[x,y,z]$ wich is one of the polynomials from Proposition 2.3.5. We then look at the image ℓ of the line $w = y = 0$ in the plane at infinity $H_\infty = \{[w:x:y:z] \in \mathbb{P}^3 \mid w = 0\}$ and apply an affine automorphism to send it back to $w = y = 0$.

In case A), $g = x + r_2(y,z) + r_3(y,z)$ for some homogeneous $r_i \in \mathbf{k}[y,z]$ of degree i. As $\deg(g) = 3$, we get $r_3 \neq 0$, and the line ℓ is given by $p_1(y,z) = 0$ for some homogeneous polynomial $p_1 \in \mathbf{k}[y,z]$ of degree 1 that divides r_3. We apply an element of $\mathrm{GL}_2(\mathbf{k})$ acting on y,z and obtain a).

In case B), $g = xy + yr_2(y,z) + z$ for a homogeneous polynomial $r_2 \in \mathbf{k}[y,z] \setminus \mathbf{k}[y]$ of degree 2. The line ℓ is given by $p_1(y,z) = 0$ for some homogeneous polynomial $p_1 \in \mathbf{k}[y,z]$ of degree 1 that divides $yr_2(y,z)$. If ℓ is the line $y = 0$ we get b). Otherwise, the line is $\alpha y + \beta z$ with $\beta \neq 0$ and $g = xy + y(\alpha y + \beta z)s_1(y,z) + z$ for some homogeneous degree 1 polynomial $s_1 \in \mathbf{k}[y,z] \setminus \{0\}$. We apply a linear coordinate change and send $\alpha y + \beta z$ and y respectively to y and z; this sends z onto $\gamma y + \delta z$ with $\gamma \in \mathbf{k}^*$, $\delta \in \mathbf{k}$, and sends g onto $xz + yzs_1'(y,z) + \gamma y + \delta z$ for some homogeneous degree 1 polynomial $s_1' \in \mathbf{k}[y,z] \setminus \{0\}$. We replace y with $\gamma^{-1}y$ and get c).

In case C), $g = xy^2 + y(z^2 + az + b) + z$ for some $a, b, \in \mathbf{k}$ and thus the line ℓ is $y = 0$. Hence we obtain d). □

Corollary 2.3.8 (Hypersurfaces isomorphic to $\mathbb{A}^2$ of degree 2)**.** *Let $f \in \boldsymbol{k}[x, y, z]$ be an irreducible polynomial of degree 2 and assume that $X = \mathrm{Spec}(\boldsymbol{k}[x, y, z]/(f))$ is isomorphic to $\mathbb{A}^2$. Then, after applying an affine automorphism, one of the following cases occur:*

(1) $f = x + y^2$;
(2) $f = x + yz$.

Proof. Since f is of degree 2, it follows from Proposition 2.3.5 that f is equal to $x + r_2(y, z)$ for a non-zero homogeneous polynomial of degree 2 up to an affine automorphism. Depending whether $r_2(y, z) = 0$ has one ore two zeros in $\mathbb{P}^1$ we are in case (1) and case (2), respectively. □

3. Families of cubic hypersurfaces of $\mathbb{A}^3$, all isomorphic to $\mathbb{A}^2$

In this section, we study families of cubic hypersurfaces of $\mathbb{A}^3$ that are isomorphic to $\mathbb{A}^2$. In order to to this we begin with linear systems on $\mathbb{P}^2$.

3.1. **Linear systems on $\mathbb{P}^2$.** To study families of hypersurfaces of $\mathbb{A}^3$, it is natural too look at the behaviour at infinity. In the following, for $d \geq 0$, we denote by $\mathbf{k}[x, y, z]_d$ the vector space of homogeneous polynomials of degree d in $\mathbf{k}[x, y, z]$ and we consider it as an affine space (of dimension $\binom{d+2}{2}$). In particular, $\mathbf{k}[x, y, z]_d$ carries the Zariski topology. Moreover, for any vector space V, we let $\mathbb{P}(V) = \mathrm{Proj}_{\mathbf{k}}(\mathrm{Sym}V^*)$ be the projectivisation of the symmetric algebra $\mathrm{Sym}V^*$ of the dual vector space V^*.

Lemma 3.1.1. *Let $f, g \in \boldsymbol{k}[x, y, z]$ be two homogeneous polynomials of degree $d \geq 1$ without common factor. The following are equivalent:*

(1) *The polynomial $\lambda f + \mu g$ is divisible by a linear factor, for all $\lambda, \mu \in \boldsymbol{k}$.*
(2) *The polynomial $\lambda f + g$ is divisible by a linear factor, for infinitely many $\lambda \in \boldsymbol{k}$.*
(3) *There are two linear polynomials $s, t \in \boldsymbol{k}[x, y, z]_1$ such that $f, g \in \boldsymbol{k}[s, t]$.*

Proof. Observe that the subset $R_d \subset \mathbf{k}[x, y, z]_d$ of elements that are divisible by a linear factor is closed. Indeed, $\mathbb{P}(R_d)$ is the image of the morphism $\mathbb{P}(\mathbf{k}[x, y, z]_1) \times \mathbb{P}(\mathbf{k}[x, y, z]_{d-1})$, $(p, q) \mapsto pq$. Hence, the set

$$\left\{ [\lambda : \mu] \in \mathbb{P}^1 \mid \lambda f + \mu g \text{ is divisible by a linear factor} \right\}$$

is a closed subset of $\mathbb{P}^1$. Thus it is infinite if and only if it is the whole $\mathbb{P}^1$. This gives the equivalence (1) $\Leftrightarrow$ (2).

Let us prove (3) $\Rightarrow$ (1). As f and g have no common factor, s, t are linearly independent. We apply a linear coordinate change and may assume that $s = x$ and $t = y$. Now, it is enough to remark that every homogeneous polynomial of $\mathbf{k}[x, y]$ is a product of linear factors.

It remains to prove (1) $\Rightarrow$ (3). We prove this by induction on $d = \deg(f) = \deg(g)$. The case where $d = 1$ holds by choosing $s = f$ and $t = g$. We consider the dominant rational map $\eta \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $[x : y : z] \mapsto [f(x, y, z) : g(x, y, z)]$. If $\mathbf{k}(\frac{f}{g})$ is separably closed in $\mathbf{k}(\frac{x}{z}, \frac{y}{z})$, then a general fibre of η is irreducible [FOV99, Theorem 3.3.17, page 105] (but not necessarily reduced). After replacing f, g with another basis of $\mathbf{k}f \oplus \mathbf{k}g$, we may thus assume that the zero locus of f and g are irreducible curves in $\mathbb{P}^2$. The assumption (1) implies that two linear factors $s, t \in \mathbf{k}[x, y, z]$ exist such that $f = s^d$ and $g = t^d$. This gives (3). If $\mathbf{k}(\frac{f}{g})$ is not separably closed in $\mathbf{k}(\frac{x}{z}, \frac{y}{z})$, then there is a rational map $\frac{a}{b}$ (where $a, b \in \mathbf{k}[x, y, z]$ are homogeneous of the same degree without common factor) such that $\mathbf{k}(\frac{f}{g}) \subsetneq \mathbf{k}(\frac{a}{b})$ is a proper algebraic field extension, by the Primitive Element Theorem. Hence, we may decompose η as $\eta = \nu \circ \eta'$, where $\nu \colon \mathbb{P}^1 \to \mathbb{P}^1$ is a finite morphism which is not an isomorphism and $\eta' \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is given by $[x : y : z] \mapsto [a(x, y, z) : b(x, y, z)]$. Note that $\deg(a) = \deg(b) < d$, since ν is not an isomorphism. As infinitely many fibres of η contain lines, the same holds for η', so (2) holds for a and b. By induction, we find two homogeneous linear polynomials $s, t \in \mathbf{k}[x, y, z]$ such that $a, b \in \mathbf{k}[s, t]$ and hence $f, g \in \mathbf{k}[s, t]$ too. □

Lemma 3.1.2. *Let $d \geq 2$ and let $V \subseteq \mathbf{k}[x, y, z]_d$ be a vector subspace such that the gcd of all elements of V is 1, and such that each element of V is divisible by a linear factor. Then, one of the following holds:*

(1) *There are two linear polynomials $s, t \in \mathbf{k}[x, y, z]_1$ such that $V \subseteq \mathbf{k}[s, t]$.*
(2) *The degree d is a power of $\mathrm{char}(\mathbf{k}) = p > 0$, and $V = \mathbf{k}x^d \oplus \mathbf{k}y^d \oplus \mathbf{k}z^d$.*

Proof. Since the gcd of all elements in V is 1, we get $\dim V \geq 2$. Suppose first that every element of V is a d-th power in $\mathbf{k}[x, y, z]$. Then up to a linear coordinate change we may assume that $x^d, y^d \in V$. Since $x^d - y^d$ is a d-th power and is divisible by $x - y$, we get $x^d - y^d = (x - y)^d$. As $d \geq 2$, this implies that $\mathrm{char}(\mathbf{k}) = p > 0$ and that d is a power of p. We get (1) if V is generated by x^d and y^d and (2) otherwise.

Suppose now that some element $f \in V$ is not a d-th power. By Lemma 3.1.1, we may apply a linear coordinate change and may assume that $f \in \mathbf{k}[x, y]$. For each element $g \in V$ that has no common factor with f, there exist two linear polynomials $s, t \in \mathbf{k}[x, y, z]_1$ such that $f, g \in \mathbf{k}[s, t]$ by Lemma 3.1.1. As $f \in \mathbf{k}[x, y]$ is not a power of an element of $\mathbf{k}[x, y, z]_1$ and as $f \in \mathbf{k}[x, y]$ is homogeneous, there are linearly independent $p_1, q_1 \in \mathbf{k}[x, y]_1$ such that f is divisible by the product $p_1 q_1$. Since $\mathbf{k}[s, t]$ is factorially closed in $\mathbf{k}[x, y, z]$ and as $f \in \mathbf{k}[s, t]$, we get $p_1, q_1 \in \mathbf{k}[s, t]$ and thus $x, y \in \mathbf{k}[s, t]$, i.e. $\mathbf{k}[x, y] = \mathbf{k}[s, t]$. In particular, $g \in \mathbf{k}[x, y]$. Since the set of elements $g \in V$ that have no common factor with f is Zariski open in V, this set spans V as a $\mathbf{k}$ vector space and so $V \subset \mathbf{k}[x, y]$. □

Lemma 3.1.3. *Assume that $\mathrm{char}(\mathbf{k}) = 2$ and let $g_1, \dots, g_n \in \mathbf{k}[x, y]_2$, such that $\mathbf{k}g_1 + \dots + \mathbf{k}g_n = \mathbf{k}x^2 \oplus \mathbf{k}y^2$. If $s \geq 0$ and $h_1, \dots, h_n \in \mathbf{k}[x, y]_s$ are such that $\sum_i \lambda_i g_i$ and $\sum_i \lambda_i h_i$ have a common non-zero linear factor for all $(\lambda_1, \dots, \lambda_n) \in \mathbf{k}^n$, then either $h_i = 0$ for all i or $s \geq 2$ and there exists $h \in \mathbf{k}[x, y]_{s-2} \setminus \{0\}$ with $h_i = h g_i$ for all i.*

Proof. Note that $n \geq 2$. After a linear coordinate change in x, y and after replacing $h_1, \dots, h_n$ and $g_1, \dots, g_n$ with certain linear combinations we may assume that $g_1 = x^2$ and $g_i = y^2$ for all $i = 2, \dots, n$. For each $i \in \{2, \dots, n\}$ and each $\alpha, \beta \in \mathbf{k}$, $(\alpha x + \beta y)^2 = \alpha^2 g_1 + \beta^2 g_i$ and $\alpha^2 h_1 + \beta^2 h_i$ have a common non-zero linear factor, so $\alpha x + \beta y$ divides $\alpha^2 h_1 + \beta^2 h_i$, which means that $\alpha^2 h_1(\beta, \alpha) + \beta^2 h_i(\beta, \alpha) = 0$. As this last equation is true for all $\alpha, \beta \in \mathbf{k}$, the polynomial $y^2 h_1 + x^2 h_i$ is zero. We get a polynomial $\tilde{h}_i$ such that $h_1 = \tilde{h}_i x^2$ and $h_i = \tilde{h}_i y^2$. The equality $h_1 = \tilde{h}_i x^2$ yields that $\tilde{h}_i$ is independent of i, so writing $h = h_i$ gives the result. □

Lemma 3.1.4. *Assume that* $\mathrm{char}(\boldsymbol{k}) > 0$ *and denote by* $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ *the Frobenius endomorphism.*

(1) *For each* $A \in \mathrm{PGL}_3$*, there exists* $v \in \mathbb{P}^2$ *such that* $A\phi(v) = v$*.*
(2) *For each* $B \in \mathrm{PGL}_3$*, there exists* $v \in \mathbb{P}^2$ *such that* $\phi(Bv) = v$*.*

Proof. We denote by $\theta\colon\ \mathrm{PGL}_3 \to \mathrm{PGL}_3$ the endomorphism that sends a matrix C to the matrix obtained from C by taking the p-th power of each entry.

We will only prove (1), as (2) follows from it by choosing $A = \theta(B)$. We then have to show that

$$\Gamma = \left\{ A \in \mathrm{PGL}_3 \mid A\phi(v) = v \text{ for some } v \in \mathbb{P}^2 \right\}$$

is equal to PGL_3. We consider

$$M = \left\{ (A, v) \in \mathrm{PGL}_3 \times \mathbb{P}^2 \mid A\phi(v) = v \right\}$$

and obtain $\Gamma = \pi_1(M)$, where $\pi_1\colon M \to \mathrm{PGL}_3$ is the first projection. As π_1 is proper, we get that Γ is closed in PGL_3 and thus we only have to show that $\dim \Gamma = 8$. We observe that the identity matrix $I \in \mathrm{PGL}_3$ belongs to Γ and that $\pi_1^{-1}(I) = \mathbb{P}^2(\mathbb{F}_p)$ is finite. By Chevalley's Upper Semi-continuity Theorem for the dimension of fibres [Gro66, Corollaire 13.1.5], the set $\left\{ A \in \Gamma \mid \dim \pi_1^{-1}(\{A\}) \geq 1 \right\}$ is closed in Γ. It then suffices to show that M is irreducible and of dimension 8.

To show this, we will prove that the second projection $\pi_2\colon M \to \mathbb{P}^2$ is a locally trivial P-bundle, where P is the parabolic subgroup of PGL_3 that fixes $[1:0:0]$. Note that $\pi_2\colon M \to \mathbb{P}^2$ is PGL_3-equivariant with respect to the natural action on $\mathbb{P}^2$ and the PGL_3-action on M given by $B \cdot (A, v) \coloneqq (BA\theta(B)^{-1}, Bv)$. We then only need to show that π_2 is a trivial P-bundle over the open subset $U = \left\{ [p_0 : p_1 : p_2] \in \mathbb{P}^2 \mid p_0 \neq 0 \right\}$. We consider the morphism $h\colon U \to \mathrm{PGL}_3$ given by

$$[p_0 : p_1 : p_2] \mapsto \begin{pmatrix} p_0 & 0 & 0 \\ p_1 & p_0 & 0 \\ p_2 & 0 & p_0 \end{pmatrix},$$

which satisfies $h(p)([1:0:0]) = p$ for each $p \in U$. We get a V-isomorphism

$$\begin{array}{ccc} P \times V & \xrightarrow{\simeq} & \pi_2^{-1}(V) \\ (A, p) & \longmapsto & (h(p)A\theta(h(p)^{-1}), p), \end{array}$$

whose inverse sends (A, p) onto $(h(p)^{-1}A\theta(h(p)), p)$. ☐

3.2. **Affine linear systems of affine spaces.** It turns out that the following definition is very useful for us:

Definition 3.2.1. Let $f_1, \ldots, f_n \in \mathbf{k}[x_1, \ldots, x_d]$. We say that a morphism

$$\begin{array}{ccc} \mathbb{A}^d & \longrightarrow & \mathbb{A}^n \\ (x_1, \ldots, x_d) & \longmapsto & (f_1(x_1, \ldots, x_d), \ldots, f_n(x_1, \ldots, x_d)) \end{array}$$

is an *affine linear system of affine spaces* if for each $\lambda_0 \in \mathbf{k}$ and each $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n \setminus \{0\}$ the polynomial $\lambda_0 + \lambda_1 f_1 + \ldots + \lambda_n f_n$ is not constant and the corresponding hypersurface in $\mathbb{A}^d$ is isomorphic to $\mathbb{A}^{d-1}$. This is equivalent to say that the preimage of every affine linear hypersurface in $\mathbb{A}^n$ under the morphism $(f_1, \ldots, f_n)\colon \mathbb{A}^d \to \mathbb{A}^n$ is isomorphic to $\mathbb{A}^{d-1}$.

We call two affine linear systems of affine spaces $(f_1, \ldots, f_n), (g_1, \ldots, g_n)\colon \mathbb{A}^d \to \mathbb{A}^n$ *equivalent* if there exist affine automorphisms $\alpha \in \mathrm{Aff}(\mathbb{A}^d)$, $\beta \in \mathrm{Aff}(\mathbb{A}^n)$ such that

$$(g_1, \ldots, g_n) = \beta \circ (f_1, \ldots, f_n) \circ \alpha\,.$$

If the preimage of every linear hypersurface in $\mathbb{A}^n$ under the morphism $f = (f_1, \ldots, f_n)\colon \mathbb{A}^d \to \mathbb{A}^n$ is isomorphic to $\mathbb{A}^{d-1}$, then we say that f is a *linear system of affine spaces*. Hence, every affine linear system of affine spaces is a linear system of affine spaces.

Remark 3.2.2. Every automorphism $f\colon \mathbb{A}^n \to \mathbb{A}^n$ is an affine linear system of affine spaces and two automorphisms $f, g\colon \mathbb{A}^n \to \mathbb{A}^n$ are equivalent, if they are the same up to affine automorphisms at the source and target.

Remark 3.2.3. Note that the notions "affine linear hypersurface" and "affine linear system of affine spaces" are not intrinsic notions of the affine space and of morphisms between them. They depend on the choice of coordinate systems of the affine spaces (up to affine automorphisms). Therefore, as mentioned in the introduction, we always make a particular choice of the coordinates of the affine spaces involved.

Example 3.2.4. Let $f_1, \ldots, f_n \in \mathbf{k}[x_1 \ldots, x_d]$. If $\deg(f_i) \leq 1$ for each i, then $f \coloneqq (f_1, \ldots, f_n)\colon \mathbb{A}^d \to \mathbb{A}^n$ is called an *affine linear morphism*. In case f is surjective, it is an affine linear system of affine spaces.

Next, we list some basic properties of affine linear systems of affine spaces.

Lemma 3.2.5. *Let $f_1, \ldots, f_n \in \boldsymbol{k}[x_1, \ldots, x_d]$ be polynomials and let $f = (f_1, \ldots, f_n)$ be the corresponding morphism $\mathbb{A}^d \to \mathbb{A}^n$.*

1. *If $f = (f_1, \ldots, f_n)$ is an affine linear system of affine spaces and if $f_{i,1}$ denotes the homogeneous part of f_i of degree 1 for $i = 1, \ldots, n$, then $f_{1,1}, \ldots, f_{n,1}$ are linearly independent over $\boldsymbol{k}$ in $\boldsymbol{k}[x_1, \ldots, x_d]_1$. In particular, $n \leq d$.*
2. *Assume that f is an affine linear system of affine spaces. Then for all automorphisms $\varphi \in \mathrm{Aut}(\mathbb{A}^d)$ and all $\alpha \in \mathrm{Aff}(\mathbb{A}^n)$, the composition $\alpha \circ f \circ \varphi\colon \mathbb{A}^d \to \mathbb{A}^n$ is an affine linear system of affine spaces.*
3. *Assume that $\deg(f) = \max_{1 \leq i \leq n} \deg(f_i) = 1$. Then f is an affine linear system of affine spaces if and only if $f\colon \mathbb{A}^d \to \mathbb{A}^n$ is surjective. In particular, if $d \geq n$, then up to equivalence there is exactly one affine linear system of affine spaces $\mathbb{A}^d \to \mathbb{A}^n$ of degree 1.*
4. *If $f_1, \ldots, f_n \in \boldsymbol{k}[x, y, z]$ are of degree ≤ 3, then $(f_1, \ldots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ defines a linear system of affine spaces if and only if it defines an* affine *linear system of affine spaces.*
5. *Let $\pi\colon \mathbb{A}^n \to \mathbb{A}^l$ be a surjective affine linear morphism. If f is an affine linear system of affine spaces, then the composition $\pi \circ f\colon \mathbb{A}^d \to \mathbb{A}^l$ as well.*
6. *Let $\rho\colon \mathbb{A}^r \to \mathbb{A}^d$ be a surjective affine linear morphism. If f is an affine linear system of affine spaces, then $f \circ \rho$ as well. If $d \leq 3$ and if $f \circ \rho$ is an affine linear system of affine spaces, then f as well.*
7. *Assume that $d = n$. If $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^n$ is an affine linear system of affine spaces, then the determinant of the Jacobian of f lies in $\boldsymbol{k}^*$.*

Proof. (1): If there exists $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n \setminus \{0\}$ such that $\sum_{i=1}^n \lambda_i f_{i,1} = 0$, we write $\lambda_0 = \sum_{i=1}^n \lambda_i f_i(0) \in \mathbf{k}$ and obtain that the polynomial $\sum_{i=1}^n \lambda_i f_i - \lambda_0$ is either 0 or defines a singular hypersurface of $\mathbb{A}^n$. In both cases $\sum_{i=1}^n \lambda_i f_i - \lambda_0$ does not define an $\mathbb{A}^{d-1}$ in $\mathbb{A}^d$.

(2): This follows directly from the definition.

(3): If f is surjective, then the statement is clear. If f is not surjective, then the image of f is contained in an affine linear hypersurface in $\mathbb{A}^n$ and thus f is not an affine linear system of affine spaces.

(4): This follows from Corollary 2.2.3.

(5): This follows, since the preimage of an affine linear hypersurface under π is again an affine linear hypersurface.

(6): Let $H \subset \mathbb{A}^n$ be an affine linear hypersurface. Then the preimage $(f\circ\rho)^{-1}(H)$ is isomorphic to $f^{-1}(H) \times \mathbb{A}^{r-d}$. Hence, the first claim follows. On the other hand, as $f^{-1}(H)$ has dimension $d-1$ and since Zariski's Cancellation Problem has an affirmative answer for the affine line (see [AHE72, Corollary 2.8]) and the affine plane (see [Fuj79, MS80] and [Rus81, Theorem 4]), the second claim follows.

(7): This follows from Lemma 3.2.6 below. □

The next Lemma is essentially due to Derksen, see [vdES97, Lemma 2.3]:

Lemma 3.2.6. *Let $f_1, \ldots, f_n \in \boldsymbol{k}[x_1, \ldots, x_n]$ and let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^n$. Then the determinant of the Jacobian of f lies in $\boldsymbol{k}^*$ if and only if the preimage of each affine linear hypersurface under f is a smooth hypersurface in $\mathbb{A}^n$.*

Proof. The determinant of the Jacobian of f does not lie in $\mathbf{k}^*$ if and only if there exist $\lambda_1, \ldots, \lambda_n \in \mathbf{k}$, not all equal to zero, and there is a point $a \in \mathbb{A}^n$ such that

$$\sum_{i=0}^{n} \lambda_i \frac{\partial f_i}{\partial x_j}(a) = 0 \quad \text{for each } j = 1, \ldots, n\,.$$

However, this last condition is equivalent to the existence of some $\lambda_0 \in \mathbf{k}$ and some $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n \setminus \{0\}$ such that either $\lambda_0 + \lambda_1 f_1 + \ldots + \lambda_n f_n$ is zero or defines a singular hypersurface in $\mathbb{A}^n$. □

In the next Proposition, we study affine linear systems of affine spaces $\mathbb{A}^2 \to \mathbb{A}^2$ of degree ≤ 3 up to affine automorphisms at the source and target.

Proposition 3.2.7. *Let $f_1, f_2 \in \boldsymbol{k}[x, y]$ of degree ≤ 3 such that $f = (f_1, f_2)\colon \mathbb{A}^2 \to \mathbb{A}^2$ is a linear system of affine spaces. Then, up to affine coordinate changes at the source and target, we get $f = (x + q(y), y)$ where $q \in \boldsymbol{k}[y]$.*

Proof. By Corollary 2.3.2, we may assume after an affine coordinate change in (x, y) that $f_1 = x + q(y)$ for some $q \in \mathbf{k}[y]$ of degree ≤ 3. Set $\psi = (x - q, y) \in \mathrm{Aut}(\mathbb{A}^2)$. The determinant of the Jacobian of $(x, f_2(x - q, y)) = f \circ \psi$ is a non-zero constant (due to Lemma 3.2.5(7)) and it is equal to the y-derivative of $f_2(x - q, y)$. Hence, $f_2(x - q, y) = ay + p(x)$ for some $a \in \mathbf{k}^*$ and $p \in \mathbf{k}[x]$, i.e. $f_2 = ay + p(x + q)$. After scaling f_2 we may assume $a = 1$. If $\deg(q) \leq 1$, then $\psi \in \mathrm{Aff}(\mathbb{A}^2)$ and since $f \circ \psi = (x, y + p(x))$, the result follows after conjugation with $(x, y) \mapsto (y, x)$. If $\deg(q) \geq 2$, then $\deg(p) \leq 1$, since otherwise $\deg(f_2) = \deg(p)\deg(q) \geq 4$. Thus $\varphi = (x, y - p(x)) \in \mathrm{Aff}(\mathbb{A}^2)$ and since $\varphi \circ f = (x + q(y), y)$, the result holds. □

3.3. **Linear systems of affine spaces of degree 3 with a conic in the base locus.** In this subsection we study linear systems $f\colon \mathbb{A}^3 \to \mathbb{A}^n$ of degree 3 such that the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^n$ which extends f contains a conic in the base locus. In fact, this study will be important in order to prove that every automorphism of degree 3 of $\mathbb{A}^3$ can be brought into standard form (Proposition 3.6.1 below). As explained in the introduction, we say that an affine linear system of affine spaces

$f\colon \mathbb{A}^3 \to \mathbb{A}^n$ is in *standard form* if $f = (xp_1+q_1, \dots, xp_n+q_n)$ for some polynomials $p_i, q_i \in \mathbf{k}[y,z]$.

Proposition 3.3.1. *Let $f_1, \dots, f_n \in \boldsymbol{k}[x,y,z]$ be polynomials and assume that $f = (f_1, \dots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ is a linear system of affine spaces of degree 3 such that there is a homogeneous irreducible polynomial of degree 2 that divides the homogeneous parts of degree 3 of $f_1, \dots, f_n$. Then f is equivalent to a linear system of affine spaces in standard form.*

Proof. For $i = 1, \dots, n$, we write $f_i = \sum_{j=0}^{3} f_{i,j}$ where $f_{i,j} \in \mathbf{k}[x,y,z]_j$. Applying an automorphism of $\mathbb{A}^n$ we may assume that $f_{i,3} \neq 0$ for each i. By assumption, there is an irreducible conic $\Gamma \subset \mathbb{P}^2$ that is contained in the zero locus of $f_{i,3}$, for each $i \in \{1, \dots, n\}$. Moreover, for each i, f_i defines an $\mathbb{A}^2$ inside $\mathbb{A}^3$, so the polynomial $f_{i,3}$ defines in $\mathbb{P}^2$ the conic Γ and a tangent line to that conic in a point q_i and the closure in $\mathbb{P}^2$ of the hypersurface given by f_i is singular at q_i (see Corollary 2.3.6). If all the points $q_1, \dots, q_n$ are the same, we can assume that these are $[1:0:0]$, and obtain the result by Lemma 2.1.1. We thus assume that two of the q_i's are distinct and derive a contradiction. We may assume that $q_1 \neq q_2$ by applying a permutation of $\mathbb{A}^n$. Applying automorphisms of $\mathbb{A}^3$, we may moreover assume that $f_1 = xy^2 + y(z^2 + az + b) + z$ for some $a, b \in \mathbf{k}$ (see Proposition 2.3.5). Hence, $q_1 = [1:0:0]$, Γ is the conic $xy + z^2 = 0$ and $q_2 \in \Gamma \setminus \{q_1\}$, so $q_2 = [-\xi^2 : 1 : \xi]$ for some $\xi \in \mathbf{k}$. Replacing f_2 with $f_2\lambda$ for some $\lambda \in \mathbf{k}^*$, we obtain

$$f_{1,3} = y(xy+z^2)\,, \quad f_{2,3} = (x - \xi^2 y + 2\xi z)(xy + z^2)\,.$$

For each $\mu \in \mathbf{k}$, the polynomial $f_2 + \mu^2 f_1$ defines a hypersurface $X_\mu \subset \mathbb{A}^3$ and its homogeneous part of degree 3 is $(x - \xi^2 y + \mu^2 y + 2\xi z)(xy+z^2)$. By Corollary 2.3.6(1), the line ℓ_μ given by $x - \xi^2 y + \mu^2 y + 2\xi z$ is tangent to Γ. Choosing $\mu = \xi$ when $\xi \neq 0$ and choosing $\mu = 1$ when $\xi = 0$ gives $\mathrm{char}(\mathbf{k}) = 2$. We may then replace f_2 with $f_2 + \xi^2 f_1$ and assume that $\xi = 0$. The point of tangency of Γ and ℓ_μ is then $p_\mu = [\mu^2 : 1 : \mu]$.

Suppose first that $f_{1,2} = f_{2,2} = 0$. We obtain

$$f_1 = y(xy+z^2) + by + z\,, \quad f_2 = x(xy+z^2) + \alpha x + \beta y + \gamma^2 z + \delta$$

for some $\alpha, \beta, \gamma, \delta \in \mathbf{k}$. The polynomial $f_2 + \gamma^2 f_1 = (x + \gamma^2 y)(xy + z^2) + \alpha x + (\beta + b\gamma^2)y + \delta$ defines an $\mathbb{A}^2$, so the same holds when we replace x and z with $x + \gamma^2 y, z + \gamma y$ respectively, hence for the polynomial $x(xy+z^2) + \alpha x + (\beta + (b+\alpha)\gamma^2)y + \delta$, impossible by Proposition 2.2.2 (applied to the polynomial obtained by exchanging x and y).

We now assume that $f_{1,2}$ and $f_{2,2}$ are not both zero. There is an affine automorphism of $\mathbb{A}^3$ that sends $f_2 + \mu^2 f_1$ onto $h = xy^2 + y(z^2 + cz + d) + z$ for some $c, d \in \mathbf{k}$ (Proposition 2.3.5). Thus, $f_2 + \mu^2 f_1$ is obtained by applying an element of $\mathrm{GL}_3(\mathbf{k})$ to $h' = h(x + \varepsilon_1, y + \varepsilon_2, z + \varepsilon_3)$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbf{k}$. As $h' = h'_0 + h'_1 + h'_2 + h'_3$ where $h'_i \in \mathbf{k}[x,y,z]_i$ and $h'_3 = y(xy+z^2)$, $h'_2 = \varepsilon_1 y^2 + cyz + \varepsilon_2 z^2$ are both singular at $[1:0:0]$, the homogeneous part of degree 2 of $f_2 + \mu^2 f_1$ is singular at p_μ.

As $f_{1,2}$ and $f_{2,2}$ are not both zero and the set $\{p_\mu \mid \mu \in \mathbf{k}\}$ is not contained in a line, there is no linear factor that divides both $f_{1,2}$ and $f_{2,2}$. However, as $f_{2,2} + \mu^2 f_{1,2}$ is divisible by a linear factor for each $\mu \in \mathbf{k}$, there exist $s, t \in \mathbf{k}[x,y,z]_1$ such that $f_{1,2}, f_{2,2} \in \mathbf{k}[s,t]$(Lemma 3.1.1). Remembering that $f_{1,2} = ayz$, we prove first that $a = 0$. Indeed, otherwise $\mathbf{k}[s,t] = \mathbf{k}[y,z]$ and $f_{2,2} + \mu^2 f_{1,2} \in \mathbf{k}[y,z]$ is singular at p_μ so is a multiple of $(\mu y + z)^2 = \mu^2 y^2 + z^2$, impossible as it contains yz for infinitely many μ. Now that $a = 0$ is proven, the polynomial $f_{2,2} + \mu^2 f_{1,2} = f_{2,2}$ is singular at each point p_μ, so $f_{2,2} = 0$, in contradiction with the above assumption. □

3.4. **Affine linear systems in characteristic 2 and 3.** We call a morphism $f\colon Y \to X$ an $\mathbb{A}^1$-*fibration* if each closed fiber is (schematically) isomorphic to $\mathbb{A}^1$. We moreover say that the $\mathbb{A}^1$-fibration f is locally trivial in the Zariski (respectively étale) topology if for each $x \in X$ there is an open neighbourhood $U \subset X$ of x (respectively an étale morphism $U \to U'$ onto an open neighbourhood U' of x in X) such that the fiber product $U \times_X Y \to U$ is isomorphic to $U \times \mathbb{A}^1$ over U.

Recall from the introduction, that an $\mathbb{A}^1$-fibration $f\colon Y \to X$ is called *trivial* if there exists an isomorphism $\varphi\colon X \times \mathbb{A}^1 \to Y$ such that the composition $f \circ \varphi\colon X \times \mathbb{A}^1 \to X$ is the projection onto the first factor.

An $\mathbb{A}^1$-bundle is then simply an $\mathbb{A}^1$-fibration that is locally trivial in the Zariski topology.

We now give two examples of linear systems of affine spaces of degree 3 that are not equivalent to linear systems in standard form.

Lemma 3.4.1. *Assume that* $\mathrm{char}(\boldsymbol{k}) = 2$ *and let*

$$f = x + z^2 + y^3 \quad \textit{and} \quad g = y + x^2$$

Then, $\pi = (f, g)\colon \mathbb{A}^3 \to \mathbb{A}^2$ *is an affine linear system of affine spaces, which is not equivalent to an affine linear system in standard form. Moreover,* π *is an* $\mathbb{A}^1$-*fibration that is not locally trivial in the étale topology.*

Proof. If $\lambda \neq 0$, then $\lambda^2 f + g = \lambda^2 x + y + (x + \lambda z)^2 + \lambda^2 y^3$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$, since the linear polynomials $\lambda^2 x + y$, $x + \lambda z$ and y are linearly independent in $\mathbf{k}[x, y, z]_1$. On the other hand, both f and g define an $\mathbb{A}^2$ in $\mathbb{A}^3$ as well. This implies that $\pi = (f, g)\colon \mathbb{A}^3 \to \mathbb{A}^2$ is a linear system of affine spaces and thus an affine linear system of affine spaces by Lemma 3.2.5(4).

Let $X, Y \subset \mathbb{P}^3$ be the closures of the hypersurfaces in $\mathbb{A}^3$ which are given by f and $f + g$, respectively. By Corollary 2.3.6(2) the singular locus of X is equal to $[0 : 1 : 0 : 0]$ and the singular locus of Y is equal to $[0 : 1 : 0 : 1]$. In particular, X, Y have no common singularity and thus, π is not equivalent to an affine linear system in standard form by Lemma 2.1.1.

It remains to see that all closed fibres of π are isomorphic to $\mathbb{A}^1$ but that π is not locally trivial in the étale topology. To simplify the situation, we apply some non-affine automorphisms at the source and the target. We first apply $(x, y + x^2, z)$ (at the source) to get $(x + z^2 + (y + x^2)^3, y)$. Applying $(x + y^3, y)$ at the target and $(x, y, z + x^3 + xy)$ at the source gives

$$\phi = (x + x^4 y + z^2, y)\colon \mathbb{A}^3 \to \mathbb{A}^2 .$$

The fibre over a point (x_0, y_0) with $y_0 = 0$ is isomorphic to $\mathbb{A}^1$, via its projection onto z. The fibre over a point $(x_0, y_0) \in \mathbb{A}^2$ with $y_0 \neq 0$ is isomorphic to $\mathbb{A}^1$, as one can apply $z \mapsto z + \sqrt{y_0} x^2$ to reduce to the previous case.

It remains to see that ϕ is not locally trivial in the étale topology. The fibre F of ϕ over the (non-closed) generic point of $\{x = 0\}$ is the scheme given by $x + x^4 y + z^2$ inside $\mathbb{A}^2_{\mathbf{k}(y)} = \mathrm{Spec}(\mathbf{k}(y)[x, z])$. By [Rus70, Corollary 2.3.1 and Lemma 1.2], F is non-isomorphic to the affine line $\mathbb{A}^1_{\mathbf{k}(y)}$ over $\mathbf{k}(y)$, however after extending the scalars to $\mathbf{k}(\sqrt{y})$ we get

$$F \times_{\mathrm{Spec}(\mathbf{k}(y))} \mathrm{Spec}(\mathbf{k}(\sqrt{2})) \simeq \mathbb{A}^1_{\mathbf{k}(\sqrt{y})} .$$

By [Rus76, Lemma 1.1] there doesn't exist any separable field extension $k(y) \subseteq K$ such that $F \times_{\mathrm{Spec}(\mathbf{k}(y))} \mathrm{Spec}(K) \simeq \mathbb{A}^1_K$. Hence, ϕ and thus π are not locally trivial in the étale topology.

□

Lemma 3.4.2. *Assume that* $\mathrm{char}(\boldsymbol{k}) = 3$ *and let*

$$f = x + z^2 + y^3 \quad \text{and} \quad g = z + x^3$$

Then, $\pi = (f, g)\colon \mathbb{A}^3 \to \mathbb{A}^2$ *is an affine linear system of affine spaces, which is not equivalent to an affine linear system in standard form. Moreover,* π *is an* $\mathbb{A}^1$*-fibration that is not locally trivial in the étale topology.*

Proof. For each $\lambda \in \mathbf{k}$, the polynomial $f + \lambda^3 g = \lambda^3 z + x + z^2 + (y + \lambda x)^3$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$: replacing y with $y - \lambda x$ and x with $x - \lambda^3 z$ gives $x + z^2 + y^3$. On the other hand, g also defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. This implies that $\pi = (f, g)\colon \mathbb{A}^3 \to \mathbb{A}^2$ is a linear system of affine spaces and thus an affine linear system of affine spaces by Lemma 3.2.5(4).

Let $X, Y \subset \mathbb{P}^3$ be the closures of the hypersurfaces of $\mathbb{A}^3$ which are given by f and g, respectively. Then the singular locus of X is only the point $[0 : 1 : 0 : 0]$ and the singular locus of Y is the line $w = x = 0$, by Corollary 2.3.6(2)). Hence, (f, g) is not equivalent to an affine linear system in standard form (see Lemma 2.1.1).

It remains to see that all closed fibres of π are isomorphic to $\mathbb{A}^1$ but that π is not a trivial $\mathbb{A}^1$-fibration. To simplify the situation, we apply some non-affine automorphisms at the source and the target. We first apply $(x, y - x^2, z - x^3)$ (at the source) to get $(x + y^3 + z^2 + x^3 z, z)$, then apply $(x - y^2, y)$ at the target to obtain

$$\phi = (x + y^3 + x^3 z, z)\colon \mathbb{A}^3 \to \mathbb{A}^2 .$$

The fibre over a point (x_0, y_0) with $y_0 = 0$ is isomorphic to $\mathbb{A}^1$, via its projection onto y. The fibre over a point $(x_0, y_0) \in \mathbb{A}^2$ with $y_0 \neq 0$ is isomorphic to $\mathbb{A}^1$, as one can apply $y \mapsto y - \sqrt[3]{y_0}x$ to reduce to the previous case.

Now, the fibre F of ϕ over the generic point of $\{z = 0\}$ is the scheme given by $x + y^3 + x^3 z$ inside $\mathbb{A}^2_{\mathbf{k}(z)} = \mathrm{Spec}(\mathbf{k}(z)[x, y])$. Using again [Rus70], we find the same way as in the proof of Lemma 3.4.1, that there exists no separable field extension $\mathbf{k}(z) \subseteq K$ such that $F \times_{\mathrm{Spec}(\mathbf{k}(z))} \mathrm{Spec}(K) \simeq \mathbb{A}^1_K$, however

$$F \times_{\mathrm{Spec}(\mathbf{k}(z))} \mathrm{Spec}(\mathbf{k}(\sqrt[3]{z})) \simeq \mathbb{A}^1_{\mathbf{k}(\sqrt[3]{z})} .$$

This implies again, that neither ϕ nor π is locally trivial in the étale topology. □

We now prove that these two examples of linear systems are unique in some sense (see Lemma 3.4.4 and 3.4.5 below).

Lemma 3.4.3. *Let* $\ell_1, \ell_2, \ell_3 \in \boldsymbol{k}[x, y, z]_1$ *be three linear polynomials such that* ℓ_2 *and* ℓ_3 *are linearly independent. Then,* $\sum_{i=1}^{3} (\ell_i)^i$ *defines an* $\mathbb{A}^2$ *in* $\mathbb{A}^3$ *if and only if* ℓ_1, ℓ_2, ℓ_3 *are linearly independent.*

Proof. If ℓ_1, ℓ_2, ℓ_3 are linearly independent, we may apply an element of $\mathrm{GL}_3(\mathbf{k})$ and assume that $\ell_1 = x$, $\ell_2 = y$, $\ell_3 = z$. Thus, $\sum_{i=1}^{3} (\ell_i)^i = x + y^2 + z^3$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. Otherwise, we may assume that $\ell_1 = ax + by$, $\ell_2 = x$, $\ell_3 = y$, so the hypersurface of $\mathbb{A}^3$ given by $\sum_{i=1}^{3} (\ell_i)^i = 0$ is isomorphic to $\Gamma \times \mathbb{A}^1$, where $\Gamma \subset \mathbb{A}^2$ is the curve given by $ax + by + x^2 + y^3 = 0$. It remains to see that Γ is not isomorphic to $\mathbb{A}^1$ (by the positive answer to Zariski's Cancellation Problem, see [AHE72, Corollary 2.8]). Indeed, the closure of Γ in $\mathbb{P}^2$ would otherwise be an irreducible curve singular at infinity, which is here not the case. $\square$

Lemma 3.4.4. *Assume that* $\mathrm{char}(\boldsymbol{k}) = 2$ *and let* $f = (f_1, \ldots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ *be an affine linear system of affine spaces. Suppose that* $f_i = \sum_{j=0}^{3} f_{i,j} \in \boldsymbol{k}[x, y, z]$ *for each* $i \in \{1, \ldots, n\}$, *where* $f_{i,j} \in \boldsymbol{k}[x, y, z]_j$ *and that*

$$\mathrm{span}_{\boldsymbol{k}}(f_{1,3}, \ldots, f_{n,3}) = \boldsymbol{k}y^3 \text{ and } \mathrm{span}_{\boldsymbol{k}}(f_{1,2}, \ldots, f_{n,2}, y^2) = \boldsymbol{k}x^2 + \boldsymbol{k}y^2 + \boldsymbol{k}z^2.$$

Then, $n = 2$ *and* f *is equivalent to the linear system* $(x + z^2 + y^3, y + x^2)$ *of Lemma* 3.4.1.

Proof. As $\mathrm{span}_{\mathbf{k}}(f_{1,2}, \ldots, f_{n,2}, y^2) = \mathbf{k}x^2 + \mathbf{k}y^2 + \mathbf{k}z^2$, we have $n \geq 2$. Applying a linear automorphism of $\mathbb{A}^n$, we may assume that $f_{1,3} = y^3$ and that $f_{i,3} = 0$ for $i \geq 2$. We may moreover assume that $\mathrm{span}_{\mathbf{k}}(f_{1,2}, f_{2,2}, y^2) = \mathbf{k}x^2 + \mathbf{k}y^2 + \mathbf{k}z^2$ by possibly adding multiples of f_i, $i \geq 2$ to f_1 and then permuting the f_i, $i \geq 2$. Hence, $f_{1,2} = \ell_1^2 + \alpha y^2$ and $f_{2,2} = \ell_2^2 + \beta y^2$, where $\ell_1, \ell_2 \in \mathbf{k}[x, z]_1$ are linearly independent and $\alpha, \beta \in \mathbf{k}$. Applying a linear automorphism at the source that fixes y, we may reduce to the case where $f_{1,2} = z^2$ and $f_{2,2} = x^2$. We may moreover assume that $f_{i,0} = 0$ for each i, by applying a translation at the target.

We then choose $a, b, c, d \in \mathbf{k}$ such that $f_{1,1} = ax + bz \mod \mathbf{k}y$ and $f_{2,1} = cx + dz \mod \mathbf{k}y$. For each $\lambda \in \mathbf{k}$, the polynomial

$$f_1 + \lambda^2 f_2 = ((a + \lambda^2 c)x + (b + \lambda^2 d)z + \zeta y) + (z + \lambda x)^2 + y^3$$

defines an $\mathbb{A}^2$ in $\mathbb{A}^3$ (where $\zeta \in \mathbf{k}$ depends on λ). This implies that $((a + \lambda^2 c)x + (b + \lambda^2 d)z + \zeta y)$, y and $z + \lambda x$ are linearly independent (Lemma 3.4.3), and thus that $(a + \lambda^2 c) + (b + \lambda^2 d)\lambda \neq 0$. As this is true for all λ, we obtain $a \neq 0$ and $b = c = d = 0$, so $f_1 = ax + \xi y + z^2 + y^3$ and $f_2 = \nu y + x^2$ for some $\xi, \nu \in \mathbf{k}$. As f_2 defines an $\mathbb{A}^2$ in $\mathbb{A}^3$, we have $\nu \neq 0$. Applying $x \mapsto \sqrt{\nu}x$ at the source and replacing f_2 by $\nu^{-1} f_2$, we may assume that $\nu = 1$. We then replace f_1 with $f_1 + \xi f_2$ and z with $z + \sqrt{\xi}x$ to assume $\xi = 0$. This gives $(f_1, f_2) = (ax + z^2 + y^3, y + x^2)$. After replacing x, y, z with $\mu x, \mu^2 y, \mu^3 z$ at the source where $\mu \in \mathbf{k}$ is chosen with $\mu^5 = a$ and after replacing f_1, f_2 with f_1/μ^6, f_2/μ^2, respectively, we may assume further that $a = 1$. This achieves the proof if $n = 2$.

It remains to see that $n \geq 3$ leads to a contradiction. We add a multiple of f_2 to f_3 and may assume that $f_{3,2}$ is equal to $\varepsilon^2 y^2 + \tau^2 z^2 = (\varepsilon y + \tau z)^2$ for some $\varepsilon, \tau \in \mathbf{k}$. For each $\lambda \in \mathbf{k}$, the polynomial $\lambda^2 f_1 + f_2 + f_3 = (\lambda^2 x + y + f_{3,1}) + (x + \varepsilon y + (\lambda + \tau)z)^2 + \lambda^2 y^3$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. Hence, for each $\lambda \in \mathbf{k}^*$, the polynomials $\lambda^2 x + y + f_{3,1}$, $x + \varepsilon y + (\lambda + \tau)z$ and y are linearly independent (Lemma 3.4.3). Writing $f_{3,1} = \alpha x + \beta z + \gamma y$, with $\alpha, \beta, \gamma \in \mathbf{k}$, the polynomials

$$(\lambda^2 + \alpha)x + \beta z \text{ and } x + (\lambda + \tau)z$$

are linearly independent, so $0 \neq (\lambda^2 + \alpha)(\lambda + \tau) + \beta = \lambda^3 + \lambda^2\tau + \lambda\alpha + (\alpha\tau + \beta)$, for each $\lambda \in \mathbf{k}^*$. Hence, $\alpha = \tau = \beta = 0$, which yields $f_3 \in \mathbf{k}[y]$. As f_3 defines an $\mathbb{A}^2$, we obtain $f_3 = \gamma y$ with $\gamma \in \mathbf{k}^*$. But then $f_2 + \gamma^{-1} f_3 = x^2$ does not define an $\mathbb{A}^2$, contradiction. $\square$

Lemma 3.4.5. *Assume that* $\mathrm{char}(\boldsymbol{k}) = 3$, *let* $f_1, \ldots, f_n \in \boldsymbol{k}[x, y, z]$ *of degree* ≤ 3 *such that* $f = (f_1, \ldots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ *is an affine linear system of affine spaces and that the linear span of the homogeneous parts of degree* 3 *of the* $f_1, \ldots, f_n$ *is a subspace of dimension* ≥ 2 *of* $\boldsymbol{k}x^3 \oplus \boldsymbol{k}y^3 \oplus \boldsymbol{k}z^3$. *Then either* f *is equivalent to a* *linear system in standard form or* $n = 2$ *and* f *is equivalent to the linear system* $(x + z^2 + y^3, z + x^3)$ *in Lemma 3.4.2.*

Proof. Let $f_{i,j} \in \mathbf{k}[x, y, z]$ be the homogeneous part of degree j of f_i for $i = 1, \ldots, n$, and let us define $V_j = \mathrm{span}_{\mathbf{k}}(f_{1,j}, \ldots, f_{n,j}) \subseteq \mathbf{k}[x, y, z]_j$ for each j. By assumption, $V_3 \subseteq \mathbf{k}x^3 \oplus \mathbf{k}y^3 \oplus \mathbf{k}z^3$, so $\sum \lambda_i f_{i,3}$ is a third power for all $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n$. We may moreover assume that $V_0 = 0$ by applying a translation at the target.

It follows from Corollary 2.3.6(2) that for each $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n$ such that $\sum \lambda_i f_{i,3} \neq 0$ (which is true for a general $(\lambda_1, \ldots, \lambda_n)$), the polynomial $\sum \lambda_i f_{i,2}$ is either zero or defines a conic in $\mathbb{P}^2$ that is singular on a point of the triple line defined by $\sum \lambda_i f_{i,3}$.

Suppose first that $\gcd(V_2) = 1$, and thus that $\dim V_2 \geq 2$. Lemma 3.1.2 gives two polynomials $s, t \in \mathbf{k}[x, y, z]_1$ such that $V_2 \subseteq \mathbf{k}[s, t]$. Changing coordinates on $\mathbb{A}^3$, we may assume that $s = y$ and $t = z$. For general $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n$, the hypersurface in $\mathbb{P}^2$ given by the homogeneous polynomial $\sum \lambda_i f_{i,2}$ is only singular at the point $p = [1 : 0 : 0] \in \mathbb{P}^2$ (as $\mathrm{char}(\mathbf{k}) \neq 2$), which is on the triple line defined by $\sum \lambda_i f_{i,3}$. This implies that $V_3 \subseteq \mathbf{k}y^3 \oplus \mathbf{k}z^3$, so f is a linear system in standard form.

We may now assume that a linear polynomial $h \in \mathbf{k}[x, y, z]_1$ divides each element of V_2. Applying an element of GL_3 at the source, we may thus assume that $h = z$. If a point $p \in \mathbb{P}^2$ is such that all elements of V_2 and V_3 vanish at p, we apply an element of GL_3 at the source to assume $p = [1 : 0 : 0]$ and obtain that f is in standard form. Hence, we may assume that the elements of V_3 do not share a common zero on the line $z = 0$.

We now prove that z^2 divides $f_{i,2}$ for each $i \in \{1, \ldots, n\}$. We suppose the converse to derive a contradiction. Applying a general element of GL_n at the target, we obtain that $f_{1,2}$ is not a multiple of z^2 and that $f_{1,3}$ and $f_{2,3}$ do not share a common zero on the line $z = 0$. Choosing $\ell_1, \ell_2 \in \mathbf{k}[x, y, z]_1$ such that $f_{1,3} = \ell_1^3$ and $f_{1,3} = \ell_1^3$, the elements ℓ_1, ℓ_2, z are linearly independent. We may thus apply an element of GL_3 and assume that $f_{1,3} = x^3$ and $f_{2,3} = y^3$. We write $f_{1,2} = z(ax + by + cz)$ $f_{2,2} = zg$ for some $a, b, c \in \mathbf{k}$ with a, b not both equal to zero and $g \in \mathbf{k}[x, y, z]_1$. For each $\lambda \in \mathbf{k}$, the polynomial $f_1 + \lambda^3 f_2$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$ and as $f_{1,3} + \lambda^3 f_{2,3} = (x + \lambda y)^3$, the hypersurface in $\mathbb{P}^2$ given by the homogeneous polynomial $f_{1,2} + \lambda^3 f_{2,2} = z(ax + by + cz + \lambda^3 g)$ is singular at a point p_λ of the line in $\mathbb{P}^2$ given by $x + \lambda y = 0$ (Corollary 2.3.6(2)). This yields $p_\lambda = [-\lambda : 1 : 0]$, and thus $-\lambda a + b + \lambda^3 g(-\lambda, 1, 0) = 0$. This being true for each λ, we get $a = b = 0$, giving the desired contradiction.

We now show that $\dim(V_3) = 2$. If $\dim(V_3) = 3$, we may assume $(f_{1,3}, f_{2,3}, f_{3,3}) = (x^3, y^3, z^3)$. By Lemma 3.1.4, there exists $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$ and $\varepsilon \neq 0$ such that $\sum \lambda_i^3 f_{i,1} = \varepsilon \ell_1$, where $\ell_1 = \lambda_1 x + \lambda_2 y + \lambda_3 z$. Hence, the polynomial $\sum \lambda_i^3 f_i$ is equal to $\varepsilon \ell_1 + \nu z^2 + (\ell_1)^3$ for some $\nu \in \mathbf{k}$ and does not define an $\mathbb{A}^2$ in $\mathbb{A}^3$: it is reducible if $\nu = 0$ or if z and ℓ_1 are collinear, and otherwise does not define an $\mathbb{A}^2$ by Lemma 3.4.3.

Now that $\dim(V_3) = 2$ and that the elements of V_3 do not share a common zero point on $z = 0$, we may apply an element of GL_3 that fixes z to get $V_3 = \mathbf{k}x^3 + \mathbf{k}y^3$. Moreover, $V_2 = \mathbf{k}z^2$ (as otherwise $V_2 = \{0\}$ would give a linear system in standard form after exchanging x and z). We apply an element of GL_n at the target and assume that $f_{1,2} = z^2$ and $f_{1,3} \neq 0$. We then add to f_2 a linear combination of the other f_i and assume that $f_{2,2} = 0$ and that $f_{2,3}$ is not a multiple of $f_{1,3}$. Applying again at the source an element of GL_3 that fixes z, we obtain $f_{1,3} = y^3$, $f_{2,3} = x^3$. We get $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbf{k}$ such that

$$f_1 = (\alpha x + \beta y + \gamma z) + z^2 + y^3\,, \quad f_2 = (\delta x + \varepsilon y + \zeta z) + x^3\,.$$

For each $\lambda \in \mathbf{k}$, the polynomial $f_1 + \lambda^3 f_3$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. This implies that $(\alpha + \lambda^3\delta)x + (\beta + \lambda^3\varepsilon)y + (\gamma + \lambda^3\zeta)z$, z and $y + \lambda x$ are linearly independent (Lemma 3.4.3). Hence, $\lambda(\beta + \lambda^3\varepsilon) - (\alpha + \lambda^3\delta) \neq 0$. This being true for each λ, we obtain $\beta = \delta = \varepsilon = 0$ and $\alpha \neq 0$. Hence $f_1 = \alpha x + \gamma z + z^2 + y^3$, $f_2 = \zeta z + x^3$, with $\alpha\zeta \neq 0$. Replacing f_1 with $f_1 - (\gamma/\zeta)\cdot f_2$ and replacing y with $y + \kappa x$ where $\kappa^3 = \gamma/\zeta$, we may assume that $\gamma = 0$. It remains then to choose $\xi \in \mathbf{k}^*$ with $\alpha^3\zeta = \xi^{15}$, to replace x, y, z with $\xi^6/\alpha x, \xi^2 y, \xi^3 z$ at the source and f_1, f_2 with f_1/ξ^6, $f_2\alpha^3/\xi^{18}$ at the target, to obtain

$$f_1 = x + z^2 + y^3\,, \quad f_2 = z + x^3\,.$$

Thus, f is the linear system of affine spaces in Lemma 3.4.2 if $n = 2$. It remains to see that $n \geq 3$ yields a contradiction. Adding to f_3 a linear combination of f_1, f_2 we obtain that $f_{3,3} = 0$. This gives $f_3 = \alpha x + \beta y + \gamma z + \theta z^2$ with $\alpha, \beta, \gamma, \theta \in \mathbf{k}$. Replacing f_3 by a multiple, we may assume that $\alpha \neq -1$ and $\theta \neq -1$. For each $\lambda \in \mathbf{k}$, the polynomial $f_1 + \lambda^3 f_2 + f_3 = (1+\alpha)x + \beta y + (\gamma + \lambda^3)z + (1+\theta)z^2 + (y + \lambda x)^3$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$, so $y + \lambda x, z, (1+\alpha)x + \beta y + (\gamma + \lambda^3)z$ are linearly independent (Lemma 3.4.3). This implies that $\beta\lambda - (1+\alpha) \neq 0$. As this is true for each λ, we get $\beta = 0$. But then the linear parts of f_1, f_2, f_3 are linearly dependent, contradicting Lemma 3.2.5(1). ∎

3.5. **Linear systems of affine spaces of degree 3 with a line in the base locus.** In the following lemma we give necessary conditions for a polynomial of degree ≤ 3 such that it defines an $\mathbb{A}^2$ in $\mathbb{A}^3$ and this hypersurface contains in its closure in $\mathbb{P}^3$ a specific line.

Lemma 3.5.1. *Let $F \in \boldsymbol{k}[w, x, y, z]$ be a homogeneous polynomial of degree 3 such that $f = F(1, x, y, z)$ satisfies $\mathrm{Spec}(\boldsymbol{k}[x, y, z]/(f)) \simeq \mathbb{A}^2$ and such that $F(0, x, 0, z) = 0$. Write F as*

$$F = wa_2(x, z) + yb_2(x, z) + w^2c_1(x, z) + wyd_1(x, z) + y^2e_1(x, z) + F_3(w, y)$$

where $a_2, b_2 \in \boldsymbol{k}[x, z]$ are homogeneous of degree 2, $c_1, d_1, e_1 \in \boldsymbol{k}[x, z]$ are homogeneous of degree 1 and $F_3 \in \boldsymbol{k}[w, y]$ is homogeneous of degree 3. Then:

(1) *The polynomial $b_2 \in \boldsymbol{k}[x, z]$ is a square;*
(2) *The polynomials $a_2, b_2 \in \boldsymbol{k}[x, z]$ have a common linear factor;*
(3) *If $b_2 = 0$, then $a_2, e_1 \in \boldsymbol{k}[x, z]$ have a common linear factor;*
(4) *If $b_2 = e_1 = 0$ and a_2 is a square, then the polynomials $a_2, d_1 \in \boldsymbol{k}[x, z]$ have a common linear factor;*
(5) *If $a_2 = b_2 = d_1 = e_1 = 0$ and $\deg(f) \geq 2$, then $c_1 \neq 0$.*

Under the additional assumption that $\deg(f) = 3$, we have:

(6) *If $b_2 = e_1 = 0$, then the polynomial $a_2 \in \boldsymbol{k}[x, z]$ is a square;*
(7) *If $b_2 = e_1 = 0$ and $(a_2, d_1) \neq (0, 0)$, then $\gcd(a_2, c_1, d_1) = 1$;*
(8) *If a_2 is not a square, then $b_2 \neq 0$ or $e_1 \neq 0$;*

Proof. The fact that $F(0, x, 0, z) = 0$ implies that F can be written in the above form. Note that $F = F_1 + F_2 + F_3$, where $F_1 = wa_2(x, z) + yb_2(x, z)$, $F_2 =$

$w^2c_1(x,z)+wyd_1(x,z)+y^2e_1(x,z)$ and F_3 are homogeneous in w,y of degree 1, 2 and 3, respectively. It remains to see that the above eight assertions hold.

First, we assume that w divides F. Then $\deg(f)<3$ and $b_2=e_1=0$, so (1), (2) and (3) hold. If in addition a_2 is a square and if a_2 and d_1 would have no common non-zero linear factor, then the homogeneous part of f of degree 2 would be $f_2=a_2+y(d_1+\lambda y)$ for some $\lambda\in\mathbf{k}$. As a_2 is a square, we may apply a linear coordinate change in x,z and assume that $a_2=x^2$. We then write $d_1=d_{1,0}x+d_{1,1}z$ with $d_{1,0},d_{1,1}\in\mathbf{k}$, and obtain

$$f_2=x^2+d_{1,0}yx+y(\lambda y+d_{1,1}z)\,.$$

Since $d_{1,1}\neq 0$, the polynomial $f_2\in\mathbf{k}[x,y,z]$ is irreducible (e.g. by the Eisenstein criterion) which contradicts Proposition 2.3.5 and therefore (4) holds. If $a_2=d_1=0$ and $\deg(f)\geq 2$, then $c_1\neq 0$, since otherwise $f\in\mathbf{k}[y]$ would not be irreducible. Hence, (5) holds.

We may now assume that w does not divide F, which implies that $\deg(f)=3$.

We observe that the group of affine automorphisms $G\subset\mathrm{Aff}(\mathbb{A}^3)\subset\mathrm{Aut}(\mathbb{P}^3)$ which preserve the line $L=\left\{[w:x:y:z]\in\mathbb{P}^3\mid w=y=0\right\}$ is generated by the following two subgroups:

$$\begin{array}{rcl} G_1 & = & \left\{\varphi_{\alpha,\beta,\gamma,\delta}\in\mathrm{Aut}(\mathbb{P}^3)\;\middle|\;\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\in\mathrm{GL}_2(\mathbf{k})\right\}\\ G_2 & = & \left\{\psi_{\varepsilon,\tau_1,\tau_2,\tau_3,\xi_1,\xi_3}\in\mathrm{Aut}(\mathbb{P}^3)\;\middle|\;\varepsilon\in\mathbf{k}^*,\tau_1,\tau_2,\tau_3,\xi_1,\xi_3\in\mathbf{k}\right\}\end{array}$$

where

$$\begin{array}{ccc}\mathbb{P}^3 & \xrightarrow{\varphi_{\alpha,\beta,\gamma,\delta}} & \mathbb{P}^3\\ {[w:x:y:z]} & \longmapsto & [w:\alpha x+\beta z:y:\gamma x+\delta z]\end{array}$$

and

$$\begin{array}{ccc}\mathbb{P}^3 & \xrightarrow{\psi_{\varepsilon,\tau_1,\tau_2,\tau_3,\xi_1,\xi_3}} & \mathbb{P}^3\\ {[w:x:y:z]} & \longmapsto & [w:x+\xi_1y+\tau_1w:\varepsilon y+\tau_2w:z+\xi_3y+\tau_3w]\,.\end{array}$$

Indeed, this follows from the facts that the action of G on L gives a group homomorphism $G\to\mathrm{Aut}(L)\simeq\mathrm{PGL}_2(\mathbf{k})$ that is surjective on G_1, and that the kernel is generated by G_2 and the homotheties of G_1. The fact that all assertions (1)-(8) hold is preserved under elements of G_1 and G_2. We may thus assume that f is of the form given in Corollary 2.3.7 and we check that the assertions (1)-(8) are satisfied.

In case a), $(a_2,b_2,c_1,d_1,e_1)=(\lambda z^2,\mu z^2,x,\varepsilon z,\nu z)$ for some $\lambda,\mu,\nu,\varepsilon\in\mathbf{k}$.

In case b), $(a_2,b_2,c_1,d_1,e_1)=(0,\mu z^2,z,x,\nu z)$ for some $\mu,\nu\in\mathbf{k}$.

In case c), $f=xz+yz(\lambda y+\mu z)+y+\delta z$ where $\lambda,\mu,\delta\in\mathbf{k}$ and $(\lambda,\mu)\neq(0,0)$, so $(a_2,b_2,c_1,d_1,e_1)=(xz,\mu z^2,\delta z,0,\lambda z)$.

In case d), $f=xy^2+y(z^2+az+b)+z$ for some $a,b\in\mathbf{k}$, so $(a_2,b_2,c_1,d_1,e_1)=(0,z^2,z,az,x)$.

In each case, b_2 is a square, and there is a linear factor that divides a_2, b_2 and a linear factor that divides a_2, e_1. Moreover, a_2 is not a square only in case c) and thus b_2 or e_1 is non-zero. This shows that (1), (2), (3) and (8) are satisfied. The equalities $a_2=b_2=d_1=e_1=0$ are only possible in case a), where $c_1=x\neq 0$, thus (5) is satisfied. The equalities $b_2=e_1=0$ are only possible in the cases a) and b); and then a_2, d_1 have a common non-zero linear factor, a_2 is a square, and if $(a_2,d_1)\neq(0,0)$, then $\gcd(a_2,c_1,d_1)=1$. Thus (4), (6) and (7) are satisfied. $\square$

Proposition 3.5.2. *Let $f_1, \dots, f_n \in \boldsymbol{k}[x, y, z]$ be polynomials and assume that $f = (f_1, \dots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ is an affine linear system of affine spaces of degree 3 such that y divides the homogeneous parts of degree 3 of $f_1, \dots, f_n$. Then, the following hold:*

(i) Either f is equivalent to a linear system of affine spaces in standard form, or $\operatorname{char}(\boldsymbol{k}) = 2$ and f is equivalent to $(x + z^2 + y^3, y + x^2)\colon \mathbb{A}^3 \to \mathbb{A}^2$.

(ii) Writing the homogeneous part of degree 3 of f_i as $y(\eta_i y^2 + y e_{i,1} + b_{i,2})$ where $\eta_i \in \boldsymbol{k}$ and $e_{i,1}, b_{i,2} \in \boldsymbol{k}[x, z]$ are homogeneous of degree 1 and 2, the polynomials $b_{1,2}, \dots, b_{n,2}$ are collinear.

Proof. For each i we denote by $F_i \in \mathbf{k}[w, x, y, z]$ a homogeneous polynomial of degree 3 such that $f_i = F_i(1, x, y, z)$ and write it as

$$w a_{i,2}(x, z) + y b_{i,2}(x, z) + w^2 c_{i,1}(x, z) + w y d_{i,1}(x, z) + y^2 e_{i,1}(x, z) + F_{i,3}(w, y)$$

where $a_{i,2}, b_{i,2} \in \mathbf{k}[x, z]$ are homogeneous of degree 2, $c_{i,1}, d_{i,1}, e_{i,1} \in \mathbf{k}[x, z]$ are homogeneous of degree 1, $F_{i,3} \in \mathbf{k}[w, y]$ is homogeneous of degree 3, and the following hold for all $(\lambda_1, \dots, \lambda_n) \in \mathbf{k}^n$ (see Lemma 3.5.1):

(1) $\sum \lambda_i b_{i,2}(x, z)$ is a square;
(2) $\sum \lambda_i a_{i,2}(x, z)$ and $\sum \lambda_i b_{i,2}(x, z)$ have a common non-zero linear factor;
(3) If $\sum \lambda_i b_{i,2}(x, z) = 0$, then $\sum \lambda_i a_{i,2}(x, z)$ and $\sum \lambda_i e_{i,1}(x, z)$ have a common non-zero linear factor;
(4) If $\sum \lambda_i b_{i,2}(x, z) = \sum \lambda_i e_{i,1}(x, z) = 0$ and $\sum \lambda_i a_{i,2}(x, z)$ is a square, then $\sum \lambda_i a_{i,2}(x, z), \sum \lambda_i d_{i,1}(x, z)$ have a common non-zero linear factor;

and if $\deg(\sum \lambda_i f_i) = 3$, then:

(8) If $\sum \lambda_i a_{i,2}(x, z)$ is not a square, then $\sum \lambda_i b_{i,2}(x, z) \neq 0$ or $\sum \lambda_i e_{i,1}(x, z) \neq 0$.

We distinguish, whether all $b_{i,2}$ are collinear (case (A)) or not (case (B)). It turns out that in fact case (B) cannot occur, which proves *(ii)*.

(A): Any two $b_{i,2}$ are collinear: After applying an element of $\mathrm{GL}_2(\mathbf{k})$ on x, z, we may assume that z^2 divides all $b_{i,2}$ by assertion (1). If z divides each $a_{i,2}$, the point $[0 : 1 : 0 : 0]$ will be a singular point of the hypersurface in $\mathbb{P}^3$ given by F_i for each i, so f is in standard form. We may thus assume that there is j such that z does not divide $a_{j,2}$. Assertion (2) then implies that $b_{i,2} = 0$ for each i.

If a linear factor divides all $a_{i,2}$, we apply an element of GL_2 on x, z and assume that z divides all $a_{i,2}$, giving again that f is in a standard form. We then assume that no linear factor divides all $a_{i,2}$. In particular, $\dim \operatorname{span}_{\mathbf{k}}(a_{1,2}, \dots, a_{n,2}) \geq 2$.

We assume that each $\sum \lambda_i a_{i,2}$ is a square, which implies that $\operatorname{char}(\mathbf{k}) = 2$ and $\operatorname{span}_{\mathbf{k}}(a_{1,2}, \dots, a_{n,2}) = \mathbf{k}x^2 \oplus \mathbf{k}z^2$. By assertion (3), we can apply Lemma 3.1.3 in order to get $e_{i,1} = 0$ for each $i = 1, \dots, n$. Then, by assertion (4) we can apply Lemma 3.1.3 once again and get $d_{i,1} = 0$ for $i = 1, \dots, n$. Hence, the result follows from Lemma 3.4.4.

We now assume that $\sum \lambda_i a_{i,2}$ is not a square for general $(\lambda_1, \dots, \lambda_n) \in \mathbf{k}^n$. Assertion (8) implies that $\sum \lambda_i e_{i,1}$ is a non-zero linear polynomial for general $(\lambda_1, \dots, \lambda_n) \in \mathbf{k}^n$, which then needs to divide $\sum \lambda_i a_{i,2}$ by Assertion (3). As no linear factor divides all $a_{i,2}$, we may apply a general element of GL_n at the target and may assume that $a_{1,2}$ and $a_{2,2}$ have no common factor, and then the same holds for $e_{1,1}$ and $e_{2,1}$ (as $e_{i,1}$ divides $a_{i,2}$ for $i = 1, 2$). We then apply GL_2 on x, z at the source to get $e_{1,1} = x$ and $e_{2,1} = z$. We get $a_{1,2} = x(\alpha x + \beta z)$, $a_{2,2} = z(\gamma x + \delta z)$ for some $\alpha, \beta, \gamma, \delta \in \mathbf{k}$. For each $\lambda \in \mathbf{k}$, the polynomial $e_{1,1} + \lambda e_{2,1} = x + \lambda z$ divides

$a_{1,2} + \lambda a_{2,2} = \alpha x^2 + (\beta + \lambda\gamma)xz + \delta\lambda z^2$, so replacing $x = \lambda$ and $z = -1$ gives $0 = \lambda^2(\alpha - \gamma) + (\delta - \beta)\lambda$. This being true for all λ, we obtain $\alpha = \gamma$ and $\beta = \delta$, contradicting the fact that $a_{1,2}$ and $a_{2,2}$ have no common factor.

(B): It remains to suppose that not all $b_{i,2}$, $i = 1, \ldots, n$ are collinear and to derive a contradiction. Since by assertion (1) each $\sum \lambda_i b_{i,2}$ is a square, we get $\mathrm{char}(\mathbf{k}) = 2$. After applying a linear automorphism at the target, we may assume that $b_{1,2} = z^2$ and $b_{2,2} = x^2$. According to (2), we can apply Lemma 3.1.3 and get $a \in \mathbf{k}$ with $a_{1,2} = az^2$, $a_{2,2} = ax^2$. Replacing y with $y + a$ at the source, we may assume $a = 0$. This gives

$$f_1 = yz^2 + \alpha x + \beta z + \varepsilon \quad \text{and} \quad f_2 = yx^2 + \gamma x + \delta z + \nu$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \nu \in \mathbf{k}[y]$ (the first four of degree ≤ 2 and the last two of degree ≤ 3). For each $\lambda \in \mathbf{k}$, the polynomial $f_1 + \lambda^2 f_2 = y(z + \lambda x)^2 + (\alpha + \lambda^2\gamma)x + (\beta + \lambda^2\delta)z + \varepsilon + \lambda^2\nu$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. Replacing z with $z + \lambda x$, the polynomial

$$R_\lambda = yz^2 + (\alpha + \lambda\beta + \lambda^2\gamma + \lambda^3\delta)x + (\beta + \lambda^2\delta)z + \varepsilon + \lambda^2\nu$$

defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. Let us write $p_\lambda = \alpha + \lambda\beta + \lambda^2\gamma + \lambda^3\delta \in \mathbf{k}[y]$.

Let us write $\alpha = \sum_{i\geq 0} \alpha_i y^i$, $\beta = \sum_{i\geq 0} \beta_i y^i$, $\gamma = \sum_{i\geq 0} \gamma_i y^i$, $\delta = \sum_{i\geq 0} \delta_i y^i$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbf{k}$ for each $i \geq 0$. If there is some $i \geq 0$ such that the coefficient of y^i of p_λ is zero for a general (or equivalently for all) $\lambda \in \mathbf{k}$, then $\alpha_i + \lambda\beta_i + \lambda^2\gamma_i + \lambda^3\delta = 0$ for each $\lambda \in \mathbf{k}$, so $\alpha_i = \beta_i = \gamma_i = \delta_i = 0$.

Suppose first that $p_\lambda \in \mathbf{k}[y] \setminus \mathbf{k}$ for a general $\lambda \in \mathbf{k}$. In this case, we may apply Proposition 2.2.2: writing $R_\lambda = xp_\lambda(y) + q_\lambda(y, z)$ with $q_\lambda \in \mathbf{k}[y, z]$, the polynomial $q_\lambda(y_0, z) \in \mathbf{k}[z]$ is of degree 1 for each root $y_0 \in \mathbf{k}$ of p_λ. As the coefficient of z^2 in $q_\lambda(y, z)$ is y, we find that 0 is the only possible root of $p_\lambda(y)$, and in fact is a root for a general λ, as we assumed $p_\lambda \in \mathbf{k}[y] \setminus \mathbf{k}$. Applying the above argument with $i = 0$ implies that $\alpha_0 = \beta_0 = \gamma_0 = \delta_0 = 0$, but then, for each $\lambda \in \mathbf{k}$ the polynomial $\beta + \lambda^2\delta$ is zero at $y = 0$, so $q_\lambda(0, z) \in \mathbf{k}[z]$ is not of degree 1.

The last case is when $p_\lambda \in \mathbf{k}$ for each $\lambda \in \mathbf{k}$. This implies (again by the above argument) that $\alpha_i = \beta_i = \gamma_i = \delta_i = 0$ for each $i \geq 1$, so $\alpha, \beta, \gamma, \delta \in \mathbf{k}$. We have $\delta \neq 0$, since otherwise $f_2 \in \mathbf{k}[x, y]$ would define in $\mathbb{A}^2_{x,y}$ a curve with two points at infinity. There exists thus $\lambda \in \mathbf{k}$ such that $p_\lambda = 0$, so R_λ does not define an $\mathbb{A}^2$ (it belongs to $\mathbf{k}[y, z]$ and the curve that it defines in $\mathbb{A}^2_{y,z}$ has two points at infinity). □

3.6. Reduction to affine linear systems of affine spaces in standard form.

Proposition 3.6.1. *Let $n \geq 1$ and let $f_1, \ldots, f_n \in \boldsymbol{k}[x, y, z]$ be polynomials of degree ≤ 3 such that $f = (f_1, \ldots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ is a linear system of affine spaces. Then either f is equivalent to a linear system of affine spaces in standard form, or f is equivalent to one of the following linear systems of affine spaces:*

1. $(x + z^2 + y^3, y + x^2)\colon \mathbb{A}^3 \to \mathbb{A}^2$ *where* $\mathrm{char}(\boldsymbol{k}) = 2$, *or*
2. $(x + z^2 + y^3, z + x^3)\colon \mathbb{A}^3 \to \mathbb{A}^2$ *where* $\mathrm{char}(\boldsymbol{k}) = 3$.

Remark 3.6.2. The families of linear systems of affine spaces in (1) and (2) from Proposition 3.6.1 are the linear systems of affine spaces from Lemmata 3.4.1 and 3.4.2. In particular, the linear systems of affine spaces in (1) and (2) are all non-equivalent to linear systems of affine spaces in standard form.

Proof of Proposition 3.6.1. If $n = 1$, the result follows from Corollary 2.1.2, so we will assume that $n \geq 2$. By Lemma 3.2.5(1), we get $n \leq 3$.

Let $d = \deg(f)$. Since the statement holds when $d = 1$, we assume $d \in \{2, 3\}$.

Let $f_{i,j} \in \mathbf{k}[x, y, z]$ be the homogeneous part of degree j of f_i for $i = 1, \ldots, n$, and let us define $V_j = \mathrm{span}_{\mathbf{k}}(f_{1,j}, \ldots, f_{n,j}) \subseteq \mathbf{k}[x, y, z]_j$ for each $j \leq d$.

First, we consider the case $d = 2$. Due to Corollary 2.3.8, each element in V_2 is reducible and due to Lemma 3.1.2 one of the following cases occur:

- There exists $h \in \mathbf{k}[x, y, z]_1$ which divides each element of V_2;
- $V_2 \subset \mathbf{k}[s, t]$ for linearly independent $s, t \in \mathbf{k}[x, y, z]_1$;
- $\mathrm{char}(\mathbf{k}) = 2$ and $V_2 = \mathbf{k}x^2 \oplus \mathbf{k}y^2 \oplus \mathbf{k}z^2$.

In the first case we may assume that $h = y$ and in the second case we may assume that $(s, t) = (y, z)$, so f is in standard form in both cases. If we are in the last case, then $n = 3$ and we may assume that $f_{1,2} = x^2$, $f_{2,2} = y^2$, $f_{3,2} = z^2$. Due to Lemma 3.1.4 there exists $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$ and $\varepsilon \neq 0$ such that $\sum \lambda_i^2 f_{i,1} = \varepsilon(\lambda_1 x + \lambda_2 y + \lambda_3 z)$ and hence we get a contradiction to the irreducibility of $\sum \lambda_i^2 f_i$.

It remains to do the case where $d = 3$. If a linear factor or an irreducible polynomial of degree 2 divides all elements of V_3, the result follows respectively from Proposition 3.5.2 (after applying an element of GL_3 at the source) and Proposition 3.3.1. By Corollary 2.3.6, no element of V_3 is irreducible, so we may assume that $\gcd(V_3) = 1$. In particular, $\dim V_3 \geq 2$.

If each element of V_3 is a third power, then $\mathrm{char}(\mathbf{k}) = 3$ and the result follows from Lemma 3.4.5. Thus we may assume that a general element in V_3 is not a third power. Now, Lemma 3.1.2 implies that there exist linearly independent $s, t \in \mathbf{k}[x, y, z]_1$ such that $V_3 \subset \mathbf{k}[s, t]$. We may assume that $(s, t) = (y, z)$. As a general element of V_3 is not a third power, then by Corollary 2.3.6(3) the closure of the cubic $\sum \lambda_i f_i = 0$ in $\mathbb{P}^3$ has a singularity at $[0 : 1 : 0 : 0]$ for general $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n$ and thus f is in standard form. $\square$

Corollary 3.6.3. *Let $1 \leq n \leq 3$ and let $f_1, \ldots, f_n \in \mathbf{k}[x, y, z]$ be polynomials of degree ≤ 3 such that $f = (f_1, \ldots, f_n) \colon \mathbb{A}^3 \to \mathbb{A}^n$ is a trivial $\mathbb{A}^{3-n}$-bundle. Then f is equivalent to a linear system of affine spaces in standard form.*

Proof. This follows directly from Proposition 3.6.1, since the linear systems of affine spaces from Lemma 3.4.1 and Lemma 3.4.2 are not trivial $\mathbb{A}^1$-bundles. $\square$

3.7. **Study of affine linear systems of affine spaces $\mathbb{A}^3 \to \mathbb{A}^2$ in standard form.** Towards the description of the automorphisms of degree ≤ 3, we study in this subsection certain affine linear systems of affine spaces $(f_1, f_2) \colon \mathbb{A}^3 \to \mathbb{A}^2$ in standard form, i.e. such that $f_i = xp_i + q_i$ for $i = 1, 2$, with $p_i, q_i \in \mathbf{k}[y, z]$.

Lemma 3.7.1. *For $i = 1, 2$, let $p_i, q_i \in \mathbf{k}[y, z]$ such that $(xp_1 + q_1, xp_2 + q_2)$ is a linear system of affine spaces. Then, $\mathbf{k}[p_1, p_2] \neq \mathbf{k}[y, z]$, i.e. $(p_1, p_2) \colon \mathbb{A}^2 \to \mathbb{A}^2$ is not an automorphism.*

Proof. If $\mathbf{k}[p_1, p_2] = \mathbf{k}[y, z]$, then we apply a (possibly non-affine) automorphism of $\mathbf{k}[y, z]$ and may assume that $p_1 = y$, $p_2 = z$. We choose $\alpha, \beta, \gamma, \delta, \varepsilon, \tau \in \mathbf{k}$ such that

$$q_1(y, z) = \alpha y + \beta z + \varepsilon \mod (y^2, yz, z^2), \quad q_2(y, z) = \gamma y + \delta z + \tau \mod (y^2, yz, z^2).$$

Proposition 2.2.2 implies that $q_1(0, z) \in \mathbf{k}[z]$ and $q_2(y, 0) \in \mathbf{k}[y]$ have degree 1, so $\beta, \gamma \in \mathbf{k}^*$. For each $\lambda \in \mathbf{k}$, the polynomial in $(xy + q_1) - \lambda(xz + q_2) = x(y - \lambda z) + (q_1 - \lambda q_2) \in \mathbf{k}[x, y, z]$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. Replacing y with $y + \lambda z$, the polynomial

$$R_\lambda = xy + q_1(y + \lambda z, z) - \lambda q_2(y + \lambda z, z)$$

defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. Proposition 2.2.2 implies that $R_\lambda(x,0,z) = R_\lambda(0,0,z) \in \mathbf{k}[z]$ is of degree 1, for each $\lambda \in \mathbf{k}$. However,

$$R_\lambda(0,0,z) = q_1(\lambda z, z) - \lambda q_2(\lambda z, z) = \alpha\lambda z + \beta z + \varepsilon - \lambda(\gamma\lambda z + \delta z + \tau) \pmod{z^2}$$

and as $\gamma \neq 0$, there is $\lambda \in \mathbf{k}$ such that the coefficient of z of $R_\lambda(0,0,z)$ is zero, contradiction. □

Lemma 3.7.2. *For $i = 1,2$, let $p_i \in \boldsymbol{k}[y]$ and $q_i \in \boldsymbol{k}[y,z]$ and assume that $f = (f_1, f_2) = (xp_1 + q_1, xp_2 + q_2)\colon \mathbb{A}^3 \to \mathbb{A}^2$ is an affine linear system of affine spaces. Then the following hold:*

(1) *If p_1 and p_2 have a common root, then they are linearly dependent.*
(2) *If $p_1 \notin \boldsymbol{k}$ and $p_2 = 0$, then $q_2 \in \boldsymbol{k}[y]$ and $\deg(q_2) = 1$.*
(3) *If $p_1 = y$ and $q_1 = ay + zr_1 + r_0$ for $a \in \boldsymbol{k}[y,z]$, $r_1 \in \boldsymbol{k}^*$, $r_0 \in \boldsymbol{k}$ and if $p_2 = 1$, then $a - q_2 \in \boldsymbol{k}[y]$.*
(4) *If $p_1 = y^2$ and $q_1 = ys(z) + z$ for some $s \in \boldsymbol{k}[z]$ and $\deg(f) \leq 3$, then:*
   (i) *If $p_2 = 1$, then $s \in \boldsymbol{k}$ and $q_2 \in \boldsymbol{k}[y]$.*
   (ii) *If $p_2 = y + 1$, then $s = -z + b$ and $q_2 = -z + r$ for some $b \in \boldsymbol{k}$ and $r \in \boldsymbol{k}[y]$ with $\deg(r) \leq 3$.*
(5) *If $p_1 = y(y+1)$ and $q_1 = s(y)z + t(y)$ for $s, t \in \boldsymbol{k}[y]$ of degree ≤ 1 and $p_2 = 1$, then $s \in \boldsymbol{k}^*$ and $q_2 \in \boldsymbol{k}[y]$.*

Proof. By assumption for each $(\lambda, \mu) \neq (0,0)$, the equation

$$\lambda f_1 + \mu f_2 = x(\lambda p_1 + \mu p_2) + \lambda q_1 + \mu q_2 = 0$$

defines an $\mathbb{A}^2$ in $\mathbb{A}^3$. Hence, by Proposition 2.2.2, for each $y_0 \in \mathbf{k}$ the following holds:

$$(*) \qquad \begin{array}{l} \text{if } \lambda p_1(y_0) + \mu p_2(y_0) = 0 \text{ and } \lambda p_1 + \mu p_2 \neq 0, \\ \text{then the degree of } \lambda q_1(y_0, z) + \mu q_2(y_0, z) \in \mathbf{k}[z] \text{ is } 1\,. \end{array}$$

We will use this fact constantly, when we consider the cases (1)-(5).

(1): After an affine coordinate change in y, we may assume that y divides p_1 and p_2. By Proposition 2.2.2 it follows that $q_i(0,z)$ is a polynomial of degree 1 in z for $i = 1,2$. Hence there exists $\mu \in \mathbf{k}$ such that $q_1(0,z) - \mu q_2(0,z)$ is constant. This, together with $(*)$, implies that $p_1 = \mu p_2$.

(2): Since $p_1 \notin \mathbf{k}$, there exists $\gamma \in \mathbf{k}$ with $p_1(\gamma) = 0$. After applying an affine coordinate change in y, we may assume that $\gamma = 0$. By $(*)$, the degree of $q_1(0,z) + \mu q_2(0,z) \in \mathbf{k}[z]$ is 1 for each $\mu \in \mathbf{k}$, so $q_2(0,z) \in \mathbf{k}$. Hence, y divides $q_2 - q_2(0,0)$ in $\mathbf{k}[y,z]$. Since $q_2 - q_2(0,0) = 0$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$, the polynomial $q_2 - q_2(0,0)$ is irreducible and thus $q_2 = \alpha y + q_2(0,0)$ for some $\alpha \in \mathbf{k}^*$.

(3): Choosing $(\lambda, \mu) = (1, -\eta)$ for some $\eta \in \mathbf{k}$, we get $\lambda p_1 + \mu p_2 = y - \eta$. Thus by $(*)$, the degree of the polynomial

$$\eta a(\eta, z) + r_1 z + r_0 - \eta q_2(\eta, z) = r_1 z + r_0 + \eta(a(\eta,z) - q_2(\eta,z)) \in \mathbf{k}[z]$$

is 1 for each $\eta \in \mathbf{k}$. This implies that $a(\eta, z) - q_2(\eta, z) \in \mathbf{k}[\eta]$.

(4)(i): Choosing $(\lambda, \mu) = (1, -\eta^2)$, we get $\lambda p_1 + \mu p_2 = (y - \eta)(y + \eta)$. By $(*)$ it follows that for all $\eta \in \mathbf{k}$ the degree of

$$\eta s(z) + z - \eta^2 q_2(\eta, z) \in \mathbf{k}[z]$$

is 1, i.e. $\eta s(z) + z - \eta^2 q_2(\eta, z) = \alpha z + \beta$ for some $\alpha \in \mathbf{k}^*$ and $\beta \in \mathbf{k}[\eta]$. In order to use

$$z\mathbf{k}[z] \oplus \mathbf{k} \oplus \eta\mathbf{k}[z] \oplus \eta^2\mathbf{k}[\eta, z] = \mathbf{k}[\eta, z]\,, \tag{$**$}$$

we write $\beta = \beta_0 + \eta\beta_1 + \eta^2\beta_2$ where $\beta_0, \beta_1 \in \mathbf{k}$, $\beta_2 \in \mathbf{k}[\eta]$ and get

$$(z, 0, \eta s(z), -\eta^2 q_2(\eta, z)) = (\alpha z, \beta_0, \eta\beta_1, \eta^2\beta_2)\,,$$

so $s = \beta_1 \in \mathbf{k}$ and $q_2(\eta, z) = -\beta_2 \in \mathbf{k}[\eta]$.

(4)(*ii*): We now choose $(\lambda, \mu) = (1 + \eta, -\eta^2)$ for some $\eta \in \mathbf{k}$ and obtain

$$\lambda p_1 + \mu p_2 = (1 + \eta)y^2 - \eta^2(y + 1) = (y - \eta)((1 + \eta)y + \eta)\,.$$

Due to ($*$), for all $\eta \in \mathbf{k}$ the degree of the polynomial

$$(1 + \eta)(\eta s(z) + z) - \eta^2 q_2(\eta, z) = z + \eta(s(z) + z) + \eta^2(s(z) - q_2(\eta, z)) \in \mathbf{k}[z]$$

is 1. Writing this polynomial as above as $\alpha z + \beta_0 + \eta\beta_1 + \eta^2\beta_2$ with $\alpha \in \mathbf{k}^*$, $\beta_0, \beta_1 \in \mathbf{k}$, $\beta_2 \in \mathbf{k}[\eta]$, the decomposition ($**$) gives

$$(z, 0, \eta(s(z) + z), \eta^2(s(z) - q_2(\eta, z))) = (\alpha z, \beta_0, \eta\beta_1, \eta^2\beta_2)\,,$$

so $s(z) + z = \beta_1 \in \mathbf{k}$ and $s(z) - q_2(\eta, z) = \beta_2 \in \mathbf{k}[\eta]$. Choosing $b = \beta_1$ and $r \in \mathbf{k}[y]$ such that $\beta_2 = b - r(\eta)$, we obtain $s(z) = -z + b$ and $q_2(y, z) = s(z) - b + r(y) = -z + r(y)$. Since $\deg(q_2) \leq 3$ it follows that $\deg(r) \leq 3$.

(5): Let $(\lambda, \mu) = (1, -\eta(\eta + 1))$. Then

$$\lambda p_1 + \mu p_2 = y(y + 1) - \eta(\eta + 1) = (y - \eta)(y + \eta + 1)$$

Due to ($*$), for all $\eta \in \mathbf{k}$, the degree of

$$s(\eta)z + t(\eta) - \eta(\eta + 1)q_2(\eta, z) \in \mathbf{k}[z]$$

is 1. This implies that the polynomial

$$h = s(\eta)z - \eta(\eta + 1)q_2(\eta, z) \in \mathbf{k}[\eta, z]$$

is of the form $\alpha z + \beta$ for some $\alpha \in \mathbf{k}^*$ and $\beta \in \mathbf{k}[\eta]$.

When we write $q_2 = \sum_{i \geq 0} q_{2,i}(y)z^i$ for $q_{2,i} \in \mathbf{k}[y]$, we obtain $q_{2,i} = 0$ for each $i \geq 2$ (as h has degree 1 in z) and $s(y) - y(y + 1)q_{2,1}(y) \in \mathbf{k}^*$. As $\deg(s) \leq 1$, this yields $q_{2,1} = 0$, and then $s(y) \in \mathbf{k}^*$. Moreover, $q_2 = q_{2,0}(y) \in \mathbf{k}[y]$. $\square$

Lemma 3.7.3. *Let $p, q \in \boldsymbol{k}[y, z]$ such that $\deg(p) \leq 1$ and $\deg(q) \leq 3$. Assume that $(x(y + z^2) + z, xp + q)\colon \mathbb{A}^3 \to \mathbb{A}^2$ is an affine linear system of affine spaces. Then*

$$p \in \boldsymbol{k}, \quad q = a \cdot (y + z^2) + b \text{ for some } a, b \in \boldsymbol{k} \quad \text{and} \quad (p, a) \neq (0, 0)\,.$$

Proof. Suppose first that $p \in \mathbf{k}$. When we write $r = q(y - z^2, z) \in \mathbf{k}[y, z]$, we obtain $q = r(y + z^2, z)$. For each $\lambda \in \mathbf{k}$, the polynomial

$$x(y + z^2) + z - \lambda(xp + q) = x(y + z^2 - \lambda p) + z - \lambda r(y + z^2, z)$$

defines an $\mathbb{A}^2$ in $\mathbb{A}^3$, so the same holds for $xy + z - \lambda r(y + \lambda p, z)$. By Proposition 2.2.2, the polynomial $z - \lambda r(\lambda p, z) \in \mathbf{k}[z]$ is of degree 1 for each $\lambda \in \mathbf{k}$. This implies that the polynomial $r(\lambda p, z) \in \mathbf{k}[\lambda, z]$ lies in $\mathbf{k}[\lambda]$. As $p \in \mathbf{k}$, either $p \neq 0$ and $r(y, z) \in \mathbf{k}[y]$ or $p = 0$ and $r(y, z) \in \mathbf{k} + y\mathbf{k}[y, z]$. The first case yields $q \in \mathbf{k}[y + z^2]$, so $q = a \cdot (y + z^2) + b$ for some $a, b \in \mathbf{k}$, since $\deg q \leq 3$. In the second case, we write $b = q(0, 0)$ and obtain that $q - b$ is irreducible, as it defines the preimage of the hyperplane $y = b$. Hence, $r(y, z) - b \in y\mathbf{k}[y, z]$ is irreducible, so equal to ay for some $a \in \mathbf{k}^*$. As before we get $q = a \cdot (y + z^2) + b$. In both cases $(p, a) \neq (0, 0)$.

It remains to see that $p \notin \mathbf{k}$ is impossible. We write $p = ay + bz + c$ for some $(a, b) \in \mathbf{k}^2 \setminus \{(0,0)\}$ and $c \in \mathbf{k}$. If $a = 0$, then $b \neq 0$ which yields $\mathbf{k}[y + z^2, p] = \mathbf{k}[y + z^2, z] = \mathbf{k}[y, z]$, impossible by Lemma 3.7.1. We may thus assume that $a = 1$. We write $q = r + z + \mu$, with $\mu \in \mathbf{k}$ and $r \in \mathbf{k}[y, z]$ such that $r(0,0) = 0$. For each $\lambda \in \mathbf{k}$, the polynomial

$$\lambda(x(y + z^2) + z) + (1 - \lambda)(xp + q - \mu) = x(y + \lambda z^2 + (1 - \lambda)(bz + c)) + z + (1 - \lambda)r$$

defines an $\mathbb{A}^2$ in $\mathbb{A}^3$, so the same holds for $xy+z+(1-\lambda)\cdot r(y-\lambda z^2+(\lambda-1)(bz+c), z)$. We again apply Proposition 2.2.2, and find that $z + (1 - \lambda) \cdot r(-\lambda z^2 + (\lambda - 1)(bz + c), z) \in \mathbf{k}[z]$ is of degree 1 for each $\lambda \in \mathbf{k}$, so the polynomial

$$R = r(-\lambda z^2 + (\lambda - 1)(bz + c), z) \in \mathbf{k}[\lambda, z]$$

is an element of $\mathbf{k}[\lambda]$ (independent of z). If $r(y, z) \notin \mathbf{k}$, then $d := \deg_y(r) \geq 1$ and we may write $r = r_0(z) + r_1(z)y + \ldots + r_d(z)y^d$ where $r_d \neq 0$. Thus we get

$$R = r(\lambda(bz + c - z^2) - (bz + c), z) = \sum_{i=0}^{d} \lambda^i q_i$$

where $q_0, \ldots, q_d \in \mathbf{k}[z]$ and $q_d = (bz + c - z^2)^d r_d(z) \in \mathbf{k}[z] \setminus \mathbf{k}$. This contradicts $R \in \mathbf{k}[\lambda]$. Hence $r(y, z) \in \mathbf{k}$, so $r = r(0,0) = 0$. This proves that $q = z + \mu$. But this is impossible, as the zero locus of the polynomial $x(y + z^2) + z - (xp + q - \mu) = x(z^2 - bz - c)$ is not isomorphic to $\mathbb{A}^2$ (it is reducible). □

3.8. **Linear systems of affine spaces of degree ≤ 3 in standard form.** We start with a lemma, which lists the possibilities for the polynomials $p_1, \ldots, p_n$ in case of a linear system of affine spaces $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3 in standard from where the polynomials $p_1, \ldots, p_n$ lie in $\mathbf{k}[y]$.

Lemma 3.8.1. *Let $n \geq 1$ and let $p_i \in \mathbf{k}[y]$, $q_i \in \mathbf{k}[y, z]$ for $i = 1, \ldots, n$ such that $f = (f_1, \ldots, f_n) = (xp_1 + q_1, \ldots, xp_n + q_n) \colon \mathbb{A}^3 \to \mathbb{A}^n$ is a linear system of affine spaces of degree ≤ 3. Let us assume that*

$$V := \mathrm{span}_{\mathbf{k}}\{p_1, \ldots, p_n\} \subseteq \mathrm{span}_{\mathbf{k}}\{1, y, y^2\}\,.$$

Then, up to affine coordinate changes in y at the source, one of the following cases holds:

case	n	V
(1)	2 *or* 3	$\mathbf{k}(y + 1) \oplus \mathbf{k}y^2$
(2)	2 *or* 3	$\mathbf{k} \oplus \mathbf{k}y^2$
(3)	2 *or* 3	$\mathbf{k} \oplus \mathbf{k}y(y + 1)$
(4)	2 *or* 3	$\mathbf{k} \oplus \mathbf{k}y$
(5)	1 , 2 *or* 3	$\mathbf{k}$
(6)	1 *or* 2	$\mathbf{k}p$ *where* $p \in \{0, y, y^2, y(y + 1)\}$

Proof. We first prove that $\mathbf{k}y \oplus \mathbf{k}y^2$ is not contained in V. Indeed, we could then assume that $p_1 = y$ and $p_2 = y^2$, but then (f_1, f_2) is not a linear system of affine spaces by Lemma 3.7.2(1). This proves in particular that $\dim V \leq 2$.

Suppose now that $\dim V \leq 1$. If $n \leq 2$, we are in case (5) or (6) up to an affine coordinate change in y. If $n = 3$ and $V = \mathbf{k}$, we obtain case (5). We then prove that $n = 3$ and $V \neq \mathbf{k}$ is impossible. Indeed, otherwise, there is $y_0 \in \mathbf{k}$ with $p_i(y_0) = 0$ for $i = 1, 2, 3$ and the Jacobian of f would be non-invertible in all points (x, y_0, z), which contradicts Lemma 3.2.5(7).

We may now assume that $\dim V = 2$, so $n \in \{2,3\}$. After a reordering of $f_1,\dots,f_n$, we get $V = \mathbf{k}p_1 \oplus \mathbf{k}p_2$. If $\deg(p_i) \le 1$ for $i = 1,2$ we are in case (4). After a possible exchange of f_1, f_2 we may assume that $\deg(p_2) = 2$. After adding a certain multiple of f_2 to f_1 we may assume that $\deg(p_1) \in \{0,1\}$. If $\deg(p_1) = 0$, then after an affine coordinate change in y at the source, we are in case (2) or (3) depending on whether p_2 is a square or not. If $\deg(p_1) = 1$, then we may assume after an affine coordinate change in y at the source that $p_1 = y$ and $p_2 = a^2y^2 + by + c^2$ for $a,b,c \in \mathbf{k}$ with $ac \neq 0$ (indeed, 0 is not a common root of p_1, p_2, as they are linearly independent, see Lemma 3.7.2(1)). After adding $-(2ac+b)f_1$ to f_2 we obtain $p_2 = (ay-c)^2$. Thus after the coordinate change $y \mapsto \frac{c}{a}(y+1)$ we get $p_2 = c^2y^2$, $p_1 = \frac{c}{a}(y+1)$ and thus we are in case (1). □

Remark 3.8.2. If $\mathrm{char}(\mathbf{k}) \neq 2$, then in case (2) of Lemma 3.8.1, one gets $V = \mathbf{k} \oplus \mathbf{k}(y+\frac{1}{2})^2$. Thus after the coordinate change $y \mapsto y - \frac{1}{2}$ we are in case (3).

In the case of a linear system of affine spaces of degree 3 of $\mathbb{A}^3$ in standard form such that one component is of the form $x(y+z^2)+z$, the remaining components are almost determined, up to affine automorphisms at the target:

Lemma 3.8.3. *Let $n \in \{2,3\}$ and let $p_i, q_i \in \boldsymbol{k}[y,z]$ for $i = 1,\dots,n$ such that $f = (f_1,\dots,f_n) = (x(y+z^2)+z, xp_2+q_2,\dots,xp_n+q_n)$ is a linear system of affine spaces of degree 3. Then, up to an affine coordinate change at the target we have:*

(1) *$n = 2$ and $f = (x(y+z^2)+z, a(y+z^2)+bx)$ for $(a,b) \in \boldsymbol{k}^2 \setminus \{0\}$ or*
(2) *$n = 3$ and $f = (x(y+z^2)+z, y+z^2, x)$.*

Proof. For $i = 2,\dots,n$, let $p_{i,2}, q_{i,3} \in \mathbf{k}[y,z]$ be the homogeneous parts of degree 2 and 3 of p_i and q_i, respectively.

We now prove that $p_{i,2}$ is divisible by z^2 for each $i \in \{2,\dots,n\}$. If $q_{i,3} = 0$, this follows from Proposition 3.5.2(*ii*), applied to the linear system $(f_1(y,x,z), f_i(y,x,z))$. Now, assume $q_{i,3} \neq 0$ and that $p_{i,2}$ is not a multiple of z^2 to derive a contradiction. Since for each $\lambda \in \mathbf{k}$ the polynomial $\lambda f_1 + f_i = x(\lambda(y+z^2)+p_i) + (\lambda z + q_i)$ defines an $\mathbb{A}^2$ in $\mathbb{A}^3$, we get that for general $\lambda \in \mathbf{k}$ the polynomial $\lambda(y+z^2)+p_i \in \mathbf{k}[y,z]$ defines a disjoint union of curves in $\mathbb{A}^2$ which are isomorphic to $\mathbb{A}^1$ (see Proposition 2.2.1). In particular, for general (and thus for all) $\lambda \in \mathbf{k}$, the polynomial $\lambda z^2 + p_{i,2}$ is a square. Since $p_{i,2}$ is not a multiple of z^2 we get that $\mathrm{char}(\mathbf{k}) = 2$ and for general $\lambda \in \mathbf{k}$, the polynomials $\lambda z^2 + p_{i,2}$ and $q_{i,3}$ in $\mathbf{k}[y,z]$ have no common non-zero linear factor (remember that $q_{i,3} \neq 0$). This implies that the homogeneous part of degree 3 of $\lambda f_1 + f_i$, which is equal to $x(\lambda z^2 + p_{i,2}) + q_{i,3}$, is irreducible for general $\lambda \in \mathbf{k}$ and thus $\lambda f_1 + f_i$ does not define an $\mathbb{A}^2$ in $\mathbb{A}^3$ (see Proposition 2.3.5), contradiction.

For each $i \in \{2,\dots,n\}$, we may now add multiples of f_1 to f_i and assume that $\deg(p_i) \le 1$. Lemma 3.7.3 implies that $p_i \in \mathbf{k}$ and gives the existence of $a_i, b_i \in \mathbf{k}$ such that

$$f_i = xp_i + a_i(y+z^2) + b_i \quad \text{and} \quad (p_i, a_i) \neq (0,0)\,.$$

After applying a translation at the target, we may assume that $b_i = 0$. If $n = 2$, then we are in case (1). Hence, we assume $n = 3$. Since f_2 and f_3 are linearly independent, it follows that $p_2a_3 - p_3a_2 \neq 0$; thus after a linear coordinate change in y, z at the target, we may assume that

$$\begin{pmatrix} a_2 & p_2 \\ a_3 & p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This proves the lemma. □

Lemma 3.8.4. *Let $n \geq 1$. For $i \in \{1, \ldots, n\}$, let $f_i = xp_i + q_i$ where $p_i, q_i \in \boldsymbol{k}[y,z]$ and $\deg(p_i) \leq 2$, $\deg(q_i) \leq 3$. If $f = (f_1, \ldots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ is a linear system of affine spaces, then one may apply affine automorphisms at the target and source and reduce to the case where $p_1, \ldots, p_n \in \boldsymbol{k}[y]$ (and still have $q_1, \ldots, q_n \in \boldsymbol{k}[y,z]$).*

Proof. Assume first that $\deg(p_i) \leq 1$ for all i. Lemma 3.7.1 implies that no two of the linear parts of $p_1, \ldots, p_n$ are linearly independent, so we reduce to the case $p_i \in \mathbf{k}[y]$ for all i by applying an automorphism on y, z.

Applying a permutation at the target we may now assume that $\deg(p_1) = 2$.

If p_1 is irreducible, we apply an affine coordinate change at the source that fixes $[0 : 1 : 0 : 0]$ and obtain one of the cases of Proposition 2.3.4 for f_1. The action of this on p_1 corresponds to the action of an affine automorphism on y, z and thus does not change the fact that p_1 is irreducible; it thus gives Case (2) of Proposition 2.3.4, namely $f_1 = x(y + z^2) + z$. We apply Lemma 3.8.3 and obtain two possible cases. Exchanging x and y at the source gives the result.

We may now assume that for each $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n \setminus \{0\}$, the polynomial $\lambda_1 p_1 + \ldots + \lambda_n p_n$ is reducible if it has degree 2. Indeed, otherwise we reduce to the previous case by applying an affine automorphism at the target.

We may moreover assume that $\deg(p_i) = 2$ for each $i \in \{1, \ldots, n\}$ by adding multiples of p_1 to the p_i for $i \geq 2$.

Let $p_{i,j} \in \mathbf{k}[y,z]$ be the homogeneous part of degree j of p_i for $i = 1, \ldots, n$, $j = 0, 1, 2$. Let $V = \operatorname{span}_{\mathbf{k}}(p_{1,2}, \ldots, p_{n,2})$. Applying Proposition 2.3.4 to each linear combination $\sum \lambda_i f_i$, we see that each element of V is a square. If $\dim(V) = 1$, then applying a linear automorphism on y, z, we get $p_{i,2} \in \mathbf{k}y^2$ for each $i \in \{1, \ldots, n\}$. For each i, the polynomial $p_i \in \mathbf{k}[y,z]$ is reducible, so $p_i \in \mathbf{k}[y]$ as desired.

It remains to see that $\dim(V) \geq 2$ leads to a contradiction. As every element of V is a square, we get $\operatorname{char}(\mathbf{k}) = 2$ and $V = \mathbf{k}y^2 + \mathbf{k}z^2$. For each $(\lambda_1, \ldots, \lambda_n) \in \mathbf{k}^n$, the polynomial $x \sum \lambda_i p_{i,2} + \sum \lambda_i q_{i,3}$ is reducible as it is the homogeneous part of degree 3 of $\sum \lambda_i f_i$ (Corollary 2.3.6), so $\sum \lambda_i p_{i,2}$ and $\sum \lambda_i q_{i,3}$ have a common linear factor. Hence, we may apply Lemma 3.1.3 to $p_{1,2}, \ldots, p_{n,2}$ and $q_{1,3}, \ldots, q_{n,3}$ and get $h \in \mathbf{k}[y,z]_1$ with $q_{i,3} = hp_{i,2}$ for $i = 1, \ldots, n$. After applying the linear automorphism $(x - h, y, z)$ at the source, we reduce to the case where $q_{i,3} = 0$ for $i = 1, \ldots, n$. The vector space generated by the homogeneous parts of degree 3 of $f_1, \ldots, f_n$ is then equal to $\mathbf{k}xy^2 + \mathbf{k}xz^2$. This is impossible, as Proposition 3.5.2(*ii*) applied to $(f_1(y,x,z), \ldots, f_n(y,x,z))$ shows. □

3.9. **The proof of Theorem 3.** In this section, we give a description of all linear systems $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3 up to composition of affine automorphisms at the source and target and prove in particular Theorem 3.

Proposition 3.9.1. *Let $n \geq 2$. For $i \in \{1, \ldots, n\}$, let $f_i = xp_i + q_i$ where $p_i, q_i \in \boldsymbol{k}[y,z]$ and $\deg(p_i) \leq 2$, $\deg(q_i) \leq 3$. If $f = (f_1, \ldots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ is a linear system of affine spaces, then $n \leq 3$ and f is equivalent to $(g_1, \ldots, g_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ with one of the following possibilities:*

(*i*) $(g_1, g_2, g_3) = (x + p(y,z), y + q(z), z)$ *where* $p \in \boldsymbol{k}[y,z]$, $q \in \boldsymbol{k}[z]$;
(*ii*) $(g_1, g_2, g_3) = (xy + ya(y,z) + z, x + a(y,z) + r(y), y)$ *where* $a \in \boldsymbol{k}[y,z]$, $r \in \boldsymbol{k}[y]$;
(*iii*) $(g_1, g_2) = (xy + ya(y,z) + z, y)$ *where* $a \in \boldsymbol{k}[y,z]$;
(*iv*) $(g_1, g_2) = (xy^2 + y(z^2 + az + b) + z, y)$ *where* $a, b \in \boldsymbol{k}$.

Proof. Using Lemma 3.8.4, we may assume that $p_i \in \mathbf{k}[y]$ for all i.

We then apply Lemma 3.8.1, and may assume that $p_3 = 0$ if $n = 3$ and that (p_1, p_2) is in one of the following cases:

(1)	$(y^2, y+1)$	(4)	$(y, 1)$
(2)	$(y^2, 1)$	(5)	$(1, 0)$
(3)	$(y(y+1), 1)$	(6)	$(p, 0)$ with $p \in \{0, y, y^2, y(y+1)\}$ and $n = 2$

We now go through the different cases.

In Cases (1)-(4), if $n = 3$ then $f_3 = q_3$ is an element of $\mathbf{k}[y]$ of degree 1. This follows from Lemma 3.7.2(2) applied to (f_1, f_3), as $p_3 = 0$ and $p_1 \in \mathbf{k}[y] \setminus \mathbf{k}$. One can then, if one needs, replace f_3 with $\alpha f_3 + \beta$ for some $\alpha, \beta \in \mathbf{k}$, $\alpha \neq 0$ and obtain $f_3 = y$.

In Cases (1)-(2), $p_1 = y^2$. There is $\alpha \in \mathrm{Aff}(\mathbb{A}^3)$ that fixes $[0 : 1 : 0 : 0]$ such that $\alpha^*(f_1)$ is one of the cases of Proposition 2.3.4. As $\alpha^*(y^2)$ is the coefficient of x in $\alpha^*(f_1)$ up to non-zero scalars, we obtain that $\alpha^*(f_1)$ is the polynomial of Case (5) in Proposition 2.3.4 and $\alpha^*(y) \in \mathbf{k}[y]$, so we reduce to the case where $p_1 = y^2$ and $q_1 = ys(z) + z$ for some $s \in \mathbf{k}[z]$ of degree ≤ 2.

(1): Here $p_2 = y + 1$, so Lemma 3.7.2(4) *(ii)* shows that $s(z) = -z + \mu$ and $q_2 = -z + r(y)$ where $\mu \in \mathbf{k}$ and $r \in \mathbf{k}[y]$ has degree ≤ 3. After performing $(x, y, z) \mapsto (x, y, z + \mu)$ at the source and adding constants at the target we may assume $\mu = 0$. Hence,

$$(f_1, f_2) = (xy^2 - zy + z, x(y+1) - z + r(y)) .$$

We apply $(x, y, z) \mapsto (z, y + 1, -x)$ at the source and get

$$(f_1, f_2) = (xy + yz(y+2) + z, x + z(y+2) + r(y+1)) .$$

This gives case *(ii)* if $n = 2$. If $n = 3$, then f_3 is still an element of $\mathbf{k}[y]$ of degree 1 and we can then assume $f_3 = y$ to obtain Case *(ii)*.

(2): Here $p_2 = 1$, so $f_2 = x + q_2(y, z)$ and if $n = 3$, then $f_3 \in \mathbf{k}[y]$ is of degree 1, so we may assume $f_3 = y$. Lemma 3.7.2(4)*(i)* gives $q_2 \in \mathbf{k}[y]$ and $s \in \mathbf{k}$, thus after a permutation of x, y, z at the source we are in case *(i)*.

(3): Here $p_1 = y(y+1)$, so $q_1 = a(y,z)y(y+1) + s(y)z + t(y)$ for polynomials $a \in \mathbf{k}[y, z]$, $s, t \in \mathbf{k}[z]$ of degree ≤ 1 with $s(0)s(-1) \neq 0$ (Proposition 2.2.2). Replacing x with $x - a(y, z)$, we may assume that $a = 0$. Lemma 3.7.2(5) then implies that $s(y) \in \mathbf{k}^*$ and $q_2(y, z) \in \mathbf{k}[y]$. Hence,

$$(f_1, f_2) = (xy(y+1) + sz + t(y), x + q_2(y))$$

and if $n = 3$, we may assume $f_3 = y$. After a permutation of x, y, z at the source and a rescaling of f_1, we are in case *(i)*.

(4): Here $p_1 = y$, so $q_1 = \tilde{a}(y, z)y + \alpha z + \beta$ where $\tilde{a} \in \mathbf{k}[y, z]$, $\alpha \in \mathbf{k}^*$ and $\beta \in \mathbf{k}$ (Proposition 2.2.2). Replacing z with $\alpha^{-1}(z - \beta)$, we get $f_1 = xy + a(y, z)y + z$ for some $a \in \mathbf{k}[y, z]$. By Lemma 3.7.2(3) there is $r(y) \in \mathbf{k}[y]$ with $f_2 = x + a(y, z) + r(y)$. Hence, we are in case *(ii)*.

(5): If $n = 2$, then according to Lemma 2.3.2 we may apply an affine automorphism in (y, z) at the source in order to get $f_2 = q_2 = y + q(z)$ and thus we are in case *(i)*. If $n = 3$, then $f = (x + q_1, q_2, q_3)$. Since $\mathbb{A}^3 \to \mathbb{A}^2$, $(x, y, z) \mapsto (q_2(y, z), q_3(y, z))$ is an affine linear system of affine spaces, by Lemma 3.2.5(6) the same holds for $(q_2, q_3)\colon \mathbb{A}^2 \to \mathbb{A}^2$. By Proposition 3.2.7, we get up to affine automorphisms in y, z at the source and target that $(q_2, q_3) = (y + q(z), z)$ for some $q \in \mathbf{k}[z]$ and thus we are again in case *(i)*.

(6): Assume first that $p = 0$. Then by Proposition 3.2.7 we may assume that $f = (y + q(z), z)$ for some $q \in \mathbf{k}[z]$. After replacing y with x and z with y we are in case (i). In all other cases $p \in \mathbf{k}[y] \setminus \mathbf{k}$ and by Lemma 3.7.2(2) we get that f_2 is a polynomial of degree 1 in $\mathbf{k}[y]$. By Proposition 2.3.4 there is $\alpha \in \mathrm{Aff}(\mathbb{A}^3)$ that fixes $[0 : 1 : 0 : 0]$ such that $\alpha^*(f_1)$ is one of the polynomials in the cases (1)-(6) of Proposition 2.3.4. Since up to scalars, $\alpha^*(p)$ is the factor of x in $\alpha^*(f_1)$ (when we consider it as a polynomial in x over $\mathbf{k}[y, z]$) and since $p \in \{y, y^2, y(y+1)\}$, it follows that $\alpha^*(f_1)$ belongs to one of the Cases (4)-(6) of Proposition 2.3.4 and $\alpha^*(y) \in \mathbf{k}[y]$. In particular, $\alpha^*(f_2)$ is a polynomial of degree 1 in y. Proposition 2.3.5 then gives $\beta \in \mathrm{Aff}(\mathbb{A}^3)$ such that $\beta^*(y) \in \mathbf{k}[y]$ and such that $\beta^*(f_1)$ is one of the polynomials in cases A), B) or C) of Proposition 2.3.5. As $\beta^*(f_2)$ is again a polynomial of degree 1 in $\mathbf{k}[y]$, we may replace it with y and get cases (i), (iii) or (iv).

□

As an immediate consequence we get

Corollary 3.9.2. *Let $n \geq 1$ and let $f\colon \mathbb{A}^3 \to \mathbb{A}^n$ be a linear system of affine spaces of degree ≤ 3. Then f is equivalent to a linear system of affine spaces in standard form if and only if f is a trivial $\mathbb{A}^{3-n}$ bundle. Moreover, the latter condition is satisfied if* $\mathrm{char}(\boldsymbol{k}) \notin \{2, 3\}$.

Proof. If $f\colon \mathbb{A}^3 \to \mathbb{A}^n$ is a trivial $\mathbb{A}^{3-n}$-bundle, then f is equivalent to a linear system of affine spaces in standard form by Corollary 3.6.3. Conversely, we assume that f is a linear system of affine spaces in standard form and prove that f is a trivial $\mathbb{A}^{3-n}$-bundle. If $n = 1$, then f is a variable of $\mathbf{k}[x, y, z]$ (Corollary 2.2.3), so it defines a trivial $\mathbb{A}^2$-bundle. If $n \geq 2$, we go through the four cases of Proposition 3.9.1. In case (i) and (ii), the morphism $(g_1, g_2, g_3)\colon \mathbb{A}^3 \to \mathbb{A}^3$ defines an automorphism and in case (iii) and (iv), Proposition 2.2.2(2) gives the existence of $g_3 \in \mathbf{k}[x, y, z]$ such that $(g_1, g_2, g_3) \in \mathrm{Aut}(\mathbb{A}^3)$. The second claim follows from Proposition 3.6.1. □

We now come to the proof of our description of linear systems of affine spaces $\mathbb{A}^3 \to \mathbb{A}^n$ of degree ≤ 3:

Proof of Theorem 3. Let $f_1, \ldots, f_n \in \mathbf{k}[x, y, z]$ such that $f = (f_1, \ldots, f_n)\colon \mathbb{A}^3 \to \mathbb{A}^n$ is a linear system of affine spaces of degree ≤ 3. If $f\colon \mathbb{A}^3 \to \mathbb{A}^n$ is not a trivial $\mathbb{A}^{3-n}$-bundle, then by Corollary 3.9.2 and Proposition 3.6.1, we are in cases (8) or (9). Thus we may assume that $f\colon \mathbb{A}^3 \to \mathbb{A}^n$ is a trivial $\mathbb{A}^{3-n}$-bundle. If $n = 1$, this means that $f = f_1$ is a variable, and the description of f follows from Proposition 2.3.5. We may then assume that $n \geq 2$, that f is in standard form (applying again Corollary 3.9.2) and then go through the different cases of Proposition 3.9.1:

(i): $(f_1, f_2) = (x+p(y, z), y+q(z))$ with $p \in \mathbf{k}[y, z]$ and $q \in \mathbf{k}[z]$, and $f_3 = z$ if $n = 3$. Since $\deg(f) \leq 3$, we may write $p = \sum_{i=0}^{3} p_i(y, z)$ and $q(z) = \sum_{i=0}^{3} q_i z^i$ where $p_i \in \mathbf{k}[y, z]$ is homogeneous of degree i and $q_i \in \mathbf{k}$. After applying a translation at the target we may assume that $p_0 = 0$ and $q_0 = 0$. After composing f with the automorphism $(x - p_1(y - q_1 z, z), y - q_1 z, z)$ at the source we are either in case (4) or (10).

(ii) and (iii): There exist $a \in \mathbf{k}[y, z]$ of degree ≤ 2 and $r \in \mathbf{k}[y]$ of degree ≤ 3 such that $g = (xy + ya(y, z) + z, x + a(y, z) + r(y), y)$ satisfies: f is either equal to g, or f is equal to $\pi \circ g$ where $\pi\colon \mathbb{A}^3 \to \mathbb{A}^2$ is one of the projections $(x, y, z) \mapsto (x, z)$ or $(x, y, z) \mapsto (x, y)$. Write $r(y) = r_0 + r_1 y + r_2 y^2 + r_3 y^3$ and $a = a_0 + a_1(y, z) + a_2(y, z)$ where $r_i \in \mathbf{k}$ and $a_i \in \mathbf{k}[y, z]$ is homogeneous of degree i. After adding constants at the target, we may assume $r_0 = 0$. After applying $(x - a_0 - a_1(y,z), y, z)$ at the source, we may further assume that $a = a_2$ is homogeneous of degree 2. After applying the permutation of the coordinates $(x,y,z) \mapsto (y,z,x)$ at the source, we have replaced g with $g = (yz + za_2(z,x) + x, y + a_2(z,x) + r_1z + r_2z^2 + r_3z^3, z)$.

If $a_2(z,x) \in \mathbf{k}[z]$ and $f_2 \neq z$, then after applying $(x, y - r_1z, z)$ at the source we are in case (4) or case (10). If $a_2(z,x) \in \mathbf{k}[z]$ and $f_2 = z$, then after exchanging y and z at the source we are again in case (4). Thus we may assume that $a_2(z,x) \notin \mathbf{k}[z]$. If $n = 2$, then we are in case (5) or (6) and if $n = 3$, then we are in case (11) after applying $(x, y - r_1z, z)$ at the target.

(iv): This is case (7). □

Next, we will show that the cases in Theorem 3 are all pairwise non-equivalent. For this we need the following lemma.

Lemma 3.9.3. *For each $r_2 \in \boldsymbol{k}[y,z] \setminus \boldsymbol{k}[y]$, homogeneous of degree 2, it is not possible to find $p \in \boldsymbol{k}[y,z]$, $\lambda \in \boldsymbol{k}$ and $\alpha \in \mathrm{Aff}(\mathbb{A}^3)$ such that*

$$\alpha^*(xy + yr_2(y,z) + z) = \lambda x + p(y,z)\,.$$

Proof. Suppose for contradiction that p, λ, α exist. We may assume that $\alpha \in \mathrm{GL}_3$, as a translation sends $\lambda x + p(y,z)$ onto $\lambda x + \tilde{p}(y,z)$ for some $\tilde{p} \in \mathbf{k}[y,z]$. Hence, the homogeneous part of degree 2 of $\alpha^*(xy + yr_2(y,z) + z)$ is $\alpha^*(xy) \in \mathbf{k}[y,z]$. This implies that $\alpha^*(x), \alpha^*(y)$ are linearly independent elements of $\mathbf{k}y + \mathbf{k}z$, as $\mathbf{k}[y,z]$ is factorially closed in $k[x,y,z]$. Replacing α by its composition with an element of GL_2 acting on y, z (which simply replaces p with another polynomial in $\mathbf{k}[y,z]$), we may assume that $\alpha^*(x) = z$ and $\alpha^*(y) = y$. Hence, $\alpha^*(z) = ax + by + cz$ for some $a, b, c \in \mathbf{k}$, $a \neq 0$. This gives

$$\lambda x + p(y,z) = \alpha^*(xy + yr_2(y,z) + z) = yz + yr_2(y, ax + by + cz) + ax + by + cz\,,$$

impossible as $r_2 \in \mathbf{k}[y,z] \setminus \mathbf{k}[y]$ and $a \neq 0$, so the coefficient of x of the right hand side is not constant. □

Proposition 3.9.4. *The eleven families in Theorem 3 define disjoint sets of equivalence classes of affine linear systems of affine spaces, i.e. if $(k), (l) \in \{(1), (2), \ldots, (11)\}$, and $f, g\colon \mathbb{A}^3 \to \mathbb{A}^n$ are equivalent affine linear systems of affine spaces as in family (k) and (l) of Theorem 3, respectively, then $(k) = (l)$.*

Proof. If f or g is a non-trivial $\mathbb{A}^1$-fibration, then both are. As $\mathrm{char}(\mathbf{k}) = 2$ in (8) and $\mathrm{char}(\mathbf{k}) = 3$ in (9), we obtain $(k) = (l) =$(8) or $k = l =$(9). We may now assume that (k) and (l) are both contained in one of the sets $\{(1), (2), (3)\}$, $\{(4), (5), (6), (7)\}$ or $\{(9), (10), (11)\}$.

We write $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$.

Assume that $f_1 = xy^2 + y(z^2 + az + b) + z$ for some $a, b \in \mathbf{k}$, i.e. $(k) \in \{(3), (7)\}$. Then for general $(\lambda_1, \ldots, \lambda_n)$, the homogeneous part of degree 3 of $\sum \lambda_i f_i$ does not factor into linear polynomials. This has to be the same for the homogeneous part of degree 3 of $\sum \lambda_i g_i$, so $(k) = (l) \in \{(3), (7)\}$ by inspecting the cases that are different from (3), (7). The same holds when $(l) \in \{(3), (7)\}$, so we may exclude these two cases.

Assume now that $f_1 = x + r_2(y,z) + r_3(y,z)$ for homogeneous polynomials $r_2, r_3 \in \mathbf{k}[y,z]$ of degree 2 and 3, respectively, i.e. $(k) \in \{(1), (4), (10)\}$. For each $(\lambda_1, \ldots, \lambda_n)$, the polynomial $\sum \lambda_i f_i$ is equal to $\lambda x + p(y,z)$ for some $\lambda \in \mathbf{k}$ and $p \in \mathbf{k}[y,z]$. Lemma 3.9.3 implies that g_1 is not equivalent to $xy + ya_2(y,z) + z$

for some $a_2 \in \mathbf{k}[y,z] \setminus \mathbf{k}[y]$, homogeneous of degree 2, so $(l) \notin \{(2), (5), (6), (11)\}$. This yields $(k) = (l) \in \{(1), (4), (10)\}$. As before, we may now exclude the cases (1), (4) and (10).

It remains to see that $(k) =$(5) and $(l) =$(6) are not equivalent. We take homogeneous polynomials $a_2, b_2 \in \mathbf{k}[x,z] \setminus \mathbf{k}[z]$ of degree 2 and $r_1, r_2, r_3 \in \mathbf{k}$ such that

$$\begin{aligned} f &= (f_1, f_2) = (yz + za_2(x,z) + x, y + a_2(x,z) + r_1 z + r_2 z^2 + r_3 z^3) \\ g &= (g_1, g_2) = (yz + zb_2(x,z) + x, z)\,. \end{aligned}$$

and prove that f, g are not equivalent. For $i = 1, 2$, denote by $f_{i,3}, g_{i,3} \in \mathbf{k}[x,y,z]$ the homogeneous part of degree 3 of f_i and g_i, respectively. If $r_3 \neq 0$, then $f_{1,3}, f_{2,3}$ are linearly independent as $a_2 \notin \mathbf{k}[z]$, but $g_{1,3}, g_{2,3}$ are not, so f and g are not equivalent. If $r_3 = 0$, as $a_2 \notin \mathbf{k}[z]$, we get that $\deg(\lambda_1 f_1 + \lambda_2 f_2) \in \{2, 3\}$ for each $(\lambda_1, \lambda_2) \neq (0,0)$. As $\deg(g_2) = 1$, f and g are not equivalent. □

Corollary 3.9.5. *Every automorphism of degree ≤ 3 of $\mathbb{A}^3$ is tame.*

Proof. As for each $a \in \mathbf{k}[x,z]$ and each $r \in \mathbf{k}[z]$ we have the decomposition

$$(x + yz + za(x,z), y + a(x,z) + r(z), z) = h_1 \circ \iota \circ h_2 \circ \iota$$

where $h_1 = (x+yz, y+r(z), z) \in \mathrm{Triang}_{\mathbf{k}}(\mathbb{A}^3)$, $h_2 = (x+a(y,z), y, z) \in \mathrm{Triang}_{\mathbf{k}}(\mathbb{A}^3)$ and $\iota = (y, x, z) \in \mathrm{Aff}_{\mathbf{k}}(\mathbb{A}^3)$, it follows from Theorem 3 that all automorphisms of degree ≤ 3 of $\mathbb{A}^3$ are tame. □

4. Dynamical degrees of automorphisms of $\mathbb{A}^3$ of degree at most 3

As an application of our description of automorphisms of $\mathbb{A}^3$ of degree ≤ 3 (see Theorem 3), we list in this section all possible dynamical degrees of these automorphisms. Recall that the dynamical degree satisfies $\lambda(f) \leq \deg(f)$ and that $\lambda(f) = \lambda(g)$ if f, g are conjugated automorphisms in $\mathrm{Aut}(\mathbb{A}^n)$ and more generally if f, g are only conjugated in the bigger group $\mathrm{Bir}(\mathbb{A}^n)$ of birational maps of $\mathbb{A}^n$.

4.1. **Affine-triangular automorphisms.** We say that an element $f \in \mathrm{Aut}(\mathbb{A}^n)$ is *affine-triangular* if $f = \alpha \circ \tau$, where $\alpha \in \mathrm{Aff}(\mathbb{A}^n)$ is an affine automorphism and $\tau \in \mathrm{Triang}_{\mathbf{k}}(\mathbb{A}^n)$ is a triangular automorphism. Note that an element $g \in \mathrm{Aut}(\mathbb{A}^n)$ is equivalent to a triangular automorphism if and only if it is conjugate to an affine-triangular automorphism by an affine automorphism. The dynamical degrees of affine-triangular automorphisms of $\mathbb{A}^3$ can be computed, using a simple algorithm described in [BvS19a]. In particular, one has the following result.

Theorem 4.1.1. [BvS19a, Theorem 1] *For each field* k *and each integer $d \geq 2$, the set of dynamical degrees of all affine-triangular automorphisms of $\mathbb{A}^3$ of degree $\leq d$ is equal to*

$$\left\{ \frac{a + \sqrt{a^2 + 4bc}}{2} \,\middle|\, (a,b,c) \in \mathbb{N}^3, a + b \leq d, c \leq d \right\} \setminus \{0\}.$$

Moreover, for all $a, b, c \in \mathbb{N}$ such that $\lambda = \frac{a+\sqrt{a^2+4bc}}{2} \neq 0$, the dynamical degree of the automorphism

$$(z + x^a y^b, y + x^c, x)$$

is equal to λ.

Corollary 4.1.2. *For each $d \geq 1$ and each field* k*, let us denote by $\Lambda_{d,\mathrm{k}} \subset \mathbb{R}$ the set of dynamical degrees of all automorphisms of $\mathbb{A}^3_{\mathrm{k}}$ of degree d. We then have*

$$\begin{array}{lcl} \Lambda_{1,\mathrm{k}} & = & \{1\} \\ \Lambda_{2,\mathrm{k}} & = & \{1, \sqrt{2}, (1+\sqrt{5})/2, 2\} \\ \Lambda_{3,\mathrm{k}} & \supseteq & \{1, \sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \sqrt{3}, 2,\ \frac{1+\sqrt{13}}{2},\ 1+\sqrt{2},\ \sqrt{6},\ \frac{1+\sqrt{17}}{2},\ 1+\sqrt{3},\ 3\}\,. \end{array}$$

Moreover, if $f \in \mathrm{Aut}(\mathbb{A}^3_{\mathrm{k}})$ is conjugated in $\mathrm{Aut}(\mathbb{A}^3_{\overline{\mathrm{k}}})$ to an affine triangular automorphism of degree ≤ 3 (where $\overline{\mathrm{k}}$ is a fixed algebraic closure of k*), then*

$$\lambda(f) \in \{1, \sqrt{2},\ (1+\sqrt{5})/2,\ \sqrt{3}, 2,\ (1+\sqrt{13})/2,\ 1+\sqrt{2},\ \sqrt{6},\ (1+\sqrt{17})/2,\ 1+\sqrt{3},\ 3\}\,.$$

Proof. Let us write

$$L_d = \left\{ \frac{a+\sqrt{a^2+4bc}}{2} \,\middle|\, (a,b,c) \in \mathbb{N}^3, a+b \leq d, c \leq d \right\} \setminus \{0\} \quad \text{for each } d \geq 1\,.$$

This gives then

$$\begin{array}{lcl} L_1 & = & \{1\} \\ L_2 & = & \{1, \sqrt{2}, (1+\sqrt{5})/2, 2\} \\ L_3 & = & \{1, \sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \sqrt{3}, 2,\ \frac{1+\sqrt{13}}{2},\ 1+\sqrt{2},\ \sqrt{6},\ \frac{1+\sqrt{17}}{2},\ 1+\sqrt{3},\ 3\}\,. \end{array}$$

For each $d \in \{1,2,3\}$ holds: If $f \in \mathrm{Aut}(\mathbb{A}^3_{\mathrm{k}})$ is conjugated in $\mathrm{Aut}(\mathbb{A}^3_{\overline{\mathrm{k}}})$ to an affine triangular automorphism of degree $\leq d$, then Theorem 4.1.1 implies that $\lambda(f) \in L_d$. In particular, $\Lambda_{1,\mathrm{k}} \subseteq L_1$ and $\Lambda_{2,\mathrm{k}} \subseteq L_2$, as every element of $\mathrm{Aut}(\mathbb{A}^3_{\mathrm{k}})$ of degree ≤ 2 is equivalent to a triangular automorphism and is thus conjugate in $\mathrm{Aut}(\mathbb{A}^3_{\overline{\mathrm{k}}})$ to an affine triangular automorphism (Theorem 3).

It remains to see that $L_d \subseteq \Lambda_{i,\mathrm{k}}$ for $d = 1,2,3$, by giving explicit examples. For $d = 1$, we simply take the identity. For $d \in \{2,3\}$, we use elements of the form

$$f_{a,b,c} = (z + x^a y^b, y + x^c, x) \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}})$$

whose dynamical degrees are equal to $\lambda(f_{a,b,c}) = (a + \sqrt{a^2+4bc})/2$ when this number is not zero (Theorem 4.1.1).

For $d = 2$, we use $f_{1,0,2}$, $f_{0,1,2}$, $f_{1,1,1}$ and $f_{1,1,2}$, which all have degree 2 and dynamical degrees $1, \sqrt{2}, (1+\sqrt{5})/2, 2$ respectively.

For $d = 3$, we first use $f_{1,0,3}$, $f_{0,1,3}$, $f_{2,0,3}$, $f_{1,1,3}$, $f_{2,1,1}$, $f_{0,2,3}$, $f_{1,2,2}$, $f_{2,1,2}$ and $f_{0,3,3}$ which all have degree 3 and dynamical degrees 1, $\sqrt{3}$, 2, $(1+\sqrt{13})/2$, $1+\sqrt{2}$, $\sqrt{6}$, $(1+\sqrt{17})/2$, $1+\sqrt{3}$ and 3, respectively. In order to obtain the values $\sqrt{2}$ and $(1+\sqrt{5})/2$, we conjugate $f_{0,1,2} = (z+y, y+x^2, x)$ and $f_{1,1,1} = (z+xy, y+x, x)$ by $(x, y+z^3, z)$ and $(x, y+z^2, z)$, respectively, to get two automorphisms of $\mathbb{A}^3$ of degree 3 having dynamical degree equal to to $\lambda(f_{0,1,2}) = \sqrt{2}$ and $\lambda(f_{1,1,1}) = (1+\sqrt{5})/2$, respectively. □

4.2. **List of dynamical degrees of all automorphisms of degree 3.** An automorphism $f \in \mathrm{Aut}(\mathbb{A}^n)$ is called *algebraically stable*, if $\deg(f^r) = \deg(f)^r$ for all $r > 0$. In this case, $\lambda(f) = \deg(f)$. Now, let $\iota\colon \mathbb{A}^n \to \mathbb{P}^n$ be the standard embedding, i.e. $\iota(x_1, \ldots, x_n) = [1 : x_1 : \cdots : x_n]$. Note that f is algebraically stable, if and only if the extension of f to a birational map $\bar{f}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ via ι satisfies the following: $\bar{f}^r$ maps the hyperplane at infinity $H_\infty = \mathbb{P}^n \setminus \iota(\mathbb{A}^n)$ not into the base locus of $\bar{f}$ for each $r > 0$ (follows for instance from [Sib99, Proposition 1.4.3] or [Bla16, Lemma 2.14]).

The computation of the dynamical degrees in Theorem 2 is heavily based on the results of [BvS19a]. Let us recall the notations and results that we need here.

Definition 4.2.1. Let $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ and $r \in \mathbb{R}_{\geq 0}$. For a polynomial $p = \sum p_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{k}[x_1, \dots, x_n]$ (where $p_{i_1,\dots,i_n} \in \mathbf{k}$) its *$\mu$-homogeneous part of degree r* is the polynomial

$$\sum_{i_1\mu_1+\dots+i_n\mu_n=r} p_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{k}[x_1, \dots, x_n]\,.$$

For each $p \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$, we define $\deg_\mu(p)$ to be the maximum of the real numbers $r \in \mathbb{R}_{\geq 0}$ such that the μ-homogeneous part of degree r of p is non-zero. We then set $\deg_\mu(0) = -\infty$.

Definition 4.2.2. Let $f = (f_1, \dots, f_n) \in \mathrm{Aut}(\mathbb{A}^n)$ and let $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$. We define the *$\mu$-degree* of f by

$$\deg_\mu(f) = \inf\left\{\, \theta \in \mathbb{R}_{\geq 0} \mid \deg_\mu(f_i) \leq \theta\mu_i \text{ for each } i \in \{1, \dots, n\} \,\right\}.$$

In particular, $\deg_\mu(f) = \infty$ if the above set is empty. If $\theta = \deg_\mu(f) < \infty$, then for each $i \in \{1, \dots, n\}$, let $g_i \in \mathbf{k}[x_1, \dots, x_n]$ be the μ-homogeneous part of degree $\theta\mu_i$ of f_i. Then $g = (g_1, \dots, g_n) \in \mathrm{End}(\mathbb{A}^n)$ is called the *μ-leading part of f*.

The following result from [BvS19a] will serve as the main technique to compute dynamical degrees.

Proposition 4.2.3. [BvS19a, Proposition A] *Let $f \in \mathrm{Aut}(\mathbb{A}^n)$ and let $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{>0})^n$ be such that $\theta = \deg_\mu(f) \in \mathbb{R}_{>1}$. If the μ-leading part $g\colon \mathbb{A}^n \to \mathbb{A}^n$ of f satisfies $g^r \neq 0$ for each $r > 0$, then the dynamical degree $\lambda(f)$ is equal to θ.*

Proposition 4.2.4. *Let $f = (f_1, f_2, f_3) = \alpha \circ g \in \mathrm{Aut}(\mathbb{A}^3)$, where $\alpha \in \mathrm{Aff}(\mathbb{A}^3)$,*

$$g = (x + yz + za(x, z) + \xi z, y + a(x, z) + r(z), z),$$

$\xi \in \boldsymbol{k}$, $a(x, z) = a_0x^2 + a_1xz + a_2z^2 + a_3x + a_4z \in \boldsymbol{k}[x, z]$, $a_0, \dots, a_4 \in \boldsymbol{k}$, $r \in \boldsymbol{k}[z]$ has degree ≤ 3 and $(a_0, a_1) \neq (0, 0)$.

If $\alpha^*(z) \in \boldsymbol{k}[z]$, then $\lambda(f) = \deg_x(a) \in \{1, 2\}$. Otherwise, either f is algebraically stable (in which case $\lambda(f) = 3$) or f is conjugate by an element of $\mathrm{Aut}(\mathbb{A}^3)$ to an affine-triangular automorphism of degree ≤ 3, or we can conjugate f by an affine automorphism and reduce to one of the following cases:

1. *$\deg(r) = 3$, $\alpha^*(x) \in \boldsymbol{k}[z]$ and the coefficient of z^3 in f_3 is zero;*
2. *$\deg(r) \leq 2$, $\alpha^*(y) \in \boldsymbol{k}[z]$ and $\alpha^*(z) \in \boldsymbol{k}[y, z]$;*
3. *$\deg(r) \leq 2$, $\alpha^*(y) \in \boldsymbol{k}[z]$, $\alpha^*(x) \in \boldsymbol{k}[y, z]$ and $a_2 = 0$;*
4. *$\deg(r) \leq 2$, $\alpha^*(x) \in \boldsymbol{k}[z]$, $\alpha^*(y) \in \boldsymbol{k}[y, z]$, $a_1 \neq 0$ and $a_2 = 0$.*

Proof. (*A*) Suppose first that $\alpha^*(z) \in \mathbf{k}[z]$. Since the dynamical degree of the automorphism $z \mapsto \alpha^*(z)$ of $\mathbb{A}^1$ is 1, by [BvS19a, Lemma 2.3.1] the dynamical degree of f is given by $\lambda(f) = \lim_{r\to\infty} \deg_{x,y}(f^r)^{\frac{1}{r}}$. If $\deg_x(a) = 1$, then $\deg_{x,y}(f^r) = 1$ for each $r \geq 1$, so $\lambda(f) = 1$. We then suppose that $\deg_x(a) = 2$ and prove that $\lambda(f) = 2$. Choosing $\mu = (1, 1, 0)$, we find $\deg_{x,y}(p) = \deg_\mu(p)$ for each $p \in \mathbf{k}[x, y, z]$. As $za(x, z)$ and $a(x, z)$ are $\mathbf{k}$-linearly independent, one finds $\deg_\mu(f_1) = \deg_\mu(f_2) = 2$ and $\deg_\mu(f_3) = 0$. Hence, $\deg_\mu(f) = 2$ and the μ-leading part of f is $g = (g_1, g_2, g_3)$, where $g_3 = f_3 \in \mathbf{k}^*z + \mathbf{k}$ and $g_1, g_2 \in (\mathbf{k}x^2z + \mathbf{k}x^2) \setminus \{0\}$. This implies by induction on r that no component of g^r is zero, for each $r \geq 1$, which implies that $\lim_{r\to\infty} \deg_\mu(f^r)^{\frac{1}{r}} = 2$ [BvS19a, Lemma 2.6.1(5)]. This gives $\lambda(f) = 2$.

We may thus assume that $\alpha^*(z) \notin \mathbf{k}[z]$ in the sequel. We denote by $\overline{f}, \overline{g} \in \mathrm{Bir}(\mathbb{P}^3)$ and $\overline{\alpha}, \overline{\tau} \in \mathrm{Aut}(\mathbb{P}^3)$ the extensions of f, g and α, τ, via the standard embedding $\mathbb{A}^3 \hookrightarrow \mathbb{P}^3$, $(x, y, z) \mapsto [1 : x : y : z]$ and denote as usual by H_∞ the hyperplane $\mathbb{P}^3 \setminus \mathbb{A}^3$ given by $w = 0$ where w, x, y, z denote the homogeneous coordinates of $\mathbb{P}^3$. Denoting by $f_{i,j}$ the homogeneous part of f_i of degree j, the restriction of $\overline{f}$ to H_∞ is given by $[0 : x : y : z] \mapsto [0 : f_{1,3}(x, y, z) : f_{2,3}(x, y, z) : f_{3,3}(x, y, z)]$.

(B) Suppose now that $\deg(r) = 3$. This implies that $\mathrm{span}_{\mathbf{k}}(f_{1,3}, f_{2,3}, f_{3,3}) \subset \mathbf{k}[x, z]_3$ has dimension 2. Hence, the image by $\overline{f}$ of H_∞ is a line $\ell \subset H_\infty$ (as $(a_0, a_1) \neq (0, 0)$) and the base-locus of f is the line $\ell_z \subset H_\infty$ given by $z = 0$. As $\overline{g}(H_\infty)$ is the line ℓ_z and as $\alpha^*(z) \notin \mathbf{k}[z]$, the line $\ell = \overline{\alpha}(\ell_z) \subset H_\infty$ satisfies $\ell \neq \ell_z$. If $\overline{f}$ restricts to a dominant rational map $\ell \dashrightarrow \ell$, then f is algebraically stable, and the same holds if $\overline{f}(\ell \setminus \ell_z)$ is a point of $\ell \setminus \ell_z$. We may thus assume that $\overline{f}(\ell \setminus \ell_z) = \ell \cap \ell_z \in H_\infty$. The fact that $\overline{f}(\ell \setminus \ell_z)$ and thus also $\overline{g}(\ell \setminus \ell_z)$ is a point implies that $\ell = \overline{\alpha}(\ell_z)$ passes through the point $[0 : 0 : 1 : 0]$ and thus ℓ is given by $x = \mu z$ for some $\mu \in \mathbf{k}$. We may conjugate f with $\kappa = (x - \mu z, y, z) \in \mathrm{Aff}(\mathbb{A}^3)$ (this replaces α with $\kappa \circ \alpha$ and g with $g \circ \kappa^{-1}$ so does not change the form of g) and assume that $\mu = 0$.

Since $\overline{f}(\ell \setminus \ell_z) = \ell \cap \ell_z = [0 : 0 : 1 : 0]$, the coefficient of z^3 of f_3 (and of f_1) is equal to zero. As $\overline{\alpha}(\ell_z)$ is the line $x = 0$, we get $\alpha^*(x) \in \mathbf{k}[z]$. We are thus in Case (1).

(C): We may now assume that $\deg(r) < 3$ (and still $\alpha^*(z) \notin \mathbf{k}[z]$). We write

$$\alpha = (\alpha_{11}x + \alpha_{12}y + \alpha_{13}z + \beta_1, \alpha_{21}x + \alpha_{22}y + \alpha_{23}z + \beta_2, \alpha_{31}x + \alpha_{32}y + \alpha_{33}z + \beta_3)$$

where $\alpha_{ij} \in \mathbf{k}$ and $\beta_i \in \mathbf{k}$ for all $i, j \in \{1, 2, 3\}$. As $\deg(r) < 3$ the vector space $\mathrm{span}_{\mathbf{k}}(f_{1,3}, f_{2,3}, f_{3,3}) \subset \mathbf{k}[x, z]_3$ has dimension 1. The image of H_∞ by $\overline{f}$ is the point $q = [0 : \alpha_{11} : \alpha_{21} : \alpha_{31}] \in H_\infty$ and the base-locus of f is the union of three lines (maybe with multiplicity). If q is not in the base-locus, then f is algebraic stable. We may thus assume that $f_{i,3}(q) = 0$ for each i. We distinguish the possible cases, depending on whether α_{11} and α_{31} are zero or not.

(C1): Assume first that $\alpha_{11} = \alpha_{31} = 0$. As $\alpha^*(z) \notin \mathbf{k}[z]$, we get $\alpha_{32} \neq 0$. Conjugating by $\kappa = (x - \alpha_{12}/\alpha_{32}z, y, z)$ (this replaces α with $\kappa \circ \alpha$ and g with $g \circ \kappa^{-1}$ so does not change the form of g), we may assume that $\alpha_{12} = 0$.

As $g = (x + yz + \xi z, y + r(z), z) \circ (x, y + a(x, z), z)$, we find

$$\begin{aligned} h &= (h_1, h_2, h_3) = (x, y + a(x, z) + r(z), z) \circ f \circ (x, y - a(x, z) - r(z), z) \\ &= (x, y + a(x, z) + r(z), z) \circ \alpha \circ (x + (y - r(z))z + \xi z, y, z) \end{aligned}$$

with

$$\begin{aligned} h_1 &= \alpha_{13}z + \beta_1 \\ h_3 &= \alpha_{32}y + \alpha_{33}z + \beta_3 \\ h_2 &= \alpha_{21}(x + (y - r(z))z + \xi z) + \alpha_{22}y + \alpha_{23}z + \beta_2 + a(h_1, h_3) + r(h_3) \end{aligned}$$

We see that h is affine-triangular of degree ≤ 3 and thus f is conjugate to an affine triangular automorphism of degree ≤ 3.

(C2): Assume now that $\alpha_{11} \neq 0$ and $\alpha_{31} = 0$. The equality $\alpha_{31} = 0$ corresponds to $\alpha^*(z) \in \mathbf{k}[y, z]$. As $\alpha^*(z) \notin \mathbf{k}[z]$, we have $\alpha_{32} \neq 0$. Conjugating by $\kappa = (x, y - \alpha_{21}/\alpha_{11}x, z)$ we may assume that $\alpha_{21} = 0$ (as before, this replaces g with $g \circ \kappa^{-1}$

and thus does not change the form of g). We then conjugate by $(x, y-\alpha_{22}/\alpha_{32}z, z)$ and may assume that $\alpha_{22}=0$, so $\alpha^*(y)\in\mathbf{k}[z]$. We are thus in Case (2).

(C3): Assume now that $\alpha_{31}\neq 0$. Conjugating by $\kappa=(x-\alpha_{11}/\alpha_{31}z, y-\alpha_{21}/\alpha_{31}z, z)$, we may assume that $\alpha_{11}=\alpha_{21}=0$, so $\alpha^*(x),\alpha^*(y)\in\mathbf{k}[y,z]$ and $q=[0:0:0:1]$. As $f_{3,3}(q)=0$ and as the coefficient of x in $\alpha^*(z)$ is non-zero, we get $a_2=0$. If $\alpha_{12}\neq 0$, we conjugate by $(x, y-\alpha_{22}/\alpha_{12}x, z)$ and may assume that $\alpha_{22}=0$, so $\alpha^*(y)\in\mathbf{k}[z]$, giving Case (3). If $\alpha_{12}=0$ and $a_1\neq 0$, we get Case (4). We may thus assume that $\alpha_{11}=\alpha_{12}=\alpha_{21}=0$ and $a_1=a_2=0$. This gives $\alpha^*(x)\in\mathbf{k}[z]$, $\alpha^*(y)\in\mathbf{k}[y,z]$ and $a(x,z)=a_0x^2+a_3x+a_4z$, with $a_0\neq 0$. Then,

$$\begin{aligned} h &= (h_1,h_2,h_3)=(x, y+a_3x+a_0x^2, z)\circ f\circ (x, y-a_3x-a_0x^2, z)\\ &= (x, y+a_3x+a_0x^2, z)\circ\alpha\circ(x+yz+a_4z^2+\xi z, y+a_4z+r(z), z)\end{aligned}$$

is such that $h_1\in\mathbf{k}[z]$, $h_2\in\mathbf{k}[y,z]$ and $h_3\in\mathbf{k}[x,y,z]$ are of degree ≤ 2. Hence, f is conjugate by an element of $\mathrm{Aut}(\mathbb{A}^3)$ to an affine-triangular automorphism of degree ≤ 2. □

Proposition 4.2.5. *The dynamical degree of any $f=\alpha\circ g$ as in the four Cases (1)-(2)-(3)-(4) of Proposition 4.2.4, is given as follows:*

(1) $\lambda(f)=\begin{cases} 1+\sqrt{2} & \text{if } a_1\neq 0;\\ (1+\sqrt{13})/2 & \text{if } a_1=0.\end{cases}$

(2) $\lambda(f)=\begin{cases} 1+\sqrt{3} & \text{if } a_0\neq 0;\\ 1+\sqrt{2} & \text{if } a_0=0.\end{cases}$

(3) *Writing the coefficient of z^2 in f_1 as ε, we obtain*

$$\lambda(f)=\begin{cases} 1+\sqrt{3} & \text{if } a_1\neq 0 & \text{and} & \varepsilon\neq 0;\\ (3+\sqrt{5})/2 & \text{if } a_1\neq 0 & \text{and} & \varepsilon= 0;\\ (1+\sqrt{17})/2 & \text{if } a_1= 0 & \text{and} & \varepsilon\neq 0;\\ 2 & \text{if } a_1= 0 & \text{and} & \varepsilon= 0.\end{cases}$$

(4) $\lambda(f)=1+\sqrt{2}$.

Proof. (1): We have $\deg(r)=3$, $\alpha^*(x)\in\mathbf{k}[z]$ and the coefficient of z^3 in f_3 is zero. This gives $f_1=f_{1,0}+f_{1,1}\in\mathbf{k}[z]$ and implies that the coefficient of z^3 in f_2 is not zero. Let θ be in the open intervall $(2,3)$ and choose $\mu=(1,3,\theta)$. The μ-degree of z^3 is bigger than any other monomial that occurs in f_1, f_2 or f_3, as $\theta>2$. We get $\deg_\mu(f_1)=\theta$, $\deg_\mu(f_2)=3\theta$, with μ-leading parts equal to ζ_1z and ζ_2z^3 for some $\zeta_1,\zeta_2\in\mathbf{k}^*$, respectively. As the coefficient of z^3 in f_3 is zero, the monomial yz occurs in f_3. Hence, the μ-leading part of f_3 belongs to $(\mathbf{k}yz+\mathbf{k}xz^2)\setminus\{0\}$. Indeed, as $\deg_\mu(y)>\deg_\mu(z)>\deg_\mu(x)$, $\deg_\mu(yz)=3+\theta$ is the biggest μ-degree of the monomials of degree ≤ 2 appearing in f; moreover $\deg_\mu(yz)>\deg_\mu(x^2z)=2+\theta$.

If $a_1\neq 0$, the coefficient of xz^2 in f_3 is not zero, so $t\in\mathbf{k}xz^2$ (since $\theta>2$). We choose $\theta=1+\sqrt{2}$ and observe that $\theta^2=2\theta+1$. Thus we obtain $\deg_\mu(f)=\theta$, with μ-leading part $g=(\zeta_1z,\zeta_2z^3,\zeta_3xz^2)$, where $\zeta_3\in\mathbf{k}^*$.

If $a_1=0$, then $t\in\mathbf{k}yz$. We choose $\theta=(1+\sqrt{13})/2$ and observe that $\theta^2=\theta+3$. Thus we obtain $\deg_\mu(f)=\theta$, with μ-leading part $g=(\zeta_1z,\zeta_2z^3,\zeta_3yz)$, where $\zeta_3\in\mathbf{k}^*$.

As g is monomial, we have $g^r\neq 0$ for each $r\geq 1$, so $\lambda(f)$ is equal to θ in both cases (Proposition 4.2.3).

(2): We have $\deg(r) \le 2$, $\alpha^*(y) \in \mathbf{k}[z]$ and $\alpha^*(z) \in \mathbf{k}[y,z]$. This gives

$$\begin{array}{rcl} f_1 &=& f_{1,0} + f_{1,1} + f_{1,2} + \zeta_1 z(a_0x^2 + a_1xz + a_2z^2), \\ f_2 &=& f_{2,0} + \zeta_2 z, \\ f_3 &=& f_{3,0} + f_{3,1} + \zeta_3(a_0x^2 + a_1xz) + \varepsilon_3 z^2, \end{array}$$

where $\zeta_1, \zeta_2, \zeta_3 \in \mathbf{k}^*$, $\varepsilon_3 \in \mathbf{k}$.

If $a_0 \neq 0$, we choose $\theta = 1 + \sqrt{3}$, $\mu = (\theta+1, 1, \theta)$ and observe that $\theta^2 = 2\theta + 2$. Then, $\deg_\mu(f) = \theta$, with μ-leading part $(\zeta_1 a_0 x^2 z, \zeta_2 z, \zeta_3 a_0 x^2)$. This gives $\lambda(f) = \theta$ by Proposition 4.2.3.

If $a_0 = 0$, then $a_1 \neq 0$. We choose $\theta = 1 + \sqrt{2}$, $\mu = (\theta + 1, 1, \theta)$ and observe that $\theta^2 = 2\theta + 1$. Then, $\deg_\mu(f) = \theta$, with μ-leading part $(\zeta_1 a_1 x z^2, \zeta_2 z, \zeta_3 a_1 xz)$. This gives $\lambda(f) = \theta$ by Proposition 4.2.3.

(3): We have $\deg(r) \le 2$, $\alpha^*(y) \in \mathbf{k}[z]$, $\alpha^*(x) \in \mathbf{k}[y,z]$ and $a_2 = 0$. This gives

$$\begin{array}{rcl} f_1 &=& f_{1,0} + f_{1,1} + \zeta_1(a_0x^2 + a_1xz) + \varepsilon_3 z^2, \\ f_2 &=& f_{2,0} + \zeta_2 z, \\ f_3 &=& f_{3,0} + f_{3,1} + f_{3,2} + \zeta_3 z(a_0x^2 + a_1xz), \end{array}$$

where $\zeta_1, \zeta_2, \zeta_3 \in \mathbf{k}^*$, $\varepsilon_3 \in \mathbf{k}$.

If $a_1 \neq 0$ and $\varepsilon_3 \neq 0$, then we choose $\theta = 1 + \sqrt{3}$, $\mu = (2, 1, \theta)$ and observe that $\theta^2 = 2\theta + 2$. Then, $\deg_\mu(f) = \theta$, with μ-leading part $(\varepsilon_3 z^2, \zeta_2 z, \zeta_3 a_1 x z^2)$. This gives $\lambda(f) = \theta$ by Proposition 4.2.3.

If $a_1 \neq 0$ and $\varepsilon_3 = 0$, then we choose $\theta = (3+\sqrt{5})/2$, $\mu = (1, \theta - 2, \theta - 1)$ and observe that $\theta^2 = 3\theta - 1$. Then $\deg_\mu(f) = \theta$, with μ-leading part $(\zeta_1 a_1 xz, \zeta_2 z, \zeta_3 a_1 x z^2)$. This gives $\lambda(f) = \theta$ by Proposition 4.2.3.

If $a_1 = 0$ and $\varepsilon_3 \neq 0$, then $a_0 \neq 0$ and we choose $\theta = (1 + \sqrt{17})/2$, $\mu = (2, 1, \theta)$. Observe that $\theta^2 = \theta + 4$. Then $\deg_\mu(f) = \theta$, with μ-leading part $(\varepsilon_3 z^2, \zeta_2 z, \zeta_3 a_0 x^2 z)$. This gives $\lambda(f) = \theta$ by Proposition 4.2.3.

If $a_1 = \varepsilon_3 = 0$, then $a_0 \neq 0$ and we choose $\theta = 2$, $\mu = (1, 1, \theta)$. Then $\deg_\mu(f) = \theta$, with μ-leading part $q = (\zeta_1 a_0 x^2 + \xi_1 z, \zeta_2 z, \zeta_3 a_0 x^2 z + \xi_3 z^2)$ for some ξ_1, $\xi_3 \in \mathbf{k}$. Let $\hat{q}\colon \mathbb{A}^2 \to \mathbb{A}^2$, $(x,z) \mapsto (\zeta_1 a_0 x^2 + \xi_1 z, \zeta_3 a_0 x^2 z + \xi_3 z^2)$ and observe that $\hat{q}$ is dominant (as $\zeta_1 a_0$ and $\zeta_3 a_0$ are both non-zero). As $\pi \circ q = \hat{q} \circ \pi$ for $\pi\colon \mathbb{A}^3 \to \mathbb{A}^2$, $(x,y,z) \mapsto (x,z)$, it follows that $q^r \neq 0$ for each $r \ge 1$. This gives $\lambda(f) = \theta$ by Proposition 4.2.3.

(4): We have $\deg(r) \le 2$, $\alpha^*(x) \in \mathbf{k}[z]$, $\alpha^*(y) \in \mathbf{k}[y,z]$, $a_1 \neq 0$ and $a_2 = 0$. This gives

$$\begin{array}{rcl} f_1 &=& f_{1,0} + \zeta_1 z, \\ f_2 &=& f_{2,0} + f_{2,1} + \zeta_2(a_0x^2 + a_1xz) + \varepsilon_2 z^2, \\ f_3 &=& f_{3,0} + f_{3,1} + f_{3,2} + \zeta_3 z(a_0x^2 + a_1xz), \end{array}$$

where $\zeta_1, \zeta_2, \zeta_3 \in \mathbf{k}^*$, $\varepsilon_2 \in \mathbf{k}$. We choose $\theta = 1 + \sqrt{2}$, $\mu = (1, 2, 1 + \sqrt{2})$ and observe that $\theta^2 = 2\theta + 1$. Then $\deg_\mu(f) = \theta$ with μ-leading part $(\zeta_1 z, \varepsilon_2 z^2, \zeta_3 a_1 x z^2)$. As $a_1 \neq 0$ and $\zeta_1, \zeta_3 \neq 0$, this gives $\lambda(f) = \theta$ by Proposition 4.2.3. $\square$

Example 4.2.6. We illustrate the different cases (1)-(4) of Proposition 4.2.4 and Proposition 4.2.5, by giving a simple example in each possible case and we give examples for the two cases where $\alpha^*(z) = z$ and the case where f is algebraically stable. All of them are of the form $\alpha \circ g$, where $\alpha \in \mathrm{Aff}(\mathbb{A}^3)$, $g = (x + yz + za(x,z), y + a(x,z) + r(z), z)$, $a = a_0x^2 + a_1xz + a_2z^2 \in \mathbf{k}[x,z] \setminus \mathbf{k}[z]$ is homogeneous of degree 2 and $r \in \mathbf{k}[z]$ is of degree ≤ 3.

Case	a	r	$f \in \mathrm{Aut}(\mathbb{A}^3)$	$\lambda(f)$
	xz	0	$(x+yz+xz^2, y+xz, z)$	1
	x^2	0	$(x+yz+x^2z, y+x^2, z)$	2
	xz	z^3	$(x+yz+xz^2, z, y+xz+z^3)$	3
(1)	xz	z^3	$(z, y+xz+z^3, x+yz+xz^2)$	$1+\sqrt{2}$
(1)	x^2	z^3	$(z, y+x^2+z^3, x+yz+x^2z)$	$(1+\sqrt{13})/2$
(2)	x^2	0	$(x+yz+zx^2, z, y+x^2)$	$1+\sqrt{3}$
(2)	xz	0	$(x+yz+xz^2, z, y+xz)$	$1+\sqrt{2}$
(3)	xz	z^2	$(y+xz+z^2, z, x+yz+xz^2)$	$1+\sqrt{3}$
(3)	xz	0	$(y+xz, z, x+yz+xz^2)$	$(3+\sqrt{5})/2$
(3)	x^2	z^2	$(y+x^2+z^2, z, x+yz+x^2z)$	$(1+\sqrt{17})/2$
(3)	x^2	0	$(y+x^2, z, x+yz+x^2z)$	2
(4)	xz	0	$(z, y+xz, x+yz+xz^2)$	$1+\sqrt{2}$

Proof of Theorem 2. Corollary 4.1.2 gives the values of $\Lambda_{1,\mathrm{k}}$ and $\Lambda_{2,\mathrm{k}}$, proves that $\Lambda_{3,\mathrm{k}}$ contains $L_3 = \{1, \sqrt{2}, \frac{1+\sqrt{5}}{2}, \sqrt{3}, 2, \frac{1+\sqrt{13}}{2}, 1+\sqrt{2}, \sqrt{6}, \frac{1+\sqrt{17}}{2}, 1+\sqrt{3}, 3\}$ and that for each $f \in \mathrm{Aut}(\mathbb{A}^3_{\mathrm{k}})$ which is conjugated in $\mathrm{Aut}(\mathbb{A}^3_{\overline{\mathrm{k}}})$ to an affine triangular automorphism of degree ≤ 3 (where $\overline{\mathrm{k}}$ is a fixed algebraic closure of k), we have $\lambda(f) \in L_3$.

Moreover, the element $(y+xz, z, x+z(y+xz)) \in \mathrm{Aut}(\mathbb{A}^3_{\mathrm{k}})$ has dynamical degree $(3+\sqrt{5})/2$ (follows from Proposition 4.2.5 as it belongs to Case (3) with $a_1 \neq 0$ and $\varepsilon = 0$, see also Example 4.2.6).

It remains then to see that each element $f \in \mathrm{Aut}(\mathbb{A}^3_{\mathrm{k}})$ of degree 3 has a dynamical degree which is either equal to $(3+\sqrt{5})/2$ or belongs to L_3. By Theorem 3, f is conjugate in $\mathrm{Aut}(\mathbb{A}^3_{\overline{\mathrm{k}}})$ either to an affine-triangular automorphism or to $f = \alpha \circ (yz + za(x,z) + x, y + a(x,z) + r(z), z)$ where $a \in \mathbf{k}[x,z] \setminus \mathbf{k}[z]$ is homogeneous of degree 2 and $r \in \mathbf{k}[z]$ is of degree ≤ 3. In the first case, $\lambda(f) \in L_3$ by Corollary 4.1.2. In the second case, Propositions 4.2.4 and 4.2.5 show that either $\lambda(f) = (3+\sqrt{5})/2$ or $\lambda(f) \in L_3$. This achieves the proof. □

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Jérémy Blanc, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland
Email address: jeremy.blanc@unibas.ch

Immanuel van Santen, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland
Email address: immanuel.van.santen@math.ch

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Andriy Regeta & Immanuel van Santen

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CHARACTERIZING SMOOTH AFFINE SPHERICAL VARIETIES VIA THE AUTOMORPHISM GROUP

by Andriy Regeta & Immanuel van Santen

Abstract. — Let G be a connected reductive algebraic group. We prove that for a quasi-affine G-spherical variety the weight monoid is determined by the weights of its non-trivial $\mathbb{G}_a$-actions that are homogeneous with respect to a Borel subgroup of G. As an application we get that a smooth affine spherical variety that is non-isomorphic to a torus is determined by its automorphism group (considered as an ind-group) inside the category of smooth affine irreducible varieties.

Résumé (Caractérisation des variétés sphériques affines lisses par le groupe des automorphismes)

Soit G un groupe réductif connexe. Nous montrons que le monoïde des poids d'une variété G-sphérique quasi-affine est déterminé par les poids de ses $\mathbb{G}_a$-actions non triviales homogènes sous l'action d'un sous-groupe de Borel de G. Comme application, nous obtenons qu'une variété sphérique affine lisse non isomorphe à un tore est déterminée par son groupe des automorphismes (considéré comme un ind-groupe) dans la catégorie des variétés irréductibles affines lisses.

Contents

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1. Introduction

In this article, we work over an algebraically closed field $\boldsymbol{k}$ of characteristic zero if it is not specified otherwise.

In [Kra17, Th. 1.1], Kraft proved that $\mathbb{A}^n$ is determined by its automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ seen as an ind-group inside the category of connected affine varieties (see [FK] for a reference of ind-groups) and in [KRvS19, Main Th.], this result was partially generalized (over the complex numbers) in case $\mathrm{Aut}(\mathbb{A}^n)$ is seen only as an abstract group. In [CRX19, Th. A], the last results are widely generalized in the following sense: $\mathbb{A}^n$ is completely characterized through the abstract group $\mathrm{Aut}(\mathbb{A}^n)$ inside the category of connected affine varieties. The result of Kraft was partially generalized (over the complex numbers) to other affine varieties than the affine space in [Reg17] and [LRU19]. More precisely, there is the following statement (formulated over the complex numbers, but valid with the same proof over $\boldsymbol{k}$):

Theorem 1 ([LRU19, Th. 1.4]). — *Let X be an affine toric variety different from the torus and let Y be an irreducible normal affine variety. If $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic as ind-groups, then X and Y are isomorphic as varieties.*

Remark 2. — In fact, in both [Kra17] and [LRU19, Th. 1.4], the authors prove the statements under the slightly weaker assumption that there is a group isomorphism $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ that preserves algebraic subgroups (see Section 5 for the definition).

A natural generalization of toric varieties are the so-called spherical varieties. Let G be a connected reductive algebraic group. Recall that a normal variety X endowed with a faithful G-action is called *G-spherical* if some (and hence every) Borel subgroup in G acts on X with an open dense orbit, see e.g. [Bri10] for a survey and [Tim11] for a reference of the topic. If G is a torus, then a G-spherical variety is the same thing as a G-toric variety. If X is G-spherical, then X has an open G-orbit which is isomorphic to G/H for some subgroup $H \subset G$. The family of G-spherical varieties is, in a sense, the widest family of G-varieties which is well-studied: in fact, G-equivariant open embeddings of G-homogeneous G-spherical varieties are classified by certain combinatorial data (analogous to the classical case of toric varieties) by Luna-Vust [LV83] (see also the work of Knop [Kno91]) and homogeneous G-spherical varieties are classified for $\boldsymbol{k}$ equal to the complex numbers by Luna, Bravi, Cupit-Foutou, Losev and Pezzini [Lun01, BP05, Bra07, Lun07, Los09b, BCF10, CF14].

In this paper, we generalize partially Theorem 1 to quasi-affine G-spherical varieties. In order to state our main results, let us introduce some notation. Let X be an irreducible G-variety for a connected algebraic group G with a fixed Borel subgroup $B \subset G$. We denote by $\mathfrak{X}(B)$ the character group of B, i.e., the group of regular group homomorphisms $B \to \mathbb{G}_m$. The *weight monoid* of X is defined by

$$\Lambda^+(X) = \{\lambda \in \mathfrak{X}(B) \mid \mathscr{O}(X)^{(B)}_\lambda \neq 0\},$$

where $\mathscr{O}(X)^{(B)}_\lambda \subset \mathscr{O}(X)$ denotes the subspace of B-semi-invariants of weight λ of the coordinate ring $\mathscr{O}(X)$ of X, i.e.,

$$\mathscr{O}(X)^{(B)}_\lambda = \{f \in \mathscr{O}(X) \mid b \cdot f = \lambda(b) f \text{ for all } b \in B\}.$$

Our main result in this article is the following:

Main Theorem A. — *Let X, Y be irreducible normal quasi-affine varieties, let $\theta\colon \mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ be a group isomorphism that preserves algebraic subgroups (see Section 5 for the definition) and let G be a connected reductive algebraic group. Moreover, we fix a Borel subgroup $B \subset G$. If X is G-spherical and not isomorphic to a torus, then the following holds:*

(1) *Y is G-spherical for the induced G-action via θ;*
(2) *the weight monoids $\Lambda^+(X)$ and $\Lambda^+(Y)$ inside $\mathfrak{X}(B)$ are the same;*
(3) *if one of the following assumptions holds*
  (i) *X, Y are smooth and affine or*
  (ii) *X, Y are affine and G is a torus,*

then X and Y are isomorphic as G-varieties.

We prove Main Theorem A(1) in Proposition 7.7, Main Theorem A(2) in Corollary 8.6 and Main Theorem A(3) in Theorem 8.7.

In case X is isomorphic to a torus and X is G-spherical, it follows that G is in fact a torus of dimension $\dim X$. Indeed, as each unipotent closed subgroup of G acts trivially on $X \simeq (\boldsymbol{k}^*)^{\dim X}$ and since G acts faithfully on X, it follows that G has no unipotent elements; hence G is a torus [Hum75, Prop. B, §21.4]. Thus $X \simeq G$. Then [LRU19, Exam. 6.17] gives an example of an affine variety Y such that there is a group isomorphism $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ that preserves algebraic subgroups, but Y is not G-toric. Thus the assumption that X is not isomorphic to a torus in Main Theorem A is essential.

Moreover, in general, we cannot drop the normality condition in Main Theorem A: We provide an example in Proposition 9.1 where the weight monoids of X and Y are different, see Section 9.

Outline of the proof of Main Theorem A(1). — We introduce *generalized root subgroups* of $\mathrm{Aut}(X)$ and study these subgroups and their weights for a G-variety X (see Section 7 for details). We show that if G is not a torus, then an irreducible normal quasi-affine variety with a faithful G-action is G-spherical if and only if the dimension of all generalized root subgroups of $\mathrm{Aut}(X)$ with respect to B is bounded (see Definition 7.1, Proposition 7.3 and Lemma 7.6). This characterization of the sphericity is stable under group isomorphisms of automorphism groups that preserve algebraic groups and thus we get Main Theorem A(1).

Outline of the proof of Main Theorem A(2). — We show that the weight monoid $\Lambda^+(X)$ of a quasi-affine G-spherical variety X is encoded in the following set:

$$D(X) = \left\{ \lambda \in \mathfrak{X}(B) \;\middle|\; \begin{array}{l} \text{there exists a non-trivial } B\text{-homogeneous} \\ \mathbb{G}_{\mathrm{a}}\text{-action on } X \text{ of weight } \lambda \end{array} \right\}$$

(see Section 4.2 for the definition of a B-homogeneous $\mathbb{G}_{\mathrm{a}}$-action). We call $D(X)$ *the set of B-homogeneous $\mathbb{G}_{\mathrm{a}}$-weights* of X. To $D(X) \subset \mathfrak{X}(B)$ we may associate its asymptotic cone $D(X)_\infty$ inside $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$ and consider the convex cone $\mathrm{Conv}(D(X)_\infty)$ of it (see Section 2 for the definitions). We prove then the following "closed formula" for the weight monoid:

Main Theorem B. — *Let G be a connected reductive algebraic group, $B \subset G$ a Borel subgroup, and X a quasi-affine G-spherical variety that is non-isomorphic to a torus. If neither G is a torus nor $\mathrm{Spec}(\mathscr{O}(X)) \not\simeq \mathbb{A}^1 \times (\mathbb{A}^1 \smallsetminus \{0\})^{\dim(X)-1}$, then*

$$\Lambda^+(X) = \mathrm{Conv}(D(X)_\infty) \cap \mathrm{Span}_{\mathbb{Z}}(D(X)),$$

where the asymptotic cones and linear spans are taken inside $\mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$.

Main Theorem B is proved in Theorem 8.2. As a consequence of this result, we get that the set of B-homogeneous $\mathbb{G}_{\mathrm{a}}$-weights determines the weight monoid, see Corollary 8.4:

Main Theorem C. — *Let G be a connected reductive algebraic group and let X, Y be quasi-affine G-spherical varieties with $D(X) = D(Y)$. Then $\Lambda^+(X) = \Lambda^+(Y)$.*

Using this last result, we get then Theorem A(2), as the existence of a group isomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ that preserves algebraic groups implies that $D(X) = D(Y)$, see Lemma 5.1.

Outline of the proof of Main Theorem A(3). — Note that the statement of Main Theorem A(3ii) is the same as Theorem 1 together with Remark 2. We mentioned it here as it is a direct consequence of Main Theorem A(2). Again using Main Theorem A(2), the statement of Main Theorem A(3i) is a direct consequence of the following beautiful result of Losev that proves Knop's Conjecture:

Theorem 3 ([Los09a, Th. 1.3]). — *If X and Y are smooth affine G-spherical varieties with $\Lambda^+(X) = \Lambda^+(Y)$, then X and Y are isomorphic as G-varieties.*

Outline of the structure of the paper. — In Section 2 we introduce the concept of the asymptotic cone D_∞ associated to a given set D in a Euclidean vector space. One can think of D_∞ as the set one receives if one looks at D from "infinitely far away". We provide in this Section several properties of (asymptotic) cones used for our study of homogeneous $\mathbb{G}_{\mathrm{a}}$-actions on toric varieties in Section 6 and also for the proof of our "closed formula" of the weight monoid in terms of the set of homogeneous $\mathbb{G}_{\mathrm{a}}$-weights, i.e., Main Theorem B.

In Sections 3, 4, 5 we gather general results about quasi-affine varieties, vector fields, automorphism groups of varieties and root subgroups. This material is constantly used in the rest of the article. For several results we don't have an appropriate reference to the literature and thus we provide full proofs.

In Section 6 we study homogeneous $\mathbb{G}_{\mathrm{a}}$-actions on quasi-affine toric varieties. Let us highlight the two main results. For this, let X be a quasi-affine toric variety described by some fan Σ of convex cones. Then the associated affine variety $X_{\mathrm{aff}} := \operatorname{Spec}(\mathscr{O}(X))$ is again toric (see Lemma 3.4) and thus can be described by some convex cone σ. Our first main result in this section (Corollary 6.7) provides a full description of the homogeneous $\mathbb{G}_{\mathrm{a}}$-actions on X in terms of the fan Σ. In our second main result (Corollary 6.9) we describe the asymptotic cone of the set $D(X)$ of homogeneous $\mathbb{G}_{\mathrm{a}}$-weights of X in terms of the convex cone σ.

In Section 7 we show that the automorphism group determines the sphericity, i.e., we prove Main Theorem A(1). As already mentioned, the idea is to characterize the sphericity in terms of so-called generalized root subgroups, see Proposition 7.3.

In Section 8 we prove Theorem 8.2 which gives the closed formula in Main Theorem B. Note that for a quasi-affine G-spherical variety X the following fact holds: the algebraic quotient $X_{\mathrm{aff}}/\!\!/U$ is an affine toric variety, where U denotes the unipotent radical of a Borel subgroup of G. Using this fact and our study of the homogeneous $\mathbb{G}_{\mathrm{a}}$-actions presented in Section 6, we prove Theorem 8.2. We then get Main Theorem C as a consequence, see Corollary 8.4. At the end of this Section we prove Theorem 8.7 which is the statement of Main Theorem A(3).

In Section 9 we provide an example that shows that the normality condition in Main Theorem A is essential.

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2. Cones and asymptotic cones

In the following section we introduce some basic facts about cones and asymptotic cones. As a reference for cones we take [Ful93, §1.2] and as a reference for asymptotic cones we take [AT03, Chap. 2].

Throughout this section V denotes a non-zero Euclidean vector space, i.e., a finite dimensional $\mathbb{R}$-vector space $V \neq \{0\}$ together with a scalar product

$$V \times V \longrightarrow \mathbb{R}, \quad (u, v) \longmapsto \langle u, v \rangle.$$

The induced norm on V we denote by $\|\cdot\| : V \to \mathbb{R}$.

A subset $C \subset V$ is a *cone* if for all $\lambda \in \mathbb{R}_{\geqslant 0}$ and for all $c \in C$ we have $\lambda \cdot c \in C$. The *asymptotic cone* D_∞ of a subset $D \subset V$ is defined as follows:

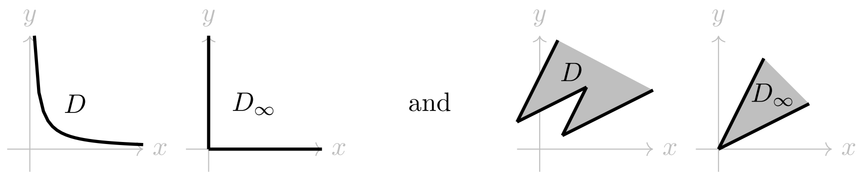
$$D_\infty = \left\{ x \in V \smallsetminus \{0\} \;\middle|\; \begin{array}{l} \text{there exists a sequence } (x_i)_i \text{ in } D \text{ with } \|x_i\| \to \infty \\ \text{such that } x_i/\|x_i\| \to x/\|x\| \end{array} \right\} \cup \{0\}.$$

The asymptotic cone satisfies the following basic properties, see e.g. [AT03, Prop. 2.1.1, Prop. 2.1.9].

Lemma 2.1 (Properties of asymptotic cones)

(1) *If* $D \subset V$*, then* $D_\infty \subset V$ *is a closed cone.*
(2) *If* $C \subset V$ *is a closed cone, then* $C_\infty = C$*.*
(3) *If* $D \subset D' \subset V$*, then* $D_\infty \subset (D')_\infty$*.*
(4) *If* $D \subset V$ *and* $v \in V$*, then* $(v + D)_\infty = D_\infty$*.*
(5) *If* $D_1, \ldots, D_k \subset V$*, then* $(D_1 \cup \cdots \cup D_k)_\infty = (D_1)_\infty \cup \cdots \cup (D_k)_\infty$*.* □

In order to illustrate the definition of the asymptotic cone, we draw the picture of two sets D in $\mathbb{R}^2$ and their asymptotic cones D_∞ in $\mathbb{R}^2$. In the first case, D is given by $xy = 1$, $x > 0$ and in the second case, D is the union of two translated copies of a cone in the plane.



Lemma 2.2 (Asymptotic cone of a δ-neighbourhood). — *Let* $D \subset V$ *and let* $\delta \in \mathbb{R}$ *with* $\delta \geqslant 0$*. Then the* δ*-neighbourhood of* D

$$D^\delta := \{x \in V \mid \text{there is } y \in D \text{ with } \|x - y\| \leqslant \delta\}$$

satisfies $(D^\delta)_\infty = D_\infty$*.*

Proof of Lemma 2.2. — We only have to show that $(D^\delta)_\infty \subset D_\infty$. Let $0 \neq x \in (D^\delta)_\infty$ and let $(x_i)_i$ be a sequence in D^δ such that $\|x_i\| \to \infty$ and $x_i / \|x_i\| \to x / \|x\|$. By definition, there is a sequence $(y_i)_i$ in D such that $\|x_i - y_i\| \leqslant \delta$. In particular, we get $\|y_i\| \to \infty$. Let $m_i := \min\{\|x_i\|, \|y_i\|\}$. Then $m_i \to \infty$ and for sufficiently big i

$$0 \leqslant \left\| \frac{x_i}{\|x_i\|} - \frac{y_i}{\|y_i\|} \right\| \leqslant \frac{\|x_i - y_i\|}{m_i} \leqslant \frac{\delta}{m_i}.$$

As $\delta / m_i \to 0$, the above inequality implies $x/\|x\| = \lim_{i \to \infty} x_i / \|x_i\| = \lim_{i \to \infty} y_i / \|y_i\|$. □

For a subset $D \subset V$ we denote by $\mathrm{int}(D)$ the topological interior of D inside the linear span of D.

Lemma 2.3 (Intersection of a cone with an affine hyperplane.) — *Let* $C \subset V$ *be a cone and let* H_1 *be an affine hyperplane in* V *such that* $0 \notin H_1$*. If* $C \cap H_1 \neq \varnothing$*, then* $\mathrm{int}(C) \cap H_1 \neq \varnothing$*.*

Proof. — Let $\pi \colon V \to \mathbb{R}$ be a linear map such that $H_1 = \pi^{-1}(1)$. By assumption, there is $c \in C \cap H_1$. We may assume that c lies in the topological boundary of C inside the linear span of C (otherwise we are finished). By the continuity of π, there is $c' \in \operatorname{int}(C)$ such that $|\pi(c) - \pi(c')| < 1$. As $\pi(c) = 1$, we get $\pi(c') > 0$. Then $\lambda = 1/\pi(c') \in \mathbb{R}_{>0}$ and thus $\lambda c' \in \operatorname{int}(C) \cap H_1$. ☐

A subset $C \subset V$ is called *convex* if for all $x, y \in C$ and all $\alpha \in [0,1]$, we have $\alpha x + (1-\alpha) y \in C$. A convex cone $C \subset V$ is called *strongly convex* if it contains no linear subspace of V except the zero subspace. For a subset $D \subset V$, we denote by $\operatorname{Conv}(D)$ the *convex cone generated by* D in V, i.e.,

$$\operatorname{Conv}(D) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \in V \mid v_1, \ldots, v_k \in D \text{ and } \lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geqslant 0}\}.$$

Lemma 2.4 (Asymptotic cone of the intersection of a closed convex cone with an affine hyperplane)

Let $C \subset V$ be a closed convex cone and let $H \subset V$ be a hyperplane. Then for each $v \in V$ such that $C \cap (v + H) \neq \varnothing$, we have

$$(C \cap (v+H))_\infty = C \cap H.$$

Proof. — We denote $D \coloneqq C \cap (v+H) \subset V$. As $D \neq \varnothing$ we can take $x \in D$. If $y \in C \cap H$, then $x + y \in C$ and $x + y \in v + H$, thus $x + y \in D$. This shows that $x + (C \cap H) \subset D$ and by Lemma 2.1, we get $C \cap H \subset D_\infty$. Now, from Lemma 2.1 we get also the reverse inclusion (here we use that C is a closed cone):

$$D_\infty = (C \cap (v+H))_\infty \subset C_\infty \cap (v+H)_\infty = C_\infty \cap H_\infty = C \cap H. \qquad ☐$$

A subset $C \subset V$ is a *convex polyhedral cone* if there is a finite subset $F \subset V$ such that

$$C = \operatorname{Conv} F.$$

For a convex polyhedral cone C in V, set

$$C^\vee = \{x \in V \mid \langle c, x\rangle \geqslant 0 \text{ for all } c \in C\}.$$

By [Ful93, Propty (1), p. 9] we have $C = (C^\vee)^\vee$. In particular C is closed in V.

A hyperplane $H \subset V$ passing through the origin is called a *supporting hyperplane* of a convex polyhedral cone $C \subset V$ if C is contained in one of the closed half spaces in V delimited by H, i.e., there is a normal vector $u \in V$ to H such that

$$C \subset \{x \in V \mid \langle u, x\rangle \geqslant 0\}.$$

A *face* of a convex polyhedral cone $C \subset V$ is the intersection of C with a supporting hyperplane of C in V. A face of dimension one of C is called an *extremal ray of C*.

For a fixed lattice $\Lambda \subset V$ (i.e., a finitely generated subgroup of $(V,+)$ of rank $\dim V$), a convex polyhedral cone $C \subset V$ is called *rational (with respect to Λ)* if there is a finite subset $F \subset \Lambda$ such that $C = \operatorname{Conv}(F)$. In case C is strongly convex, then C is rational if and only if each extremal ray of C is generated by some element from $C \cap \Lambda$, see [Ful93, p. 14]. Note that a face of a rational convex polyhedral cone is again a rational convex polyhedral cone, see [Ful93, Prop. 2].

Lemma 2.5. — *Let $\Lambda \subset V$ be a lattice.*

(1) *If $C \subset V$ is a rational convex polyhedral cone, then $C = (C \cap \Lambda)_\infty$.*

(2) *If $v_0 \in V$, $S \subset V$ denotes the unit sphere with center* 0 *and* $\rho\colon V \smallsetminus \{0\} \to S$ *denotes the map given by* $v \mapsto v/\|v\|$, *then* $\rho((v_0+\Lambda)\smallsetminus\{0\})$ *is dense in* S.

Proof

(1) Let $v_1, \ldots, v_r \in C \cap \Lambda$ such that $C = \operatorname{Conv}(v_1, \ldots, v_r)$. Then

$$K \coloneqq \left\{ \textstyle\sum_{i=1}^r t_i v_i \in V \mid 0 \leqslant t_i \leqslant 1 \text{ for } i = 1, \ldots, r \right\}$$

is a compact subset of C. In particular, there is a real number $\delta \geqslant 0$ such that $\|v\| \leqslant \delta$ for all $v \in K$. Now, let $c \in C$. Then there exist $m_1, \ldots, m_r \in \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant t_1, \ldots, t_r \leqslant 1$ such that

$$c = \underbrace{\sum_{i=1}^r m_i v_i}_{\in C\cap\Lambda} + \underbrace{\left(\sum_{i=1}^r t_i v_i\right)}_{\in K}.$$

This shows that c is contained in the δ-neighbourhood $(C\cap\Lambda)^\delta$. In summary, we get $C \cap \Lambda \subset C \subset (C\cap\Lambda)^\delta$ and by using Lemmas 2.1 and 2.2 the statement follows.

(2) By (1) applied to $C = V$ and Lemma 2.1 we get $V = \Lambda_\infty = (v_0 + \Lambda)_\infty$. By definition of the asymptotic cone, thus for every $v \neq 0$ in V there exists a sequence $(\lambda_i)_i$ in Λ such that $\|v_0 + \lambda_i\| \to \infty$ and $\rho(v) = \lim_{i\to\infty} \rho(v_0 + \lambda_i)$. This shows that $\rho((v_0+\Lambda)\smallsetminus\{0\})$ is dense in $S = \rho(V \smallsetminus \{0\})$. ☐

Proposition 2.6. — *Let $\Lambda \subset V$ be a lattice, let $C \subset V$ be a convex polyhedral cone, let $H \subset V$ be a hyperplane, and let $H' \coloneqq \gamma + H$ for some $\gamma \in \Lambda \smallsetminus H$.*

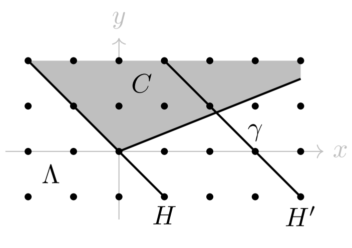
(1) *If $C \cap H' \neq \varnothing$, $\dim C\cap H = \dim H$ and H is rational, then $\operatorname{int}(C)\cap H' \cap \Lambda \neq \varnothing$.*

(2) *If $\operatorname{int}(C)\cap H'\cap\Lambda \neq \varnothing$ and $C \cap H$ is a rational convex polyhedral cone, then*

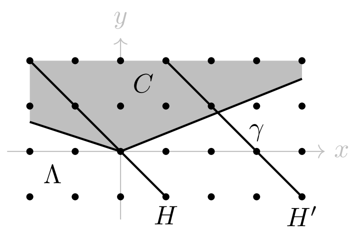
$$C \cap H = (\operatorname{int}(C) \cap H' \cap \Lambda)_\infty.$$

The pictures below illustrate the setups of Proposition 2.6 in the two cases.

(1)



(2)



Proof

(1) If $\dim V = 1$, then $H = \{0\}$. Thus $C \cap H' \neq \varnothing$ gives $C \cap H' = \{\gamma\} \subset \Lambda \smallsetminus \{0\}$ and the statement follows. Hence, we assume $\dim V \geqslant 2$.

As $\gamma \notin H$, we get $0 \notin H'$. Since $C \cap H' \neq \varnothing$ there is thus $x \in \operatorname{int}(C) \cap H'$ by Lemma 2.3. As $\dim(C \cap H) = \dim H$, the linear span of $C \cap H$ is H and we get that $\operatorname{int}(C\cap H)$ is a non-empty open subset of H. Set

$$D \coloneqq x + (\operatorname{int}(C\cap H) \smallsetminus \{0\}) \subset (\operatorname{int}(C) \cap H') \smallsetminus \{x\}.$$

Denote by S the unit sphere in H with center 0 and consider

$$\pi\colon H' \smallsetminus \{x\} \longrightarrow S, \quad w \longmapsto \frac{w-x}{\|w-x\|}.$$

For all $h \in \operatorname{int}(C \cap H)$ we have $\mathbb{R}_{>0}h \subset \operatorname{int}(C \cap H)$ and thus

$$\pi^{-1}(\pi(D)) = D.$$

As $\dim H \geqslant 1$ (note that $\dim V \geqslant 2$), we get that D is a non-empty open subset of $H' \smallsetminus \{x\}$, and thus the same is true for $\pi(D)$ in S (because π is open). As $\gamma \in \Lambda$ and $H' = \gamma + H$, it follows that $H' \cap \Lambda = \gamma + (H \cap \Lambda)$. Using that H is rational, we get that $\pi((H' \cap \Lambda) \smallsetminus \{x\})$ is dense in S by Lemma 2.5(2) applied to the point $\gamma - x \in H$ and the lattice $H \cap \Lambda$ in H. By the openness of $\pi(D)$ in S, there is $\lambda \in (H' \cap \Lambda) \smallsetminus \{x\} \subset \Lambda$ with $\pi(\lambda) \in \pi(D)$. In particular, $\lambda \in \pi^{-1}(\pi(D)) = D$ and thus $\lambda \in D \cap \Lambda \subset \operatorname{int}(C) \cap H' \cap \Lambda$.

(2) By assumption, there is $y \in \operatorname{int}(C) \cap H' \cap \Lambda$. Thus we get

$$y + (C \cap H \cap \Lambda) \subset \operatorname{int}(C) \cap H' \cap \Lambda.$$

This implies by Lemma 2.1

$$(C \cap H \cap \Lambda)_\infty \subset (\operatorname{int}(C) \cap H' \cap \Lambda)_\infty \subset (C \cap H')_\infty. \tag{a}$$

By Lemma 2.4 we get

$$(C \cap H')_\infty = C \cap H. \tag{b}$$

By Lemma 2.5(1) applied to the rational convex polyhedral cone $C \cap H \subset V$ we get

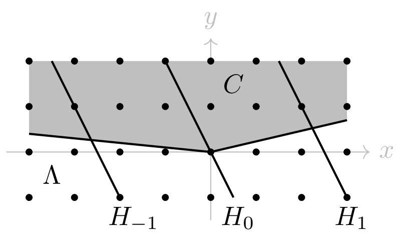
$$C \cap H = (C \cap H \cap \Lambda)_\infty. \tag{c}$$

Combining (a), (b) and (c) yields the result. ☐

Proposition 2.7. — *Let $\Lambda \subset V$ be a lattice, $C \subset V$ a convex polyhedral cone and $H_0 \subset V$ a hyperplane such that $C \cap H_0$ is a rational convex polyhedral cone. Let $H_1 \subset V$ be an affine hyperplane parallel to H_0 and set $H_{-1} \coloneqq -H_1$. If $C \cap H_i \neq \varnothing$ for each $i \in \{\pm 1\}$, then*

$$C \cap H_{-1} \cap \Lambda \neq \varnothing \quad \Longleftrightarrow \quad C \cap H_1 \cap \Lambda \neq \varnothing.$$

The picture below illustrates the setup of Proposition 2.7.



Proof. — If $H_0 = H_1$, then $H_0 = H_{-1}$ and the statement is trivial. Thus we assume that $H_0 \neq H_1$, whence $H_0 \neq H_{-1}$ and $H_1 \neq H_{-1}$.

Since $C \cap H_{\pm 1} \neq \varnothing$ and since H_0, H_1 and H_{-1} are pairwise disjoint, there exist $c_{\pm 1} \in \operatorname{int}(C) \cap H_{\pm 1}$ by Lemma 2.3. As C is convex, the line segment in V that connects c_1 and c_{-1} lies in $\operatorname{int}(C)$ and thus $\operatorname{int}(C) \cap H_0 \neq \varnothing$. Let $B \subset C$ be the union of the proper faces of C, i.e., B is the topological boundary of C inside the linear span of C, see [Ful93, Propty (7), p. 10]. If $C \cap H_0 \cap \Lambda \subset B$, then by Lemma 2.5(1) applied to the rational convex polyhedral cone $C \cap H_0$ in V we get $C \cap H_0 = (C \cap H_0 \cap \Lambda)_\infty \subset B_\infty = B$, a contradiction to $\operatorname{int}(C) \cap H_0 \neq \varnothing$. In particular, we may choose

$$\gamma_0 \in (C \cap H_0 \cap \Lambda) \smallsetminus B.$$

By exchanging H_1 and H_{-1}, it is enough to prove "$\Rightarrow$" of the statement. For this, let $\gamma_{-1} \in C \cap H_{-1} \cap \Lambda$. Since C is a convex polyhedral cone in V (and thus in the linear span $\operatorname{Span}_{\mathbb{R}}(C)$), there is a finite set $E \subset \operatorname{Span}_{\mathbb{R}}(C) \smallsetminus \{0\}$ with

$$C = \bigcap_{u \in E} \{v \in \operatorname{Span}_{\mathbb{R}}(C) \mid \langle u, v \rangle \geqslant 0\},$$

see [Ful93, Propty (8), p. 11]. Since $\gamma_0 \in C \smallsetminus B$, we get $\langle u, \gamma_0 \rangle > 0$ for all $u \in E$. In particular, we may choose an integer $m \geqslant 0$ big enough so that

$$\langle u, m\gamma_0 - \gamma_{-1} \rangle = m\langle u, \gamma_0 \rangle - \langle u, \gamma_{-1} \rangle \geqslant 0$$

for all $u \in E$, i.e., $m\gamma_0 - \gamma_{-1} \in C$. As $\gamma_0 \in H_0 \cap \Lambda$, we get $m\gamma_0 - \gamma_{-1} \in C \cap H_1 \cap \Lambda$. $\square$

3. Quasi-affine varieties

To any variety X, we can naturally associate an affine scheme

$$X_{\mathrm{aff}} \coloneqq \operatorname{Spec} \mathscr{O}(X).$$

Moreover this scheme comes equipped with the so-called canonical morphism

$$\iota \colon X \longrightarrow X_{\mathrm{aff}}$$

which is induced by the natural isomorphism $\mathscr{O}(X) = \mathscr{O}(X_{\mathrm{aff}})$.

Remark 3.1. — For any variety X, the canonical morphism $\iota \colon X \to X_{\mathrm{aff}}$ is dominant. Indeed, let $X' \coloneqq \overline{\iota(X)} \subset X_{\mathrm{aff}}$ be the closure of the image of ι (endowed with the induced reduced subscheme structure). Since the composition

$$\mathscr{O}(X) = \mathscr{O}(X_{\mathrm{aff}}) \longrightarrow \mathscr{O}(X') \longrightarrow \mathscr{O}(X)$$

is the identity on $\mathscr{O}(X)$, it follows that the surjection $\mathscr{O}(X) = \mathscr{O}(X_{\mathrm{aff}}) \to \mathscr{O}(X')$ is injective and thus $X' = X$.

Lemma 3.2 ([Gro61, §5, Prop. 5.1.2]). — *Let X be a variety. Then X is quasi-affine if and only if the canonical morphism $\iota \colon X \to X_{\mathrm{aff}}$ is an open immersion.* $\square$

If X is quasi-affine and endowed with an algebraic group action, then this action uniquely extends to an algebraic group action on X_{aff}:

Lemma 3.3. — *Let X be a quasi-affine H-variety for some algebraic group H. Then X_{aff} is an affine scheme that has a unique H-action that extends the H-action on X via the canonical open immersion $X \hookrightarrow X_{\mathrm{aff}}$.*

Proof. — By Lemma 3.2, the canonical morphism $X \to X_{\mathrm{aff}}$ is an open immersion of schemes and there is a unique action of H on X_{aff} that extends the H-action on X, see e.g. [KRvS19, Lem. 5]. $\square$

Now, we compare the G-sphericity of X and X_{aff}.

Lemma 3.4. — *Let G be a connected reductive algebraic group and let X be a quasi-affine G-variety. Then*

$$X \text{ is } G\text{-spherical} \iff X_{\mathrm{aff}} \text{ is an affine } G\text{-spherical variety.}$$

Proof. — If X_{aff} is an affine G-spherical variety, then X is G-spherical by Lemma 3.2.

For the other implication, assume that X is G-spherical. It follows that $\mathscr{O}(X)$ is a finitely generated algebra over the ground field by [Kno93] and thus $X_{\mathrm{aff}} = \operatorname{Spec} \mathscr{O}(X)$ is an affine variety. Since X is irreducible, X_{aff} is irreducible by Remark 3.1. Moreover, for each $x \in X$, the local ring $\mathscr{O}_{X,x}$ is integrally closed and thus $\mathscr{O}(X) = \bigcap_{x\in X} \mathscr{O}_{X,x}$ is integrally closed, i.e., X_{aff} is normal. Since X is an open subset of X_{aff}, and since a Borel subgroup of G acts with an open orbit on X, the same is true for X_{aff}. $\square$

For the rest in this section, we recall two classical facts from invariant theory.

Proposition 3.5 (see [Sha94, Lem. 1.4, part II] and [Kra84, §2.4 Lem.])

Let X be any variety endowed with an H-action for some algebraic group H. The natural action of H on $\mathscr{O}(X)$ satisfies the following: If $f \in \mathscr{O}(X)$, then $\operatorname{Span}_{\boldsymbol{k}}(Hf)$ is a finite dimensional H-invariant subspace of $\mathscr{O}(X)$ and H acts regularly on it. $\square$

Proposition 3.6 (see [Sha94, Th. 3.3, part II]). — *Let H be a connected solvable algebraic group and let X be an irreducible quasi-affine H-variety. Then, for every H-invariant rational map $f\colon X \dashrightarrow \boldsymbol{k}$ there exist H-semi-invariants $f_1, f_2 \in \mathscr{O}(X)$ such that $f = f_1/f_2$.* $\square$

4. Vector fields

4.1. Generalities on vector fields. — Let X be any variety. We denote by $\operatorname{Vec}(X)$ the vector space of all algebraic vector fields on X, i.e., all algebraic sections of the tangent bundle $TX \to X$. Note that $\operatorname{Vec}(X)$ is in a natural way an $\mathscr{O}(X)$-module.

Now, assume X is endowed with a regular action of an algebraic group H. Then, $\operatorname{Vec}(X)$ is an H-module, via the following action: Let $h \in H$ and $\xi \in \operatorname{Vec}(X)$, then $h \cdot \xi$ is defined via

$$(h \cdot \xi)(x) = \mathrm{d}\varphi_h(\xi(\varphi_{h^{-1}}(x))) \quad \text{for each } x \in X,$$

where φ_h denotes the automorphism of X given by multiplication with h and $d\varphi_h$ denotes the differential of φ_h. For a fixed character $\lambda\colon H \to \mathbb{G}_m$ we say that a vector field $\xi \in \operatorname{Vec}(X)$ is *normalized by H with weight λ* if ξ is a H-semi-invariant of weight λ, i.e., for all $h \in H$ the following diagram commutes

$$\begin{tikzcd} TX \arrow[r, "\mathrm{d}\varphi_h"] & TX \\ X \arrow[u, "\xi"] \arrow[r, "\varphi_h"] & X \arrow[u, "\lambda(h)\xi"'] \end{tikzcd}$$

We denote by $\operatorname{Vec}(X)_{\lambda,H}$ the subspace in $\operatorname{Vec}(X)$ of all vector fields which are normalized by H with weight λ. If it is clear which action on X is meant, we drop the index H and simply write $\operatorname{Vec}(X)_\lambda$. Note that $\operatorname{Vec}(X)_\lambda$ is in a natural way an $\mathscr{O}(X)^H$-module, where $\mathscr{O}(X)^H$ denotes the H-invariant regular functions on X. We denote by $\operatorname{Vec}^H(X)$ the subspace of all H-invariant vector fields in $\operatorname{Vec}(X)$, i.e., $\operatorname{Vec}^H(X) = \operatorname{Vec}(X)_0$, where 0 denotes the trivial character of H.

Now, assume that X is affine. There is a $\boldsymbol{k}$-linear map

$$\operatorname{Vec}(X) \longrightarrow \operatorname{Der}_{\boldsymbol{k}}(\mathscr{O}(X)), \quad \xi \longmapsto D_\xi,$$

where $D_\xi \colon \mathscr{O}(X) \to \mathscr{O}(X)$ is given by $D_\xi(f)(x) \coloneqq \xi(x)(f)$ (here we identify the tangent space of X at x with the $\boldsymbol{k}$-derivations $\mathscr{O}_{X,x} \to \boldsymbol{k}$ in x). In fact, $\operatorname{Vec}(X) \to \operatorname{Der}_{\boldsymbol{k}}(\mathscr{O}(X))$ is an isomorphism: Indeed, as X is affine, we have

$$\operatorname{Vec}(X) = \left\{ \eta\colon X \longrightarrow TX \;\middle|\; \begin{array}{l} \eta \text{ is a set-theoretical section and for all } f \in \mathscr{O}(X) \\ \text{the map } x \mapsto \eta(x)(f) \text{ is a regular function on } X \end{array} \right\},$$

see [FK, §3.2].

4.2. Homogeneous $\mathbb{G}_{\mathrm{a}}$-actions and vector fields. — The material of this small subsection is contained in [FK, §6.5], however formulated for all varieties.

Let P be an algebraic group that acts regularly on a variety X. Then we get a $\boldsymbol{k}$-linear map $\operatorname{Lie} P \to \operatorname{Vec}(X)$, $A \mapsto \xi_A$, where the vector field ξ_A is given by

$$(\ominus) \qquad \xi_A \colon X \longrightarrow TX, \quad x \longmapsto (\mathrm{d}_e\mu_x)A$$

and $\mu_x \colon P \to X$, $p \mapsto px$ denotes the orbit morphism in x: Indeed $(\ominus)$ is a morphism as it is the composition of the morphisms

$$X \longrightarrow T_eP \times TX,\ x \longmapsto (A, 0_x) \quad \text{and} \quad d\mu|_{T_eP\times TX} \colon T_eP \times TX \longrightarrow TX,$$

where $0_x \in TX$ denotes the zero vector inside T_xX and $\mu\colon P \times X \to X$ denotes the P-action.

Lemma 4.1. — *If P is an algebraic group that acts faithfully on a variety X, then the $\boldsymbol{k}$-linear map $\operatorname{Lie} P \to \operatorname{Vec}(X)$, $A \mapsto \xi_A$ is injective.*

Proof. — For each $x \in X$, the kernel of the differential $\mathrm{d}_e\mu_x \colon \operatorname{Lie} P \to T_xX$ of the orbit morphism $\mu_x \colon P \to X$, $p \mapsto px$ is equal to $\operatorname{Lie} P_x$, where P_x denotes the stabilizer of x in P. If $A \in \operatorname{Lie} P$ satisfies $\xi_A = 0$, then $(\mathrm{d}_e\mu_x)A = 0$ for each $x \in X$, i.e., $A \in \operatorname{Lie} P_x$ for each $x \in X$. As P acts faithfully on X, we get $\{e\} = \bigcap_{x\in X} P_x$ and thus $\{0\} = \operatorname{Lie}(\bigcap_{x\in X} P_x) = \bigcap_{x\in X} \operatorname{Lie}(P_x)$ which implies $A = 0$. □

Let H be an algebraic group. A $\mathbb{G}_{\mathrm{a}}$-action on an H-variety X is called *H-homogeneous of weight* $\lambda \in \mathfrak{X}(H)$ if

$$h \circ \varepsilon(t) \circ h^{-1} = \varepsilon(\lambda(h) \cdot t) \quad \text{for all } h \in H \text{ and all } t \in \mathbb{G}_{\mathrm{a}},$$

where $\varepsilon \colon \mathbb{G}_{\mathrm{a}} \to \mathrm{Aut}(X)$ is the group homomorphism induced by the $\mathbb{G}_{\mathrm{a}}$-action.

Lemma 4.2. — *Let H be an algebraic group, X an H-variety and ρ an H-homogeneous $\mathbb{G}_{\mathrm{a}}$-action on X of weight $\lambda \in \mathfrak{X}(H)$. Then the image of the previously introduced $\boldsymbol{k}$-linear map $\mathrm{Lie}\,\mathbb{G}_{\mathrm{a}} \to \mathrm{Vec}(X)$, $A \mapsto \xi_A$ associated to ρ lies in $\mathrm{Vec}(X)_{\lambda,H}$.*

Proof. — As ρ is H-homogeneous, we get for each $x \in X$ and each $h \in H$ the following commutative diagram

$$\begin{tikzcd} \mathbb{G}_{\mathrm{a}} \arrow[r, "t \mapsto \lambda(h)t"] \arrow[d, "\mu_x"'] & \mathbb{G}_{\mathrm{a}} \arrow[d, "\mu_{hx}"] \\ X \arrow[r, "\varphi_h"] & X \end{tikzcd}$$

where $\varphi_h \colon X \to X$ denotes multiplication by h. Taking differentials in the neutral element $e \in \mathbb{G}_{\mathrm{a}}$ gives $\mathrm{d}_x\varphi_h \mathrm{d}_e\mu_x = \lambda(h)\mathrm{d}_e\mu_{hx}$ for each $A \in \mathrm{Lie}\,\mathbb{G}_{\mathrm{a}}$. This implies that $h \cdot \xi_A(x) = \lambda(h)\xi_A(x)$ for each $A \in \mathrm{Lie}\,\mathbb{G}_{\mathrm{a}}$ and thus the statement follows. $\square$

Lemma 4.3. — *Let H be an algebraic group and let $N \subset H$ be a normal subgroup such that the character group $\mathfrak{X}(N)$ is trivial. If X is an irreducible H-variety, then*

$$D_H(X) = \left\{ \lambda \in \mathfrak{X}(H) \;\middle|\; \begin{array}{l} \textit{there is a non-trivial } H\textit{-homogeneous} \\ \mathbb{G}_{\mathrm{a}}\textit{-action on } X \textit{ of weight } \lambda \end{array} \right\}$$

is contained in the set of H-weights of non-zero vector fields in $\mathrm{Vec}^N(X)$ that are normalized by H.

Proof. — Let $\rho \colon \mathbb{G}_{\mathrm{a}} \times X \to X$ be a non-trivial $\mathbb{G}_{\mathrm{a}}$-action on X. By Lemmas 4.1 and 4.2 there is a non-zero $\xi \in \mathrm{Vec}(X)$ such that for each $h \in H$ we have $h \cdot \xi = \lambda(h)\xi$. Moreover, since $\mathfrak{X}(N) = 0$, ξ is N-invariant. Thus $D_H(X)$ is contained in the set of H-weights of non-zero vector fields in $\mathrm{Vec}^N(X)$ that are normalized by H. $\square$

Now, assume that X is an affine variety and fix some non-zero element $A_0 \in \mathrm{Lie}\,\mathbb{G}_{\mathrm{a}}$. Moreover, denote by $\mathrm{LND}_{\boldsymbol{k}}(\mathscr{O}(X)) \subset \mathrm{Der}_{\boldsymbol{k}}(\mathscr{O}(X))$ the cone of locally nilpotent derivations on $\mathscr{O}(X)$, i.e., the cone in $\mathrm{Der}_{\boldsymbol{k}}(\mathscr{O}(X))$ of $\boldsymbol{k}$-derivations D of $\mathscr{O}(X)$ such that for all $f \in \mathscr{O}(X)$ there is a $n = n(f) \geqslant 1$ such that $D^n(f) = 0$, where D^n denotes the n-fold composition of D. There is a map

$$\{\mathbb{G}_{\mathrm{a}}\text{-actions on } X\} \overset{1:1}{\longleftrightarrow} \mathrm{LND}_{\boldsymbol{k}}(\mathscr{O}(X)), \quad \rho \longmapsto D_{\xi_{A_0}},$$

where ξ_{A_0} is defined as in ($\ominus$) with respect to the $\mathbb{G}_{\mathrm{a}}$-action ρ. As for each $f \in \mathscr{O}(X)$ we have that $D_{\xi_{A_0}}(f)$ is the morphism $x \mapsto A_0(f \circ \mu_x)$ (we interpret A_0 as a $\boldsymbol{k}$-derivation of $\mathscr{O}_{\mathbb{G}_{\mathrm{a}},e} \to \boldsymbol{k}$ in e), it follows from [Fre17, §1.5] that the above map is in fact a bijection.

4.3. Finiteness results on modules of vector fields. — Let G be an algebraic group. For this subsection, let X be a G-variety. Note that $\mathrm{Vec}(X)$ is an $\mathscr{O}(X)$-G-*module* via the $\mathscr{O}(X)$- and G-module structures given in §4.1, i.e., $\mathrm{Vec}(X)$ is a G-module, it is an $\mathscr{O}(X)$-module, and both structures are compatible in the sense that

$$g \cdot (f \cdot \xi) = (g \cdot f) \cdot (g \cdot \xi) \quad \text{for all } g \in G,\ f \in \mathscr{O}(X) \text{ and } \xi \in \mathrm{Vec}(X).$$

Lemma 4.4. — *Assume that X is a quasi-affine G-variety and that $\mathscr{O}(X)$ is finitely generated as a $\boldsymbol{k}$-algebra. Then the $\mathscr{O}(X)$-G-module* $\mathrm{Vec}(X)$ *is finitely generated and rational, i.e.,* $\mathrm{Vec}(X)$ *is finitely generated as an $\mathscr{O}(X)$-module and the G-representation* $\mathrm{Vec}(X)$ *is a sum of finite dimensional rational G-subrepresentations.*

Proof. — Since $\mathscr{O}(X)$ is finitely generated, $X_{\mathrm{aff}} = \operatorname{Spec} \mathscr{O}(X)$ is an affine variety that is endowed with a natural G-action, see Lemma 3.3. By [Kra84, Satz 2, II.2.S] there is a rational G-representation V and a G-equivariant closed embedding $X_{\mathrm{aff}} \subseteq V$. We denote by

$$\iota \colon X \longrightarrow V$$

the composition of the canonical open immersion $X \subset X_{\mathrm{aff}}$ with $X_{\mathrm{aff}} \subset V$. Note that the image of ι is locally closed in V and that ι induces an isomorphism of X onto that locally closed subset of V. Thus, $\mathrm{d}\iota \colon TX \to TV|_X$ is a G-equivariant closed embedding over X which is linear on each fiber of $TX \to X$. Thus we get an $\mathscr{O}(X)$-G-module embedding

$$\mathrm{Vec}(X) \longrightarrow \Gamma(TV|_X), \quad \xi \longmapsto \mathrm{d}\iota \circ \xi,$$

where $\Gamma(TV|_X)$ denotes the $\mathscr{O}(X)$-G-module of sections of $TV|_X \to X$. However, since the vector bundle $TV|_X \to X$ is trivial, there is a $\mathscr{O}(X)$-G-module isomorphism

$$\Gamma(TV|_X) \simeq \mathrm{Mor}(X, V),$$

where G acts on $\mathrm{Mor}(X, V)$ via $g \cdot \eta = (x \mapsto g\eta(g^{-1}x))$. Now, the $\mathscr{O}(X)$-G-module $\mathrm{Mor}(X, V) \simeq \mathscr{O}(X) \otimes_{\boldsymbol{k}} V$ is finitely generated and rational (see Proposition 3.5), and thus the statement follows. □

For the next result we recall the following definition.

Definition 4.5. — Let G be an algebraic group. A closed subgroup $H \subset G$ is called a *Grosshans subgroup* if G/H is quasi-affine and $\mathscr{O}(G/H) = \mathscr{O}(G)^H$ is a finitely generated $\boldsymbol{k}$-algebra.

Let G be a connected reductive algebraic group. Examples of Grosshans subgroups of G are unipotent radicals of parabolic subgroups of G, see [Gro97, Th. 16.4]. In particular, the unipotent radical U of a Borel subgroup $B \subset G$ is a Grosshans subgroup in G (see also [Gro97, Th. 9.4]). A very important property of Grosshans subgroups is the following:

Proposition 4.6 ([Gro97, Th. 9.3]). — *Let A be a finitely generated $\boldsymbol{k}$-algebra and let G be a connected reductive algebraic group that acts via $\boldsymbol{k}$-algebra automorphisms on A such that A becomes a rational G-module. If $H \subset G$ is a Grosshans subgroup, then the ring of H-invariants*

$$A^H = \{a \in A \mid ha = a \text{ for all } h \in H\}$$

is a finitely generated $\boldsymbol{k}$-subalgebra of A. □

Proposition 4.7. — *Let R be a finitely generated $\boldsymbol{k}$-algebra and assume that a connected reductive algebraic group G acts via $\boldsymbol{k}$-algebra automorphisms on R such that R becomes a rational G-module. Let $H \subset G$ be a Grosshans subgroup. If M is a finitely generated rational R-G-module, then M^H is a finitely generated R^H-module.*

Proof. — We consider the $\boldsymbol{k}$-algebra $A = R \oplus \varepsilon M$, where the multiplication on A is defined via

$$(r + \varepsilon m) \cdot (q + \varepsilon n) = rq + \varepsilon(rn + qm).$$

Since R is a finitely generated $\boldsymbol{k}$-algebra and since M is a finitely generated R-module, A is a finitely generated $\boldsymbol{k}$-algebra. Moreover, since R and M are rational G-modules, A is a rational G-module. Moreover, G acts via $\boldsymbol{k}$-algebra automorphisms on A. Since H is a Grosshans subgroup of G, it now follows by Proposition 4.6 that

$$A^H = R^H \oplus \varepsilon M^H$$

is a finitely generated $\boldsymbol{k}$-algebra. Thus one can choose finitely many elements $m_1, \dots, m_k \in M^H$ such that $\varepsilon m_1, \dots, \varepsilon m_k$ generate A^H as an R^H-algebra. However, since $\varepsilon^2 = 0$, it follows that $m_1, \dots, m_k$ generate M^H as an R^H-module.

As M is a rational G-module, it follows that M^H is a rational H-module. □

As an application of Lemma 4.4 and Proposition 4.7 we get the following finiteness result of $\mathrm{Vec}^H(X)$ for a Grosshans subgroup H of a connected reductive algebraic group.

Corollary 4.8. — *Let H be a Grosshans subgroup of a connected reductive algebraic group G. If X is a quasi-affine G-variety such that $\mathscr{O}(X)$ is finitely generated as a $\boldsymbol{k}$-algebra, then $\mathrm{Vec}^H(X)$ is a finitely generated $\mathscr{O}(X)^H$-module.* □

4.4. Vector fields normalized by a group action with an open orbit. — For this subsection, let H be an algebraic group and let X be an irreducible H-variety which contains an open H-orbit. Moreover, fix a character λ of H. We provide an upper bound on the dimension of $\mathrm{Vec}(X)_\lambda = \mathrm{Vec}(X)_{\lambda,H}$.

Lemma 4.9. — *Fix $x_0 \in X$ that lies in the open H-orbit and let H_{x_0} be the stabilizer of x_0 in H. Then, there exists an injection of $\mathrm{Vec}(X)_\lambda$ into the H_{x_0}-eigenspace of the tangent space $T_{x_0}X$ of weight $\lambda|_{H_{x_0}}$ given by*

$$\xi \longmapsto \xi(x_0).$$

In particular, the dimension of $\mathrm{Vec}(X)_\lambda$ *is smaller than or equal to the dimension of the* H_{x_0}*-eigenspace of weight* $\lambda|_{H_{x_0}}$ *of* $T_{x_0}X$.

Proof. — Let $\xi \in \mathrm{Vec}(X)_\lambda$. By definition we have for all $h \in H$

$$\lambda(h)\xi(hx_0) = \mathrm{d}\varphi_h\xi(x_0), \tag{Φ}$$

where $\varphi_h\colon X \to X$ denotes the automorphism given by multiplication with h. Since x_0 lies in the open H-orbit and X is irreducible, ξ is uniquely determined by $\xi(x_0)$. Moreover, (Φ) implies that $\xi(x_0)$ is an H_{x_0}-eigenvector of weight $\lambda|_{H_{x_0}}$ of $T_{x_0}X$. $\square$

5. Automorphism group of a variety and root subgroups

Let X be a variety and denote by $\mathrm{Aut}(X)$ its automorphism group. A subgroup $H \subset \mathrm{Aut}(X)$ is called an *algebraic subgroup of* $\mathrm{Aut}(X)$ if H has the structure of an algebraic group such that the action $H \times X \to X$ is a regular action of the algebraic group H on X. It follows from [Ram64] (see also [KRvS19, Th. 2.9]) that this algebraic group structure on H is unique in the following sense: if H_1, H_2 are algebraic groups with group isomorphisms $\iota_i\colon H_i \to H$ for $i = 1, 2$ such that the induced actions $H_i \times X \to X$ are morphisms for $i = 1, 2$, then $\iota_2^{-1} \circ \iota_1\colon H_1 \to H_2$ is an isomorphism of algebraic groups.

Let X, Y be varieties. We say that a group homomorphism $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ *preserves algebraic subgroups* if for each algebraic subgroup $H \subset \mathrm{Aut}(X)$ its image $\theta(H)$ is an algebraic subgroup of $\mathrm{Aut}(Y)$ and if the restriction $\theta|_H\colon H \to \theta(H)$ is a homomorphism of algebraic groups. We say that a group isomorphism $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ *preserves algebraic subgroups* if both homomorphisms θ and θ^{-1} preserve algebraic subgroups.

Assume now that X is an H-variety for some algebraic group H and that $U_0 \subset \mathrm{Aut}(X)$ is a one-parameter unipotent subgroup, i.e., an algebraic subgroup of $\mathrm{Aut}(X)$ that is isomorphic to $\mathbb{G}_\mathrm{a}$. If for some isomorphism $\mathbb{G}_\mathrm{a} \simeq U_0$ of algebraic groups the induced $\mathbb{G}_\mathrm{a}$-action on X is H-homogeneous of weight $\lambda \in \mathfrak{X}(H)$, then we call U_0 a *root subgroup with respect to* H *of weight* λ (see §4.2). Note that this definition does not depend on the choice of the isomorphism $\mathbb{G}_\mathrm{a} \simeq U_0$. This notion goes back to Demazure [Dem70].

Lemma 5.1. — *Let* X, Y *be* H*-varieties for some algebraic group* H. *If* $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ *is a group homomorphism that preserves algebraic subgroups and if* θ *is compatible with the* H*-actions in the way that*

$$\begin{tikzcd} & H \arrow[dl] \arrow[dr] & \\ \mathrm{Aut}(X) \arrow[rr, "\theta"] & & \mathrm{Aut}(Y) \end{tikzcd}$$

commutes, then for any root subgroup $U_0 \subset \mathrm{Aut}(X)$ *with respect to* H, *the image* $\theta(U_0)$ *is either the trivial group or a root subgroup with respect to* H *of the same weight as* U_0.

Proof. — We can assume that $\theta(U_0)$ is not the trivial group. Hence, $\theta(U_0)$ is a one-parameter unipotent group.

Let $\varepsilon\colon \mathbb{G}_{\mathrm{a}} \simeq U_0 \subset \mathrm{Aut}(X)$ be an isomorphism and let $\lambda\colon H \to \mathbb{G}_m$ be the weight of U_0. Then we have for each $t \in \mathbb{G}_{\mathrm{a}}$

$$h \circ \theta(\varepsilon(t)) \circ h^{-1} = \theta(h \circ \varepsilon(t) \circ h^{-1}) = \theta(\varepsilon(\lambda(h) \cdot t)).$$

Since $\theta|_{U_0}\colon U_0 \to \theta(U_0)$ is a surjective homomorphism of algebraic groups that are both isomorphic to $\mathbb{G}_{\mathrm{a}}$ (and since the ground field is of characteristic zero), $\theta|_{U_0}$ is in fact an isomorphism. Thus $\theta \circ \varepsilon\colon \mathbb{G}_{\mathrm{a}} \simeq \theta(U_0) \subset \mathrm{Aut}(X)$ is an isomorphism and hence λ is the weight of $\theta(U_0)$ with respect to H. $\square$

6. Homogeneous $\mathbb{G}_{\mathrm{a}}$-actions on quasi-affine toric varieties

In this section, we provide a description of the homogeneous $\mathbb{G}_{\mathrm{a}}$-actions on a quasi-affine toric variety. Throughout this section, we denote by T an algebraic torus. Recall that a T-toric variety is a T-spherical variety. A $\mathbb{G}_{\mathrm{a}}$-action is called *homogeneous* if it is T-homogeneous of some weight $\lambda \in \mathfrak{X}(T)$, see §4.2.

Let X be a toric variety. In case X is affine, Liendo [Lie10] gave a full description of all homogeneous $\mathbb{G}_{\mathrm{a}}$-actions. In case X is quasi-affine, $X_{\mathrm{aff}} = \mathrm{Spec}(\mathscr{O}(X))$ is an affine T-toric variety by Lemma 3.4. Moreover, every homogeneous $\mathbb{G}_{\mathrm{a}}$-action on X extends uniquely to a homogeneous $\mathbb{G}_{\mathrm{a}}$-action on X_{aff} by Lemma 3.3. Thus we are led to the problem of describing the homogeneous $\mathbb{G}_{\mathrm{a}}$-actions on X_{aff} that preserve the open subvariety X.

This requires some preparation. First, we provide a description of X_{aff} in case X is toric and provide a characterization, when X is quasi-affine. For this, let us introduce some basic terms from toric geometry. As a reference we take [Ful93] and [CLS11].

Note that $M = \mathfrak{X}(T) = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, where N denotes the free abelian group of rank $\dim T$ of the regular group homomorphisms $\mathbb{G}_m \to T$ and denote by $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ the extensions to $\mathbb{R}$. Moreover, let

$$M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad (u, v) \longmapsto \langle u, v\rangle$$

be the canonical $\mathbb{R}$-bilinear form. Denote by $\boldsymbol{k}[M]$ the $\boldsymbol{k}$-algebra with basis χ^m for all $m \in M$ and multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. Note that there is an identification

$$T = \mathrm{Spec}\, \boldsymbol{k}[M].$$

Let $\sigma \subset N_{\mathbb{R}}$ be a *strongly convex rational polyhedral cone* in $N_{\mathbb{R}}$, i.e., it is a convex rational polyhedral cone with respect to the lattice $N \subset N_{\mathbb{R}}$ and σ contains no non-zero linear subspace of $N_{\mathbb{R}}$. Then its dual

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} \mid \langle u, v\rangle \geqslant 0 \text{ for all } v \in \sigma\}$$

is a convex rational polyhedral cone in $M_{\mathbb{R}}$. Denote by $\sigma_M^{\vee}$ the intersection of $\sigma^{\vee}$ with M inside $M_{\mathbb{R}}$. We can associate to σ a toric variety

$$X_\sigma = \mathrm{Spec}\, \boldsymbol{k}[\sigma_M^{\vee}], \quad \text{where} \quad \boldsymbol{k}[\sigma_M^{\vee}] = \bigoplus_{m \in \sigma_M^{\vee}} \boldsymbol{k}\chi^m \subset \boldsymbol{k}[M].$$

The torus T acts on X_σ with an open orbit where this action is induced by the coaction $\boldsymbol{k}[\sigma_M^\vee] \to \boldsymbol{k}[\sigma_M^\vee] \otimes_{\boldsymbol{k}} \boldsymbol{k}[M]$, $\chi^u \mapsto \chi^u \otimes \chi^u$. Note that we have an order-reversing bijection between the faces of σ and the faces of its dual $\sigma^\vee$:

$$\{\text{faces of } \sigma\} \stackrel{1:1}{\longleftrightarrow} \{\text{faces of } \sigma^\vee\}, \quad \tau \longmapsto \sigma^\vee \cap \tau^\perp,$$

where $\tau^\perp$ consists of those $u \in M_{\mathbb{R}}$ that satisfy $\langle u, v\rangle = 0$ for all $v \in \tau$, see [Ful93, Propty (10), p. 12]. Moreover, each face $\tau \subset \sigma$ determines an orbit of dimension $n - \dim(\tau)$ of the T-action on X_σ (see [Ful93, §3.1]). We denote its closure in X_σ by $V(\tau)$. In particular, $V(\tau)$ is an irreducible closed T-invariant subset of X_σ.

More generally, for a fan Σ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ we denote by X_Σ its associated toric variety, which is covered by the open affine toric subvarieties X_σ, where σ runs through the cones in Σ.

Lemma 6.1. — *Let $X = X_\Sigma$ be a toric variety for a fan Σ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$. Denote by $\sigma_1, \dots, \sigma_r \subset N_{\mathbb{R}}$ the maximal cones in Σ and set*

$$\sigma = \operatorname{Conv} \bigcup_{i=1}^r \sigma_i \subset N_{\mathbb{R}}.$$

Then:

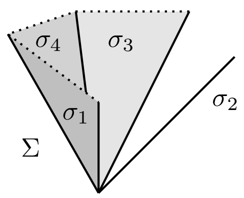
(1) *We have $X_{\mathrm{aff}} = X_\sigma$ and the canonical morphism $\iota\colon X \to X_{\mathrm{aff}}$ is induced by the embeddings $\sigma_i \subset \sigma$ for $i = 1, \dots, r$.*(1)

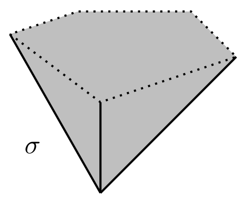
(2) *The toric variety X is quasi-affine if and only if each σ_i is a face of σ. Moreover, if X is quasi-affine, then σ is strongly convex.*

(3) *If X is quasi-affine, then the irreducible components of $X_{\mathrm{aff}} \smallsetminus X$ are the closed sets of the form $V(\tau)$, where τ is a minimal face of σ with $\tau \notin \Sigma$.*

(4) *If X is quasi-affine, then each face τ of σ with $\tau \notin \Sigma$ has dimension at least 2. In particular, $X_{\mathrm{aff}} \smallsetminus X$ is a closed subset of codimension at least 2 in X_{aff}.*

Below we draw a picture where the fan Σ with maximal cones $\sigma_1, \dots, \sigma_4$ defines a 3-dimensional quasi-affine variety with associated cone $\sigma = \operatorname{Conv} \bigcup_{i=1}^4 \sigma_i$:





Proof of Lemma 6.1

(1) Since the affine toric varieties $X_{\sigma_1}, \dots, X_{\sigma_r}$ cover X, we get inside $\mathscr{O}(T) = \boldsymbol{k}[M]$:

$$\mathscr{O}(X) = \bigcap_{i=1}^r \mathscr{O}(X_{\sigma_i}) = \bigcap_{i=1}^r \boldsymbol{k}[(\sigma_i)_M^\vee] = \boldsymbol{k}\left[\left(\bigcap_{i=1}^r \sigma_i^\vee\right) \cap M\right].$$

(1)Note that we defined X_σ only for strongly convex rational polyhedral cones σ. However, the definition X_σ makes sense for every convex rational polyhedral cone σ. In this case, the torus T may act non-faithfully on X_σ.

Since

$$\sigma^\vee = \left(\operatorname{Conv} \bigcup_{i=1}^{r} \sigma_i\right)^\vee = \{u \in M_\mathbb{R} \mid \langle u, v\rangle \geqslant 0 \text{ for all } v \in \sigma_i \text{ and all } i\} = \bigcap_{i=1}^{r} \sigma_i^\vee,$$

we get $\mathscr{O}(X) = \mathscr{O}(X_\sigma)$ which implies the first claim.

For the second claim, denote by $\iota_i\colon X_{\sigma_i} \to (X_{\sigma_i})_{\mathrm{aff}}$ the canonical morphism of X_{σ_i} (which is in fact an isomorphism). Then we have for each $i = 1, \dots, r$ the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] & X_{\mathrm{aff}} \arrow[r, equal] & X_\sigma \\ X_{\sigma_i} \arrow[u, phantom, "\bigcup"] \arrow[r, "\iota_i"] & (X_{\sigma_i})_{\mathrm{aff}} \arrow[u, "\eta"'] & \end{tikzcd}$$

where η is induced by the inclusion $\boldsymbol{k}[\sigma_M^\vee] \subset \boldsymbol{k}[(\sigma_i)_M^\vee]$. As $\eta \circ \iota_i\colon X_{\sigma_i} \to X_{\mathrm{aff}} = X_\sigma$ is induced by the inclusion $\sigma_i \subset \sigma$, the second claim follows.

(2) If $\sigma_i \subset \sigma$ is a face, then the induced morphism $X_{\sigma_i} \to X_\sigma$ is an open immersion (see [Ful93, §1.3 Lem.]). Now, if each σ_i is a face of σ, then by (1) the canonical morphism $\iota\colon X \to X_\sigma$ is an open immersion, i.e., X is quasi-affine (see Lemma 3.2).

On the other hand, if X is quasi-affine, then $\iota\colon X \to X_\sigma$ is an open immersion (again by Lemma 3.2) and by (1), the morphism $X_{\sigma_i} \to X_\sigma$ induced by $\sigma_i \subset \sigma$ is also an open immersion. It now follows from [Ful93, §1.3 Exer. p. 18] that σ_i is a face of σ.

If X is quasi-affine, then $X_{\mathrm{aff}} = X_\sigma$ is a toric variety by Lemma 3.4 and thus σ is strongly convex.

(3) We claim that $X_{\mathrm{aff}} \smallsetminus X$ is the union of all $V(\tau)$, where $\tau \subset \sigma$ is a face with $\tau \notin \Sigma$.

Let $\tau \subset \sigma$ be a face such that $\tau \notin \Sigma$. In particular we have for all i that $\tau \not\subset \sigma_i$. Since X is quasi-affine, σ_i is a face of σ by (2). Hence, there is a $u_i \in \sigma_M^\vee$ with $u_i^\perp \cap \sigma = \sigma_i$ and

$$X_{\sigma_i} = X_\sigma \smallsetminus Z_{X_\sigma}(\chi^{u_i})$$

by [Ful93, §1.3 Lem.], where $Z_{X_\sigma}(\chi^{u_i})$ denotes the zero set of $\chi^{u_i} \in \mathscr{O}(X_\sigma)$ inside X_σ. As $\tau \subset \sigma$, but $\tau \not\subset \sigma_i$, we get $\tau \not\subset u_i^\perp$ and thus $u_i \in \sigma_M^\vee \smallsetminus \tau^\perp$. By [Ful93, §3.1], the closed embedding $V(\tau) \subset X_\sigma$ corresponds to the surjective $\boldsymbol{k}$-algebra homomorphism

$$\boldsymbol{k}[\sigma_M^\vee] \longrightarrow \boldsymbol{k}[\sigma_M^\vee \cap \tau^\perp], \quad \chi^m \longmapsto \begin{cases} \chi^m & \text{if } m \in \tau^\perp, \\ 0 & \text{if } m \in \sigma_M^\vee \smallsetminus \tau^\perp. \end{cases}$$

In particular, χ^{u_i} vanishes on $V(\tau)$ and thus $V(\tau)$ and X_{σ_i} are disjoint for all $i = 1, \dots, r$, i.e., $V(\tau) \subset X_{\mathrm{aff}} \smallsetminus X$. On the other hand, if $\eta \subset \sigma$ is a face with $\eta \in \Sigma$, then there is a $i \in \{1, \dots, r\}$ such that η is a face of σ_i. Then by [Ful93, §3.1, p. 53], it follows that $V(\eta)$ and X_{σ_i} do intersect. In particular, $V(\eta) \not\subset X_{\mathrm{aff}} \smallsetminus X$. Since $X_{\mathrm{aff}} \smallsetminus X$ is a closed T-invariant subset, it is the union of some $V(\varepsilon)$ for some faces ε of σ. This implies then the claim.

Statement (3) now follows from the claim, since the minimal faces $\tau \subset \sigma$ with $\tau \notin \Sigma$ correspond to the maximal $V(\tau)$ in $X_{\mathrm{aff}} \smallsetminus X$.

(4) Since X is quasi-affine it follows from (2) that each σ_i is a face of σ. Since σ is the convex hull of the σ_i, we get thus that the extremal rays of σ are the same as the extremal rays of all the σ_i. Hence the extremal rays of σ are the same as the cones of dimension one in Σ. In particular, each face τ of σ with $\tau \notin \Sigma$ has dimension at least 2. $\square$

For the description of the homogeneous $\mathbb{G}_{\mathrm{a}}$-actions, let us set up the following notation. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. If $\rho \subset \sigma$ is an extremal ray and $\tau \subset \sigma$ a face, we denote

$$\tau_\rho \coloneqq \mathrm{Conv}(\text{extremal rays in } \tau \text{ except } \rho) \subset N_{\mathbb{R}}.$$

In the picture below, we draw a picture of τ and τ_ρ:

In particular, if ρ is not an extremal ray of τ, then $\tau_\rho = \tau$. Let us mention the following easy observations of this construction for future use:

Lemma 6.2. — *Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone, $\tau \subset \sigma$ a face and $\rho \subset \sigma$ an extremal ray. Then*

(1) *τ_ρ is a face of σ_ρ;*
(2) *If $\dim \tau_\rho < \dim \tau$, then τ_ρ is a face of τ.*

Proof

(1) By definition, there is $u \in \sigma^\vee$ with $\tau = \sigma \cap u^\perp$. Hence $\tau_\rho \subset \sigma_\rho \cap u^\perp \subset \tau$. Since $u \in (\sigma_\rho)^\vee$, $\sigma_\rho \cap u^\perp$ is a face of σ_ρ. If $\rho \not\subset \tau$, then $\tau_\rho = \tau$ and thus $\tau_\rho = \sigma_\rho \cap u^\perp$ is a face of σ_ρ. If $\rho \subset \tau$, then $\sigma_\rho \cap u^\perp$ is the convex cone generated by the extremal rays in τ, except ρ, i.e., $\tau_\rho = \sigma_\rho \cap u^\perp$. Thus τ_ρ is a face of σ_ρ.

(2) As $\dim \tau_\rho < \dim \tau$, we get $\rho \subset \tau$ and

$$\mathrm{Span}_{\mathbb{R}}(\tau) = \mathbb{R}\rho \oplus \mathrm{Span}_{\mathbb{R}}(\tau_\rho). \tag{$\boxdot$}$$

Hence, there is $u \in M$ such that $\mathrm{Span}_{\mathbb{R}}(\tau_\rho) = u^\perp \cap \mathrm{Span}_{\mathbb{R}}(\tau)$. After possibly replacing u by $-u$, we may assume $\langle u, v_\rho \rangle \geqslant 0$, where $v_\rho \in \rho$ denotes the unique primitive generator. As $\tau_\rho \subset u^\perp$, we get now $u \in \tau^\vee$. Moreover,

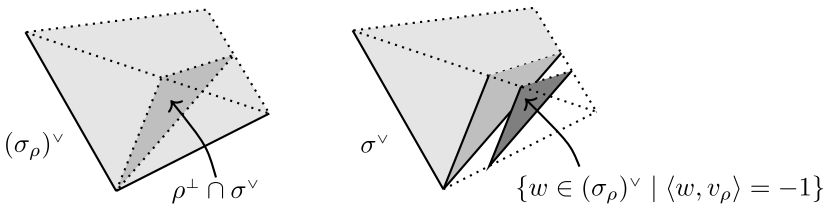
$$u^\perp \cap \tau = (u^\perp \cap \mathrm{Span}_{\mathbb{R}}(\tau)) \cap \tau = \mathrm{Span}_{\mathbb{R}}(\tau_\rho) \cap \tau = \tau_\rho,$$

where the third equality follows from ($\boxdot$) as one may write each element in τ as $\lambda v_\rho + \mu w$ for $w \in \tau_\rho$ and $\lambda, \mu \geqslant 0$. Thus τ_ρ is a face of τ. $\square$

For each extremal ray ρ in a strongly convex rational polyhedral cone σ, let

$$S_\rho \coloneqq \{w \in (\sigma_\rho)^\vee \mid \langle w, v_\rho \rangle = -1\} \cap M,$$

where $v_\rho \in \rho$ denotes the unique primitive generator. In [Lie10, after Def. 2.3] there is an illuminating picture that shows the situation. We provide below our own picture of the situation. In the first picture we draw $(\sigma_\rho)^\vee$ in light gray whereas in the second picture we draw $\sigma^\vee$ in light gray.



Remark 6.3 (see also [Lie10, Rem. 2.5]). — The set S_ρ is non-empty. Indeed, apply Proposition 2.6(1) to the convex polyhedral cone $C = \sigma_\rho^\vee$ and the hyperplanes $H = \rho^\perp$, $H' = \{u \in M_\mathbb{R} \mid \langle u, v_\rho\rangle = -1\}$ inside $V = M_\mathbb{R}$.

Now, we come to the promised description of the homogeneous $\mathbb{G}_\mathrm{a}$-actions on toric varieties due to Liendo:

Proposition 6.4 ([Lie10, Lem. 2.6, Th. 2.7]). — *Let $\sigma \subset N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Then for any extremal ray ρ in σ and any $e \in S_\rho$, the $\boldsymbol{k}$-linear map*

$$\partial_{\rho,e}\colon \boldsymbol{k}[\sigma_M^\vee] \longrightarrow \boldsymbol{k}[\sigma_M^\vee], \quad \chi^m \longmapsto \langle m, v_\rho\rangle \chi^{e+m}$$

is a homogeneous locally nilpotent derivation of degree e, and every homogeneous locally nilpotent derivation of $\boldsymbol{k}[\sigma_M^\vee]$ is a constant multiple of some $\partial_{\rho,e}$. $\square$

Remark 6.5. — The weight of the homogeneous $\mathbb{G}_\mathrm{a}$-action induced by $\partial_{\rho,e}$ is $e \in M$. The kernel of the locally nilpotent derivation $\partial_{\rho,e}$ is $\boldsymbol{k}[\sigma_M^\vee \cap \rho^\perp]$.

The following lemma is the key for the description of the homogeneous $\mathbb{G}_\mathrm{a}$-actions on a quasi-affine toric variety.

Proposition 6.6. — *Let $\sigma \subset N_\mathbb{R}$ be a strongly convex rational polyhedral cone, $\tau \subset \sigma$ a face, $\rho \in \sigma$ an extremal ray and $e \in S_\rho$. Then the $\mathbb{G}_\mathrm{a}$-action on X_σ corresponding to the locally nilpotent derivation $\partial_{\rho,e}$ leaves $V(\tau)$ invariant if and only if*

$$\rho \not\subset \tau \quad \textit{or} \quad e \notin \tau_\rho^\perp.$$

Proof. — As in the proof of Lemma 6.1 (3), the embedding $\iota\colon V(\tau) \subset X_\sigma$ corresponds to the surjective $\boldsymbol{k}$-algebra homomorphism

$$\iota^*\colon \boldsymbol{k}[\sigma_M^\vee] \longrightarrow \boldsymbol{k}[\sigma_M^\vee \cap \tau^\perp], \quad \chi^m \longmapsto \begin{cases} \chi^m & \text{if } m \in \tau^\perp, \\ 0 & \text{if } m \in \sigma_M^\vee \smallsetminus \tau^\perp. \end{cases}$$

Thus the $\mathbb{G}_\mathrm{a}$-action on X_σ corresponding to $\partial_{\rho,e}$ preserves $V(\tau)$ if and only if

$$\partial_{\rho,e}(\ker \iota^*) \subset \ker \iota^*,$$

see [Fre17, §1.5]. Since $\langle m, v_\rho\rangle = 0$ for all $m \in \rho^\perp$ and since $e + m \in \sigma_M^\vee$ for all $m \in \sigma_M^\vee \smallsetminus \rho^\perp$, this last condition is equivalent to

$$(\odot) \qquad m \in \sigma_M^\vee \smallsetminus (\tau^\perp \cup \rho^\perp) \quad \Longrightarrow \quad e + m \notin \tau^\perp.$$

We now distinguish two cases:

(1) Assume $\rho \not\subset \tau$. Then $\tau \subset \sigma_\rho$. In particular, we get $\langle e, v\rangle \geqslant 0$ for all $v \in \tau$. Let $m \in \sigma_M^\vee \smallsetminus \tau^\perp$. Then we get $\langle m, v\rangle > 0$ for some $v \in \tau$ and hence

$$\langle e + m, v\rangle > 0 \quad \text{for some } v \in \tau,$$

which in turn implies $e + m \notin \tau^\perp$. Thus $(\odot)$ is satisfied.

(2) Assume $\rho \subset \tau$. In particular, we have $\tau^\perp \subset \rho^\perp$. We distinguish two cases:

- $e \notin \tau_\rho^\perp$: Then there exists an extremal ray $\rho' \subset \tau$ with $\rho' \neq \rho$ such that $e \notin (\rho')^\perp$ and the unique primitive generator $v_{\rho'} \in \rho'$ satisfies
$$\langle e + m, v_{\rho'}\rangle = \underbrace{\langle e, v_{\rho'}\rangle}_{>0} + \underbrace{\langle m, v_{\rho'}\rangle}_{\geqslant 0} > 0 \quad \text{for all } m \in \sigma_M^\vee.$$
In particular $e + m \notin \tau^\perp$ for all $m \in \sigma_M^\vee$ and thus $(\odot)$ is satisfied.
- $e \in \tau_\rho^\perp$: Now, we want to apply Proposition 2.7. For this we fix the lattice $\Lambda = M \cap \tau_\rho^\perp$ inside $V = \tau_\rho^\perp$. Since τ_ρ is a face of σ_ρ (see Lemma 6.2(1)), $C = (\sigma_\rho)^\vee \cap \tau_\rho^\perp$ is a rational convex polyhedral cone in $M_\mathbb{R}$ and thus also in V. Moreover, we set $H_0 = \rho^\perp \cap V = \tau^\perp$ and $H_{\pm 1} = \{u \in V \mid \langle u, v_\rho\rangle = \pm 1\}$. Since $e \in (S_\rho \cap \tau_\rho^\perp) \smallsetminus H_0$, H_0 is a hyperplane in V and $C \cap H_{-1} \cap \Lambda = S_\rho \cap \tau_\rho^\perp \neq \varnothing$. Since $H_0 \subsetneq V$ we get thus $\dim \tau_\rho < \dim \tau$. Now, by Lemma 6.2(2), τ_ρ is a face of τ and therefore $\sigma^\vee \cap \tau_\rho^\perp \supsetneq \sigma^\vee \cap \tau^\perp$ by the order-reversing bijection between faces of σ and $\sigma^\vee$. Hence, there is $u \in (\sigma^\vee \cap \tau_\rho^\perp) \smallsetminus \tau^\perp$ and in particular $u \in C \smallsetminus H_0$. As $\langle u, v_\rho\rangle > 0$, after scaling u with a real number > 0, we may assume $u \in C \cap H_1$ and hence $C \cap H_1 \neq \varnothing$. Now, as $C \cap H_0 = \sigma^\vee \cap \tau^\perp$ is rational in $M_\mathbb{R}$ and thus also in V, we may apply Proposition 2.7 and get an element
$$m_1 \in (\sigma_\rho)^\vee \cap \tau_\rho^\perp \cap \{m \in M \mid \langle m, v_\rho\rangle = 1\}.$$
Hence, $m_1 \in \sigma_M^\vee \smallsetminus \rho^\perp$. Since $e, m_1 \in \tau_\rho^\perp$, we get $e + m_1 \in \tau_\rho^\perp$. Since
$$\langle e + m_1, v_\rho\rangle = \langle e, v_\rho\rangle + \langle m_1, v_\rho\rangle = -1 + 1 = 0,$$
we get thus $e + m_1 \in \tau^\perp$. This implies that $(\odot)$ is not satisfied. $\square$

We can use this lemma to provide a full description of all homogeneous $\mathbb{G}_\mathrm{a}$-actions on a quasi-affine toric variety $X = X_\Sigma$. Recall that $X_{\mathrm{aff}} = X_\sigma$, where σ is the cone in $N_\mathbb{R}$ generated by all maximal cones in Σ, see Lemma 6.1. Moreover, $X_{\mathrm{aff}} \smallsetminus X$ is the union of the sets of the form $V(\tau)$, where $\tau \subset \sigma$ runs through the minimal faces with the property that $\tau \notin \Sigma$ (again by Lemma 6.1). In the next corollaries (Corollary 6.7-Corollary 6.10), we use this notation freely.

Corollary 6.7. — *Let $X = X_\Sigma$ be a quasi-affine toric variety, let $X_{\mathrm{aff}} = X_\sigma$ and let $\tau_1, \ldots, \tau_s \subset \sigma$ be the minimal faces of σ which do not belong to Σ. Then, the homogeneous $\mathbb{G}_\mathrm{a}$-actions on X are the restricted homogeneous $\mathbb{G}_\mathrm{a}$-actions on X_{aff}*

that are induced by the constant multiples of $\partial_{\rho,e} \in \mathrm{LND}_{\boldsymbol{k}}(\mathscr{O}(X))$ *such that for all* $i = 1, \dots, s$ *we have*

$$(\circledast) \qquad \rho \not\subset \tau_i \quad \text{or} \quad e \notin (\tau_i)_\rho^\perp.$$

Proof. — Assume that $\partial_{\rho,e}$ is a locally nilpotent derivation of $\mathscr{O}(X)$ such that $(\circledast)$ is satisfied for all $i = 1, \dots, s$. Then by Proposition 6.6, the sets $V(\tau_1), \dots, V(\tau_s) \subset X_{\mathrm{aff}}$ are left invariant by the homogeneous $\mathbb{G}_{\mathrm{a}}$-action $\varepsilon_{\rho,e}\colon \mathbb{G}_{\mathrm{a}} \times X_{\mathrm{aff}} \to X_{\mathrm{aff}}$ which is induced by $\partial_{\rho,e}$. In particular, $X = X_{\mathrm{aff}} \smallsetminus (V(\tau_1) \cup \dots V(\tau_s))$ (see Lemma 6.1) is left invariant by $\varepsilon_{\rho,e}$.

On the other hand, let $\varepsilon\colon \mathbb{G}_{\mathrm{a}} \times X \to X$ be a homogeneous $\mathbb{G}_{\mathrm{a}}$-action on X. By Lemma 3.3 and Proposition 6.4 this $\mathbb{G}_{\mathrm{a}}$-action extends to a homogeneous $\mathbb{G}_{\mathrm{a}}$-action $\varepsilon_{\rho,e}\colon \mathbb{G}_{\mathrm{a}} \times X_{\mathrm{aff}} \to X_{\mathrm{aff}}$ which is induced by some locally nilpotent derivation $\lambda \cdot \partial_{\rho,e} \in \mathrm{LND}_{\boldsymbol{k}}(\mathscr{O}(X))$ for some constant $\lambda \in \boldsymbol{k}$, some extremal ray ρ in σ and some $e \in S_\rho$. Since $\varepsilon_{\rho,e}$ extends ε, the subset $V(\tau_1) \cup \dots \cup V(\tau_s) = X_{\mathrm{aff}} \smallsetminus X$ is left invariant by $\varepsilon_{\rho,e}$. Since the $V(\tau_1), \dots, V(\tau_s)$ are the irreducible components of $X_{\mathrm{aff}} \smallsetminus X$ and since $\mathbb{G}_{\mathrm{a}}$ is an irreducible algebraic group, it follows that $\varepsilon_{\rho,e}$ preserves each $V(\tau_i)$. By Proposition 6.6 we get that for each $i = 1, \dots, s$ the condition $(\circledast)$ is satisfied. $\square$

For the next consequences of Corollary 6.7 we recall the following notation from Section 2: For a subset $E \subset M_{\mathbb{R}}$ we denote by $\mathrm{int}(E)$ the topological interior of E inside the linear span of E. In these consequences we provide a closer description of the weights in M arising from homogeneous $\mathbb{G}_{\mathrm{a}}$-actions on quasi-affine toric varieties and compute the asymptotic cone of these weights.

Corollary 6.8. — *Let* $X = X_\Sigma$ *be a quasi-affine toric variety, let* $X_{\mathrm{aff}} = X_\sigma$*, let* $\rho \subset \sigma$ *be an extremal ray and let* $D_\rho(X)$ *be the set of weights* $e \in S_\rho$ *such that the locally nilpotent derivation* $\partial_{\rho,e}$ *of* $\mathscr{O}(X)$ *induces a homogeneous* $\mathbb{G}_{\mathrm{a}}$*-action on* X*. Then*

$$S_\rho \cap \mathrm{int}(\sigma_\rho^\vee) \subset D_\rho(X) \subset S_\rho.$$

Proof. — Let $e \in S_\rho \subset M$ such that e is contained in $\mathrm{int}(\sigma_\rho^\vee) \subset M_{\mathbb{R}}$. Let $\tau_1, \dots, \tau_s$ be the minimal faces of σ which are not contained in Σ. According to Corollary 6.7 it is enough to show that for each τ_i with $\rho \subset \tau_i$ we have $e \notin (\tau_i)_\rho^\perp$. By Lemma 6.1 (4) we get that $\dim \tau_i \geqslant 2$ for every i. Hence, $\dim(\tau_i)_\rho \geqslant 1$ and thus $(\tau_i)_\rho^\perp \cap \sigma_\rho^\vee$ is a proper face of $\sigma_\rho^\vee$. As $e \in \mathrm{int}(\sigma_\rho^\vee)$, we get $e \notin (\tau_i)_\rho^\perp \cap \sigma_\rho^\vee$ and thus $e \notin (\tau_i)_\rho^\perp$. $\square$

Corollary 6.9. — *Let* $X = X_\Sigma$ *be a quasi-affine toric variety. Let* $X_{\mathrm{aff}} = X_\sigma$ *and let* $D(X)$ *be the set of homogeneous* $\mathbb{G}_{\mathrm{a}}$*-weights on* X*. Then the asymptotic cone of* $D(X) \subset M_{\mathbb{R}}$ *satisfies*

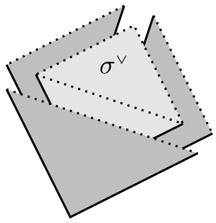
$$D(X)_\infty = \sigma^\vee \smallsetminus \mathrm{int}(\sigma^\vee).$$

By Corollary 6.8, the set $D(X)$ is contained in the set

$$S \coloneqq \bigcup_{\substack{\rho \text{ is an extr.} \\ \text{ray of } \sigma}} \{w \in (\sigma_\rho)^\vee \mid \langle w, v_\rho \rangle = -1\}.$$

Below, we illustrate the dual cone of σ in light gray and the set S associated to σ in dark gray:

$D(X) \subset S$



Intuitively (and rigorously with Lemma 2.4 applied to the convex polyhedral cone $(\sigma_\rho)^\vee$ and the hyperplane $\rho^\perp$ for each ρ) it follows that the asymptotic cone of $D(X)$ is contained in $\bigcup_\rho \{w \in (\sigma_\rho)^\vee \mid \langle w, v_\rho\rangle = 0\}$. This last set is equal to $\sigma^\vee \smallsetminus \operatorname{int}(\sigma^\vee)$. Now, we provide a detailed proof.

Proof. — By [Ful93, Propty (7), p. 10] we have

$$\sigma^\vee \smallsetminus \operatorname{int}(\sigma^\vee) = \bigcup_{\substack{\rho \text{ is an extr.}\\ \text{ray of } \sigma}} \sigma^\vee \cap \rho^\perp.$$

Since $D(X)$ is the union of the $D_\rho(X)$ for the extremal rays $\rho \subset \sigma$ (with the definition of $D_\rho(X)$ from Corollary 6.8), we get by Lemma 2.1 that

$$D(X)_\infty = \bigcup_{\substack{\rho \text{ is an extr.}\\ \text{ray of } \sigma}} D_\rho(X)_\infty.$$

Hence, it is enough to show that $\sigma^\vee \cap \rho^\perp = D_\rho(X)_\infty$ for every extremal ray ρ of σ.

In order to do this, we want to apply Proposition 2.6. For this we fix the lattice $\Lambda = M$ inside $V = M_\mathbb{R}$ and consider the convex polyhedral cone $C = \sigma_\rho^\vee$ inside V and the hyperplane $H = \rho^\perp \subset V$. Note that $C \cap H = \sigma^\vee \cap \rho^\perp$ is a rational convex polyhedral cone in V of dimension $\dim H$ and that H is rational. Moreover, setting $H' = \{u \in M_\mathbb{R} \mid \langle u, v_\rho\rangle = -1\}$, where $v_\rho \in \rho$ denotes the unique primitive generator, there exists $m_{-1} \in M \smallsetminus H$ such that $H' = m_{-1} + H$ (as the coordinates of v_ρ are coprime after identifying N with $\mathbb{Z}^{\operatorname{rank} N}$). Since ρ is an extremal ray of σ, it follows that $\sigma_\rho \subsetneq \sigma$ and thus $\sigma_\rho^\vee \supsetneq \sigma^\vee = \sigma_\rho^\vee \cap \{u \in M_\mathbb{R} \mid \langle u, v_\rho\rangle \geqslant 0\}$. This implies that there is $u \in C$ with $\langle u, v_\rho\rangle < 0$. Since C is a cone, we get that $C \cap H'$ is non-empty. Now, Proposition 2.6 applied to $\Lambda, C, H, H' \subset V$ implies that

$$\sigma^\vee \cap \rho^\perp = \sigma_\rho^\vee \cap \rho^\perp = (S_\rho \cap \operatorname{int}(\sigma_\rho^\vee))_\infty. \tag{d}$$

By Corollary 6.8, Lemma 2.1 and Lemma 2.4 we get

$$(S_\rho \cap \operatorname{int}(\sigma_\rho^\vee))_\infty \subset D_\rho(X)_\infty \subset (S_\rho)_\infty \subset (\sigma_\rho^\vee \cap (m_{-1} + \rho^\perp))_\infty \subset \sigma_\rho^\vee \cap \rho^\perp. \tag{e}$$

Combining (d) and (e) yields $\sigma^\vee \cap \rho^\perp = D_\rho(X)_\infty$ which implies the result. □

Corollary 6.10. — *Let X be a quasi-affine toric variety and let $D(X)$ be the set of homogeneous $\mathbb{G}_\mathrm{a}$-weights. If $X \not\simeq T$, then $D(X)$ generates M as a group.*

Proof. — Let $X_{\mathrm{aff}} = X_\sigma$. Since X is quasi-affine and $X \not\simeq T$, the cone σ is strongly convex and non-zero by Lemma 6.1(2). In particular it has an extremal ray ρ. Corollary 6.10 follows thus from the next lemma, since $S_\rho \cap \operatorname{int}(\sigma_\rho^\vee) \subset D(X)$ (see Corollary 6.8). □

Lemma 6.11. — *Let $\sigma \subset N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Then for every extremal ray $\rho \subset \sigma$, the set $S_\rho \cap \operatorname{int}(\sigma_\rho^\vee)$ generates M as a group.*

Proof. — Denote by $v_\rho \in \rho$ the unique primitive generator. By Remark 6.3, S_ρ is non-empty. Thus by Proposition 2.6(1) applied to the convex polyhedral cone $C = \sigma_\rho^\vee$ and the hypersurfaces $H = \rho^\perp$, $H' = \{u \in V \mid \langle u, v_\rho\rangle = -1\}$ in $V = M_\mathbb{R}$ we get $S_\rho \cap \operatorname{int}(\sigma_\rho^\vee) \neq \varnothing$. Let $A = S_\rho \cap \operatorname{int}(\sigma_\rho^\vee)$ and choose $a \in A$. By definition of S_ρ,

$$a + (\sigma_M^\vee \cap \rho^\perp) \subset A.$$

Since $v_\rho \in N$ is primitive, we may choose a basis of $N = \mathbb{Z}^n$ (where $n = \operatorname{rank} N$) such that $v_\rho = (1, 0, \dots, 0)$. We then identify $M = \operatorname{Hom}_\mathbb{Z}(N, \mathbb{Z})$ with $\mathbb{Z}^n$ by choosing the dual basis of $N = \mathbb{Z}^n$. Since $\sigma^\vee \cap \rho^\perp$ is a convex rational polyhedral cone of dimension $\dim \rho^\perp$ in $\rho^\perp$, there is $m \in \sigma_M^\vee \cap \rho^\perp$ such that the closed ball of radius 1 and center m in $\rho^\perp$ is contained in $\sigma^\vee \cap \rho^\perp$. In particular, $m + e_i \in \sigma_M^\vee \cap \rho^\perp$ for $i = 2, \dots, n$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ and 1 is at position i. In particular,

$$e_i = (a + m + e_i) - (a + m) \in \operatorname{Span}_\mathbb{Z}(A) \quad \text{for } i = 2, \dots, n.$$

Since $v_\rho = (1, 0, \dots, 0)$ and $\langle a, v_\rho\rangle = -1$, it follows that $a = (-1, a_2, \dots, a_n)$ for certain $a_2, \dots, a_n \in \mathbb{Z}$. In particular, $(1, 0, \dots, 0) = -a + \sum_{i=2}^n a_i e_i \in \operatorname{Span}_\mathbb{Z}(A)$. Thus, $\operatorname{Span}_\mathbb{Z}(A) = M$. □

7. The automorphism group determines sphericity

Our first goal in this section is to provide a criterion for a solvable algebraic group B to act with an open orbit on a quasi-affine B-variety. For this, we introduce the notion of *generalized root subgroups*:

Definition 7.1. — Let H be an algebraic group and let X be an H-variety. We call an algebraic subgroup $U_0 \subseteq \operatorname{Aut}(X)$ of dimension m which is isomorphic to $(\mathbb{G}_\mathrm{a})^m$ a *generalized root subgroup (with respect to H)* if there exists a character $\lambda \in \mathfrak{X}(H)$, called the *weight of* U_0 such that

$$h \circ \varepsilon(t) \circ h^{-1} = \varepsilon(\lambda(h) \cdot t) \quad \text{for all } h \in H \text{ and all } t \in (\mathbb{G}_\mathrm{a})^m,$$

where $\varepsilon\colon (\mathbb{G}_\mathrm{a})^m \xrightarrow{\sim} U_0$ is a fixed isomorphism.

Using that a group automorphism of $(\mathbb{G}_\mathrm{a})^m$ is $\boldsymbol{k}$-linear, we see that the weight of a generalized root subgroup U_0 does not depend on the choice of an isomorphism $\varepsilon\colon (\mathbb{G}_\mathrm{a})^m \simeq U_0$.

Remark 7.2. — Let H be an algebraic group and let X be an H-variety. Using again that algebraic group automorphisms of $(\mathbb{G}_{\mathrm{a}})^m$ are $\boldsymbol{k}$-linear, one can see the following: An algebraic subgroup $U_0 \subset \operatorname{Aut}(X)$ which is isomorphic to $(\mathbb{G}_{\mathrm{a}})^m$ for some $m \geqslant 1$ is a generalized root subgroup with respect to H if and only if each one-dimensional closed subgroup of U_0 is a root subgroup of $\operatorname{Aut}(X)$ with respect to H. In particular, root subgroups are generalized root subgroups of dimension one.

Proposition 7.3. — *Let B be a connected solvable algebraic group that contains non-trivial unipotent elements and let X be an irreducible quasi-affine variety with a faithful B-action. Then, the following statements are equivalent:*

(1) *The variety X has an open B-orbit;*

(2) *There is a constant C such that $\dim \operatorname{Vec}(X)_\lambda \leqslant C$ for all weights $\lambda \in \mathfrak{X}(B)$;*

(3) *There exists a constant C such that $\dim U_0 \leqslant C$ for each $U_0 \subseteq \operatorname{Aut}(X)$ that is a generalized root subgroup with respect to B.*

Proof

(1) $\Longrightarrow$ (2) By Lemma 4.9 we get $\dim \operatorname{Vec}(X)_\lambda \leqslant \dim T_{x_0}X$, where $x_0 \in X$ is a fixed element of the open B-orbit.

(2) $\Longrightarrow$ (3) Let $U_0 \subseteq \operatorname{Aut}(X)$ be a generalized root subgroup of weight $\lambda \in \mathfrak{X}(B)$. By Lemma 4.1, the $\boldsymbol{k}$-linear map $\operatorname{Lie}(U_0) \to \operatorname{Vec}(X)$, $A \mapsto \xi_A$ is injective. Now, take $A \in \operatorname{Lie}(U_0)$ which is non-zero. Then there is a one-parameter unipotent subgroup $U_{0,A} \subset U_0$ such that $\operatorname{Lie}(U_{0,A})$ is generated by A. By definition, $U_{0,A}$ is a root subgroup with respect to B of weight λ. By Lemma 4.2, it follows that ξ_A lies in $\operatorname{Vec}(X)_\lambda$. Thus the whole image of $\operatorname{Lie}(U_0) \to \operatorname{Vec}(X)$ lies in $\operatorname{Vec}(X)_\lambda$ and we get $\dim U_0 \leqslant \dim \operatorname{Vec}(X)_\lambda$.

(3) $\Longrightarrow$ (1) Assume that X admits no open B-orbit. This implies by Rosenlicht's Theorem [Ros56, Th. 2] that there is a B-invariant non-constant rational map $f\colon X \dashrightarrow \boldsymbol{k}$. By Proposition 3.6, there exist B-semi-invariant regular functions $f_1, f_2\colon X \to \boldsymbol{k}$ such that $f = f_1/f_2$ and since f is B-invariant, the weights of f_1 and f_2 under B are the same, say $\lambda_0 \in \mathfrak{X}(B)$.

Moreover, there exists no non-zero homogeneous polynomial p in two variables with $p(f_1, f_2) = 0$. Indeed, otherwise there exist $m > 0$ and a non-zero tuple $(a_0, \ldots, a_m) \in \boldsymbol{k}^{m+1}$ such that $\sum_{i=0}^m a_i (f_1)^i (f_2)^{m-i} = 0$ and hence $\sum_{i=0}^m a_i f^i = 0$. Since f is non-constant, we get a contradiction, as $\boldsymbol{k}$ is algebraically closed.

Since B contains non-trivial unipotent elements, the center of the unipotent radical in B is non-trivial. Since this center is normalized by B, there exists a one-dimensional closed subgroup U of this center that is normalized by B. Let $\rho\colon \mathbb{G}_{\mathrm{a}} \times X \to X$ be the $\mathbb{G}_{\mathrm{a}}$-action on X corresponding to U. Hence ρ is B-homogeneous for some weight $\lambda_1 \in \mathfrak{X}(B)$. Thus for any $m \geqslant 0$, we get a faithful $(\mathbb{G}_{\mathrm{a}})^{m+1}$-action on X given by

$$\mathbb{G}_{\mathrm{a}}^{m+1} \times X \longrightarrow X, \quad ((t_0, \ldots, t_m), x) \longmapsto \rho\biggl(\sum_{i=0}^m t_i (f_1^i \cdot f_2^{m-i})(x), x\biggr),$$

since $\sum_{i=0}^{m} t_i f_1^i \cdot f_2^{m-i} \neq 0$ for all non-zero $(t_0, \dots, t_m)$. The corresponding subgroup U_0 in $\mathrm{Aut}(X)$ is then a generalized root subgroup of dimension $m+1$ with respect to B of weight $\lambda_1 + m\lambda_0 \in \mathfrak{X}(B)$. As m was arbitrary, (3) is not satisfied. $\square$

Example 7.4. — If the connected solvable algebraic group B does not contain unipotent elements, then Proposition 7.3 is in general false: Let $B = \mathbb{G}_m$ act on the product $X = \mathbb{G}_m \times C$ via $t \cdot (s, c) = (ts, c)$, where C is any affine curve of genus > 1. Then X has no open B-orbit.

On the other hand, X admits no non-trivial $\mathbb{G}_a$-action and thus property (3) of Proposition 7.3 is satisfied. Indeed, if there is a $\mathbb{G}_a$-action on X with a non-trivial orbit $\mathbb{G}_a \simeq O \subset X$, then one of the restrictions of the projections

$$\mathrm{pr}_1|_O \colon O \longrightarrow \mathbb{G}_m,\ (s,c) \longmapsto s \quad \text{or} \quad \mathrm{pr}_2|_O \colon O \longrightarrow C,\ (s,c) \longmapsto c$$

is non-constant, contradiction.

Lemma 7.5. — *Let T be an algebraic torus and let X be a quasi-affine T-toric variety such that $X \not\simeq T$. Then there exists a non-trivial T-homogeneous $\mathbb{G}_a$-action on X and a subtorus $T' \subset T$ of codimension one such that the induced $\mathbb{G}_a \rtimes T'$-action on X has an open orbit.*

Proof. — Since $X \not\simeq T$, there is a non-trivial T-homogeneous $\mathbb{G}_a$-action on X by Corollary 6.10. Denote by $U \subset \mathrm{Aut}(X)$ the corresponding root subgroup.

Let $x_0 \in X$ such that $Tx_0 \subset X$ is open in X and let S be the connected component of the stabilizer in $U \rtimes T$ of x_0. As $\dim U \rtimes T = \dim X + 1$, we get $\dim S = 1$. If S would be contained in U, then $S = U$ and thus $ux_0 = x_0$ for all $u \in U$. From this we would get for all $t \in T$, $u \in U$ that

$$(tut^{-1}) \cdot (tx_0) = tx_0$$

and hence U would fix each element of the open orbit Tx_0, contradiction. Hence, $S \not\subset U$, which implies that there is a codimension one subtorus $T' \subset T$ with $S \not\subset U \rtimes T'$. This implies that $(U \rtimes T') \cap S$ is finite and thus $(U \rtimes T')x_0$ is dense in X. As orbits are locally closed, we get that $(U \rtimes T')x_0$ is open in X. $\square$

For the sake of completeness let us recall the following well-known fact from the theory of algebraic groups:

Lemma 7.6. — *Let G be a connected reductive algebraic group and let $B \subset G$ be a Borel subgroup. If G is not a torus, then B contains non-trivial unipotent elements.*

Proof. — If B contains no non-trivial unipotent elements, then B is a torus and it follows from [Hum75, Prop. 21.4B] that $G = B$, contradiction. $\square$

Now, we prove that one can recognize the sphericity of an irreducible quasi-affine normal G-variety from its automorphism group.

Proposition 7.7. — *Let G be a connected reductive algebraic group and let X, Y be irreducible quasi-affine normal varieties. Assume that there is a group isomorphism $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ that preserves algebraic subgroups. If X is non-isomorphic to a torus and G-spherical, then Y is G-spherical for the induced G-action via θ.*

Proof. — We denote by $B \subseteq G$ a Borel subgroup and by $T \subseteq G$ a maximal torus. We distinguish two cases:

- $G \neq T$: By Lemma 7.6 the Borel subgroup B contains unipotent elements and thus we may apply Proposition 7.3 in order to get a bound on the dimension of every generalized root subgroup with respect to B of $\mathrm{Aut}(X)$. Since the generalized root subgroups of $\mathrm{Aut}(X)$ (with respect to B) correspond bijectively to the generalized root subgroups of $\mathrm{Aut}(Y)$ (with respect to $\theta(B)$) via θ (see Remark 7.2 and Lemma 5.1), it follows by Proposition 7.3 that Y is $\theta(G)$-spherical.
- $G = T$: In this case X is T-toric. Since X is not isomorphic to a torus, we may apply Lemma 7.5 in order to get a codimension one subtorus $T' \subset T$ and a root subgroup $V \subset \mathrm{Aut}(X)$ with respect to T such that $V \cdot T'$ acts with an open orbit on X. As before, it follows from Proposition 7.3 that $\theta(V) \cdot \theta(T')$ acts with an open orbit on Y. This implies that $\dim(Y) \leqslant \dim(V) + \dim(T') = \dim(T)$. On the other hand, since $\theta(T)$ acts faithfully on Y, we get $\dim(T) \leqslant \dim(Y)$. In summary, $\dim(Y) = \dim(T)$ and thus Y is $\theta(T)$-toric. □

8. Relation between the set of homogeneous $\mathbb{G}_{\mathrm{a}}$-weights and the weight monoid

Throughout the whole section we fix the following

Notation. — We denote by G a connected reductive algebraic group, by $B \subset G$ a Borel subgroup and by $T \subset B$ a maximal torus. By convention G is non-trivial. We denote by $U \subset B$ the unipotent radical of B. Moreover, we denote $\mathfrak{X}(B)_{\mathbb{R}} = \mathfrak{X}(B) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathfrak{X}(B)$ is the character group of B. For a G-variety X let us recall the definition of the set of B-homogeneous $\mathbb{G}_{\mathrm{a}}$-weights:

$$D(X) = \left\{ \lambda \in \mathfrak{X}(B) \;\middle|\; \begin{array}{l} \text{there exists a non-trivial } B\text{-homogeneous} \\ \mathbb{G}_{\mathrm{a}}\text{-action on } X \text{ of weight } \lambda \end{array} \right\}$$

(see Section 4.2 for the definition of a B-homogeneous $\mathbb{G}_{\mathrm{a}}$-action).

In this section we provide for a quasi-affine G-spherical variety X a description of the weight monoid $\Lambda^+(X)$ in terms of $D(X)$, see Theorem 8.2 below.

Proposition 8.1. — *Let X be an irreducible quasi-affine variety with a faithful G-action such that $\mathscr{O}(X)$ is a finitely generated $\boldsymbol{k}$-algebra. If $G \neq T$, then there is a $\lambda \in D(X)$ with*

$$\lambda + \Lambda^+(X) \subset D(X) \quad \textit{and} \quad \Lambda^+(X)_\infty = D(X)_\infty,$$

where the asymptotic cones are taken inside $\mathfrak{X}(B)_{\mathbb{R}}$.

Proof. — We denote $D = D(X)$. By Lemma 4.3 we have

$$D \subset \left\{ \lambda \in \mathfrak{X}(B) \;\middle|\; \begin{array}{l} \text{there is a non-zero vector field in } \mathrm{Vec}^U(X) \\ \text{that is normalized by } B \text{ of weight } \lambda \end{array} \right\} =: D'.$$

By Corollary 4.8 we know that $\mathrm{Vec}^U(X)$ is finitely generated as an $\mathscr{O}(X)^U$-module. Hence, there are finitely many non-zero B-homogeneous $\xi_1, \ldots, \xi_k \in \mathrm{Vec}^U(X)$ such that the B-module homomorphism

$$\pi\colon \bigoplus_{i=1}^k \mathscr{O}(X)^U \xi_i \longrightarrow \mathrm{Vec}^U(X), \quad (r_1\xi_1, \ldots, r_k\xi_k) \longmapsto r_1\xi_1 + \cdots + r_k\xi_k$$

is surjective. Let $\lambda \in D'$ and let $\eta \in \mathrm{Vec}^U(X)$ be a non-zero vector field that is normalized by B of weight λ. Thus $M = \pi^{-1}(\boldsymbol{k}\eta)$ is a rational B-submodule of $\bigoplus_{i=1}^k \mathscr{O}(X)^U \xi_i$ (see Proposition 3.5). As each element in M can be written as a sum of T-semi-invariants, as U acts trivially on M and as $\mathfrak{X}(U)$ is trivial, it follows that each element in M can be written as a sum of B-semi-invariants. Hence, there is a non-zero B-semi-invariant $\xi \in M$ such that $\pi(\xi) = \eta$. As a consequence, the weight of ξ is λ. Thus we proved that D' is contained in the weights of non-zero B-semi-invariants of $\bigoplus_{i=1}^k \mathscr{O}(X)^U \xi_i$, i.e.,

$$D' \subset \bigcup_{i=1}^k \left(\lambda_i + \Lambda^+(X)\right),$$

where $\lambda_i \in \mathfrak{X}(B)$ denotes the weight of ξ_i.

Since $G \neq T$, we get by Lemma 7.6 that $U \neq \{e\}$. Since G (and therefore U) acts faithfully on X, there is a non-trivial B-homogeneous $\mathbb{G}_\mathrm{a}$-action $\rho\colon \mathbb{G}_\mathrm{a} \times X \to X$ of a certain weight $\lambda \in D$ associated to a root subgroup with respect to B in the center of U. Now, we claim that

$$\lambda + \Lambda^+(X) \subset D.$$

Indeed, this follows since for every non-zero B-semi-invariant $r \in \mathscr{O}(X)^U$ of weight $\lambda' \in \mathfrak{X}(B)$, the $\mathbb{G}_\mathrm{a}$-action

$$\mathbb{G}_\mathrm{a} \times X \longrightarrow X, \quad (t, x) \longmapsto \rho(r(x)t, x)$$

is non-trivial and B-homogeneous of weight $\lambda + \lambda' \in \mathfrak{X}(B)$.

In summary, we have proved

$$\lambda + \Lambda^+(X) \subset D \subset D' \subset \bigcup_{i=1}^k \left(\lambda_i + \Lambda^+(X)\right) \subset \mathfrak{X}(B)_\mathbb{R}.$$

From Lemma 2.1 it now follows that $\Lambda^+(X)_\infty \subset D_\infty \subset D'_\infty = \Lambda^+(X)_\infty$. $\square$

Theorem 8.2. — *Let X be a quasi-affine G-spherical variety which is non-isomorphic to a torus. If $G \neq T$ or $X_\mathrm{aff} \not\simeq \mathbb{A}^1 \times (\mathbb{A}^1 \smallsetminus \{0\})^{\dim(X)-1}$, then $D(X)$ is non-empty and*

$$(\circledcirc) \qquad \Lambda^+(X) = \mathrm{Conv}(D(X)_\infty) \cap \mathrm{Span}_\mathbb{Z}(D(X)),$$

where the asymptotic cones and linear spans are taken inside $\mathfrak{X}(B)_\mathbb{R}$. Moreover, $\dim \mathrm{Conv}(D(X)_\infty) = \dim \mathrm{Span}_\mathbb{R}(D(X))$.

In case X is isomorphic to a torus, $D(X)$ is empty and thus $\mathrm{Span}_{\mathbb{Z}}(D(X)) = \{0\}$. In particular, (◎) is not satisfied (as G is non-trivial). In case $G = T$ and $X_{\mathrm{aff}} \simeq \mathbb{A}^1 \times (\mathbb{A}^1 \smallsetminus \{0\})^{\dim(X)-1}$, Remark 8.5 below implies that (◎) is not satisfied.

Proof of Theorem 8.2. — As in the last proof, we set $D = D(X)$. We get $D \neq \varnothing$. Indeed: if $G \neq T$, this follows from Lemma 7.6 and if $G = T$, this follows from Corollary 6.10 (as X is not a torus).

Since X is a quasi-affine G-spherical variety, it follows from Lemma 3.4 that $X_{\mathrm{aff}} = \operatorname{Spec} \mathscr{O}(X)$ is an affine G-spherical variety. In particular, $\mathscr{O}(X)$ is an integrally closed domain, that is finitely generated as a $\boldsymbol{k}$-algebra. Hence $\mathscr{O}(X)^U$ is integrally closed and it is finitely generated as a $\boldsymbol{k}$-algebra (by Proposition 4.6). Since B acts with an open orbit on X_{aff}, the algebraic quotient $X_{\mathrm{aff}} /\!\!/ U = \operatorname{Spec} \mathscr{O}(X)^U$ is an affine T'-toric variety, where T' is a quotient torus of T. Thus we get a natural inclusion of character groups

$$\mathfrak{X}(T') \subset \mathfrak{X}(T) = \mathfrak{X}(B),$$

where we identify $\mathfrak{X}(B)$ with $\mathfrak{X}(T)$ via the restriction homomorphism. Using the above inclusion, $\Lambda^+(X)$ is contained inside $\mathfrak{X}(T')$ and it is equal to the set of T'-weights of non-zero T'-semi-invariants of $\mathscr{O}(X)^U$. As $X_{\mathrm{aff}} /\!\!/ U$ is T'-toric, $\Lambda^+(X)$ is a finitely generated semi-group and $\mathrm{Conv}(\Lambda^+(X))$ is a convex rational polyhedral cone inside $\mathfrak{X}(T')_{\mathbb{R}} \subset \mathfrak{X}(B)_{\mathbb{R}}$. Moreover, $\Lambda^+(X)$ generates $\mathfrak{X}(T')$ as a group inside $\mathfrak{X}(B)$ and $\Lambda^+(X)$ is saturated in $\mathfrak{X}(T')$, i.e.,

$$\Lambda^+(X) = \mathrm{Conv}(\Lambda^+(X)) \cap \mathfrak{X}(T')$$

(see [CLS11, Ex. 1.3.4 (a)]). Using the inclusion $\mathfrak{X}(T') \subset \mathfrak{X}(T) = \mathfrak{X}(B)$ again, we get $D \subset \mathfrak{X}(T')$, since each B-homogeneous $\mathbb{G}_{\mathrm{a}}$-action on X induces a T'-homogeneous $\mathbb{G}_{\mathrm{a}}$-action on $X_{\mathrm{aff}} /\!\!/ U$. We distinguish two cases:

• $G \neq T$. By Proposition 8.1, we get inside $\mathfrak{X}(B)_{\mathbb{R}}$

$$\Lambda^+(X)_\infty = D_\infty$$

and there is a $\lambda \in D$ with $\lambda + \Lambda^+(X) \subset D \subset \mathfrak{X}(T')$. Since $\Lambda^+(X)$ generates the group $\mathfrak{X}(T')$, we get thus $\mathrm{Span}_{\mathbb{Z}}(D) = \mathfrak{X}(T')$. As $\mathrm{Conv}(\Lambda^+(X))$ is a rational convex polyhedral cone, we get $\mathrm{Conv}(\Lambda^+(X)) = \mathrm{Conv}(\Lambda^+(X)_\infty)$. In summary, we have

$$\begin{aligned}\Lambda^+(X) = \mathrm{Conv}(\Lambda^+(X)) \cap \mathfrak{X}(T') &= \mathrm{Conv}(\Lambda^+(X)_\infty) \cap \mathfrak{X}(T') \\ &= \mathrm{Conv}(D_\infty) \cap \mathrm{Span}_{\mathbb{Z}}(D)\end{aligned}$$

and thus (◎) holds. The second statement now follows from

$$\dim \mathrm{Span}_{\mathbb{R}}(D) = \dim T' = \operatorname{rank} \Lambda^+(X) \leqslant \dim \mathrm{Conv}(D_\infty) \leqslant \dim T'.$$

• $G = T$. In particular, T acts faithfully with an open orbit on X. Thus $T' = T$ and both varieties X, $X_{\mathrm{aff}} = X_{\mathrm{aff}} /\!\!/ U$ are T-toric.

Denote by $\sigma \subset \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}(T), \mathbb{R})$ the strongly convex rational polyhedral cone that describes X_{aff} and let $\sigma^\vee \subset \mathfrak{X}(T)_{\mathbb{R}}$ be the dual of σ. By Corollary 6.9

$$D_\infty = \sigma^\vee \smallsetminus \mathrm{int}(\sigma^\vee), \tag{$\triangle$}$$

where $\operatorname{int}(\sigma^\vee)$ denotes the interior of $\sigma^\vee$ inside $\mathfrak{X}(T)_\mathbb{R}$. By assumption,

$$X_{\text{aff}} \not\simeq \mathbb{A}^1 \times (\mathbb{A}^1 \smallsetminus \{0\})^{\dim(X)-1}.$$

This implies that $\dim\sigma > 1$ and we may write $\sigma^\vee = C \times W$, where $C \subset \mathfrak{X}(T)_\mathbb{R}$ is a strongly convex polyhedral cone of dimension > 1 and $W \subset \mathfrak{X}(T)_\mathbb{R}$ is a linear subspace. Hence, C is the convex hull of its codimension one faces and thus the same holds for $\sigma^\vee$. Using ($\triangle$), we get

$$\operatorname{Conv}(D_\infty) = \sigma^\vee = \operatorname{Conv}(\Lambda^+(X)).$$

Since $\Lambda^+(X)$ is saturated in $\mathfrak{X}(T)$, the above equality implies that

$$\Lambda^+(X) = \operatorname{Conv}(\Lambda^+(X)) \cap \mathfrak{X}(T) = \operatorname{Conv}(D_\infty) \cap \mathfrak{X}(T).$$

It follows from Corollary 6.10 that $\mathfrak{X}(T) = \operatorname{Span}_\mathbb{Z}(D)$ (here we use that $X \not\simeq T$) and thus ($\circledcirc$) holds. The second statement now follows from

$$\dim \operatorname{Span}_\mathbb{R}(D) = \dim T = \operatorname{rank}\Lambda^+(X) \leqslant \dim \operatorname{Conv}(D_\infty) \leqslant \dim T. \qquad \square$$

Remark 8.3. — Assume that $G = T$ and that X is a T-toric quasi-affine variety. Then one could recover the extremal rays of the strongly convex rational polyhedral cone that describes X_{aff} from $D(X)$ in a similar way as in [LRU19, Lem. 6.11] by using Corollary 6.8. In particular, one could then recover $\Lambda^+(X)$ from $D(X)$. However, we wrote Theorem 8.2 in order to have a nice "closed formula" of $\Lambda^+(X)$ in terms of $D(X)$ for almost all quasi-affine G-spherical varieties.

Corollary 8.4. — *For a quasi-affine G-spherical variety X, exactly one of the following cases holds (the linear spans and asymptotic cones are taken inside $\mathfrak{X}(B)_\mathbb{R}$):*

(1) $\dim \operatorname{Conv}(D(X)_\infty) = \dim \operatorname{Span}_\mathbb{R}(D(X))$, *$D(X)$ is non-empty and*

$$\Lambda^+(X) = \operatorname{Conv}(D(X)_\infty) \cap \operatorname{Span}_\mathbb{Z}(D(X));$$

(2) $\dim \operatorname{Conv}(D(X)_\infty) < \dim \operatorname{Span}_\mathbb{R}(D(X))$, *$D(X)$ is non-empty, $D(X)_\infty$ is a hyperplane in* $\operatorname{Span}_\mathbb{R}(D(X))$ *and*

$$\Lambda^+(X) = H^+ \cap \operatorname{Span}_\mathbb{Z}(D(X)),$$

where $H^+ \subset \operatorname{Span}_\mathbb{R}(D(X))$ is the closed half space with boundary $D(X)_\infty$ that does not intersect $D(X)$;

(3) *$D(X)$ is empty and $\Lambda^+(X) = \mathfrak{X}(T)$.*

In particular, the following holds: If Y is another quasi-affine G-spherical variety with $D(Y) = D(X)$, then $\Lambda^+(Y) = \Lambda^+(X)$.

Proof. — If X is a torus, then $D(X)$ is empty. In particular, $G = T$ by Lemma 7.6 and thus $X \simeq T$. Hence, $\Lambda^+(X) = \mathfrak{X}(T)$ and we are in case (3). Thus we may assume that X is not a torus.

If $G \neq T$ or $X_{\text{aff}} \not\simeq \mathbb{A}^1 \times (\mathbb{A}^1 \smallsetminus \{0\})^{\dim(X)-1}$, then Theorem 8.2 implies that we are in case (1).

Thus we may assume that $G = T$ and $X_{\text{aff}} \simeq \mathbb{A}^1 \times (\mathbb{A}^1 \smallsetminus \{0\})^{\dim(X)-1}$. In particular, $D(X)$ is non-empty and by Corollary 6.10 we get $\mathfrak{X}(T) = \operatorname{Span}_\mathbb{Z}(D(X))$. Denote by $\sigma \subset \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}(T), \mathbb{R})$ the closed strongly convex rational polyhedral cone that describes X_{aff}. In this case σ is a single ray and thus $\sigma^\vee$ is a closed half space in $\mathfrak{X}(T)_{\mathbb{R}}$. As $D(X)_\infty = \sigma^\vee \setminus \mathrm{int}(\sigma^\vee)$ (see Corollary 6.9), it follows that $D(X)_\infty$ is a hyperplane in $\mathrm{Span}_{\mathbb{R}}(D(X))$. By definition $\Lambda^+(X) = \sigma^\vee \cap \mathrm{Span}_{\mathbb{Z}}(D(X))$ and $\sigma^\vee$ is in fact the closed half space with boundary $D(X)_\infty$ that does not intersect $D(X)$ (see Corollary 6.8). In particular, $\dim \mathrm{Conv}(D(X)_\infty) < \dim T = \dim \mathrm{Span}_{\mathbb{R}}(D(X))$ and thus we are in case (2). □

Remark 8.5. — The proof of Corollary 8.4 shows that in case $G = T$ and $X_{\mathrm{aff}} \simeq \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{\dim(X)-1}$ we are in case 2. In particular,

$$\Lambda^+(X) \neq \mathrm{Conv}(D(X)_\infty) \cap \mathrm{Span}_{\mathbb{Z}}(D(X)).$$

As a consequence of Corollary 8.4 we prove that for a G-spherical variety X the weight monoid $\Lambda^+(X) \subseteq \mathfrak{X}(B)$ is determined by its automorphism group.

Corollary 8.6. — *Let X, Y be irreducible quasi-affine normal varieties. Assume that X is G-spherical, X is different from an algebraic torus and that there exists an isomorphism of groups $\theta\colon \mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ that preserves algebraic subgroups. Then Y is G-spherical for the G-action induced by θ and $\Lambda^+(X) = \Lambda^+(Y)$.*

Proof. — The first claim follows from Proposition 7.7. To show that $\Lambda^+(X) = \Lambda^+(Y)$ let us denote by $D(X), D(Y) \subset \mathfrak{X}(B)$ the set of B-weights of non-trivial B-homogeneous $\mathbb{G}_a$-actions on X and Y, respectively. We get $D(X) = D(Y)$ from Lemma 5.1. Now, Corollary 8.4 implies $\Lambda^+(X) = \Lambda^+(Y)$. □

Theorem 8.7. — *Let X and Y be irreducible normal affine varieties. Assume that X is G-spherical and that X is not isomorphic to a torus. Moreover, we assume that there is an isomorphism of groups $\theta\colon \mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ that preserves algebraic subgroups. We consider Y as a G-variety by the induced action via θ. Then X, Y are isomorphic as G-varieties, provided one of the following statements holds*

(a) *X and Y are smooth or*
(b) *$G = T$ is a torus.*

Proof. — By Corollary 8.6, Y is G-spherical and the weight monoids $\Lambda^+(X)$ and $\Lambda^+(Y)$ coincide. In case X and Y are smooth, the statement now follows from Losev's result, i.e., Theorem 3. In case G is a torus, it is classical, that from the weight monoid $\Lambda^+(X)$ one can reconstruct the toric variety X up to G-equivariant isomorphisms, see e.g. [Ful93, §1.3]. □

We end this Section with the following natural question concerning Theorem 8.7:

Question 8.8. — *Does the conclusion of Theorem* 8.7 *also hold without the extra assumptions* (a) *and* (b)*?*

9. A counterexample

For the rest of this article, we give an example which shows that we cannot drop the normality condition in Main Theorem A. The example is borrowed from [Reg17].

Let $\mu_d \subset \boldsymbol{k}^*$ be the finite cyclic subgroup of order d and let it act on $\mathbb{A}^n$ via $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots tx_n)$. The algebraic quotient $\mathbb{A}^n/\mu_d$ has the coordinate ring

$$\mathscr{O}(\mathbb{A}^n/\mu_d) = \bigoplus_{k \geqslant 0} \boldsymbol{k}[x_1, \ldots, x_n]_{kd} \subset \boldsymbol{k}[x_1, \ldots, x_n],$$

where $\boldsymbol{k}[x_1, \ldots, x_n]_i \subset \boldsymbol{k}[x_1, \ldots, x_n]$ denotes the subspace of homogeneous polynomials of degree i. For each $s \geqslant 2$, let

$$A^s_{d,n} = \operatorname{Spec}\Bigg(\boldsymbol{k} \oplus \bigoplus_{k \geqslant s} \boldsymbol{k}[x_1, \ldots, x_n]_{kd}\Bigg).$$

Proposition 9.1. — *For $n, s \geqslant 2$, $d \geqslant 1$ and the algebraic quotient $\pi \colon \mathbb{A}^n \to \mathbb{A}^n/\mu_d$ holds:*

(1) *The variety $\mathbb{A}^n/\mu_d$ is $\mathrm{SL}_n(\boldsymbol{k})$-spherical for the induced $\mathrm{SL}_n(\boldsymbol{k})$-action on $\mathbb{A}^n$ and $\mathbb{A}^n/\mu_d$ is smooth outside $\pi(0, \ldots, 0)$;*

(2) *There is an $\mathrm{SL}_n(\boldsymbol{k})$-action on $A^s_{d,n}$ such that the morphism $\eta \colon \mathbb{A}^n/\mu_d \to A^s_{d,n}$ which is induced by the natural inclusion $\mathscr{O}(A^s_{d.n}) \subset \mathscr{O}(\mathbb{A}^n/\mu_d)$ is $\mathrm{SL}_n(\boldsymbol{k})$-equivariant. Moreover, η is the normalization morphism and it is bijective;*

(3) *The natural group homomorphism $\mathrm{Aut}(A^s_{d,n}) \to \mathrm{Aut}(\mathbb{A}^n/\mu_d)$ is a group isomorphism that preserves algebraic subgroups;*

(4) *The variety $A^s_{d,n}$ is not normal;*

(5) *The weight monoids $\Lambda^+(A^s_{d,n})$ and $\Lambda^+(\mathbb{A}^n/\mu_d)$ inside $\mathfrak{X}(B)$ are distinct when we fix a Borel subgroup $B \subset \mathrm{SL}_n(\boldsymbol{k})$.*

Proof

(1) As the natural $\mathrm{SL}_n(\boldsymbol{k})$-action on $\mathbb{A}^n$ commutes with the μ_d-action, we get an induced $\mathrm{SL}_n(\boldsymbol{k})$-action on $\mathbb{A}^n/\mu_d$ such that π is $\mathrm{SL}_n(\boldsymbol{k})$-equivariant and $\mathbb{A}^n/\mu_d$ is $\mathrm{SL}_n(\boldsymbol{k})$-spherical. As $\mathrm{SL}_n(\boldsymbol{k})$ acts transitively on $\mathbb{A}^n \smallsetminus \{0\}$, the projection π induces a finite étale morphism $\mathbb{A}^n \smallsetminus \{(0, \ldots, 0)\} \to (\mathbb{A}^n/\mu_d) \smallsetminus \{\pi(0, \ldots, 0)\}$. This shows that $(\mathbb{A}^n/\mu_d) \smallsetminus \{\pi(0, \ldots, 0)\}$ is smooth.

(2) As $\mathrm{SL}_n(\boldsymbol{k})$ acts linearly on $\mathbb{A}^n$, we get an $\mathrm{SL}_n(\boldsymbol{k})$-action on $A^s_{d,n}$ such that $\eta \colon \mathbb{A}^n/\mu_d \to A^s_{d,n}$ is $\mathrm{SL}_n(\boldsymbol{k})$-equivariant.

As $\mathbb{A}^n$ is normal, the algebraic quotient $\mathbb{A}^n/\mu_d$ is normal. As $\mathscr{O}(A^s_{d,n})$ has finite codimension in $\mathscr{O}(\mathbb{A}^n/\mu_d)$, the ring extension $\mathscr{O}(A^s_{d,n}) \subset \mathscr{O}(\mathbb{A}^n/\mu_d)$ is integral. Moreover, for each monomial $f \in \boldsymbol{k}[x_1, \ldots, x_n]$ of degree sd, we get an equality by localizing, namely $\mathscr{O}(A^s_{d,n})_f = \mathscr{O}(\mathbb{A}^n/\mu_d)_f$, and thus η is birational. This shows that η is the normalization morphism. Moreover, as η is $\mathrm{SL}_n(\boldsymbol{k})$-equivariant and as $\mathrm{SL}_n(\boldsymbol{k})$ acts transitively on $(\mathbb{A}^d/\mu_d) \smallsetminus \{\pi(0, \ldots, 0)\}$, we get that $A^s_{d,n} \smallsetminus \{\eta(\pi(0, \ldots, 0))\}$ is smooth and as η is the normalization, it is an isomorphism over the complement of $\eta(\pi(0, \ldots, 0))$. Moreover, $\eta^{-1}(\eta(\pi(0, \ldots, 0))) = \{\pi(0, \ldots, 0)\}$ and thus η is bijective.

(3) Each automorphism ε of $A^s_{d,n}$ lifts uniquely to an automorphism $\widetilde{\varepsilon}$ of $\mathbb{A}^n/\mu_d$ via the normalization morphism $\mathbb{A}^n/\mu_d \to A^s_{d,n}$ and therefore

$$\theta\colon \operatorname{Aut}(A^s_{d,n}) \longrightarrow \operatorname{Aut}(\mathbb{A}^n/\mu_d), \quad \varepsilon \longmapsto \widetilde{\varepsilon}$$

is an injective group homomorphism.

Now we prove that θ is surjective. For this, let $\varphi \in \operatorname{Aut}(\mathbb{A}^n/\mu_d)$. As $n \geqslant 2$, the algebraic quotient $\mathbb{A}^n \to \mathbb{A}^n/\mu_d$ is in fact the Cox realization of the toric variety $\mathbb{A}^n/\mu_d$ (see [AG10, Th. 3.1]). By [Ber03, Cor. 2.5, Lem. 4.2], φ lifts via $\mathbb{A}^n \to \mathbb{A}^n/\mu_d$ to an automorphism ψ of $\mathbb{A}^n$ and there is an integer $c \geqslant 1$ which is coprime to d such that for each $t \in \mu_d$ and each $(a_1, \ldots, a_n) \in \mathbb{A}^n$ we have

$$\psi(ta_1, \ldots, ta_n) = t^c \psi(a_1, \ldots, a_n).$$

This implies that for each $i \in \{1, \ldots, n\}$,

$$\psi^*(x_i) \in \bigoplus_{k \geqslant 0} \boldsymbol{k}[x_1, \ldots, x_n]_{kd+c}.$$

As ψ is an automorphism of $\mathbb{A}^n$, we get $c = 1$ and thus ψ is μ_d-equivariant (see also [Reg17, Prop. 4]). Hence, $\psi^*\colon \boldsymbol{k}[x_1, \ldots, x_n] \to \boldsymbol{k}[x_1, \ldots, x_n]$ maps $\mathscr{O}(A^s_{d,n})$ onto itself and by construction restricts to φ^* on $\mathscr{O}(\mathbb{A}^n/\mu_d)$. Therefore, there is an endomorphism $\widetilde{\varphi}\colon A^s_{d,n} \to A^s_{d,n}$ that induces $\varphi \in \operatorname{Aut}(\mathbb{A}^n/\mu_d)$ via the normalization morphism $\eta\colon \mathbb{A}^n/\mu_d \to A^s_{d,n}$. As η and φ are bijective, $\widetilde{\varphi}$ is bijective as well; hence $\widetilde{\varphi}$ is an automorphism of $A^s_{d,n}$ by [Kal05, Lem. 1] and thus θ is surjective.

Since $\theta\colon \operatorname{Aut}(A^s_{d,n}) \to \operatorname{Aut}(\mathbb{A}^n/\mu_d)$ is a group isomorphism and as it is induced by the normalization morphism $\mathbb{A}^n/\mu_d \to A^s_{d,n}$, it follows that θ is an isomorphism of ind-groups, see [FK, Prop. 12.1.1]. In particular, θ is a group isomorphism that preserves algebraic subgroups.

(4) The normalization morphism $\mathbb{A}^n/\mu_d \to A^s_{d,n}$ is not an isomorphism, since the inclusion $\mathscr{O}(A^s_{d,n}) \subset \mathscr{O}(\mathbb{A}^n/\mu_d)$ is proper (note that $s \geqslant 2$).

(5) We may assume that $B \subset \mathrm{SL}_n(\boldsymbol{k})$ is the Borel subgroup of upper triangular matrices. Denote by $U \subset B$ the unipotent radical, i.e., the upper triangular matrices with 1 on the diagonal. Then the subrings of U-invariant functions satisfy $\mathscr{O}(\mathbb{A}^n/\mu_d)^U = \bigoplus_{k \geqslant 0} \boldsymbol{k} x_n^{kd}$ and $\mathscr{O}(A^s_{d,n})^U = \boldsymbol{k} \oplus \bigoplus_{k \geqslant s} \boldsymbol{k} x_n^{kd}$. Denote by $\chi_n\colon B \to \mathbb{G}_m$ the character which is the projection to the entry (n, n). Then we get

$$\Lambda^+(\mathbb{A}^n/\mu_d) = \{\chi_n^{kd} \mid k \geqslant 0\} \quad \text{and} \quad \Lambda^+(A^s_{d,n}) = \{\chi_n^{kd} \mid k = 0 \text{ or } k \geqslant s\}$$

inside $\mathfrak{X}(B)$ and as $s \geqslant 2$, these monoids are distinct. ☐

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Andriy Regeta, Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität Jena
Ernst-Abbe-Platz 2, DE-07743 Jena, Germany
E-mail : andriyregeta@gmail.com
Url : https://andriyregeta.wixsite.com/homepage

Immanuel van Santen, Departement Mathematik und Informatik, Universität Basel,
Spiegelgasse 1, CH-4051 Basel, Switzerland
E-mail : immanuel.van.santen@math.ch
Url : https://algebra.dmi.unibas.ch/vansanten/

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Is the Affine Space Determined by Its Automorphism Group?

Hanspeter Kraft[1,*], Andriy Regeta[2], and Immanuel van Santen (born Stampfli)[1]

[1]Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, Basel CH-4051, Switzerland and [2]Mathematisches Institut, Universität Köln, Weyertal 86-90, Köln D-50931, Germany

[*]*Correspondence to be sent to: e-mail: hanspeter.kraft@unibas.ch*

In this note we study the problem of characterizing the complex affine space $\mathbb{A}^n$ via its automorphism group. We prove the following. Let X be an irreducible quasi-projective n-dimensional variety such that $\mathrm{Aut}(X)$ and $\mathrm{Aut}(\mathbb{A}^n)$ are isomorphic as abstract groups. If X is either quasi-affine and toric or X is smooth with Euler characteristic $\chi(X) \neq 0$ and finite Picard group $\mathrm{Pic}(X)$, then X is isomorphic to $\mathbb{A}^n$.

The main ingredient is the following result. Let X be a smooth irreducible quasi-projective variety of dimension n with finite $\mathrm{Pic}(X)$. If X admits a faithful $(\mathbb{Z}/p\mathbb{Z})^n$-action for a prime p and $\chi(X)$ is not divisible by p, then the identity component of the centralizer $\mathrm{Cent}_{\mathrm{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a torus.

1 Introduction

In 1872, Felix Klein suggested as part of his Erlangen Programm to study geometrical objects through their symmetries. In the spirit of this program it is natural to ask to which extent a geometrical object is determined by its automorphism group. This is the case for compact and locally Euclidean manifolds as shown by Whittaker [30]. It also holds for differentiable manifolds, for symplectic manifolds, and for contact manifolds; see [30], [6], [27], and [28].

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We will study this question in the algebraic setting, that is, for complex algebraic varieties. For such a variety X we denote by $\mathrm{Aut}(X)$ the group of regular automorphisms of X. As this automorphism group is usually quite small, it almost never determines the variety. However, if $\mathrm{Aut}(X)$ is large, like for affine n-space $\mathbb{A}^n$, $n \geq 2$, this might be true. Our guiding question is the following.

Question. Let X be a variety. Assume that $\mathrm{Aut}(X)$ is isomorphic to the group $\mathrm{Aut}(\mathbb{A}^n)$. Does this imply that X is isomorphic to $\mathbb{A}^n$?

This question cannot have a positive answer for *all* varieties X. For example, $\mathrm{Aut}(\mathbb{A}^n)$ and $\mathrm{Aut}(\mathbb{A}^n \times Z)$ are isomorphic for any complete variety Z with a trivial automorphism group. Similarly, $\mathrm{Aut}(\mathbb{A}^n)$ and $\mathrm{Aut}(\mathbb{A}^n \dot\cup Y)$ are isomorphic for any variety Y with a trivial automorphism group. Thus, we have to impose certain assumptions on X.

In case X is affine, the group $\mathrm{Aut}(X)$ has the structure of a so-called ind-group. Using this extra structure one has the following result; see [17]. *If X is a connected affine variety, then every isomorphism of ind-groups between* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(\mathbb{A}^n)$ *is induced by an isomorphism $X \xrightarrow{\sim} \mathbb{A}^n$ of varieties.* For some generalizations of this result we refer to [25].

In dimension 2, it is shown in [22] that *if X is an irreducible normal surface and Y is an affine toric surface, then X is isomorphic to Y if the automorphism groups* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(Y)$ *are isomorphic.*

Our main result in this paper is the following.

Main Theorem. Let X be a complex irreducible quasi-projective variety of dimension n such that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$. Then $X \simeq \mathbb{A}^n$ if one of the following conditions holds.

1. X is smooth, the Euler characteristic $\chi(X)$ is nonzero and the Picard group $\mathrm{Pic}(X)$ is finite.
2. X is toric and quasi-affine.

As an immediate application we get the following result.

Corollary. If $S \subset \mathbb{A}^n$ is a closed subvariety such that $\chi(S) \neq 1$, then $\mathrm{Aut}(\mathbb{A}^n \setminus S) \not\simeq \mathrm{Aut}(\mathbb{A}^n)$.

In fact, $X := \mathbb{A}^n \setminus S$ is smooth and quasi-projective, $\chi(X) = \chi(\mathbb{A}^n) - \chi(S) \neq 0$ (Lemma 2.14(1)), and $\mathrm{Pic}(X)$ is trivial.

Outline of Proof

Let $\theta\colon \operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \operatorname{Aut}(X)$ be an isomorphism. First we show that if a torus of $\operatorname{Aut}(\mathbb{A}^n)$ of maximal dimension n is mapped onto an algebraic subgroup of $\operatorname{Aut}(X)$ and if X is quasi-affine, then $X \simeq \mathbb{A}^n$ (Proposition 4.1). Our main result in order to achieve these conditions is the following. (For the definition of the topology on $\operatorname{Aut}(X)$ see Section 2.2.)

Theorem 1.1. Let Y and Z be irreducible quasi-projective varieties, and let $\theta\colon \operatorname{Aut}(Y) \xrightarrow{\sim} \operatorname{Aut}(Z)$ be an isomorphism. Assume that $n := \dim Y \geq \dim Z$ and that the following conditions are satisfied:

(i) Y is quasi-affine and toric.
(ii) Z is smooth, $\chi(Z) \neq 0$, and $\operatorname{Pic}(Z)$ is finite.

Then $\dim Z = n$, and for each n-dimensional torus $T \subseteq \operatorname{Aut}(Y)$, the identity component of the image $\theta(T)^\circ$ is a closed torus of dimension n. Furthermore, Z is quasi-affine.

From this and Proposition 4.1 we can deduce our Main Theorem by setting $Y := \mathbb{A}^n$ and $Z := X$ in case (1) and $Y := X$ and $Z := \mathbb{A}^n$ in case (2); see Section 4.2.

For the proof of Theorem 1.1 we first remark that every torus $T \subseteq \operatorname{Aut}(X)$ of maximal dimension $n = \dim X$ is self-centralizing (Lemma 2.10). For any prime p the torus T contains a unique subgroup μ_p isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$. In particular, $T \subseteq \operatorname{Cent}_{\operatorname{Aut}(X)}(\mu_p)$, and thus the image of T under $\theta\colon \operatorname{Aut}(X) \xrightarrow{\sim} \operatorname{Aut}(Y)$ is mapped to a subgroup of the centralizer of $\theta(\mu_p)$.

Our strategy is then to show that the identity component of the centralizer $\operatorname{Cent}_{\operatorname{Aut}(Y)}(\theta(\mu_p))$ is an algebraic group. Our main result in this direction is the following generalization of [19, Proposition 3.4].

Theorem 1.2. Let X be a smooth, irreducible, quasi-projective variety of dimension n with finite Picard group $\operatorname{Pic}(X)$. Assume that X carries a faithful $(\mathbb{Z}/p\mathbb{Z})^n$-action for some prime p that does not divide $\chi(X)$. Then the centralizer $C := \operatorname{Cent}_{\operatorname{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a closed subgroup of $\operatorname{Aut}(X)$ and its identity component C° is a closed torus of dimension $\leq n$.

For the proof we first show that the fixed-point set $X^{(\mathbb{Z}/p\mathbb{Z})^n}$ contains an isolated point x_0. This follows from the smoothness of X and the assumption that p does not divide $\chi(X)$. Now we study the tangent representation of $(\mathbb{Z}/p\mathbb{Z})^n$ in x_0 and show that the homomorphism $C^\circ \to \operatorname{GL}(T_{x_0}X)$ is regular and has a finite kernel.

2 Preliminary Results

Throughout this note we work over the field $\mathbb{C}$ of complex numbers. A variety will be a reduced separated scheme of finite type over $\mathbb{C}$.

2.1 Quasi-affine varieties

Let us recall some well-known results about quasi-affine varieties.

Lemma 2.1 ([10, Chapter II, Proposition 5.1.2]). A variety X is quasi-affine if and only if the canonical morphism $\eta\colon X \to \operatorname{Spec} \mathcal{O}(X)$ is a dominant open immersion of schemes.

Lemma 2.2 ([5, Chapter I, Section 2, Proposition 2.6]). Let X and Y be varieties. Then the natural homomorphism

$$\mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$$

is an isomorphism of $\mathbb{C}$-algebras.

Lemma 2.3. Let X and Y be varieties where X is quasi-affine. Then every morphism $Y \times X \to X$ extends uniquely to a morphism $Y \times \operatorname{Spec} \mathcal{O}(X) \to \operatorname{Spec} \mathcal{O}(X)$. In particular, every regular action of an algebraic group on X extends to a regular action on $\operatorname{Spec} \mathcal{O}(X)$.

Proof. We can assume that Y is affine. By Lemma 2.2 we have $\mathcal{O}(Y \times X) = \mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{O}(X)$. Hence, $Y \times X \to X$ induces a homomorphism of $\mathbb{C}$-algebras $\mathcal{O}(X) \to \mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{O}(X)$ that in turn gives the desired extension $Y \times \operatorname{Spec} \mathcal{O}(X) \to \operatorname{Spec} \mathcal{O}(X)$. ■

2.2 Algebraic structure on the group of automorphisms

In this subsection, we recall some basic results about the automorphism group $\operatorname{Aut}(X)$ of a variety X. The survey [2] and the article [24] will serve as references. Recall that a *morphism* $\nu\colon A \to \operatorname{Aut}(X)$ is a map from a variety A to $\operatorname{Aut}(X)$ such that the associated map

$$\tilde{\nu}\colon A \times X \to X, \quad (a, x) \mapsto ax := \nu(a)(x)$$

is a morphism of varieties. We get a topology on $\operatorname{Aut}(X)$, called *Zariski topology*, by declaring a subset $F \subset \operatorname{Aut}(X)$ to be *closed*, if for every variety A the preimage $\nu^{-1}(F)$ under every morphism $\nu\colon A \to \operatorname{Aut}(X)$ is closed in A. In particular, a morphism $\nu\colon A \to \operatorname{Aut}(X)$ is continuous with respect to the Zariski topology.

Similarly, a *morphism* $\nu = (\nu_1, \nu_2) \colon A \to \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ is a map from a variety A into $\mathrm{Aut}(X) \times \mathrm{Aut}(X)$ such that ν_1 and ν_2 are morphisms. Thus, we get analogously as before a topology on $\mathrm{Aut}(X) \times \mathrm{Aut}(X)$. Note that for morphisms $\nu, \nu_1, \nu_2 \colon A \to \mathrm{Aut}(X)$ the following maps are again morphisms

$$A \to \mathrm{Aut}(X),\ a \mapsto \nu_1(a) \circ \nu_2(a),$$

$$A \to \mathrm{Aut}(X),\ a \mapsto \nu(a)^{-1},$$

and that $\nu^{-1}(\Delta)$ is closed in A where $\Delta \subset \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ denotes the diagonal. It follows that $\mathrm{Aut}(X)$ behaves like an algebraic group.

Lemma 2.4. For any variety X the maps

$$\mathrm{Aut}(X) \times \mathrm{Aut}(X) \to \mathrm{Aut}(X)\,,\ (\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2$$

$$\mathrm{Aut}(X) \to \mathrm{Aut}(X)\,,\ \varphi \mapsto \varphi^{-1}$$

are continuous, and the diagonal Δ is closed in $\mathrm{Aut}(X) \times \mathrm{Aut}(X)$.

Example 2.5. For any set $S \subseteq \mathrm{Aut}(X)$ the centralizer $\mathrm{Cent}(S)$ is a closed subgroup of $\mathrm{Aut}(X)$. This is a consequence of Lemma 2.4.

Definition 2.6. For a subset $S \subseteq \mathrm{Aut}(X)$ its *dimension* is defined by

$$\dim S := \sup \left\{ d \;\middle|\; \begin{array}{l} \text{there exists a variety } A \text{ of dimension } d \text{ and an} \\ \text{injective morphism } \nu \colon A \to \mathrm{Aut}(X) \text{ with image in } S \end{array} \right\}.$$

The following lemma generalizes the classical dimension estimate to morphisms $A \to \mathrm{Aut}(X)$.

Lemma 2.7. If $\nu \colon A \to \mathrm{Aut}(X)$ is a morphism, then $\dim \nu(A) \leq \dim A$.

Proof. Let $\eta \colon B \to \mathrm{Aut}(X)$ be an injective morphism such that $\eta(B) \subseteq \nu(A)$. We have to show that $\dim B \leq \dim A$. For this consider the fiber product

$$\begin{array}{ccc} A \times_{\mathrm{Aut}(X)} B & \xrightarrow[\text{surjective}]{\bar{\nu}} & B \\ {\scriptstyle \bar{\eta}}\big\downarrow{\scriptstyle \text{injective}} & & {\scriptstyle \eta}\big\downarrow{\scriptstyle \text{injective}} \\ A & \xrightarrow{\nu} & \mathrm{Aut}(X). \end{array}$$

By definition, we have $A \times_{\mathrm{Aut}(X)} B := \{(a,b) \in A \times B \mid \nu(a) = \eta(b)\}$. Since $\nu \times \eta \colon A \times B \to \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ is a morphism, hence continuous, and $\Delta \subset \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ is closed, it follows that $A \times_{\mathrm{Aut}(X)} B \subset A \times B$ is closed. Thus, the fiber product is a variety, and the two maps $\bar{\nu}$ and $\bar{\eta}$ are morphisms. By assumption, $\bar{\nu}$ is surjective and $\bar{\eta}$ is injective, and the claim follows. ■

For a subgroup $G \subseteq \mathrm{Aut}(X)$, the *identity component* $G^\circ \subseteq G$ is defined by

$$G^\circ = \left\{ g \in G \;\middle|\; \begin{array}{l} \text{there exists an irreducible variety } A \text{ and a morphism} \\ \nu \colon A \to \mathrm{Aut}(X) \text{ with image in } G \text{ such that } g, e \in \nu(A) \end{array} \right\}.$$

We call a subgroup $G \subseteq \mathrm{Aut}(X)$ *connected* if $G = G^\circ$. In the next proposition, we list several properties of the identity component of a subgroup of $\mathrm{Aut}(X)$. If G is an ind-group, then these properties are known; see [9, Proposition 2.2.1].

Proposition 2.8. Let X be a variety, and let $G \subseteq \mathrm{Aut}(X)$ be a subgroup. Then the following holds.

1. G° is a normal subgroup of G.
2. The cosets of G° in G are the equivalence classes under the relation

$$g_1 \sim g_2 \iff \left\{ \begin{array}{l} \text{there exists an irreducible variety } A \\ \text{and a morphism } \nu \colon A \to \mathrm{Aut}(X) \\ \text{with image in } G \text{ such that } g_1, g_2 \in \nu(A)\,. \end{array} \right.$$

3. For each morphism $\nu \colon A \to \mathrm{Aut}(X)$ with image in G the preimage $\nu^{-1}(G^\circ)$ is closed in A. In particular, if G is closed in $\mathrm{Aut}(X)$, then G° is also closed in $\mathrm{Aut}(X)$.
4. If X is quasi-projective and G is closed in $\mathrm{Aut}(X)$, then the index of G° in G is countable.

Proof. (1) This follows immediately from the definition of G°.

(2) We have to show that "$\sim$" is an equivalence relation on G. Reflexivity and symmetry are obvious. For the transitivity, let $g \sim h$ and $h \sim k$. By definition, there exist irreducible varieties A and B, morphisms $\nu \colon A \to \mathrm{Aut}(X)$ and $\eta \colon B \to \mathrm{Aut}(X)$ with image

in G, and $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $\nu(a_1) = g$, $\nu(a_2) = h$, $\eta(b_1) = h$, $\eta(b_2) = k$. Then the map

$$A \times B \to \mathrm{Aut}(X), \quad (a,b) \mapsto \nu(a) \circ h^{-1} \circ \eta(b)$$

is a morphism with image in G that sends (a_1, b_1) to g and (a_2, b_2) to k. Thus, $g \sim k$, proving the transitivity.

(3) Let

$$\bigcup_{i=1}^{k} B_i = \overline{\nu^{-1}(G^\circ)} \subseteq A$$

be the decomposition of the closure of $\nu^{-1}(G^\circ)$ into irreducible components $B_1, \dots, B_k$. Thus, $B_i \cap \nu^{-1}(G^\circ)$ is nonempty. Since ν has image in G it follows from the transitivity of "$\sim$" that $\nu(B_i) \subseteq G^\circ$. Thus, $B_i \subseteq \nu^{-1}(G^\circ)$ for all i. Hence, $\nu^{-1}(G^\circ)$ is closed in A.

(4) Let $\nu\colon A \to \mathrm{Aut}(X)$ be a morphism. Since $\nu^{-1}(G) \subseteq A$ is closed, it has only finitely many irreducible components. This implies that its image $\nu(A)$ meets only finitely many cosets of G° in G. The claim follows if we show that there exist countably many morphisms of varieties into $\mathrm{Aut}(X)$ whose images cover $\mathrm{Aut}(X)$.

Since X is quasi-projective, there exists a projective variety $\overline{X}$ and an open embedding $X \subseteq \overline{X}$. For each polynomial $p \in \mathbb{Q}[x]$ we denote by Hilb^p the Hilbert scheme of $\overline{X} \times \overline{X}$ associated with the Hilbert polynomial p and denote by $\mathcal{U}^p \subseteq \mathrm{Hilb}^p \times \overline{X} \times \overline{X}$ the universal family, which is by definition flat over Hilb^p. By [14, Theorem 3.2], Hilb^p is a projective scheme over $\mathbb{C}$. For $i = 1, 2$ consider the following morphisms:

$$q_i\colon (\mathrm{Hilb}^p \times X \times X) \cap \mathcal{U}^p \to \mathrm{Hilb}^p \times X, \quad (h, x_1, x_2) \mapsto (h, x_i),$$

which are defined over Hilb^p. By [12, Proposition 9.6.1], the points $h \in \mathrm{Hilb}^p$ where the restriction

$$q_i|_{\{h\}}\colon (\{h\} \times X \times X) \cap \mathcal{U}^p \to \{h\} \times X$$

is an isomorphism form a constructible subset S^p of Hilb^p. Now choose locally closed subsets S^p_j, $j = 1, \dots, k_p$ of Hilb^p that cover S^p. We equip each S^p_j with the underlying reduced scheme structure of Hilb^p. Note that $(\mathrm{Hilb}^p \times X \times X) \cap \mathcal{U}^p$ and $\mathrm{Hilb}^p \times X$ are both flat over Hilb^p. Therefore, we can apply [13, Proposition 5.7] and we get that q_i restricts to an isomorphism over S^p_j. Thus, for each j we get a morphism of varieties

$$S^p_j \times X \xrightarrow{(q_1|_{S^p_j})^{-1}} (S^p_j \times X \times X) \cap \mathcal{U}^p \xrightarrow{q_2|_{S^p_j}} S^p_j \times X \longrightarrow X,$$

which defines a morphism $S_j^p \to \mathrm{Aut}(X)$. For each automorphism φ in $\mathrm{Aut}(X)$, the closure in $\overline{X} \times \overline{X}$ of the graph $\Gamma_\varphi \subseteq X \times X$ defines a (closed) point in the Hilbert scheme Hilb^p for a certain rational polynomial p, which belongs to S^p. Thus, the images of the morphisms $S_j^p \to \mathrm{Aut}(X)$ cover $\mathrm{Aut}(X)$. Since there are only countably many rational polynomials, the claim follows. ∎

We say that G is an *algebraic subgroup* of $\mathrm{Aut}(X)$ if there exists a morphism $\nu\colon H \to \mathrm{Aut}(X)$ of an algebraic group H with image G, which is a homomorphism of groups.

The next result gives a criterion for a subgroup of $\mathrm{Aut}(X)$ to be algebraic. The main argument is due to Ramanujam [24].

Theorem 2.9. Let X be an irreducible variety, and let $G \subseteq \mathrm{Aut}(X)$ be a subgroup. Then the following statements are equivalent:

(1) G is an algebraic subgroup of $\mathrm{Aut}(X)$.
(2) There exists a morphism of a variety into $\mathrm{Aut}(X)$ with image G.
(3) $\dim G$ is finite and G° has finite index in G.
(4) There is a structure of an algebraic group on G such that for each irreducible variety A we get a bijection

$$\left\{ \begin{array}{c} \text{morphisms } A \to \mathrm{Aut}(X) \\ \text{with image in } G \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{c} \text{morphisms of} \\ \text{varieties } A \to G \end{array} \right\}.$$

Proof. The implication (1) ⇒ (2) follows from the definition.

Assume that there is a morphism $\eta\colon A \to \mathrm{Aut}(X)$ with image equal to G. By Lemma 2.7 we get $\dim G \le \dim A$; hence, $\dim G$ is finite. Since A has only finitely many irreducible components it follows from Proposition 2.8 2 that G° has finite index in G. This proves (2) ⇒ (3).

The implication (3) ⇒ (4) is proved in [24, Theorem, p. 26] in case $G = G^\circ$. This implies that G° carries the structure of an algebraic group with the required property. Since G° has finite index in G we obtain a unique structure of an algebraic group on G extending the given structure on G°. It remains to see that the required property holds for G.

By construction, the canonical inclusion $\iota\colon G \to \mathrm{Aut}(X)$ is a morphism, and thus each morphism of varieties $A \to G$ yields a morphism $A \to \mathrm{Aut}(X)$ by composing with ι. For the reverse, let $\nu\colon A \to \mathrm{Aut}(X)$ be a morphism with image in G. Since A is irreducible there is $g \in G$ such that the image of ν lies in gG° (Proposition 2.8(2)). Thus, the composition $\lambda_{g^{-1}} \circ \nu\colon A \to \mathrm{Aut}(G)$ is a morphism with image in G° where $\lambda_g \in \mathrm{Aut}(X)$ is the left multiplication with g. It follows that ν corresponds to a morphism $A \to G$ of varieties, proving (3) $\Rightarrow$ (4).

The remaining implication (4) $\Rightarrow$ (1) is obvious. ■

2.3 Ingredients from toric geometry

Recall that a toric variety is a normal irreducible variety X together with a regular faithful action of a torus of dimension $\dim X$. For details concerning toric varieties we refer to [8].

Lemma 2.10. Let X be a toric variety, and let T be a torus of dimension $\dim X$ that acts faithfully on X. Then the centralizer of T in $\mathrm{Aut}(X)$ is equal to T. In particular, the image of T in $\mathrm{Aut}(X)$ is closed.

Proof. Let $g \in \mathrm{Aut}(X)$ such that $gt = tg$ for all $t \in T$. By definition, there is an open, dense T-orbit in X, say U. Since $gU \cap U$ is nonempty, there exists $x \in U$ such that $gx \in U$. Using that $U = Tx$ we find $t_0 \in T$ with $gx = t_0x$. Thus, for each $t \in T$ we get

$$gtx = tgx = tt_0x = t_0tx\,.$$

Using that $U = Tx$ is dense in X, we get $g = t_0$. ■

Lemma 2.11. Let X be a toric variety. Then the coordinate ring $\mathcal{O}(X)$ is finitely generated and integrally closed.

Proof. This is a special case of a result of Knop; see [16, Satz, p. 33]. ■

The next proposition is based on the study of homogeneous $\mathbb{G}_a$-actions on affine toric varieties in [21]. Recall that a group action $\nu\colon G \to \mathrm{Aut}(X)$ on a toric variety is called *homogeneous* if the torus normalizes the image $\nu(G)$. Note that for any homogeneous $\mathbb{G}_a$-action ν there is a well-defined character $\chi\colon T \to \mathbb{G}_m$, defined by the formula

$$t\,\nu(s)\,t^{-1} = \nu(\chi(t)\cdot s) \quad \text{for } t \in T, s \in \mathbb{C}.$$

Proposition 2.12. Let X be an n-dimensional quasi-affine toric variety. If X is not a torus, then there exist homogeneous $\mathbb{G}_a$-actions

$$\eta_1, \dots, \eta_n \colon \mathbb{G}_a \times X \to X$$

such that the corresponding characters $\chi_1, \dots, \chi_n$ are linearly independent in the character group of T.

The proof needs some preparation. Denote by Y the spectrum of $\mathcal{O}(X)$. By Lemma 2.11, the variety Y is normal, and the faithful torus action on X extends uniquely to a faithful torus action on Y, by Lemma 2.3.

The following notation is taken from [21]. Let N be a lattice of rank n, $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, we have a natural pairing $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$, $(m, n) \mapsto \langle m, n\rangle$. Let $\sigma \subset N_{\mathbb{Q}}$ be the strongly convex polyhedral cone that describes Y and let $\sigma_M^\vee$ be the intersection of the dual cone $\sigma^\vee$ in $M_{\mathbb{Q}}$ with M. Thus, $Y = \operatorname{Spec} R$, where

$$R := \mathbb{C}[\sigma_M^\vee] = \bigoplus_{m \in \sigma_M^\vee} \mathbb{C}\chi^m \subseteq \mathbb{C}[M]\,.$$

For each extremal ray $\rho \subset \sigma$, denote by $\rho^\perp$ the elements $u \in M_{\mathbb{Q}}$ with $\langle u, v\rangle = 0$ for all $v \in \rho$. Moreover, let $\tau_M = \rho^\perp \cap \sigma_M^\vee$ and let

$$S_\rho = \{\, e \in M \mid e \notin \sigma_M^\vee\,,\; e + m \in \sigma_M^\vee \text{ for all } m \in \sigma_M^\vee \setminus \tau_M \,\}\,.$$

By [21, Remark 2.5] we have $S_\rho \neq \varnothing$ and $e + m \in S_\rho$ for all $e \in S_\rho$ and all $m \in \tau_M$. Let us recall the description of the homogeneous locally nilpotent derivations on R.

Proposition 2.13 ([21, Lemma 2.6 and Theorem 2.7]). Let ρ be an extremal ray in σ and let $e \in S_\rho$. Then

$$\partial_{\rho,e} \colon R \to R\,, \quad \chi^m \mapsto \langle m, \rho\rangle \chi^{e+m}$$

is a homogeneous locally nilpotent derivation of degree e, and every homogeneous locally nilpotent derivation of R is a constant multiple of some $\partial_{\rho,e}$.

Proof of Proposition 2.12. Since X is not a torus, Y is also not a torus. Thus, σ contains extremal rays, say $\rho_1, \dots, \rho_k$ and $k \geq 1$. Recall that associated to these extremal rays, there exist torus-invariant divisors $V(\rho_1), \dots, V(\rho_k)$ in Y. Again, since X is not a torus, one of these divisors does intersect X. Let us assume that $\rho = \rho_1$ is an extremal ray such that $V(\rho) \cap X$ is nonempty. Then using the orbit-cone correspondence, one can see that $Y \setminus X$ is contained in the union $Z = \bigcup_{i=2}^{k} V(\rho_i)$; see [8, Section 3.1]. Let $e \in S_\rho$ be fixed. We claim that the $\mathbb{G}_a$-action on Y associated with the locally nilpotent derivation $\partial_{\rho,e+m'}$ of Proposition 2.13 fixes Z for all $m' \in \tau_M \setminus \bigcup_{i\geq 2} \rho_i^\perp$.

Let us fix $m' \in \tau_M$ with $\langle m', v\rangle > 0$ for all $v \in \bigcup_{i\geq 2} \rho_i$. Note that the fixed-point set of the $\mathbb{G}_a$-action on Y corresponding to $\partial_{\rho,e+m'}$ is the zero set of the ideal generated by the image of $\partial_{\rho,e+m'}$. The divisor $V(\rho_i)$ is the zero set of the kernel of the canonical $\mathbb{C}$-algebra surjection

$$p_i \colon \mathbb{C}[\sigma_M^\vee] \to \mathbb{C}[\sigma_M^\vee \cap \rho_i^\perp], \quad \chi^m \mapsto \begin{cases} \chi^m, & \text{if} m \in \rho_i^\perp \\ 0, & \text{otherwise} \end{cases};$$

see [8, Section 3.1]. Thus, we have to prove that for all $i = 2, \dots, k$ the composition

$$\mathbb{C}[\sigma_M^\vee] \xrightarrow{\partial_{\rho,e+m'}} \mathbb{C}[\sigma_M^\vee] \xrightarrow{p_i} \mathbb{C}[\sigma_M^\vee \cap \rho_i^\perp]$$

is the zero map. Since, by definition, $\partial_{\rho,e+m'}$ vanishes on $\tau_M = \rho^\perp \cap \sigma_M^\vee$, we only have to show that for all $m \in \sigma_M^\vee \setminus \tau_M$ the following holds:

$$\langle e + m' + m, v\rangle > 0 \quad \text{for all } v \in \rho_i, i = 2, \dots, k.$$

This is satisfied because $\langle m', v\rangle > 0$ and $\langle e + m, v\rangle \geq 0$ (note that $e \in S_\rho$ implies $e + m \in \sigma_M^\vee$). This proves the claim.

Since τ_M spans a hyperplane in M and $e \notin \tau_M$, we can choose $m'_1, \dots, m'_n \in \tau_M \setminus \bigcup_{i\geq 2} \rho_i^\perp$ such that $e + m'_1, \dots, e + m'_n$ are linearly independent in $M_{\mathbb{Q}}$. Hence, the homogeneous locally nilpotent derivations

$$\partial_{\rho,e+m'_i}, \quad i = 1, \dots, n$$

define $\mathbb{G}_a$-actions on Y that fix Z and thus restrict to $\mathbb{G}_a$-actions on X. Moreover, the character of $\partial_{\rho,e+m'_i}$ is $\chi_i = \chi^{e+m'_i}$. In particular, $\chi_1, \dots, \chi_n$ are linearly independent, finishing the proof of Proposition 2.12. ■

2.4 The Euler characteristic

For a variety X, the *Euler characteristic* is defined by

$$\chi(X) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H^i(X,\mathbb{Q}),$$

where $H^i(X,\mathbb{Q})$ denotes the i-th singular cohomology group with rational coefficients. The following results can be found in [18, Appendix].

Lemma 2.14. The Euler characteristic has the following properties.

(1) If X is a variety and $Y \subseteq X$ is a closed subvariety, then $\chi(X) = \chi(Y)+\chi(X\setminus Y)$.

(2) If $X \to Y$ is a fiber bundle, which is locally trivial in the étale topology with fiber F, then $\chi(X) = \chi(Y)\chi(F)$.

2.5 Results about the fixed-point variety

The next result gives a criterion for the existence of fixed points under the action of a finite p-group.

Proposition 2.15. Let p be a prime, and let G be finite p-group acting on a variety X. If p does not divide the Euler characteristic $\chi(X)$, then the fixed-point variety X^G is nonempty.

Proof. Assume that X^G is empty, that is, every G-orbit has cardinality p^k for some $k > 0$. We prove by induction on the dimension of X that p divides $\chi(X)$. Let $X' \subset X$ be a dense smooth open affine subset. By intersecting the G-translates gX' for $g \in G$ we can in addition assume that X' is G-invariant. Denote by $\pi : X' \to X'/G$ the algebraic quotient, that is, the morphism corresponding to the inclusion of the invariant ring $\mathcal{O}(X')^G$ in $\mathcal{O}(X')$. It follows from Luna's slice theorem [23, Chapter II, Section 2] that there is a smooth open dense subset $U \subset X'/G$ such that π restricts to a fiber bundle $\pi^{-1}(U) \to U$, which is locally trivial in the étale topology. Now Lemma 2.14(2) implies that p divides $\chi(\pi^{-1}(U))$. Using Lemma 2.14(1) and $\dim X \setminus \pi^{-1}(U) < \dim X$ the claim follows by induction. ■

Remark 1. The proposition above is a purely topological result and holds in a much more general setting; see, for example, [3, Chapter III, Theorem 4.4] or [4, Section III.7].

The next result is essentially due to Fogarty; see [7, Theorem 5.2].

Proposition 2.16. Let G be a reductive group acting on a variety X. Assume that X is smooth at a point $x \in X^G$. Then X^G is smooth at x and the tangent space satisfies $T_x(X^G) = (T_x X)^G$.

Remark 2. Assume that $(\mathbb{Z}/p\mathbb{Z})^n$ acts faithfully on a smooth quasi-projective variety X. If p does not divide $\chi(X)$, then $\dim X \geq n$.

In fact, by Proposition 2.15 there is a fixed-point $x \in X$, and the action of $(\mathbb{Z}/p\mathbb{Z})^n$ on the tangent space $T_x X$ is faithful [19, Lemma 2.2]; hence, $n \leq \dim T_x X = \dim X$.

3 Proof of Theorems 1.1 and 1.2

Definition 3.1. Let X be a variety and $M \subseteq \mathrm{Aut}(X)$ a subset. A map $\eta\colon M \to Z$ into a variety Z is called *regular* if for every morphism $\nu\colon A \to \mathrm{Aut}(X)$ with image in M, the composition $\eta \circ \nu\colon A \to Z$ is a morphism of varieties.

3.1 Semi-invariant functions

Lemma 3.2. Let X be an irreducible normal variety, and let $f \in \mathcal{O}(X)$ be a non-constant function such that the zero set $Z := \mathcal{V}_X(f) \subset X$ is an irreducible hypersurface. Let $G \subseteq \mathrm{Aut}(X)$ be a connected subgroup that stabilizes Z. Then the function f is a G-semi-invariant, that is,

$$f(gx) = \chi(g)^{-1} \cdot f(x) \quad \text{for } x \in X \text{ and } g \in G,$$

where $\chi\colon G \to \mathbb{C}^*$ is a character and a regular map.

For the proof we need the following description of the invertible functions on a product variety, which is due to Rosenlicht [26, Theorem 2]. For a variety X we denote by $\mathcal{O}(X)^*$ the group of invertible functions on X.

Lemma 3.3. Let X_1 and X_2 be irreducible varieties. Then $\mathcal{O}(X_1 \times X_2)^* = \mathcal{O}(X_1)^* \cdot \mathcal{O}(X_2)^*$.

Proof of Lemma 3.2. Since X is normal, the local ring $R = \mathcal{O}_{X,Z}$ is a discrete valuation ring. Let $\mathfrak{m}$ be the maximal ideal of R. By assumption, $fR = \mathfrak{m}^k$ for some $k > 0$. Since $\mathfrak{m}$ is stable under G, the same is true for $\mathfrak{m}^k$. Hence, for every $g \in G$, there exists a unit $r_g \in R^*$ such that $gf = r_g \cdot f$ in R. Since f and gf have no zeroes in $X \setminus Z$, it follows that

r_g is regular and nonzero in $X \setminus Z$. Moreover, the open set where $r_g \in R$ is defined and nonzero meets Z; hence, $r_g \in \mathcal{O}(X)^*$. Consider the homomorphism

$$\chi \colon G \to \mathcal{O}(X)^*, \; g \mapsto r_g .$$

For all $x \in X \setminus Z$, $g \in G$ we get $f(gx) = \chi(g)(x)^{-1} f(x)$, and $f(gx)$ and $f(x)$ are both nonzero. Since for each morphism $\nu \colon A \to \mathrm{Aut}(X)$ with image in G, the map $\tilde{\nu} \colon A \times X \to X$, $(a, x) \mapsto \nu(a)(x)$ is a morphism, we see that

$$A \times (X \setminus Z) \to \mathbb{C}^*, \; (a, x) \mapsto \chi(\nu(a))(x) = f(x) \cdot f(\tilde{\nu}(a, x))^{-1}$$

is a morphism. If A is irreducible, then, by Lemma 3.3, there exist invertible functions $q \in \mathcal{O}(A)^*$ and $p \in \mathcal{O}(X \setminus Z)^*$ such that $\chi(\nu(a))(x) = q(a)p(x)$. If, moreover, $\nu(a_0) = e \in G$ for some $a_0 \in A$, then

$$1 = r_e(x) = \chi(\nu(a_0))(x) = q(a_0)p(x) \quad \text{for all } x \in X \setminus Z,$$

that is, $p \in \mathbb{C}^*$; hence, the composition $\chi \circ \nu \colon A \mapsto \mathcal{O}(X)^*$ has image in $\mathbb{C}^*$. Since G is connected, this implies that $\chi(G) \subseteq \mathbb{C}^*$ and that $\chi \colon G \to \mathbb{C}^*$ is a character.

It remains to see that χ is regular. Choose $x_0 \in X \setminus Z$. As before, for each morphism $\nu \colon A \to \mathrm{Aut}(X)$ with image in G, the map

$$A \to \mathbb{C}^*, \quad a \mapsto \chi(\nu(a)) = f(x_0) \cdot f(\nu(a)(x_0))^{-1}$$

is also a morphism. ■

Lemma 3.4. Let X be an irreducible normal variety, and let $G \subseteq \mathrm{Aut}(X)$ be a connected subgroup. Assume that $f_1, \dots, f_n \in \mathcal{O}(X)$ have the following properties.

(1) $Z_i := \mathcal{V}_X(f_i)$, $i = 1, \dots, n$, are irreducible G-invariant hypersurfaces.

(2) $\bigcap_i Z_i$ contains an isolated point.

If $\chi_i \colon G \to \mathbb{C}^*$ is the character of f_i (Lemma 3.2), then

$$\chi := (\chi_1, \dots, \chi_n) \colon G \to (\mathbb{C}^*)^n$$

is a regular homomorphism with finite kernel.

Proof. Let G act on $\mathbb{A}^n$ by

$$g(a_1,\ldots,a_n) := (\chi_1(g)^{-1}\cdot a_1,\ldots,\chi_n(g)^{-1}\cdot a_n).$$

Then the map $f := (f_1,\ldots,f_n)\colon X \to \mathbb{A}^n$ is G-equivariant. Let $Y \subseteq \mathbb{A}^n$ be the closure of $f(X)$. By assumption, $f^{-1}(0) = \bigcap_i Z_i$ contains an isolated point; hence, $f\colon X \to Y$ has a finite degree, that is, the field extension $\mathbb{C}(X) \supset \mathbb{C}(Y)$ is finite. This implies that the kernel K of $\chi\colon G \to (\mathbb{C}^*)^n$ is finite because K embeds into $\mathrm{Aut}_{\mathbb{C}(Y)}(\mathbb{C}(X))$. By Lemma 3.2, χ is regular. ■

3.2 Another centralizer result

For an irreducible normal variety X, we denote by $\mathrm{CH}^1(X)$ the *first Chow group*, that is, the free group of integral Weil divisors modulo linear equivalence [15, Chapter II, Section 6].

Proposition 3.5. Let X be an irreducible normal variety of dimension n with a faithful action of $(\mathbb{Z}/p\mathbb{Z})^n$. Assume that $\mathrm{CH}^1(X)$ is finite and that there exists a fixed-point x which is a smooth point of X. Then the centralizer $\mathrm{Cent}_{\mathrm{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a closed subgroup of $\mathrm{Aut}(X)$, and its identity component is a closed torus of dimension $\le n$.

Proof. We denote $G := \mathrm{Cent}_{\mathrm{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$. By [19, Lemma 2.2] we get a faithful representation of $(\mathbb{Z}/p\mathbb{Z})^n$ on T_xX, and thus we can find generators $\sigma_1,\ldots,\sigma_n$ such that $(T_xX)^{\sigma_i} \subset T_xX$ is a hyperplane for each i and that $(T_xX)^{(\mathbb{Z}/p\mathbb{Z})^n} = 0$. By Proposition 2.16, the hypersurface $X^{\sigma_i} \subset X$ is smooth at x, with tangent space $T_x(X^{\sigma_i}) = (T_xX)^{\sigma_i}$. Hence, there is a unique irreducible hypersurface $Z_i \subseteq X$ which contains x and is contained in X^{σ_i}. It follows that Z_i is G°-stable, and that x is an isolated point of $\bigcap_i Z_i$, because $(T_xX)^{(\mathbb{Z}/p\mathbb{Z})^n} = 0$. Since a multiple of Z_i is zero in $\mathrm{CH}^1(X)$, there exist G°-semi-invariant functions $f_i \in \mathcal{O}(X)$ such that $\mathcal{V}_X(f_i) = Z_i$ (Lemma 3.2), and the corresponding characters χ_i define a regular homomorphism

$$\chi = (\chi_1,\ldots,\chi_n)\colon G^\circ \to (\mathbb{C}^*)^n$$

with a finite kernel (Lemma 3.4). It follows that $\dim G^\circ \le n$. Indeed, if $\nu\colon A \to \mathrm{Aut}(X)$ is an injective morphism with image in G°, then $\chi\circ\nu\colon A \to (\mathbb{C}^*)^n$ is a morphism with finite fibers, and so $\dim A \le n$. This implies, by Theorem 2.9, that $G^\circ \subseteq \mathrm{Aut}(X)$ is an algebraic subgroup and that χ is a homomorphism of algebraic groups with a finite kernel. Hence, G° is a torus. Since G is closed in $\mathrm{Aut}(X)$ the same holds for G°, see Proposition 2.8. ■

3.3 Proof of Theorem 1.2

Now we can prove Theorem 1.2 that has the same conclusion as the proposition above, but under different assumptions. We have to show that the assumptions of Proposition 3.5 are satisfied. Since X is smooth, it follows that $\mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$ is finite, and Proposition 2.15 implies that the fixed-point variety $X^{(\mathbb{Z}/p\mathbb{Z})^n} \subseteq X$ is nonempty. Now the claims follow from Proposition 3.5.

3.4 Images of maximal tori under group isomorphisms

Proposition 3.6. Let X and Y be irreducible quasi-projective varieties such that $n := \dim X \geq \dim Y$. Assume that the following conditions are satisfied:

(1) X is quasi-affine and toric.
(2) Y is smooth, $\chi(Y) \neq 0$, and $\mathrm{Pic}(Y)$ is finite.

If $\theta\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$ is an isomorphism, then $\dim Y = n$, and for each n-dimensional torus $T \subseteq \mathrm{Aut}(X)$ the identity component of the image $\theta(T)^\circ \subset \mathrm{Aut}(Y)$ is a closed torus of dimension n.

Proof. Let $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ be an isomorphism. Since $\chi(Y) \neq 0$ it follows that there is a prime p that does not divide $\chi(Y)$.

Let $T \subset \mathrm{Aut}(X)$ be a torus of dimension n. We have $T = \mathrm{Cent}_{\mathrm{Aut}(X)}(T)$ (Lemma 2.10), and thus $\theta(T)$ is a closed subgroup of $\mathrm{Aut}(Y)$. Let $\mu_p \subset T$ be the subgroup generated by the elements of order p, and let $G := \mathrm{Cent}_{\mathrm{Aut}(Y)}(\theta(\mu_p))$ that is closed in $\mathrm{Aut}(Y)$. By Remark 2, we have $\theta(T) \subseteq G$ and $\dim Y = n$. Now Theorem 1.2 implies that $G^\circ \subset \mathrm{Aut}(Y)$ is a closed torus of dimension $\leq n$, and by Proposition 2.8 and Theorem 2.9, we see that $\theta(T)^\circ$ is a closed connected algebraic subgroup of G°.

In order to show that $\dim \theta(T)^\circ \geq n$ we construct closed subgroups $\{1\} = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$ with the following properties:

(i) $\dim T_i = i$ for all i.
(ii) $\theta(T_i)$ is closed in $\theta(T)$ for all i.

It then follows that $\theta(T_i)^\circ$ is a connected algebraic subgroup of $\theta(T)^\circ$. Since the index of $\theta(T_i)^\circ$ in $\theta(T_i)$ is countable (Proposition 2.8), but the index of T_i in T_{i+1} is not countable, we see that $\dim \theta(T_{i+1})^\circ > \dim \theta(T_i)^\circ$, and so $\dim \theta(T)^\circ \geq n$.

(a) Assume first that X is a torus. Then $\mathrm{Aut}(X)$ contains a copy of the symmetric groups $\mathcal{S}_n$, and we can find cyclic permutations $\tau_i \in \mathrm{Aut}(X)$ such that $T_i := \mathrm{Cent}_T(\tau_i)$ is a closed subtorus of dimension i, and $T_i \subset T_{i+1}$ for all $0 < i < n$. It then follows that $\theta(T_i) = \mathrm{Cent}_{\theta(T)}(\theta(\tau_i))$ is closed in $\theta(T)$, and we are done.

(b) Now assume that X is not a torus. By Proposition 2.12 there exist one-dimensional unipotent subgroups $U_1, \ldots, U_n$ of $\mathrm{Aut}(X)$ normalized by T such that the corresponding characters $\chi_1, \ldots, \chi_n \colon T \to \mathbb{C}^*$ are linearly independent. Since

$$\ker(\chi_i) = \{t \in T \mid t \circ u_i \circ t^{-1} = u_i \text{for all} u_i \in U_i\} = \mathrm{Cent}_T(U_i)$$

it follows that

$$T_i := \bigcap_{k=1}^{n-i} \ker(\chi_k) = \mathrm{Cent}_T(U_1 \cup \cdots \cup U_{n-i}) \subseteq T$$

is a closed algebraic subgroup of T of dimension i. It follows that the image $\theta(T_i) = \mathrm{Cent}_{\theta(T)}(\theta(U_1) \cup \cdots \cup \theta(U_n))$ is closed in $\theta(T)$, and the claim follows also in this case. ■

3.5 Proof of Theorem 1.1

Using Proposition 3.6, it is enough to show that a smooth toric variety Y with finite (and hence trivial) Picard group is quasi-affine.

For proving this, let $\Sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ be the fan that describes Y where N is a lattice of rank n. Let $N' \subseteq N$ be the sublattice spanned by $\Sigma \cap N$, and let Y' be the toric variety corresponding to the fan Σ in $N'_{\mathbb{Q}} = N' \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows from [8, p. 29] that

$$Y \simeq Y' \times (\mathbb{C}^*)^k,$$

where $k = \operatorname{rank} N/N'$. Thus, Y' is a smooth toric variety with trivial Picard group. Hence, it is enough to prove that Y' is quasi-affine and therefore we can assume $k = 0$, that is, Σ spans $N_{\mathbb{Q}}$. By [8, Proposition in Section 3.4] we get

$$0 = \operatorname{rank} \operatorname{Pic}(Y) = d - n,$$

where d is the number of edges in Σ. Let $\sigma \subset N_{\mathbb{Q}}$ be the convex cone spanned by the edges of Σ and let $\sigma^\vee$ denote the dual cone of σ in $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ where $M = \mathrm{Hom}(N, \mathbb{Z})$. Since $d = n$, the edges of Σ are linearly independent in $\mathbb{N}_{\mathbb{Q}}$ and thus σ is a simplex. From the inclusion of the cones of Σ in σ we get a morphism $f \colon Y \to \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ by [8, Section 1.4], and since each cone in Σ is a face of σ it is locally an open immersion

[8, Lemma in Section 1.3]. This implies that f is quasi-finite and birational and thus by Zariski's Main Theorem [11, Corollaire 4.4.9] it is an open immersion.

4 Proof of the Main Theorem

4.1 A first characterization

Proposition 4.1. Let X be an irreducible quasi-affine variety. If $\mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$ is an isomorphism that maps an n-dimensional torus in $\mathrm{Aut}(\mathbb{A}^n)$ to an algebraic subgroup, then $X \simeq \mathbb{A}^n$ as a variety.

Proof. Since all n-dimensional tori in $\mathrm{Aut}(\mathbb{A}^n)$ are conjugate [1], all n-dimensional tori are sent to algebraic subgroups of $\mathrm{Aut}(X)$ via θ. The standard maximal torus T in $\mathrm{Aut}(\mathbb{A}^n)$ acts via conjugation on the subgroup of standard translations $\mathrm{Tr} \subset \mathrm{Aut}(\mathbb{A}^n)$ with a dense orbit $O \subset T$ and thus we get $\mathrm{Tr} = O \circ O$.

This implies that $S := \theta(T)$ acts on $U := \theta(\mathrm{Tr})$ via conjugation and we get $U = \theta(O) \circ \theta(O)$. Hence, for fixed $u_0 \in \theta(O) \subset U$ the morphism

$$S \times S \to \mathrm{Aut}(X)\,, \quad (s_1, s_2) \mapsto s_1 \circ u_0 \circ s_1^{-1} \circ s_2 \circ u_0 \circ s_2^{-1}$$

has image equal to U. Now it follows from Theorem 2.9 that U is a closed (commutative) algebraic subgroup of $\mathrm{Aut}(X)$ with no nontrivial element of finite order, hence a unipotent subgroup.

We claim that U has no nonconstant invariants on X. Indeed, let $\rho\colon \mathbb{G}_a \times X \to X$ be the $\mathbb{G}_a$-action on X coming from a nontrivial element of U. If $f \in \mathcal{O}(X)^U$ is a U-invariant, then it is easy to see that

$$\rho_f(s, x) := \rho(f(x) \cdot s, x) \qquad (*)$$

is a $\mathbb{G}_a$-action commuting with U. Since U is self-centralizing, we see that $\rho_f(s) \in U$ for all $s \in \mathbb{G}_a$. Moreover, formula $(*)$ shows that for every finite dimensional subspace $V \subset \mathcal{O}(X)^U$ the map $V \to U$, $f \mapsto \rho_f(1)$, is a morphism, which is injective. Indeed, $\rho_f(1) = \rho_h(1)$ implies that $\rho(f(x), x) = \rho(h(x), x)$ for all $x \in X$; hence, $f(x) = h(x)$ for all $x \in X \setminus X^\rho$. It follows that $\mathcal{O}(X)^U$ is finite-dimensional. Since X is irreducible, $\mathcal{O}(X)^U$ is an integral domain and hence equal to $\mathbb{C}$, as claimed.

Since X is irreducible and quasi-affine, the unipotent group U has a dense orbit that is closed, and so X is isomorphic to an affine space $\mathbb{A}^m$. Since m is the maximal number such that there exists a faithful action of $(\mathbb{Z}/2\mathbb{Z})^m$ on $\mathbb{A}^m$ (Remark 2), we finally get $m = n$. ■

If X is an affine variety, then $\mathrm{Aut}(X)$ has the structure of a so-called affine ind-group; see, for example, [20], [29], and [9] for more details. The following result is a special case of [17, Theorem 1.1]. It is an immediate consequence of Proposition 4.1 above because a homomorphism of affine ind-groups sends algebraic groups to algebraic groups.

Corollary 4.2. Let X be an irreducible affine variety. If there is an isomorphism $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ of affine ind-groups, then $X \simeq \mathbb{A}^n$ as a variety.

Corollary 4.3. Let X be a smooth, irreducible quasi-projective variety such that $\chi(X) \neq 0$ and $\mathrm{Pic}(X)$ is finite. If there is an isomorphism $\mathrm{Aut}(\mathbb{A}^n) \simeq \mathrm{Aut}(X)$ of abstract groups and if $\dim X \leq n$, then $X \simeq \mathbb{A}^n$ as a variety.

Proof. Theorem 1.1 shows that for an isomorphism $\theta\colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$ and any n-dimensional torus $T \subseteq \mathrm{Aut}(\mathbb{A}^n)$, the identity component of the image $S := \theta(T)^\circ$ is a closed torus of dimension n in $\mathrm{Aut}(X)$ and $\dim X = n$, and X is quasi-affine. Thus, we can apply Theorem 1.1 to $\theta^{-1}\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{A}^n)$ and get that $\theta^{-1}(S)^\circ$ is a closed torus of dimension n in $\mathrm{Aut}(\mathbb{A}^n)$. Since

$$\theta^{-1}(S)^\circ \subseteq \theta^{-1}(S) \subseteq T\,,$$

it follows that $\theta^{-1}(S) = T$, that is, $\theta(T) = S$ is a closed n-dimensional torus in $\mathrm{Aut}(X)$. The assumptions of Proposition 4.1 are now satisfied for the isomorphism $\theta\colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$, and the claim follows. ■

4.2 Proof of the Main Theorem

If the assumptions (1) of the Main Theorem hold, that is, X is a smooth, irreducible, quasi-projective variety of dimension n such that $\chi(X) \neq 0$ and $\mathrm{Pic}(X)$ is finite, then the claim follows from Corollary 4.3.

Now assume that the assumptions (2) are satisfied, that is, X is quasi-affine and toric of dimension n. Let $T \subseteq \mathrm{Aut}(X)$ be a torus of maximal dimension. We can apply Theorem 1.1 to an isomorphism $\theta\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{A}^n)$ and find that $S := \theta(T)^\circ \subset \mathrm{Aut}(\mathbb{A}^n)$ is a closed torus of dimension n. Since the index of the standard n-dimensional torus in its normalizer in $\mathrm{Aut}(\mathbb{A}^n)$ is finite and since all n-dimensional tori in $\mathrm{Aut}(\mathbb{A}^n)$ are conjugate [1], it follows that S has finite index in $\theta(T)$. Hence, $\theta^{-1}(S)$ has finite index in T. Since T is a divisible group, $\theta^{-1}(S) = T$ is an algebraic group. Thus, we can apply Proposition 4.1 to the isomorphism $\theta^{-1}\colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$ and find that $X \simeq \mathbb{A}^n$ as a variety.

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UNIQUENESS OF EMBEDDINGS OF THE AFFINE LINE INTO ALGEBRAIC GROUPS

PETER FELLER AND IMMANUEL VAN SANTEN

Abstract

Let Y be the underlying variety of a complex connected affine algebraic group. We prove that two embeddings of the affine line $\mathbb{C}$ into Y are the same up to an automorphism of Y provided that Y is not isomorphic to a product of a torus $(\mathbb{C}^*)^k$ and one of the three varieties $\mathbb{C}^3$, SL_2, and PSL_2.

Contents

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1. Introduction

In this paper, *varieties* are understood to be (reduced) algebraic varieties over the field of complex numbers $\mathbb{C}$, carrying the Zariski topology. We say that two closed[1] embeddings of varieties $f, g\colon X \to Y$ are *equivalent* or *the same up to an automorphism of* Y if there exists an automorphism $\varphi\colon Y \to Y$ such that $\varphi \circ f = g$. We consider embeddings of the affine line $\mathbb{C}$ into varieties Y that arise as underlying varieties of affine algebraic groups and study these embeddings up to automorphisms of Y. Recall that an *affine algebraic group* is a closed subgroup of the complex general linear group GL_n for some n. In this paper, all groups are affine and algebraic. Our main result is the following.

Theorem 1.1. *Let Y be the underlying variety of a connected affine algebraic group. Then two embeddings of the affine line $\mathbb{C}$ into Y are the same up to an automorphism of Y provided that Y is not isomorphic to a product of a torus $(\mathbb{C}^*)^k$ and one of the three varieties $\mathbb{C}^3$, SL_2, and PSL_2.*

In particular, $\mathbb{C}$ embeds uniquely (up to automorphisms) into the underlying variety of any affine algebraic group without non-trivial characters of dimension other than three; compare with Remark 8.3. Note also that connectedness is not a restriction since any connected component of an affine algebraic group G is itself isomorphic (as a variety) to the connected component of the identity element.

Let us put Theorem 1.1 in context. Embedding problems are most classically considered for $Y = \mathbb{C}^n$; compare, e.g., the overviews by Kraft and van den Essen [Kra96, vdE04]. We recall what is known about uniqueness of embeddings of $\mathbb{C}$ into $\mathbb{C}^n$. If $n = 2$, there is a unique embedding (up to automorphisms) by the Abhyankar-Moh-Suzuki Theorem [AM75, Suz74]. For $n \geq 4$, again there is a unique embedding (up to automorphisms) by work of Craighero; see [Cra86]. More generally, Kaliman [Kal88, Kal91], Nori (unpublished), and Srinivas [Sri91] proved that smooth affine varieties of dimension d embed uniquely into $\mathbb{C}^n$ whenever $n \geq 2d+2$. This result improved a previously established bound obtained by Jelonek [Jel87] and in case of embeddings of affine spaces a previously established bound obtained by Craighero [Cra86] and Jelonek [Jel87]. The existence of non-equivalent embeddings $\mathbb{C} \to \mathbb{C}^3$ is a long-standing open problem; see [Kra96]. There are various potential examples of non-equivalent embeddings of $\mathbb{C}$ into $\mathbb{C}^3$; see, e.g., [Sha92].

The above-mentioned results of Craighero, Jelonek, and Kaliman, Nori, and Srinivas are established by cleverly projecting to different linear coordinates.

[1] All embeddings in this paper are closed. In fact, closedness is automatic for embeddings of the affine line $\mathbb{C}$ into quasi-affine varieties, which is the setting we are considering.

The second author was able to use projections to coordinates to establish that there is a unique embedding of $\mathbb{C}$ into the underlying variety of SL_n (up to automorphisms) for all integers $n \geq 3$; see [Sta17]. For affine algebraic groups in general, projections to coordinates are no longer available. Our approach to embeddings of $\mathbb{C}$ is to study projections onto quotients by (unipotent) subgroups.

For a different point of view we consider the notion of flexible varieties as studied by Arzhantsev, Flenner, Kaliman, Kutzschebauch, and Zaidenberg in [AFK$^+$13]. Flexible varieties can be seen as a generalization of connected affine algebraic groups without non-trivial characters. Smooth irreducible affine flexible varieties of dimension at least two have the property that all embeddings of a fixed finite set are equivalent [AFK$^+$13, Theorem 0.1]. Theorem 1.1 states that in the underlying variety of most affine algebraic groups even all embeddings of $\mathbb{C}$ are equivalent. In light of Theorem 1.1, the following question is natural in this context.

Question 1.2. Let Y be a smooth irreducible affine flexible variety of dimension at least 4. Is there at most one embedding of $\mathbb{C}$ into Y up to automorphisms?

There exist smooth irreducible flexible affine surfaces that contain non-equivalent embeddings of $\mathbb{C}$; see Example 2.1. Since in dimension three there is the long-standing open problem, whether all embeddings of $\mathbb{C}$ into $\mathbb{C}^3$ are equivalent, we ask Question 1.2 only for varieties of dimension ≥ 4. In Example 2.2, we provide a contractible smooth affine irreducible surface S such that $S \times \mathbb{C}^n$ contains non-equivalent embeddings of $\mathbb{C}$ for all integers $n \geq 1$. These examples of varieties that contain non-equivalent embeddings of $\mathbb{C}$ are the content of Section 2.

Note that some sort of "flexibility" is required to prove results such as Theorem 1.1 in case one has "many" embeddings of $\mathbb{C}$. For example, if every pair of points in an affine variety Y can be connected by a chain of embedded affine lines[2] and Y admits a non-trivial $\mathbb{C}^+$-action, then flexibility of Y is a necessary condition for the equivalence of all embeddings $\mathbb{C} \to Y$; see [AFK$^+$13, Theorem 0.1].

Theorem 1.1 can be seen as covering all cases of embeddings of $\mathbb{C}$ into the underlying variety of any connected affine algebraic group without non-trivial characters except the well-known open problem of embeddings into $\mathbb{C}^3$ and embeddings into the underlying variety of SL_2 and PSL_2. As argued by the second author in [Sta17], the underlying variety of SL_2 (and in fact similarly for PSL_2) allows for many embeddings of $\mathbb{C}$ and perceivably their equivalence

[2]Compare the notion of $\mathbb{A}^1$-chain connectedness in [AM11] and rationally chain connectedness in [Kol96].

or non-equivalence up to automorphism might be as challenging as for the $\mathbb{C}^3$ case. In Section 3, we report on these examples of embeddings into $\mathbb{C}^3$ and the underlying variety of SL_2 and PSL_2.

1.1. Conventions. Let G be an affine algebraic group. During this article, an *automorphism of* G is an automorphism of the underlying variety of G. A *group automorphism of* G is an automorphism of the underlying variety of G that in addition also preserves the group structure of G.

1.2. Tools for the proof of Theorem 1.1. In Section 4, notions and basic facts from the theory of affine algebraic groups, their principal bundles and homogeneous spaces are introduced.

In order to prove equivalence of embeddings we need a good way to construct automorphisms. This is the content of Section 5. Let us expand on that. While we are only interested in showing uniqueness of embeddings up to automorphisms of the *underlying variety* of an affine algebraic group, we will heavily depend on the group structure to construct automorphisms. The following shearing-tool follows readily by using the group structure; see Proposition 5.1. It is our main tool to construct automorphisms of the underlying variety of an affine algebraic group.

> **Shearing-tool.** Let X and X' be affine lines embedded in an affine algebraic group G and let $H \subseteq G$ be a closed subgroup such that G/H is quasi-affine. If $\pi\colon G \to G/H$ restricts to an embedding on X and X' and if $\pi(X) = \pi(X')$, then there exists a π-fiber-preserving automorphism of G that maps X to X'.

This could be seen as an analog to a fact used in proving the earlier mentioned results of Craighero, Jelonek, and Kaliman, Nori and Srinivas about embeddings into $\mathbb{C}^n$: given two embeddings σ, σ' of an affine line (or in fact any affine variety) into $\mathbb{C}^n$ such that the last $m < n$ coordinate functions agree and yield an embedding into $\mathbb{C}^m$, then there exists a shear ϕ of $\mathbb{C}^n$ with respect to the projection to the last m coordinates such that $\phi \circ \sigma = \sigma'$; see [Kal91] and [Sri91].

In Section 6, we show that all embeddings $\mathbb{C} \to G$ with image a one-parameter unipotent subgroup of G are equivalent. A one-parameter unipotent group is an algebraic group that is isomorphic to the additive group of the field of complex numbers $\mathbb{C}^+$.

Thus in order to prove equivalence of all embeddings $\mathbb{C} \to G$, it suffices to show that every affine line in G can be moved via an automorphism of G into a one-parameter unipotent subgroup.

In Section 7, we introduce another tool. In view of the above shearing-tool, given a curve X in G, we are interested in having many closed subgroups H

such that X projects isomorphically (or at least birationally) to G/H. We establish several results in that direction and we call them generic projection results. In this context our main result is the following; see Proposition 7.4. It is based on an elegant formula that relates the dimension of the conjugacy class C of a unipotent element in a semisimple group with the dimension of the intersection of C with a maximal unipotent subgroup; see [Ste76] and [Hum95, §6.7].

> **Main generic projection result.** If G is a simple affine algebraic group of rank at least two, and H a closed unipotent subgroup, then for any curve $X \subseteq G$ that is isomorphic to $\mathbb{C}$ there exists an automorphism φ of G such that for generic $g \in G$ the quotient map $G \to G/gHg^{-1}$ restricts to an embedding on $\varphi(X)$.

1.3. Outline of the proof of Theorem 1.1. In Section 8, we reduce Theorem 1.1 to the case of a semisimple group. In a bit more detail: let G be an affine algebraic group satisfying the assumptions of Theorem 1.1. We note that G is isomorphic (as a variety) to $G^u \times (\mathbb{C}^*)^k$ for some integer $k \geq 0$, where G^u denotes the normal subgroup of G generated by unipotent elements. Embeddings of $\mathbb{C}$ into $G^u \times (\mathbb{C}^*)^n$ are necessarily constant on the second factor; thus we study embeddings into G^u. We have that G^u is isomorphic as a variety to $R_u(G^u) \times G^u/R_u(G^u)$, where $R_u(G^u)$ denotes the unipotent radical—the largest normal unipotent subgroup of G^u. If $R_u(G^u)$ is non-trivial nor equal to G^u, then the non-trivial product structure on G^u allows us to show equivalence of all embedded affine lines; see Proposition 8.6. If $R_u(G^u) = G^u$, then $G^u \cong \mathbb{C}^n$ for some $n \neq 3$, and the result follows by Jelonek's work (for $n \geq 4$) and by the Abhyankar-Moh-Suzuki Theorem (for $n = 2$). This leaves the case where $R_u(G^u)$ is trivial; i.e., G^u is semisimple.

In Section 9, we prove Theorem 1.1 for the case of a semisimple, but not simple group G. We use the fact that G is isomorphic to a quotient of the product of at least two simple groups by a finite central subgroup. Part of the argument relies on the fact that simple groups have sufficiently many unipotent elements. To ensure this, the classification of simple groups of small rank is invoked; see Lemma B.6.

Finally, in Section 10, we prove Theorem 1.1 in the case of a simple group G. This constitutes the technical heart of the proof. Besides using several results from previous sections about embeddings into products and generic projection results, we use the language of affine algebraic group theory to define an interesting subvariety E of G. In fact, E is the preimage of the (unique) Schubert curve under the projection to G/P, where P is a maximal parabolic subgroup of G. We show that any embedding of the affine line in G can be moved into E by an automorphism of G; compare Subsection 10.4. This is in fact the key step in our proof. Let us expand on this.

Let P^- be an opposite parabolic subgroup to P and denote by $\pi\colon G \to G/R_u(P^-)$ the quotient map. We establish that the restriction of π to E is a locally trivial $\mathbb{C}$-bundle over $\pi(E)$ and $\pi(E)$ is a big open subset of $G/R_u(P^-)$, i.e., the complement is a closed subset of codimension at least two in $G/R_u(P^-)$; see Proposition 10.2. Now, one can move X into E via the following steps:

- Using our main generic projection result, we can achieve that π restricts to an embedding on X.
- Using that $\pi(E)$ is a big open subset of $G/R_u(P^-)$, we can move X into $\pi^{-1}(\pi(E))$ by left multiplication with a group element. In particular, π still restricts to an embedding on X, by G-equivariancy.
- Since $E \to \pi(E)$ is a locally trivial $\mathbb{C}$-bundle, it has a section $X' \subseteq E$ over $\pi(X) \cong \mathbb{C}$. Therefore, we can move X into X' with our shearing-tool.

Next we exploit that $E = KP$ for a certain non-trivial closed subgroup K of G and the parabolic subgroup P used to define E. Under the assumption that the rank of G is at least two, i.e., G is different from SL_2 and PSL_2, we show the following. Via an automorphism of G one can move any affine line in E to an affine line in E such that the quotient map $E \to K\backslash E$ restricts to an embedding on this affine line; see Proposition 10.7. Using this result and the fact that the product map $K \times P \to E$ is a principal $K \cap P$-bundle we can move any affine line in E into an affine line in P. Since P is a proper subgroup of G, one can move any affine line in P into a one-parameter unipotent subgroup of G. This implies Theorem 1.1 in this last case.

1.4. Overview of the appendices. We have three appendix sections, which contain results that are used in the proof of Theorem 1.1, but that are either classical or the proofs are independent of the general idea of the proof of Theorem 1.1. Appendix A provides a proof of the fact that principal G-bundles over the affine line are trivial for all affine algebraic groups G. In Appendix B, we provide generalities on the Weyl group, on parabolic subgroups of reductive groups, and on Schubert varieties, as needed in Section 10. In Appendix C, we provide results about $\mathbb{C}^+$-equivariant morphisms of surfaces as needed in the proof of Proposition 10.7 (which constitutes the most technical part of Section 10).

2. Examples of varieties that contain non-equivalent embeddings of $\mathbb{C}$

In the first example we provide an irreducible smooth affine flexible surface that contains non-equivalent embeddings of $\mathbb{C}$. This example is due to Decaup and Dubouloz. For a deeper study of this example see [DD18].

Example 2.1. Let $S = \mathbb{P}^2 \setminus Q$, where Q is a smooth conic in $\mathbb{P}^2$. Clearly, S is irreducible, smooth and affine. Let $[x : y : z]$ be a homogeneous coordinate system of $\mathbb{P}^2$. We can assume without loss of generality that Q is given by the homogeneous equation $xz = y^2$ in $\mathbb{P}^2$.

Let L_1 be the curve $S \cap \{z = 0\}$ and let L_2 be the curve $S \cap \{xz - y^2 = z^2\}$. One can see that $\mathrm{Pic}(S \setminus L_1)$ is trivial, whereas $\mathrm{Pic}(S \setminus L_2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence there are non-equivalent embeddings of $\mathbb{C} \cong L_1 \cong L_2$ into S.

To establish the flexibility of S, we have to show that $\mathrm{SAut}(S)$ acts transitively on S where $\mathrm{SAut}(S)$ denotes the subgroup of $\mathrm{Aut}(S)$ that is generated by all automorphisms coming from $\mathbb{C}^+$-actions on S; see [AFK$^+$13, Theorem 0.1]. Consider the $\mathbb{C}^+$-action $t \cdot [x : y : z] = [x : y + tx : z + 2yt + t^2 x]$ on S. A computation shows that every orbit of this $\mathbb{C}^+$-action intersects the curve L_2. Since L_2 is an orbit of the $\mathbb{C}^+$-action $t \cdot [x : y : z] = [x + 2yt + t^2 z : y + tz : z]$ on S, it follows that $\mathrm{SAut}(S)$ acts transitively on S.

Next, we give in any dimension ≥ 3 an example of an irreducible smooth contractible affine variety that contains non-equivalent embeddings of $\mathbb{C}$. Note that for any irreducible smooth contractible affine variety, the ring of regular functions is a unique factorization domain and all invertible functions on it are constant; see, e.g., [Kal94, Proposition 3.2].

Example 2.2. Let S be an irreducible smooth contractible affine surface of logarithmic Kodaira dimension one that contains a copy C of the affine line. For example, by [tDP90, Theorem A] the affine hypersurface in $\mathbb{C}^3$ defined by

$$z^2 x^3 + 3zx^2 + 3x - zy^2 - 2y = 1$$

is smooth, contractible, and of logarithmic Kodaira dimension one, and $z = 0$ inside this hypersurface defines a copy of $\mathbb{C}$. Since S is smooth, affine, and of logarithmic Kodaira dimension one, there exists no $\mathbb{C}^+$-action on S, by [MS80, Lemma 1.3]. In other words, the Makar-Limanov invariant of S is equal to the ring of regular functions on S. Now, by [Cra04, Corollary 5.20], it follows that the Makar-Limanov invariant of

$$S \times \mathbb{C}^n$$

is equal to the ring of regular functions on S. In particular, every automorphism of $S \times \mathbb{C}^n$ maps fibers of the canonical projection $\pi \colon S \times \mathbb{C}^n \to S$ to fibers of it. Thus any copy of $\mathbb{C}$ inside $S \times \mathbb{C}^n$ that lies in some fiber of π is non-equivalent to the section $C \times \{0\} \subseteq S \times \mathbb{C}^n$ of π over C. In summary, we proved that $S \times \mathbb{C}^n$ is irreducible, affine, smooth, contractible and contains non-equivalent copies of $\mathbb{C}$, provided that $n \geq 1$.

To compare Examples 2.1 and 2.2, note that there exists no smooth irreducible affine surface that is *contractible* and contains two non-equivalent copies of $\mathbb{C}$. Indeed, smooth homology planes of logarithmic Kodaira dimension one or two, contain at most one copy of $\mathbb{C}$ and smooth homology planes of logarithmic Kodaira dimension zero do not exist; see, e.g., [GM92]. If the logarithmic Kodaira dimension of a smooth, contractible affine surface is $-\infty$, then it must be $\mathbb{C}^2$ by Miyanishi's characterization of the affine plane; see [Miy75] and [Miy84]. Thus, the Abhyankar-Moh-Suzuki Theorem implies our claim.

3. Examples of embeddings of $\mathbb{C}$ into $\mathbb{C}^3$, SL_2, and PSL_2

In this section we discuss what is known about embeddings of $\mathbb{C}$ into $\mathbb{C}^3$ and give embeddings of $\mathbb{C}$ into SL_2 and PSL_2 arising from embeddings of $\mathbb{C}$ into $\mathbb{C}^3$.

3.1. Embeddings into $\mathbb{C}^3$. After Abyankar and Moh and, independently, Suzuki established uniqueness of embeddings of $\mathbb{C}$ into $\mathbb{C}^3$, many examples of embeddings of $\mathbb{C}$ into $\mathbb{C}^3$ that are potentially different (up to automorphisms) from the standard embedding $\mathbb{C} \to \mathbb{C}^3$, $t \mapsto (t, 0, 0)$ where suggested; many of these have since been proven to be standard; compare, e.g., [vdE04]. However, examples due to Shastri, which are based on the idea of using embeddings with real coefficients such that the restriction map $\mathbb{R} \to \mathbb{R}^3$ is knotted, seem among the most promising to be non-standard. Concretely, the embeddings $\mathbb{C} \to \mathbb{C}^3$

$$t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t) \quad \text{and} \quad t \mapsto (t^3 - 3t, t(t^2 - 1)(t^2 - 4), t^7 - 42t)\,,$$

which restrict to embeddings $\mathbb{R} \to \mathbb{R}^3$ of a trefoil knot and a figure eight knot, respectively, are not known to be standard; see [Sha92].

3.2. Comparison of embeddings into $\mathbb{C}^3$ and SL_2. Embeddings of $\mathbb{C}$ into SL_2 are less studied. Following an example of the second author (compare [Sta17]), we briefly discuss how embeddings into $\mathbb{C}^3$ give rise to embeddings into SL_2. In fact, for any embedding h of $\mathbb{C}$ into $\mathbb{C}^3$ there exists an automorphism φ of $\mathbb{C}^3$ such that

$$t \mapsto \begin{pmatrix} f_1(t) & (f_1(t)f_3(t) - 1)/f_2(t) \\ f_2(t) & f_3(t) \end{pmatrix} \tag{3.1}$$

defines an embedding of $\mathbb{C}$ into SL_2 where f_1, f_2, and f_3 are the components of $f = \varphi \circ h$. In fact, it suffices to arrange that f_2 divides $f_1 f_3 - 1$ in $\mathbb{C}[t]$, which is explicitly done in [Sta17].

On the other hand, if we start with an embedding g of $\mathbb{C}$ into SL_2, then there exists an automorphism ψ of SL_2 such that $p \circ \psi \circ g$ is an embedding of $\mathbb{C}$ into $\mathbb{C}^3$ where $p \colon \mathrm{SL}_2 \to \mathbb{C}^3$ is the projection to three coordinate functions of SL_2; see [Sta17, Lemma 10].

3.3. Comparison of embeddings into SL_2 and PSL_2. In this subsection we construct a natural surjective map from the set of all embeddings of $\mathbb{C}$ into PSL_2 to the set of all embeddings of $\mathbb{C}$ into SL_2 where we consider the embeddings up to automorphisms. Thus, using Subsection 3.2, every embedding of $\mathbb{C}$ into $\mathbb{C}^3$ gives rise to an embedding of $\mathbb{C}$ into PSL_2.

By Hurwitz's Theorem, every finite étale morphism $E \to \mathbb{C}$ is trivial in the sense that every connected component of E maps isomorphically onto $\mathbb{C}$; see, e.g., [Har77, Ch. IV, Corollary 2.4]). In particular, every embedding of $\mathbb{C}$ into PSL_2 lifts via the canonical quotient $\eta \colon \mathrm{SL}_2 \to \mathrm{PSL}_2$ to two embeddings into SL_2, which are the same up to the involution $X \mapsto -X$ of SL_2. Since every automorphism of PSL_2 lifts to an automorphism of SL_2 via η (see [Ser58, Proposition 20]), we constructed a well-defined map

$$\begin{aligned} \Xi \colon & \{ \text{Embeddings of } \mathbb{C} \text{ into } \mathrm{PSL}_2 \text{ up to automorphisms of } \mathrm{PSL}_2 \} \\ & \to \{ \text{Embeddings of } \mathbb{C} \text{ into } \mathrm{SL}_2 \text{ up to automorphisms of } \mathrm{SL}_2 \} . \end{aligned}$$

We claim that Ξ is surjective. For this, let $f \colon \mathbb{C} \to \mathrm{SL}_2$ be an embedding. It is enough to prove that there exists an automorphism φ of SL_2 such that $\eta \circ \varphi \circ f$ is an embedding into PSL_2. Since $\eta \circ \varphi \circ f$ is always immersive and proper, we only have to prove injectivity of $\eta \circ \varphi \circ f$. Let $\pi_i \colon \mathrm{SL}_2 \to \mathbb{C}^2 \setminus \{0\}$ be the projection to the ith column. We can assume, after composing f with an automorphism of SL_2, that $\pi_1 \circ f \colon \mathbb{C} \to \mathbb{C}^2 \setminus \{0\}$ is immersive; see [Sta17, Lemma 10]. Let C be the image of $\pi_1 \circ f$, which is closed in $\mathbb{C}^2 \setminus \{0\}$. There is a commutative diagram

$$\begin{tikzcd} \mathrm{SL}_2 \arrow[r, "\pi_1"] \arrow[d, "\eta"'] & \mathbb{C}^2 \setminus \{0\} \arrow[d, "\rho"] \\ \mathrm{PSL}_2 \arrow[r] & V \end{tikzcd}$$

where $\rho \colon \mathbb{C}^2 \setminus \{0\} \to V$ denotes the quotient by the $\mathbb{Z}/2\mathbb{Z}$-action $(x, z) \mapsto (-x, -z)$ on $\mathbb{C}^2 \setminus \{0\}$. Let $Z = \rho(C)$. Since the morphism ρ is étale, it follows that $\rho \circ \pi_1 \circ f \colon \mathbb{C} \to Z$ is immersive and hence birational. Let $Z_0 \subseteq Z$ be a finite subset such that $\rho \circ \pi_1 \circ f$ restricts to an isomorphism $\mathbb{C} \setminus (\rho \circ \pi_1 \circ f)^{-1}(Z_0) \cong Z \setminus Z_0$. Let T be the finite set $(\rho \circ \pi_1 \circ f)^{-1}(Z_0)$. For a polynomial $p \in \mathbb{C}[x, z]$, let $\varphi_p \colon \mathrm{SL}_2 \to \mathrm{SL}_2$ be the automorphism given by

$$\varphi_p \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y + xp(x,z) \\ z & w + zp(x,z) \end{pmatrix}$$

and let $g_p = \varphi_p \circ f$. Note that

$$\pi_1 \circ g_p = \pi_1 \circ f \quad \text{and} \quad \pi_2 \circ g_p = \pi_2 \circ f + (p \circ \pi_1 \circ f) \cdot (\pi_1 \circ f)\,.$$

Thus for all $t \neq s$ in T, the condition $(\eta \circ g_p)(t) \neq (\eta \circ g_p)(s)$ is satisfied if

$$\begin{aligned} &(\pi_2 \circ f)(t) + p((\pi_1 \circ f)(t)) \cdot (\pi_1 \circ f)(t) \\ &\neq \pm \left[(\pi_2 \circ f)(s) + p((\pi_1 \circ f)(s)) \cdot (\pi_1 \circ f)(s)\right]\,. \end{aligned} \tag{3.2}$$

There exists a polynomial $p \in \mathbb{C}[x,z]$ such that for all $t \neq s$ in T the condition (3.2) is satisfied, since for all $t \neq s$ in T the negation of condition (3.2) is defined by the union of two proper affine linear subspaces of the vector space $\mathbb{C}[x,z]$. Since $\rho \circ \pi_1 \circ f$ restricts to an isomorphism $\mathbb{C} \setminus T \cong Z \setminus Z_0$, it follows that $\eta \circ g$ restricted to $\mathbb{C} \setminus T$ is injective. By (3.2), we have $(\eta \circ g)(t) \neq (\eta \circ g)(s)$ for all $t \neq s$ in T and thus $\eta \circ g$ restricted to T is injective. Since the images under $\eta \circ g$ of $\mathbb{C} \setminus T$ and T are disjoint, it follows that $\eta \circ g$ is injective, which implies our claim.

4. Notation and generalities on affine algebraic groups and their principal bundles

4.1. Affine algebraic groups. For the basic results on affine algebraic groups we refer to [Hum75] and for the basic results about Lie algebras and root systems we refer to [Hum78]. In order to set up conventions, let us recall the basic terms. A connected non-trivial affine algebraic group G is called *semisimple* if it has a trivial radical $R(G)$, where $R(G)$ is the largest connected normal solvable subgroup of G. An affine algebraic group G is called *reductive* if it has a trivial unipotent radical $R_u(G)$, where $R_u(G)$ is the closed normal subgroup of $R(G)$ consisting of all unipotent elements. A non-commutative connected affine algebraic group G is called *simple* if it contains no non-trivial closed connected normal subgroup. Note that for a simple affine algebraic group G, the quotient $G/\mathrm{Z}(G)$ by the center $\mathrm{Z}(G)$ is simple as an abstract group (see [Hum75, Corollary 29.5]); i.e., it contains no proper normal subgroup.

For any connected affine algebraic group G, we denote by $\mathcal{U}_G$ the subset of unipotent elements in G. It is irreducible and closed in G; see [Hum95, Theorem 4.2]. We denote by $\mathrm{rank}(G)$ the dimension of a maximal torus of G.

By [Hum95, §4.2] we have for any reductive group G

$$\dim \mathcal{U}_G = \dim G - \operatorname{rank} G$$

and using the Levi decomposition (see [OV90, Theorem 4, Ch. 6]) this formula holds more generally for every connected affine algebraic group G. Moreover, we denote by G^u the subgroup that is generated by all unipotent elements of G. It is normal, connected, and closed in G. Normal is immediate since conjugates of unipotent elements are unipotent. For connected and closed see [Hum75, Proposition 7.5]. We note that if $\dim(\mathcal{U}_G) \leq 1$, then $\mathcal{U}_G = G^u$ since $\mathcal{U}_G$ is (by irreducibility) either a one-parameter unipotent subgroup or equal to $\{e\}$. For any semisimple G, we have $G = G^u$; see [Hum75, Theorem 27.5]. In particular, $\dim \mathcal{U}_G \geq 2$ for semisimple G.

We use $\mathfrak{g}$ to denote the Lie algebra of an affine algebraic group G. Moreover, we denote by $\mathcal{N}_{\mathfrak{g}}$ the closed irreducible cone of nilpotent elements inside $\mathfrak{g}$. Note that the exponential $\exp\colon \mathfrak{g} \to G$ restricts to an isomorphism of affine varieties $\exp\colon \mathcal{N}_{\mathfrak{g}} \to \mathcal{U}_G$.

4.2. Principal bundles. Our general reference for principal bundles is [Ser58]. Again, in order to set up conventions, let us recall the basic terms. Let G be any affine algebraic group. A *principal* G-bundle is a variety P with a right G-action together with a G-invariant morphism $\pi\colon P \to X$ such that locally on X, π becomes a trivial principal G-bundle after a finite étale base change. If one can choose these étale base changes to be open injective immersions, then we say π is a *locally trivial principal G-bundle*.

The most prominent example of a principal bundle in this article is the following: let G be an affine algebraic group and let H be a closed subgroup. Then $G \to G/H$ is a principal H-bundle; see [Ser58, Proposition 3].

For any affine algebraic group G, any principal G-bundle over $\mathbb{C}$ is trivial; see Appendix A.

4.3. Homogeneous varieties. Let G be an affine algebraic group and let $H \subseteq G$ be a closed subgroup. If H has no non-trivial character, then the quotient G/H is quasi-affine (see [Tim11, Example 3.10]) and if H is normal in G or reductive, then G/H is affine (see [Tim11, Theorem 3.8]). In this article we frequently use that G/H is quasi-affine in case H is unipotent.

5. Construction of automorphisms of an affine algebraic group

In this section we introduce a construction of automorphisms of affine algebraic groups that we use throughout this article.

Let G be an affine algebraic group. Let $H \subseteq G$ be a closed subgroup and let $\pi\colon G \to G/H$ be the quotient by left H-cosets. For any morphism $f\colon G/H \to H$, the map

$$\varphi_f\colon G \longrightarrow G\,, \quad g \mapsto gf(\pi(g)) \tag{5.1}$$

is an automorphism of G that preserves the quotient π. Let $\rho\colon G \to H\backslash G$ be the quotient by right H-cosets. Analogously to φ_f, we define for any morphism $d\colon H\backslash G \to H$ the automorphism

$$\psi_d\colon G \longrightarrow G\,, \quad g \mapsto d(\rho(g))g\,. \tag{5.2}$$

We frequently use these constructions in the following situation.

Proposition 5.1. *Let G be an affine algebraic group and let $X \subseteq G$ be a closed curve that has only one place at infinity. Moreover, we assume that there is a closed subgroup H of G such that $\pi\colon G \to G/H$ restricts to an embedding on X and that X' is another section of $\pi^{-1}(\pi(X)) \to \pi(X)$. If*

(1) *H is unipotent or*
(2) *$X \cong \mathbb{C}$ and G/H is quasi-affine,*

then there is an automorphism of G that preserves π and maps X onto X'.

Remark 5.2. A curve X has only one place at infinity if there exists a projective curve $\bar{X}$ that contains X as an open subset such that $\bar{X}\setminus X$ consists only of one point and this point is a smooth point of $\bar{X}$.

Remark 5.3. The analog of Proposition 5.1 for right coset spaces also holds.

Proof of Proposition 5.1. Note that G/H is quasi-affine, in case H is unipotent (see Subsection 4.3). Thus G/H is in both cases quasi-affine. The curve $\pi(X)$ is closed in G/H since $\pi(X)$ has only one place at infinity and therefore is closed in any affine variety that contains G/H as an open subset. Denote by $s\colon \pi(X) \to X$ and $s'\colon \pi(X) \to X'$ the inverse maps of $\pi|_X\colon X \to \pi(X)$ and $\pi|_{X'}\colon X' \to \pi(X)$, respectively. Consider the morphism

$$\pi(X) \longrightarrow H\,, \quad v \mapsto (s(v))^{-1} \cdot s'(v)\,. \tag{5.3}$$

If one can extend this morphism to a morphism $f\colon G/H \to H$, then the automorphism φ_f of (5.1) preserves π and satisfies $\varphi_f(X) = X'$. Thus we only have to show that such an extension of (5.3) exists.

- If H is unipotent, then it is isomorphic to some $\mathbb{C}^n$ as a variety and the desired extension $f\colon G/H \to H$ of (5.3) exists.
- If $X \cong \mathbb{C}$, then $\pi(X) \cong X \cong \mathbb{C}$. Hence there exists a retraction $r\colon G/H \to \pi(X)$ of G/H onto $\pi(X)$. Composing r with the morphism (5.3) yields the desired extension $f\colon G/H \to H$. □

6. Embeddings of $\mathbb{C}$ with unipotent image

The following result says that two embeddings f_1 and f_2 of $\mathbb{C}$ into an affine algebraic group G are the same up to an automorphism of G, provided that $f_1(\mathbb{C})$ and $f_2(\mathbb{C})$ are one-parameter unipotent subgroups of G.

Proposition 6.1. *Let G be any affine algebraic group and let U, V be one-parameter unipotent subgroups. For any isomorphism of varieties $\sigma\colon U \to V$, there exists an automorphism φ of G such that $\varphi|_U = \sigma$.*

Proof. If G^u is one-dimensional, then $G^u = R_u(G)$ and G is isomorphic to $G^u \times G/G^u$ as a variety by the Levi decomposition of G (see [OV90, Ch. 6, Theorem 4]). In particular, $U = V = G^u$ and every automorphism of U extends to G. Thus, we can assume that G^u is at least two-dimensional and hence we can assume that $V \neq U$. This implies $V \cap U = \{e\}$ and therefore multiplication $V \times U \to VU \subseteq G$ is an embedding. Hence, the quotient map $\pi\colon G \to G/U$ restricts to an embedding on V. Since G/U is quasi-affine, the morphism

$$\pi(V) \xrightarrow{(\pi|_V)^{-1}} V \xrightarrow{\sigma^{-1}} U$$

extends to a morphism $f\colon G/U \to U$. Hence the automorphism φ_f of G (see (5.1) in Section 5) satisfies $\varphi_f(v) = v \cdot \sigma^{-1}(v)$ for all $v \in V$. Using the quotient $\rho\colon G \to V\backslash G$ one can similarly construct an automorphism ψ_d of G such that $\psi_d(u) = \sigma(u) \cdot u$ for all $u \in U$ (see (5.2) in Section 5). It follows that $\varphi = \varphi_f^{-1} \circ \psi_d$ restricts to σ on U. □

7. Generic projection results

The aim of this section is to prove results, which enable us to quotient by unipotent subgroups such that the projection restricts to a closed embedding or to a birational map on a given fixed curve. These projection results will be applied in Sections 8 and 9 to reduce Theorem 1.1 to the case of semisimple and simple groups, respectively. In Section 10 we use these results in the heart of the proof of Theorem 1.1, namely, for the case of embeddings into simple groups.

Let V be a variety. Throughout this paper we say that a property is satisfied *for generic* $v \in V$ if there exists a dense open subset O in V such that the property is satisfied for all v in O.

7.1. Quotients that restrict to closed embeddings on a fixed curve. Our first result in this section deals with arbitrary affine algebraic groups and quotients by one-parameter unipotent subgroups.

Lemma 7.1 (Communicated by Winkelmann)**.** *Let G be an affine algebraic group and let $X \subseteq G$ be a closed smooth curve that has only one place at infinity. If the set of unipotent elements $\mathcal{U}_G$ has dimension at least four, then, for a generic one-parameter unipotent subgroup $U \subseteq G$, the quotient $G \to G/U$ restricts to a closed embedding on X.*

Remark 7.2. Using the exponential map $\exp\colon \mathcal{N}_{\mathfrak{g}} \to \mathcal{U}_G$, we consider the set of one-parameter unipotent subgroups of G as the image of $\mathcal{N}_{\mathfrak{g}} \setminus \{0\}$ under the quotient $\mathfrak{g} \setminus \{0\} \to \mathbb{P}(\mathfrak{g})$. Note that this image is closed in $\mathbb{P}(\mathfrak{g})$ and therefore we can speak of a "generic one-parameter unipotent subgroup".

Proof of Lemma 7.1. As already mentioned, the exponential restricts to an isomorphism of affine varieties $\exp\colon \mathcal{N}_{\mathfrak{g}} \to \mathcal{U}_G$. We denote by F the set of all elements in G of the form $y^{-1}x$ with $x, y \in X$ and $x \neq y$. Let

$$F' = \exp(\operatorname{cone}(\exp^{-1}(F \cap \mathcal{U}_G))) \subseteq \mathcal{U}_G,$$

where $\operatorname{cone}(M)$ denotes the union of all lines in $\mathcal{N}_{\mathfrak{g}}$ that pass through the origin and intersect M, for any subset M of $\mathcal{N}_{\mathfrak{g}}$. Let $U \subseteq G$ be a one-parameter unipotent subgroup. Thus $G \to G/U$ maps X injectively onto its image if and only if $U \cap F' = \{e\}$. However, F' is a constructible subset of $\mathcal{U}_G$ of dimension at most three.

Let $S \subseteq \mathfrak{g}$ be the union of all lines $Dl_{x^{-1}}(T_xX)$, $x \in X$, where $l_g\colon G \to G$ denotes left multiplication by $g \in G$. Let $U \subseteq G$ be a one-parameter unipotent subgroup. Thus $G \to G/U$ maps X immersively onto its image if and only if $\mathfrak{u} \cap S \cap \mathcal{N}_{\mathfrak{g}} = \{0\}$ where $\mathfrak{u}$ denotes the Lie algebra of U. Clearly, $S \cap \mathcal{N}_{\mathfrak{g}}$ is a constructible subset of $\mathcal{N}_{\mathfrak{g}}$ of dimension at most two.

Since G/U is quasi-affine, the quotient $G \to G/U$ maps X properly onto its image, as long as the image is not a single point, since X has only one place at infinity.

In summary, we proved that the restriction of $G \to G/U$ to X is injective, immersive, and proper for a generic one-parameter unipotent subgroup U in G. $\square$

Remark 7.3. The proof of Lemma 7.1 shows that we can replace $\mathcal{U}_G$ by some closed subset W of $\mathcal{U}_G$ that is a union of unipotent subgroups and such that each irreducible component of W has dimension at least four, in order to prove that for a generic one-parameter unipotent subgroup U in W the quotient $G \to G/U$ restricts to a closed embedding on X.

Our second result deals with simple groups and quotients by arbitrary unipotent subgroups.

Proposition 7.4. *Let G be a simple affine algebraic group of rank at least two and let $U \subseteq G$ be a unipotent subgroup. If $X \subseteq G$ is a closed smooth curve with only one place at infinity, then there exists an automorphism φ of*

G such that for generic $g \in G$ the projection $G \to G/gUg^{-1}$ restricts to a closed embedding on $\varphi(X)$.

In order to prove this result, we have to show that for generic $g \in G$ the projection $G \to G/gUg^{-1}$ restricts to an injective and immersive map on $\varphi(X)$ for a suitable automorphism φ. If this is the case, then this restriction is automatically proper, since X has only one place at infinity.

Lemma 7.5 (Immersivity). *Let G be a connected reductive affine algebraic group and let $U \subseteq G$ be a closed unipotent subgroup. If $X \subseteq G$ is a closed irreducible smooth curve such that $e \in X$ and T_eX contains non-nilpotent elements of the Lie algebra $\mathfrak{g}$, then for generic $g \in G$ the projection $\pi_g \colon G \to G/gUg^{-1}$ restricts to an immersion on X.*

Proof of Lemma 7.5. Denote by $\mathfrak{u}$ the Lie algebra of U. The kernel of the differential of π_g in $e \in G$ is the sub Lie algebra $\mathrm{Ad}(g)\mathfrak{u}$ of $\mathfrak{g}$, where $\mathrm{Ad}(g)$ denotes the linear isomorphism of $\mathfrak{g}$ induced by the differential in e of the automorphism of G that is given by $h \mapsto ghg^{-1}$. Consider the morphism

$$G \times (\mathfrak{u} \setminus \{0\}) \to \mathbb{P}(\mathfrak{g}) , \quad (g, v) \mapsto [\mathrm{Ad}(g)v] , \tag{7.1}$$

where $[w]$ denotes the line through $0 \neq w \in \mathfrak{g}$. Since G is not unipotent, the set of non-nilpotent elements is a dense open subset of $\mathfrak{g}$, which maps via the projection $\mathfrak{g} \setminus \{0\} \to \mathbb{P}(\mathfrak{g})$ to a dense open subset O. Since $\mathrm{Ad}(g)v$ is nilpotent for all $v \in \mathfrak{u}$, the open set O lies in the complement of the image of the morphism in (7.1). Let

$$S = \bigcup_{x \in X} \mathbb{P}(T_e(x^{-1}X)) \subseteq \mathbb{P}(\mathfrak{g}) ,$$

which is a locally closed irreducible curve in $\mathbb{P}(\mathfrak{g})$. Hence, π_g restricted to X is immersive for $g \in G$ if and only if $S \cap \mathbb{P}(\mathrm{Ad}(g)\mathfrak{u})$ is empty. By assumption $S \cap O$ is non-empty and thus there exists a finite subset F of S such that $S \setminus F \subseteq O$, since S is irreducible. Thus $(S \setminus F) \cap \mathbb{P}(\mathrm{Ad}(g)\mathfrak{u})$ is empty for all $g \in G$. We claim that

$$\bigcap_{g \in G} \mathrm{Ad}(g)\mathfrak{u} = \{0\} . \tag{7.2}$$

Using the isomorphism $\exp \colon \mathcal{N}_{\mathfrak{g}} \to \mathcal{U}_G$, (7.2) is equivalent to the intersection

$$\bigcap_{g \in G} gUg^{-1} \tag{7.3}$$

being trivial. Let v be in the intersection in (7.3) and let N be the smallest closed subgroup of G that contains all conjugates gvg^{-1} of v. Clearly, $N \subseteq U$. By [Hum75, Proposition 7.5], N is connected and normal in G. Since the unipotent radical of G is trivial, N is trivial. Thus, $v = e$, which proves our claim. As a consequence of (7.2), the intersection $F \cap \mathbb{P}(\mathrm{Ad}(g)\mathfrak{u})$ is empty for generic $g \in G$. This proves the lemma. $\square$

Lemma 7.6 (Injectivity)**.** *Let G be a simple affine algebraic group of rank ≥ 2 and let $U \subseteq G$ be a unipotent subgroup. If $X \subseteq G$ is a closed irreducible curve such that $e \in X$ and X contains non-unipotent elements, then for generic $g \in G$, the projection $\pi_g \colon G \to G/gUg^{-1}$ restricts to an injection on X.*

Proof of Lemma 7.6. The strategy of the proof resembles the one of the proof of Lemma 7.5 and Lemma 7.1. Consider the morphism

$$G \times U \to G\,, \quad (g, u) \mapsto gug^{-1}\,.$$

Since G is not unipotent, $G \setminus \mathcal{U}_G$ is dense and open in G, and it is contained in the complement of the image of the above morphism. Let us denote this open subset by O. Let

$$F = \{\, x^{-1}y \in G \mid x \neq y \in X \,\}\,.$$

Hence, π_g is injective if and only if $F \cap gUg^{-1}$ is empty. By assumption $F \cap O$ is non-empty and thus there exists a curve (or finite set) $C \subseteq F$ consisting of unipotent elements such that $F \setminus C \subseteq O$ since F is irreducible. Hence, $(F \setminus C) \cap gUg^{-1}$ is empty for all $g \in G$. Therefore it is enough to show that $C \cap gUg^{-1}$ is empty for generic $g \in G$. This can be achieved by showing that for all $v \in \mathcal{U}_G \setminus \{e\}$ the set

$$A_v = \{\, g \in G \mid v \in gUg^{-1} \,\}$$

has codimension ≥ 2 in G. Indeed, if $\mathrm{codim}_G(A_v) \geq 2$ for all $v \neq e$, then the dimension of

$$A = \{\, (v, g) \in C \times G \mid g \in A_v \,\}$$

is less than the dimension of G. Hence, A maps to a subset of codimension ≥ 1 in G via the natural projection $C \times G \to G$, which then implies that $C \cap gUg^{-1}$ is empty for generic $g \in G$.

So let us prove that $\mathrm{codim}_G A_v \geq 2$. Denote by $\mathrm{Cl}_G(v)$ the conjugacy class of v in G. By using the orbit map $G \to \mathrm{Cl}_G(v)$, $g \mapsto g^{-1}vg$ one can see that $\mathrm{codim}_G A_v$ is the same as the codimension of $U \cap \mathrm{Cl}_G(v)$ in $\mathrm{Cl}_G(v)$. Since G is semisimple, by [Hum95, Proposition 6.7] we have

$$\dim U \cap \mathrm{Cl}_G(v) \leq \frac{1}{2} \dim \mathrm{Cl}_G(v)\,.$$

Hence, it remains to show that $\mathrm{Cl}_G(v)$ has dimension ≥ 3, since the dimension of $\mathrm{Cl}_G(v)$ is even by [Hum95, Proposition 6.7]. This is in fact equivalent to the statement that the centralizer $\mathrm{C}_G(v) = \{g \in G \mid gvg^{-1} = v\}$ of v has codimension ≥ 3 in G. The latter is true by the following argument. The unipotent radical $R_u(\mathrm{C}_G(v))$ is not trivial since the one-parameter unipotent subgroup which contains $v \neq e$ is normal in $\mathrm{C}_G(v)$ (note that by definition the subgroup generated by v is normal in $C_G(v)$ and therefore also its closure). Clearly, $\mathrm{C}_G(v)$ lies inside the normalizer $N_G(R_u(\mathrm{C}_G(v))$. However, this normalizer is contained in some parabolic subgroup P that itself is the normalizer of some non-trivial unipotent subgroup of G; see [Hum75, Corollary 30.3A]. Since G is reductive, this implies that P is a proper subgroup of G. Since $G/\mathrm{C}_G(v) \to G$, $g \mapsto g^{-1}vg$ is injective, G is an affine variety, and G/P is projective and of positive dimension, it follows that $\mathrm{C}_G(v)$ must be a proper subgroup of P. Since P is connected, we have $\dim \mathrm{C}_G(v) < \dim P$. Since G is simple and since the rank of G is at least two, it follows from Lemma B.6 that $\dim R_u(P^-) \geq 2$. Here P^- is the opposite parabolic subgroup to P with respect to some maximal torus that is contained in some Borel subgroup which in turn is contained in P; see Appendix B.2. This implies that the codimension of P in G is at least 2 by Lemma B.5. This in turn implies that $\mathrm{C}_G(v)$ has codimension ≥ 3 in G, which proves the lemma. □

Proof of Proposition 7.4. Since G is simple, it is a so-called flexible variety; see [AFK$^+$13, §0]. Hence, there exists an automorphism φ of G such that $\varphi(X)$ contains non-unipotent elements, $e \in \varphi(X)$ and the tangent space T_eX contains non-nilpotent elements of the Lie algebra $\mathfrak{g}$; see [AFK$^+$13, Theorem 4.14, Remark 4.16, and Theorem 0.1]. By Lemmas 7.5 and 7.6, for generic $g \in G$ the projection $\pi_g \colon G \to G/gUg^{-1}$ restricted to $\varphi(X)$ is immersive and injective. As already mentioned, if this is the case, then $\pi_g|_{\varphi(X)}$ is proper. This finishes the proof. □

7.2. Quotients that restrict to birational maps on a fixed curve. Let us introduce the following notation. If G is an affine algebraic group, then for any $u \in \mathcal{U}_G \setminus \{e\}$ we denote by $\mathbb{C}^+(u)$ the one-parameter unipotent subgroup of G that contains u. Roughly speaking the next lemma says: Under certain assumptions, a curve C in an affine homogeneous G-variety Y projects birationally onto its image if we quotient Y by $\mathbb{C}^+(u)$ where u belongs to a dense subset of $\mathcal{U}_G$.

Lemma 7.7. *Let Y be an affine homogeneous G-variety where G is a connected affine algebraic group acting from the right. We assume that generic elements in $\mathcal{U}_G$ act without fixed point on Y. Moreover, we assume that for all y in Y, every fiber of the morphism*

$$\rho_y \colon \mathcal{U}_G \to Y\,, \quad u \mapsto yu$$

has codimension at least three in $\mathcal{U}_G$. If $C \subseteq Y$ is a closed curve, then there exists a dense subset in $\mathcal{U}_G$ consisting of elements u such that $\mathbb{C}^+(u)$ acts without fixed point on Y and the algebraic quotient $S_u \to S_u/\!/\mathbb{C}^+(u)$ restricts *to a birational morphism on C, where S_u denotes the smallest closed affine surface in Y that contains all $\mathbb{C}^+(u)$-orbits passing through C.*

Remark 7.8. The algebraic quotient $S_u/\!/\mathbb{C}^+(u)$ is the spectrum of the ring of functions on S_u that are invariant under the action of $\mathbb{C}^+(u)$. In fact, $S_u/\!/\mathbb{C}^+(u)$ is an irreducible affine curve; see [Mat86, Theorem 11.7] and [OY82, Corollary 1.2, Theorem 3.2].

Proof of Lemma 7.7. Let $c_0 \in C$ and let K_{c_0} be the union of the orbits $c_0\mathbb{C}^+(u)$, $u \in \mathcal{U}_G \setminus \{e\}$ where $c_0\mathbb{C}^+(u)$ is either equal to $\{c_0\}$ or it contains points of C different from c_0. In other words,

$$K_{c_0} = \bigcup_{e \neq u \in \mathcal{U}_G \text{ such that } c_0 u \in C} c_0\mathbb{C}^+(u)\,.$$

With the aid of the exponential map $\exp\colon \mathcal{N}_{\mathfrak{g}} \to \mathcal{U}_G$ we define

$$N_{c_0} = \bigcup_{e \neq u \in \rho_{c_0}^{-1}(C)} \mathbb{C}^+(u) = \exp(\operatorname{cone}(\exp^{-1}(\rho_{c_0}^{-1}(C)))) \subseteq \mathcal{U}_G\,.$$

One can see that $N_{c_0} = \rho_{c_0}^{-1}(K_{c_0})$. In particular, we have for $u \in \mathcal{U}_G \setminus N_{c_0}$ that $c_0\mathbb{C}^+(u)$ intersects C only in the point c_0. Since all the fibers of ρ_{c_0} have codimension at least three in $\mathcal{U}_G$ and since $\dim C = 1$, it follows that $\dim \rho_{c_0}^{-1}(C) \leq \dim \mathcal{U}_G - 2$. By the construction of N_{c_0} we now get

$$\dim N_{c_0} \leq \dim \mathcal{U}_G - 1\,.$$

Take a countably infinite subset $C_0 \subseteq C$. Since our ground field is uncountable, the intersection $\bigcap_{c_0 \in C_0} \mathcal{U}_G \setminus N_{c_0}$ is dense in $\mathcal{U}_G$. Let $u \in \mathcal{U}_G$ be an element that acts without fixed point on Y and such that $u \notin \bigcup_{c_0 \in C_0} N_{c_0}$. Since a fiber of $S_u \to S_u/\!/\mathbb{C}^+(u)$ over a generic point of $S_u/\!/\mathbb{C}^+(u)$ is a $\mathbb{C}^+(u)$-orbit, it follows that infinitely many fibers of $C \to S_u \to S_u/\!/\mathbb{C}^+(u)$ consist only of one point. Thus, C is mapped birationally onto the algebraic quotient. $\square$

Remark 7.9. The proof of the Lemma 7.7 shows the following: If there exist infinitely many c_0 in C such that $\rho_{c_0}^{-1}(C) \leq \dim \mathcal{U}_G - 2$, then the statement of the lemma holds. In particular, the statement of the lemma holds if there are infinitely many $c_0 \in C$ such that all fibers of $\rho_{c_0}\colon \mathcal{U}_G \to Y$ have codimension at least two in $\mathcal{U}_G$ and $c_0\mathcal{U}_G \cap C$ is finite.

Corollary 7.10. *Let G be a connected affine algebraic group such that $\dim G \geq 3$, $\dim \mathcal{U}_G \geq 2$, and $G = G^u$. If $C \subseteq G$ is a closed irreducible curve, then there exists an automorphism φ of G and a dense subset of $\mathcal{U}_G$ consisting of elements u such that $G \to G/\mathbb{C}^+(u)$ maps $\varphi(C)$ birationally onto its image.*

Proof. If G is a unipotent group, the statement is clear, since $\dim G \geq 3$. Thus we can assume that $\mathcal{U}_G$ is a proper subset of G. Since $G = G^u$, the variety G is flexible. Fix some point c_0 in C. By [AFK$^+$13, Theorem 0.1] there exists an automorphism φ of G that fixes c_0 and the image $\varphi(C)$ intersects

$c_0\mathcal{U}_G$ only in finitely many points. Thus we can assume that $c_0\mathcal{U}_G \cap C$ is finite. The fiber over $c \in C$ of the morphism

$$\{\,(c,u) \in C \times \mathcal{U}_G \mid cu \in C\,\} \to C\,, \quad (c,u) \mapsto c \tag{7.4}$$

is isomorphic to $c\mathcal{U}_G \cap C$. Since C is irreducible, the subset of C given by

$$C' := \{\, c \in C \mid C \subseteq c\mathcal{U}_G\,\}$$

consists of exactly those points for which the fiber of (7.4) is not finite. Note that C' is closed in C. Since $c_0\mathcal{U}_G \cap C$ is finite, C' is a proper subset of C. Since C is irreducible, it follows now that a general fiber of (7.4) is finite, i.e., $c\mathcal{U}_G \cap C$ is finite for generic c in C. Since $\dim \mathcal{U}_G \geq 2$, it follows that for all $c \in C$ the fibers of the map $\rho_c \colon \mathcal{U}_G \to G$, $\rho_c(u) = cu$ have codimension at least two in $\mathcal{U}_G$. The corollary follows from Remark 7.9 applied to the homogeneous G-variety $Y = G$. □

8. Reduction to semisimple groups

In this section we reduce the proof of Theorem 1.1 to semisimple groups.

Lemma 8.1. *Let G be a connected affine algebraic group with $G = G^u$ and let X be an affine variety that admits no non-constant invertible function $X \to \mathbb{C}^*$. Moreover, let n be a non-negative integer. Then, all closed embeddings of X into $G \times (\mathbb{C}^*)^n$ are equivalent if and only if all closed embeddings of X into G are equivalent.*

Proof. Let $f_i \colon X \to G \times (\mathbb{C}^*)^n$, $i = 1, 2$ be two closed embeddings. By assumption, $f_i(X)$ lies in some fiber of $\pi \colon G \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ for $i = 1, 2$. After multiplying with a suitable element of $G \times (\mathbb{C}^*)^n$ we can assume that $f_1(X)$ and $f_2(X)$ lie in the same fiber of π. Since any automorphism of one fiber can be extended to $G \times (\mathbb{C}^*)^n$, this proves the if-part of the lemma.

The other direction works pretty much in the same way by using the fact that every automorphism of $G \times (\mathbb{C}^*)^n$ permutes the fibers of π, since $G = G^u$, and thus there are no non-constant invertible functions $G \to \mathbb{C}^*$. □

Lemma 8.2. *Let G be a connected affine algebraic group. Then G is isomorphic to $G^u \rtimes (\mathbb{C}^*)^n$ as an algebraic group for a certain non-negative integer n.*

Proof. Note that G/G^u is a torus, since it is connected and contains only semisimple elements; see [Hum78, Proposition 21.4B and Theorem 19.3]. Let T be a maximal torus of G. Since G^u is normal in G, and since G^u and T generate G (see [Hum75, Theorem 27.3]) we have $TG^u = G$. In particular,

T is mapped surjectively onto the torus G/G^u via the canonical projection $\pi\colon G \to G/G^u$. Thus we get a short exact sequence

$$1 \longrightarrow G^u \cap T \longrightarrow T \xrightarrow{\pi|_T} G/G^u \longrightarrow 1\,.$$

By Lemma 8.4, $G^u \cap T$ is a torus. Thus the above short exact sequence splits; see [Hum75, §16.2]. In particular, the associated section yields a section of the homomorphism $\pi\colon G \to G/G^u$, which proves the lemma. □

Remark 8.3. Lemma 8.2 implies that for a connected affine algebraic group G, the following statements are equivalent:

(i) $G = G^u$;
(ii) G has no non-trivial character;
(iii) each invertible function on the underlying variety of G is constant;
(iv) there is no variety V and no integer $k > 0$ such that $G \cong V \times (\mathbb{C}^*)^k$ as a variety.

Lemma 8.4. *Let G be any connected affine algebraic group and let H be a closed connected normal subgroup of G. If T is a maximal torus of G, then $T \cap H$ is a maximal torus of H.*

Proof. Let $T' \subseteq H$ be a maximal torus that contains the connected component of the identity element $(T \cap H)^\circ$ which is also a torus. Since all maximal tori in G are conjugate, there exists $g \in G$ such that $g^{-1}T'g \subseteq T$. By the normality of H we get $g^{-1}T'g \subseteq (T \cap H)^\circ$. Hence

$$g^{-1}(T \cap H)^\circ g \subseteq g^{-1}T'g \subseteq (T \cap H)^\circ\,.$$

Thus $(T \cap H)^\circ = g^{-1}T'g$ is a maximal torus of H (note that all maximal tori of H are conjugate, since H is connected). Now, if there exists $x \in T \cap H \setminus (T \cap H)^\circ$, then clearly x is semisimple and centralizes the torus $(T \cap H)^\circ$. However, this implies that $\{x\} \cup (T \cap H)^\circ$ lies in a torus of H, since H is connceted (see [Hum75, Corollary B, §22.3]). This contradicts the maximality of $(T \cap H)^\circ$ and thus $T \cap H = (T \cap H)^\circ$ is a maximal torus of H. □

We are now in position to formulate our main result of this section.

Theorem 8.5. *Let G be a connected affine algebraic group with $G = G^u$. If G is not semisimple and not isomorphic to $\mathbb{C}^3$ as a variety, then all embeddings of $\mathbb{C}$ into G are equivalent.*

Using Lemma 8.1 and Lemma 8.2, Theorem 8.5 reduces the proof of Theorem 1.1 to the case that the group under consideration is semisimple and not isomorphic to SL_2 or PSL_2; compare with the proof of Theorem 1.1 in Section 10.

The rest of this section is devoted to the proof of Theorem 8.5. First we have to do some preliminary work.

Proposition 8.6. *Let K be a connected affine algebraic group that contains non-trivial unipotent elements and let H be a semisimple group* (*which is non-trivial by convention*). *Then all embeddings of $\mathbb{C}$ into $K \times H$ are equivalent.*

Proof of Proposition 8.6. Let $\mathbb{C} \cong X \subseteq K \times H$ be an embedding. We can assume that the canonical projection $\pi_H \colon K \times H \to H$ maps X birationally onto its image; compare Lemma 8.7 below. We can apply Corollary 7.10 to the group H and the curve $\pi_H(X)$, since H is a (non-trivial) semisimple group. Hence we can assume that there exists a one-parameter unipotent subgroup $U \subseteq H$ such that the composition

$$\rho \colon K \times H \xrightarrow{\pi_H} H \longrightarrow H/U$$

restricts to a birational morphism $X \to \rho(X)$. Let E be the finite subset of elements z in H/U such that the fiber over z of $\rho|_X$ contains more than one element. Moreover, let X' be the finite subset of X of critical points of $\rho|_X$. For a morphism $f \colon K \to U$ consider the two properties:

(i) For every $z \in E$ and for every pair $(k,h),(k',h')$ in $\rho^{-1}(z) \cap X$ with $(k,h) \neq (k',h')$ we have

$$hf(k) \neq h'f(k')\,.$$

(ii) For every $x' \in X'$ the differential of

$$\eta_f \colon X \longrightarrow H\,, \quad x \mapsto \pi_H(x) f(\pi_K(x))$$

in x' is non-vanishing.

If we consider U as a one-dimensional vector space, then for every pair of points $(k,h) \neq (k',h')$ in $\rho^{-1}(z) \cap X$, the expression $hf(k) = h'f(k')$ defines a non-trivial affine linear equation for f in the vector space of maps $K \to U$ (note that by assumption $(h')^{-1}h$ lies in U). Moreover, we claim that for every $x' \in X'$ the vanishing of the differential $D_{x'}\eta_f$ defines a non-trivial affine linear equation for f in the vector space of maps $K \to U$. Indeed, let $x' \in X'$ and let W be an open neighborhood of $\rho(x')$ in H/U in the Euclidean topology such that $H \to H/U$ gets trivial over W. Then the map η_f can be written in a Euclidean neighborhood $U_{x'}$ in X around x' as

$$U_{x'} \longrightarrow W \times U\,, \quad x \mapsto (\rho(x), q(x) f(\pi_K(x)))\,,$$

where $q \colon U_{x'} \to U$ defines a holomorphic map (that does not depend on f) with the following property: If the differential $D_{x'}q$ vanishes, then the differential $D_{x'}(\pi_K|_X)$ is non-vanishing. Now the vanishing of the differential of η_f in x' is equivalent to the vanishing of the linear map

$$D_{x'}q + D_{\pi_K(x')}f \circ D_{x'}(\pi_K|_X) \colon T_{x'}X \to U\,,$$

where we consider again U as a one-dimensional vector space. However, this last condition defines a non-trivial affine linear equation for f. This proves the claim.

In summary, we showed that there exists $f_0 \colon K \to U$ such that (i) and (ii) are satisfied. Define

$$\psi_0 \colon K \times H \to K \times H\,, \quad (k,h) \mapsto (k, hf_0(k))\,.$$

Then, the restriction of π_H onto $\psi_0(X)$ is injective and immersive, since f_0 satisfies (i) and (ii). Since $X \cong \mathbb{C}$, the map π_H restricts to an embedding on $\psi_0(X)$. Hence, after composing ψ_0 with an automorphism of $K \times H$ we can assume that X lies in $\{e\} \times H$; see Proposition 5.1. Let $V \subseteq K$ be any one-parameter unipotent subgroup and let $f_1 \colon H \to V$ be a morphism that restricts to an isomorphism on X. Let ψ_1 be defined as

$$\psi_1 \colon K \times H \to K \times H\,, \quad (k,h) \mapsto (kf_1(h), h)\,.$$

It follows that π_K maps $\psi_1(X)$ isomorphically onto V. Hence, there exists an automorphism of $K \times H$ that sends $\psi_1(X)$ into $V \times \{e\}$; see Proposition 5.1. Thus the proposition follows from Proposition 6.1. $\square$

Lemma 8.7. *Let H be an affine algebraic group with* $\dim H^u \geq 2$ *and let K be any affine variety. For any closed curve $X \subset K \times H$ that is isomorphic to $\mathbb{C}$, there exists an automorphism ψ of $K \times H$ such that the canonical projection $\pi_H \colon K \times H \to H$ restricts to a birational map $X \to \psi(X)$.*

Proof of Lemma 8.7. We only consider the case that K has dimension at least one (otherwise π_H restricts to an embedding on X). By the same argument, we can assume that the canonical projection $\pi_K \colon K \times H \to K$ is non-constant on X. We will use automorphisms of the form

$$\psi_f \colon K \times H \to K \times H\,, \quad (k,h) \mapsto (k, f(k)h)\,, \tag{8.1}$$

where $f \colon K \to U$ is a morphism to a one-parameter unipotent subgroup U of H.

Let us first consider the case where $\pi_H(X)$ is zero-dimensional; i.e., $\pi_H(X)$ is a point, and show that we can change that by applying an automorphism of the form (8.1). Without loss of generality, we may assume that the point $\pi_H(X)$ is the identity element e of H; i.e., X lies in $K \times \{e\}$. Choose any non-trivial one-parameter unipotent subgroup $U \subseteq H$ and let $f \colon K \to U$ be a morphism that is non-constant on X. Thus $\pi_H(\psi_f(X))$ is one-dimensional.

By the above we may assume that $\pi_H(X)$ is one-dimensional. We consider a regular value $h \in \pi_H(X)$ of the map $\pi_H|_X \colon X \to \pi_H(X)$ in the smooth locus of $\pi_H(X)$. Since $\pi_K|_X$ is non-constant, we can assume that the differential of $\pi_K|_X$ is non-vanishing in every point of the fiber $(\pi_H|_X)^{-1}(h)$. As before we may assume that h is the identity element e of H. Denote by

$$x_1 = (k_1, e)\,, \dots, x_n = (k_n, e)$$

the elements of the fiber $(\pi_H|_X)^{-1}(e)$. Note that for $i = 1, \dots, n$ the lines $D_{x_i}\pi_H(T_{x_i}X)$ are all the same in T_eH (otherwise e does not lie in the smooth part of $\pi_H(X)$). Let us denote this line in T_eH by l.

For an automorphism ψ_f of the form (8.1), we denote $Y = \psi_f(X)$. We next establish that we may choose ψ_f such that

- $\psi_f(x_i) = x_i$ for $i = 1, \dots, n$,
- $Y \cap \pi_H^{-1}(e) = \{x_1, \dots, x_n\}$, and
- $D_{x_i}\pi_H(T_{x_i}Y)$ and $D_{x_j}\pi_H(T_{x_j}Y)$ are different lines for $i \neq j$.

Since $\dim H^u \geq 2$, we can find a one-parameter unipotent subgroup $U \subseteq H$ such that T_eU differs from l and such that $\pi_H(X) \cap U$ is finite. The first two conditions are arranged by choosing an $f\colon K \to U$ with

$$f(k) = e \qquad \text{for all } k \in \pi_K\left(\{x_1, \dots, x_n\} \cup \pi_H^{-1}(\pi_H(X) \cap U)\right). \tag{8.2}$$

Let $t_i = v_i \oplus w_i \in T_{x_i}X \subset T_{k_i}K \oplus T_eH$ be non-zero tangent vectors to X at x_i for all $1 \leq i \leq n$. We calculate $D_{x_i}(\pi_H \circ \psi_f)(t_i)$ for any f satisfying (8.2). In fact, by writing $T_{(k_i,e)}(K \times H) = T_{k_i}K \oplus T_eH$, we get that

$$D_{(k_i,e)}\psi_f = \begin{pmatrix} \mathrm{id} & 0 \\ D_{k_i}f & \mathrm{id} \end{pmatrix},$$

and thus

$$D_{x_i}(\pi_H \circ \psi_f)(t_i) = D_{k_i}f(v_i) + w_i\,.$$

Since $D_{k_i}f(v_i) \in T_eU$, $v_i \neq 0$, $0 \neq w_i \in l$, and $l \neq T_eU$, we see that we may choose f (by prescribing its derivative at k_i for all $1 \leq i \leq n$) such that

$$D_{x_i}(\pi_H \circ \psi_f)(t_i) \quad \text{and} \quad D_{x_j}(\pi_H \circ \psi_f)(t_j)$$

are linearly independent for all $1 \leq i < j \leq n$. This choice of f ensures the third condition.

We conclude the proof by observing that $\varepsilon\colon Y \to \pi_H(Y)$ is birational. Indeed, let $Z = \pi_H(Y)$ and let $\eta\colon \tilde{Z} \to Z$ be the normalization, which is birational. Note that Z is closed in H as Z has only one place at infinity. As Y is smooth, ε factorizes as $Y \to \tilde{Z} \to Z$. Since η factorizes through the blow-up of Z in e and the tangent directions of the branches of Z in e are all different, it follows that $\eta^{-1}(e)$ consists of n points, say $v_1, \dots, v_n$. Note that $Y \to \tilde{Z}$ is surjective since Y has only one place at infinity. Hence, after reordering the $v_1, \dots, v_n$, we can assume that $Y \to \tilde{Z}$ maps x_i to v_i for all i. Since $Y \to Z$ is immersive in x_i, it follows that $Y \to \tilde{Z}$ is étale in x_i for all i. Thus the fiber of $Y \to \tilde{Z}$ over v_i consists only of x_i and it is reduced for all i.

Since $Y \cong \mathbb{C}$, it follows that $\tilde{Z} \cong \mathbb{C}$ and therefore $Y \to \tilde{Z}$ is an isomorphism. This proves that ε is birational. □

Proof of Theorem 8.5. Note that if F is a connected reductive group, then F^u is semisimple or trivial. Indeed, by [Bor91, Proposition 14.2] the derived group $[F, F]$ is semisimple (or trivial) and in fact, $F^u = [F, F]$, since $[F, F]$ contains all root subgroups with respect to any maximal torus of F.

Let $G = R_u(G) \rtimes L$ be a Levi decomposition where L is a Levi factor (see [OV90, Theorem 4, Ch. 6]). By definition, L is connected and reductive and since $G = G^u$, we get $L = L^u$. Thus L is semisimple or trivial by the preceding paragraph. Now, we distinguish two cases:

(i) $G \neq R_u(G)$. Since G is not semisimple by assumption, the radical $R_u(G)$ is not trivial. Thus we can apply Proposition 8.6 to the non-trivial groups $K = R_u(G)$ and $H = L \cong G/R_u(G)$ and get the result.

(ii) $G = R_u(G)$. Thus G is isomorphic as a variety to $\mathbb{C}^n$ where n is a non-negative integer $\neq 3$. Clearly, we can assume that $n > 1$. If $n = 2$, then the result follows from the Abhyankar-Moh-Suzuki theorem [AM75, Theorem 1.2], [Suz74] and if $n \geq 4$, then the result follows from Jelonek's theorem [Jel87, Theorem 1.1]. □

9. Reduction to simple groups

The aim of this section is to reduce our problem to the case of a simple group.

Proposition 9.1. *Let G be a semisimple affine algebraic group that is not simple. Then, two embeddings of the affine line into G are the same up to an automorphism of G.*

For the proof we need three lemmata, which we also use later on.

Lemma 9.2. *Let G be a connected affine algebraic group and let K, H be closed connected subgroups such that KH is closed in G and $K \backslash G$ is quasi-affine. If $X \subseteq KH$ is a closed curve that is isomorphic to $\mathbb{C}$ and if the canonical projection $G \to K\backslash G$ restricts to an embedding on X, then there exists an automorphism ψ of G with $\psi(X) \subseteq H$.*

Proof of Lemma 9.2. Let $K \times^{K \cap H} H \to K/K \cap H$ be the bundle associated to the principal $K \cap H$-bundle $K \to K/K \cap H$ with fiber H; compare Appendix A. The natural morphism $K \times^{K\cap H} H \to KH$ is bijective and since KH is a smooth irreducible variety (note that KH is closed in G), it follows from Zariski's Main Theorem [Gro61, Corollaire 4.4.9] that $K \times^{K\cap H} H \to KH$ is an isomorphism. Thus, multiplication $m\colon K \times H \to KH$ is a principal $K \cap H$-bundle; see [Ser58, Proposition 4].

Since $\mathbb{C} \cong X \subseteq KH$, there exists a section $Y \subseteq K \times H$ over X by Theorem A.1. Denote by $\mathrm{pr}_K \colon K \times H \to K$ the canonical projection to K and by $\rho \colon G \to K\backslash G$ the quotient morphism. By assumption, $\rho \circ m|_Y \colon Y \to \rho(X)$ is an isomorphism. Since $\rho(X) \cong \mathbb{C}$ and since $K\backslash G$ is quasi-affine, we have a retraction $K\backslash G \to \rho(X)$. Thus the morphism

$$\rho(X) \to K\,, \quad v \mapsto \left(\mathrm{pr}_K \circ (\rho \circ m|_Y)^{-1}(v)\right)^{-1}$$

extends to a morphism $d \colon K\backslash G \to K$. Let ψ_d be the automorphism of G constructed in (5.2), Section 5. One can easily see that $\psi_d(X) \subseteq H$. $\square$

Lemma 9.3. *Let G be an affine algebraic group with $G = G^u$ and let K be a closed proper subgroup of G. Assume that $\mathcal{U}_G$ has dimension at least four. If $X \subseteq K$ is a closed curve that is isomorphic to $\mathbb{C}$, then there exists an automorphism φ of G such that $\varphi(X)$ is a one-parameter unipotent subgroup of G.*

Proof of Lemma 9.3. Note that the connected components of K/K^u are tori. Since X is the affine line, it lies in some fiber of $K \to K/K^u$. Hence, after multiplying from the left with a suitable element of K, we can assume that $X \subseteq K^u$. Since K^u does not contain all unipotent elements of G (otherwise $K = G$, since $G = G^u$), by Lemma 7.1 there exists a one-parameter unipotent subgroup $U \subseteq G$ such that $U \cap K^u = \{e\}$ and $\pi \colon G \to G/U$ induces an embedding on X.

Choose an isomorphism $\pi(X) \cong U$ and let $f \colon G/U \to U$ be an extension of it. The automorphism φ_f of G (see (5.1) in Section 5) leaves K^uU invariant. Since $U \cap K^u = \{e\}$, there is a canonical projection $K^uU \to U$. Since $X \subseteq K^u$, the composition

$$X \xrightarrow{\varphi_f} \varphi_f(X) \subseteq K^uU \longrightarrow U$$

is an isomorphism. In particular, we can assume that $X \subseteq K^uU$ and that $\rho \colon G \to K^u\backslash G$ induces an embedding on X. Note that $K^u\backslash G$ is quasi-affine since K^u has no non-trivial character; see Subsection 4.3. Now, we can apply Lemma 9.2 to the group G and the closed connected subgroups K^u and U to get an automorphism φ of G such that $\varphi(X) = U$. $\square$

Lemma 9.4. *Let K, H be non-trivial connected affine algebraic groups with $K = K^u$, $H = H^u$ and let $Z \subseteq K \times H$ be a finite central subgroup. Assume that $\dim \mathcal{U}_H \geq 4$. If $X \subseteq (K \times H)/Z$ is a closed curve that is isomorphic to $\mathbb{C}$, then there exists an automorphism φ of $(K \times H)/Z$ such that $\varphi(X)$ is a one-parameter unipotent subgroup of $(K \times H)/Z$.*

Proof of Lemma 9.4. Denote $G' = (K \times H)/Z$ and consider the following subgroups:

$$K' = (K \times \{e\})/((K \times \{e\}) \cap Z)\,,$$
$$H' = (\{e\} \times H)/((\{e\} \times H) \cap Z)\,.$$

We claim that the projection

$$p \colon G' \to K' \backslash G'$$

restricts to an embedding on X after applying a suitable automorphism of G'. This can be seen as follows.

Let $U \subseteq H$ be a one-parameter unipotent subgroup such that $\pi \colon G' \to G'/U$ restricts to an embedding on X; see Lemma 7.1 and Remark 7.3. Here we consider U as a subgroup of G' via the isomorphism $\rho|_U \colon U \to \rho(U)$ where $\rho \colon H \to G'$ denotes the composition of the natural inclusion $H \to K \times H$ with the natural projection $K \times H \to G'$. Let $Z_H \subset H$ be the image of Z under the natural projection $K \times H \to H$ and denote by $\mathrm{pr} \colon G' \to H/Z_H$ the morphism which is induced by $K \times H \to H$. Since Z is finite and central in $K \times H$, the same holds for Z_H in H. In particular, H/Z_H is an affine algebraic group. Since $U \cap Z_H = \{e\}$, the algebraic group U is mapped isomorphically onto its image via $\mathrm{pr} \colon G' \to H/Z_H$ and thus we can and will identify this image with U. We have a commutative diagram

$$\begin{tikzcd} G' \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"] & H/Z_H \arrow[d] \\ G'/U \arrow[r, "\overline{\mathrm{pr}}"] & (H/Z_H)/U \end{tikzcd}$$

where $\overline{\mathrm{pr}} \colon G'/U \to (H/Z_H)/U$ is defined by the commutativity. Note that both vertical arrows are principal U-bundles and that $\mathrm{pr} \colon G' \to H/Z_H$ is U-equivariant. Note that G'/U and $(H/Z_H)/U$ are quasi-affine; see Subsection 4.3. Since $\mathbb{C} \cong \pi(X)$, it follows that $\overline{\mathrm{pr}}(\pi(X))$ is either a point or a curve with one place at infinity. In particular, $\pi(X)$ is closed in G'/U and $\overline{\mathrm{pr}}(\pi(X))$ is closed in $(H/Z_H)/U$. Hence, the diagram above restricts to the following commutative diagram of affine varieties:

$$\begin{tikzcd} \pi^{-1}(\pi(X)) \arrow[r] \arrow[d] & \mathrm{pr}(\pi^{-1}(\pi(X))) \arrow[d] \\ \pi(X) \arrow[r] & \overline{\mathrm{pr}}(\pi(X)) \end{tikzcd}$$

where the vertical arrows are principal U-bundles. Since $\mathbb{C}^+$ is a special group in the sense of Serre [Ser58, §4], both principal bundles are locally trivial. Since the base varieties $\pi(X)$ and $\overline{\mathrm{pr}}(\pi(X))$ are affine, both principal bundles are trivial; see Remark A.4. Therefore, we can choose U-equivariant morphisms $f\colon \pi^{-1}(\pi(X)) \to U$ and $g\colon \mathrm{pr}(\pi^{-1}(\pi(X))) \to U$ such that the diagram

$$\begin{tikzcd} & U & \\ \pi^{-1}(\pi(X)) \arrow[u, "f"] \arrow[rr] & & \mathrm{pr}(\pi^{-1}(\pi(X))) \arrow[ull, "g"'] \end{tikzcd} \tag{9.1}$$

commutes, where U acts on itself by right multiplication. We find then a section $Y \subseteq \pi^{-1}(\pi(X))$ of π over $\pi(X)$ that is mapped isomorphically onto U via f (embed $Y \cong \mathbb{C}$ "diagonally" into $\pi^{-1}(\pi(X))$). By the commutativity of (9.1) it follows that $\mathrm{pr}\colon G' \to H/Z_H$ maps Y isomorphically onto its image. By Proposition 5.1 there exists an automorphism of G which moves X into Y along the fibers of π and thus we can assume that $\mathrm{pr}\colon G' \to H/Z_H$ restricts to an embedding on X. Since $\mathrm{pr}\colon G' \to H/Z_H$ factors through the projection $p\colon G' \to K'\backslash G'$, this proves the claim.

Note that K' is normal in G' and therefore $K'\backslash G'$ is affine; see Subsection 4.3. Since $p\colon G' \to K'\backslash G'$ restricts to an embedding on X, we can apply Lemma 9.2 to the affine algebraic group G' and the closed connected subgroups K', H' and hence assume that $X \subseteq H'$. Since $K = K^u$ and $H = H^u$ it follows that $G' = (G')^u$. Hence we can apply Lemma 9.3 to G' and the proper subgroup H' to get an automorphism φ of G' such that $\varphi(X)$ is a one-parameter unipotent subgroup of G'. ☐

Proof of Proposition 9.1. Since G is a semisimple affine algebraic group, there exist simple affine algebraic groups $G_1, \ldots, G_n$ and an epimorphism

$$G_1 \times \cdots \times G_n \to G$$

with finite kernel; see [Hum75, Theorem 27.5]. As $G_1 \times \cdots \times G_n$ is connected, this kernel is central. By assumption, $n \geq 2$. If the Lie type of G is equal to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, then G is isomorphic as a variety to one of the groups

$$\mathrm{SL}_2 \times \mathrm{SL}_2, \quad \mathrm{SL}_2 \times \mathrm{PSL}_2, \quad \text{or} \quad \mathrm{PSL}_2 \times \mathrm{PSL}_2\,.$$

Indeed, if we consider the quotients of $\mathrm{SL}_2 \times \mathrm{SL}_2$ by subgroups of the center

$$\mathrm{Z}(\mathrm{SL}_2 \times \mathrm{SL}_2) = \{(E,E), (E,-E), (-E,E), (-E,-E)\}\,,$$

we get

$$\frac{\mathrm{SL}_2 \times \mathrm{SL}_2}{\langle (E,E) \rangle} \cong \mathrm{SL}_2 \times \mathrm{SL}_2, \qquad \frac{\mathrm{SL}_2 \times \mathrm{SL}_2}{\mathrm{Z}(\mathrm{SL}_2 \times \mathrm{SL}_2)} \cong \mathrm{PSL}_2 \times \mathrm{PSL}_2\,,$$

and

$$\frac{\mathrm{SL}_2 \times \mathrm{SL}_2}{\langle(-E,-E)\rangle} \cong \frac{\mathrm{SL}_2 \times \mathrm{SL}_2}{\langle(E,-E)\rangle} \cong \frac{\mathrm{SL}_2 \times \mathrm{SL}_2}{\langle(-E,E)\rangle} \cong \mathrm{SL}_2 \times \mathrm{PSL}_2,$$

where the first isomorphism of the last line is induced by the automorphism

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SL}_2 \times \mathrm{SL}_2, \quad (A,B) \mapsto (AB,B)\,.$$

By Proposition 8.6 all embeddings of $\mathbb{C}$ into one of these groups are equivalent. Hence, we can assume that the Lie type of G is not equal to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Therefore one can find $J \subseteq I = \{1,\dots,n\}$ such that $\varnothing \neq J \neq I$ and such that the Lie-type of $H := \prod_{j\in J} G_j$ is not $\mathfrak{sl}_2$. If $|J| = 1$, then H is simple and we have $\dim \mathcal{U}_H \geq 4$ by Lemma B.6. If $|J| > 1$, then $\dim \mathcal{U}_H \geq 4$, since for any simple group the variety of unipotent elements has dimension ≥ 2. For $K = \prod_{i\in I\setminus J} G_i$ we have $G \cong (K \times H)/Z$ where Z is a central finite subgroup of $K \times H$. If $X \subseteq (K \times H)/Z$ is a closed curve that is isomorphic to $\mathbb{C}$, then we can apply Lemma 9.4 to K, H, and $Z \subseteq K \times H$ to find an automorphism that maps X into a one-parameter unipotent subgroup. Thus Proposition 6.1 implies the result. $\square$

Remark 9.5. Note that $\mathrm{SL}_2 \times \mathrm{SL}_2 /\langle(-E,-E)\rangle$ and $\mathrm{SL}_2 \times \mathrm{PSL}_2$ are not isomorphic as algebraic groups, since $(A,B) \mapsto (B,A)$ is a group automorphism of $\mathrm{SL}_2 \times \mathrm{SL}_2 /\langle(-E,-E)\rangle$ that is not inner; however, all group automorphisms of $\mathrm{SL}_2 \times \mathrm{PSL}_2$ are inner, since (by a calculation)

$$\mathrm{Aut}_{\mathrm{grp.}}(\mathrm{SL}_2 \times \mathrm{PSL}_2) \cong \mathrm{Aut}_{\mathrm{grp.}}(\mathrm{SL}_2) \times \mathrm{Aut}_{\mathrm{grp.}}(\mathrm{PSL}_2)$$

and since all group automorphisms of SL_2 and PSL_2 are inner; see [Hum75, Theorem 27.4].

10. Embeddings into simple groups

In this section, we prove the hardest part of Theorem 1.1:

Theorem 10.1. *Let G be a simple affine algebraic group of rank at least two. Then two embeddings of the affine line into G are the same up to an automorphism of G.*

We remark that Theorems 10.1 and 8.5 and Proposition 9.1 imply Theorem 1.1. We do this in detail:

Proof of Theorem 1.1. By Lemma 8.2, G is isomorphic to $G^u \times (\mathbb{C}^*)^n$ as a variety, where n is some non-negative integer. By Lemma 8.1, all embeddings of $\mathbb{C}$ into G are equivalent if and only if all embeddings of $\mathbb{C}$ into G^u are equivalent. Hence, it suffices to consider embeddings of $\mathbb{C}$ into G^u. If G^u is not semisimple and not isomorphic as a variety to $\mathbb{C}^3$, then all embeddings of $\mathbb{C}$ into G^u are equivalent by Theorem 8.5. If G^u is semisimple but not simple, then all embeddings of $\mathbb{C}$ into G^u are equivalent by Proposition 9.1. Finally, if G^u is simple and different from SL_2 and PSL_2, then G^u has rank at least two; thus all embeddings of $\mathbb{C}$ into G^u are equivalent by Theorem 10.1. $\square$

10.1. Outline of the proof of Theorem 10.1. In light of Proposition 6.1, it is enough to prove that any closed curve $X \subseteq G$ which is isomorphic to $\mathbb{C}$ can be moved into a one-parameter unipotent subgroup of G via an automorphism of G. In a first step we move our X into a naturally defined subvariety E (see Section 10.4) and in a second step we move it into a proper subgroup (see Section 10.5). By Lemma 9.3 we are then able to move X into a one-parameter unipotent subgroup of G, which then finishes the proof of Theorem 10.1.

The subvariety E is defined using classical theory of affine algebraic groups. The necessary notion is set up in the next subsection.

10.2. Notation and basic facts. Let us fix the following notation for the whole section. By G we denote a simple affine algebraic group, by $B \subseteq G$ a fixed Borel subgroup, and by $T \subseteq B$ a fixed maximal torus. Let Φ be the irreducible root system of G with respect to T. Moreover, we denote by W the Weyl group with respect to T and we denote by Δ the base of Φ with respect to B. Note that W is generated by the reflections associated to elements of Δ. We denote by w_0 the unique longest word in W with respect to Δ and by B^- the opposite Borel subgroup of B that contains T, i.e., $B^- = w_0 B w_0$; see Examples B.1 and B.3.

We fix a maximal parabolic subgroup P that contains B; i.e., we fix a simple root $\alpha \in \Delta$ such that $P = BW_I B$ where $I = \Delta \setminus \{\alpha\}$ and W_I denotes the subgroup in W generated by the reflections corresponding to the roots in I. We denote the reflection corresponding to α by s_α. Furthermore, we denote by P^- the unique opposite parabolic subgroup to P with respect to T; see Appendix B.2. Again by Appendix B.2, $P^- = B^- W_I B^-$, PP^- is open in G and $PP^- = R_u(P)P^- = PR_u(P^-)$.

The quotient of G by the unipotent radical of P^- will play a crucial role in the proof. We denote this quotient throughout this section by

$$\pi \colon G \to G/R_u(P^-)\,.$$

Since $R_u(P^-)$ is a special group in the sense of Serre [Ser58, §4], π is a locally trivial principal $R_u(P^-)$-bundle.

For the main step of the proof of Theorem 10.1 we use Schubert varieties in G/P. For their basic properties we refer the reader to [Spr09], [BK05, Ch. 2], and [BL03]. In Appendix B.4, we summarize the basic facts needed for this article. Since P is a maximal parabolic subgroup of G, there exists a unique Schubert curve C in G/P; i.e., C is the closure of the one-dimensional B-orbit through s_α in G/P. In fact, C is the disjoint union of the B-orbit through s_α and the class of the identity element e in G/P; see Corollary B.10. We denote by $E \subseteq G$ the inverse image of C under the natural projection $G \to G/P$. Thus E is the union of the two disjoint subsets $Bs_\alpha P$ and P in G.

10.3. The restriction of π to E. Recall that E denotes the inverse image of the unique Schubert curve C in G/P and $\pi\colon G \to G/R_u(P^-)$ denotes the canonical projection. The following result describes the restriction of π to E. It is the key ingredient that enables us to move our curve into E.

Proposition 10.2. *The complement of $\pi(E)$ in $G/R_u(P^-)$ is closed and has codimension at least two in $G/R_u(P^-)$. Moreover, the restriction of π to E turns E into a locally trivial $\mathbb{C}$-bundle over $\pi(E)$.*

Proof of Proposition 10.2. For the first statement it is enough to show that $\pi^{-1}(\pi(E)) = EP^-$ is open in G and that $G \setminus EP^-$ has codimension at least two in G. We have the following inclusion inside G:

$$BP^- \cup Bs_\alpha P^- \subseteq PP^- \cup Bs_\alpha PP^- = EP^- \, .$$

Since $BP^- = PP^-$ is open in G, it follows that EP^- is open in G. More precisely, $G \setminus BP^-$ is an irreducible closed hypersurface in G. This follows from the fact that $(G/P^-) \setminus Be$ is the translate by w_0 of the unique Schubert divisor in G/P^- with respect to B^-; see Corollary B.10. Since BP^- and $Bs_\alpha P^-$ are disjoint we have a proper inclusion $G \setminus EP^- \subsetneq G \setminus BP^-$. Thus $G \setminus EP^-$ has codimension at least two in G.

For proving the second statement, we first show that all fibers of $\pi|_E \colon E \to \pi(E)$ are reduced and isomorphic to $\mathbb{C}$. In fact, the schematic fiber over $\pi(g)$ is the schematic intersection $E \cap gR_u(P^-)$ for all $g \in E$. Since Schubert varieties are normal (see [RR85, Theorem 3]) and rational, it follows that $C \cong \mathbb{P}^1$. For each $g \in G$, consider the following commutative diagram:

$$\begin{array}{ccccc}
E \cap gR_u(P^-) & \longrightarrow & E & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
gR_u(P^-) & \longrightarrow & G & \longrightarrow & G/P
\end{array}$$

Note that all squares are pullback diagrams. Since $gR_u(P^-) \to G \to G/P$ is an open injective immersion, the same holds for $E \cap gR_u(P^-) \to E \to C$. Note that the image of $gR_u(P^-)$ inside G/P is equal to $gB^-e \subseteq G/P$. Since E is the inverse image of C under $G \to G/P$ we get an isomorphism

$$E \cap gR_u(P^-) \cong C \cap gB^-e \, .$$

Let $C^{\mathrm{op}} \subseteq G/P$ be the opposite Schubert variety to C; i.e., C^{op} is the closure of the B^--orbit through s_α inside G/P. By Corollary B.10 we have a disjoint union

$$C^{\mathrm{op}} \cup B^- e = G/P\,.$$

It follows from Lemma 10.3 below that for all $g \in G$ the subset $C \cap gC^{\mathrm{op}}$ consists of a single point or $C \subseteq gC^{\mathrm{op}}$. Hence

$$C \setminus (C \cap gC^{\mathrm{op}}) = C \cap gB^- e$$

is either isomorphic to $\mathbb{C}$ or it is empty. This proves that all fibers of $\pi|_E \colon E \to \pi(E)$ are reduced and isomorphic to $\mathbb{C}$.

Since C is smooth and since $G \to G/P$ is a smooth morphism, it follows that E is smooth; see [GR04, Ch. II, Proposition 3.1]. Moreover, $\pi(E)$ is smooth as an open subset of the smooth variety $G/R_u(P^-)$. Since all fibers of $\pi|_E$ have the same dimension, the morphism $\pi|_E$ is faithfully flat. Since π is affine as a locally trivial principal $R_u(P^-)$-bundle, the restriction $\pi|_E$ is also affine. It follows from [KW85] or [KR14, Theorem 5.2] that $\pi|_E$ is a locally trivial $\mathbb{C}$-bundle. ☐

Lemma 10.3. *Let $C^{\mathrm{op}} \subset G/P$ be the opposite Schubert variety to C, i.e., C^{op} is the closure of the B^--orbit through s_α. Then for all $g \in G$ either $gC \cap C^{\mathrm{op}}$ is a reduced point of G/P or $gC \subseteq C^{\mathrm{op}}$.*

Remark 10.4. Compare the proof of this lemma with [Har77, Ch. III, Proof of Theorem 10.8].

Proof of Lemma 10.3. Consider the following pullback diagram:

$$\begin{tikzcd}
(G \times C) \times_{G/P} C^{\mathrm{op}} \arrow[r] \arrow[d] & C^{\mathrm{op}} \arrow[d] \\
G \times C \arrow[r] & G/P
\end{tikzcd}$$

where $G \times C \to G/P$ denotes the map $(g, c) \mapsto gc$. Note that the vertical arrows are closed embeddings. Since $G \times C$ is smooth, by generic smoothness [Har77, Ch. III, Corollary 10.7] and G-equivariance, the morphism $G \times C \to G/P$ is smooth. Since C^{op} is reduced, it follows that the fiber product $(G \times C) \times_{G/P} C^{\mathrm{op}}$ is reduced [GR04, Ch. II, Proposition 3.1]. Let q be the following composition:

$$q \colon (G \times C) \times_{G/P} C^{\mathrm{op}} \to G \times C \to G,$$

where the last map is the projection onto G. Note that the fiber of q over $g \in G$ is isomorphic to the scheme theoretic intersection $gC \cap C^{\mathrm{op}}$. Since C is projective, the morphism q is projective and thus by [Eis95, Theorem 14.8] the subset

$$V = \{\, g \in G \mid gC \cap C^{\mathrm{op}} \text{ is finite} \,\}$$

is open in G. Let $q' = q|_{q^{-1}(V)} \colon q^{-1}(V) \to V$. By definition, q' is quasi-finite. Since q is projective (and thus q' also), it follows that q' is finite; see

[Gro66, Théorème 8.11.1]. We claim that q is birational. Indeed, this can be seen as follows. The fiber of q over $e \in G$ is isomorphic to $C \cap C^{\mathrm{op}}$. By [Ram85, Theorem 3 and Remark 3] this last scheme is reduced and by [Ric92, Theorem 3.7] it is irreducible and of dimension zero; cf. also [BL03]. Thus the fiber of q over e is a reduced point. Hence, the tangent space of the fiber satisfies

$$0 = T_{x_0} q^{-1}(e) = \ker d_{x_0} q,$$

where $\{x_0\} = q^{-1}(e)$. Therefore q is immersive at x_0. Hence $q^{-1}(V)$ is smooth at x_0 by dimension reasons and q' is étale in x_0. Let S be the set of points in $q^{-1}(V)$, where q' is not étale. By [GR04, Ch. I, Proposition 4.5] the set S is closed in $q^{-1}(V)$. As q' is finite, $q(S)$ is closed in V. Clearly, q' restricts to a finite étale morphism

$$q^{-1}(V \setminus q(S)) \longrightarrow V \setminus q(S)\,. \tag{10.1}$$

Since $\{x_0\}$ is a fiber of q and since q' is étale at x_0, it follows that $q(x_0) \notin q(S)$, i.e., $x_0 \in q^{-1}(V \setminus q(S))$. This implies that the morphism (10.1) is of degree one and therefore it is an isomorphism. Since V is irreducible and since q' is finite, it follows that $q^{-1}(V \setminus q(S))$ is dense in $q^{-1}(V)$. Since (10.1) is an isomorphism, $q^{-1}(V)$ is irreducible. This implies that q' is birational. Since V is smooth and irreducible and since q' is finite and birational, it follows that q' is an isomorphism by Zariski's Main Theorem [Gro61, Corollaire 4.4.9]). This implies the lemma. ☐

10.4. Moving a curve into E.

Proposition 10.5. *If $X \subseteq G$ is a closed curve that is isomorphic to $\mathbb{C}$, then there exists an automorphism φ of G such that $\varphi(X) \subseteq E$.*

Proof of Proposition 10.5. If $\operatorname{rank}(G) = 1$, then $E = G$ and there is nothing to prove. Thus we assume that $\operatorname{rank}(G) \geq 2$. Therefore, we can apply Proposition 7.4 to G and the unipotent subgroup $R_u(P^-)$ to get an automorphism φ of G such that $\pi\colon G \to G/R_u(P^-)$ restricts to an embedding on $\varphi(X)$. Let us replace X by $\varphi(X)$. Since the complement of $\pi(E)$ in $G/R_u(P^-)$ is closed and has codimension at least two in $G/R_u(P^-)$ by Proposition 10.2, there exists by Kleiman's Theorem $g \in G$ such that $g\pi(X)$ lies inside $\pi(E)$; see [Kle74, Theorem 2]. Since π is G-equivariant, it restricts to an isomorphism $gX \to \pi(gX)$. Hence, we can replace X by gX and assume in addition that $\pi(X) \subseteq \pi(E)$. Since π restricts to a locally trivial $\mathbb{C}$-bundle $\pi|_E\colon E \to \pi(E)$ by Proposition 10.2 and since $\pi(X) \cong \mathbb{C}$, there exists a section σ of $\pi|_E$ over $\pi(X)$; see, e.g., [BCW77].

Recall that $G/R_u(P^-)$ is quasi-affine since $R_u(P^-)$ is a closed unipotent subgroup; compare Subsection 4.3. Therefore, by Proposition 5.1 there exists an automorphism of G that moves X to the section $\sigma(\pi(X)) \subset E$ and fixes $\pi\colon G \to G/R_u(P^-)$. This implies the result. $\square$

10.5. Moving a curve in E into a proper subgroup. The aim of this section is to prove the following result.

Proposition 10.6. *Assume that* $\operatorname{rank} G \geq 2$. *If $X \subseteq E$ is a closed curve that is isomorphic to $\mathbb{C}$, then there exists an automorphism φ of G such that $\varphi(X)$ lies in a proper subgroup of G.*

Proposition 10.6 is based on the following rather technical result.

Proposition 10.7. *Assume that* $\operatorname{rank}(G) \geq 2$. *Let K be a closed connected reductive subgroup of G such that KP is closed in G. Assume that $K \cap P$ is connected and solvable and, moreover, that $R_u(K \cap P)$ has dimension one and lies in $R_u(P)$. If $X \subseteq KP$ is a closed curve that is isomorphic to $\mathbb{C}$, then there exists an automorphism φ of G such that $G \to K\backslash G$ restricts to an embedding on $\varphi(X)$ and $\varphi(KP) = KP$.*

Before proving Proposition 10.7, we show how it implies Proposition 10.6.

Proof of Proposition 10.6. Let $K = \mathrm{C}_G((\ker\alpha)^\circ)$ be the centralizer in G of the connected component of the identity element of the kernel of the root $\alpha\colon T \to \mathbb{C}^*$. By definition, T and the root subgroups $U_{\pm\alpha}$ lie inside K. By [Hum75, Theorem 22.3, Corollary 26.2B], the group K is connected, reductive, the semisimple rank is one, and the Lie algebra of K decomposes as $\mathfrak{t} \oplus \mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha}$, where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{u}_{\pm\alpha}$ is the Lie algebra of $U_{\pm\alpha}$. Since K is connected and not solvable, TU_α is connected and solvable, and TU_α is of codimension one in K, it follows that TU_α is a Borel subgroup of K. Since $TU_\alpha \subseteq K \cap P \subseteq K$, the subgroup $K \cap P$ is parabolic in K and in particular it is connected; see [Hum75, Corollary 23.1B]. We have $K \cap P \neq K$, since otherwise P would contain the root subgroup $U_{-\alpha}$ and thus we would have $P = G$; see [Hum75, Theorem 27.3]. Hence

$$K \cap P = TU_\alpha\,.$$

Moreover, we have by [Hum75, §30.2]

$$R_u(K \cap P) = U_\alpha \subseteq R_u(P)\,.$$

We claim that $U_\alpha s_\alpha P = Bs_\alpha P$ inside G. Indeed, otherwise $U_\alpha s_\alpha P = s_\alpha P$, since $\dim Bs_\alpha P = \dim E = 1 + \dim P$. Therefore $U_{-\alpha} = s_\alpha U_\alpha s_\alpha \subseteq P$, a contradiction. It follows that

$$E = U_\alpha s_\alpha P \cup P\,.$$

Since T and $U_{\pm\alpha}$ generate K, it follows that K lies inside the minimal parabolic subgroup $P_{\{\alpha\}} = Bs_\alpha B \cup B$. By [Bor91, Theorem 13.18] the reflection

s_α generates the Weyl group of K and, in particular, every representative of s_α lies in K. In summary, we get

$$E \subseteq KP \subseteq P_{\{\alpha\}}P = Bs_\alpha P \cup P = E\,,$$

which proves $E = KP$.

Now, we can apply Proposition 10.7 and thus we can assume that $G \to K\backslash G$ restricts to an embedding on X. Since $E = KP$, applying Lemma 9.2 to G and the closed connected subgroups K and P yield the desired result (note that $K\backslash G$ is affine since K is reductive). $\square$

The rest of this subsection is devoted to the proof of Proposition 10.7. First we provide an estimation of the dimension of the intersection of every translate of the torus T with the variety $\mathcal{U}_G$ of unipotent elements in case G is of rank two. Note that by the classification of simple groups of rank two, G is either of type A_2, B_2, or G_2.

Lemma 10.8. *Assume that* $\operatorname{rank}(G) = 2$. *Then the following hold:*

(i) *If G is of type A_2, then $Tp \cap \mathcal{U}_G$ is finite for all $p \in P$.*
(ii) *If G is of type B_2, then* $\dim(Tg \cap \mathcal{U}_G) \leq 1$ *for all $g \in G$.*

Remark 10.9. To complement (i), note that for some $g \in G$ the intersection $Tg \cap \mathcal{U}_G$ is not finite. For example, if $G = \mathrm{SL}_3$, T is the diagonal torus in G and

$$g = \begin{pmatrix} 3 & 0 & -4 \\ 2 & 0 & -3 \\ 0 & 1 & 0 \end{pmatrix}\,,$$

then a calculation shows that $Tg \cap \mathcal{U}_G$ is one-dimensional.

Proof of Lemma 10.8. To every simple group H there exists a simply connected simple group $\tilde{H}$ and an isogeny $\tilde{H} \to H$, i.e., an epimorphism with finite kernel; see [Che05, §23.1, Proposition 1]. Two simply connected simple groups with the same root system are always isomorphic by [Hum75, Theorem 32.1]. Therefore it is enough to prove (i) for the simply connected group $G = \mathrm{SL}_3$ and to prove (ii) for the simply connected group $G = \mathrm{Sp}_4$; see [Che05, §20.1, §22.1] and [Hum75, Corollary 21.3C].

Assume G is SL_3. We can assume that T is the subgroup of G of diagonal matrices and B is the subgroup of upper triangular matrices. Moreover, we can assume without loss of generality that P is the maximal parabolic subgroup

$$P = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \subseteq \mathrm{SL}_3\,.$$

An element $a \in \mathrm{SL}_3$ is unipotent if and only if its characteristic polynomial χ_a is $(t-1)^3$. We have

$$\chi_a(t) = t^3 - \mathrm{tr}(a)t^2 + s(a)t - 1,$$

where

$$s(a) = (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}a_{33} - a_{13}a_{31}) + (a_{22}a_{33} - a_{23}a_{32})$$

and a_{ij} denotes the ijth entry of a. Let $p \in P$. The variety $Tp \cap \mathcal{U}_G$ is isomorphic to

$$S = \{\, t \in T \mid tp \in \mathcal{U}_G \,\}.$$

Let x, y, z denote the entries on the diagonal of a 3×3-diagonal matrix. The set S can be realized as the closed subvariety of $\mathbb{C}^3$ given by the equations

$$3 = xp_{11} + yp_{22} + zp_{33}, \tag{10.2}$$

$$3 = xyp_{11}p_{22} + xzp_{11}p_{33} + yz(p_{22}p_{33} - p_{23}p_{32}), \tag{10.3}$$

$$1 = xyz\,. \tag{10.4}$$

Clearly, p_{11} is non-zero. Inserting (10.2) into (10.4) yields the irreducible equation

$$p_{11} = (3 - yp_{22} - zp_{33})yz\,. \tag{10.5}$$

Inserting (10.2) into (10.3) yields a non-trivial equation of degree ≤ 2 in y and z. If p_{22} or p_{33} is non-zero, then (10.5) is an equation of degree three, and thus S is finite. If $p_{22} = p_{33} = 0$, then S is realized as the closed subset of $\mathbb{C}^2$ given by the equations

$$3 = -yzp_{23}p_{32} \quad \text{and} \quad p_{11} = 3yz\,.$$

However, since p has determinant equal to 1, we get $-p_{11}p_{23}p_{32} = 1$. Hence, S is empty in case $p_{22} = p_{33} = 0$. This proves (i).

Assume that G is Sp_4. Since all non-degenerate alternating bilinear forms on an even dimensional vector space are equivalent, we can choose Ω as the matrix with entries $1, 1, -1, -1$ on the antidiagonal and all other entries equal to zero, and then define Sp_4 as those 4×4-matrices g that satisfy $g^t\Omega g = \Omega$. Thus we can choose for the maximal torus T the subgroup of Sp_4 consisting of diagonal matrices with entries t_1, t_2, t_2^{-1}, t_1^{-1} on the diagonal for arbitrary non-zero t_1 and t_2. If an element in GL_4 is unipotent, then its trace is equal to 4. Let $g \in \mathrm{Sp}_4$. One can see that

$$\{\, t \in T \mid \mathrm{tr}(tg) = 4 \,\}$$

is a proper closed subset of the torus T and thus $Tg \cap \mathcal{U}_{\mathrm{Sp}_4}$ is properly contained in Tg, which proves (ii). $\square$

Lemma 10.10. *Assume that* $\operatorname{rank}(G) \geq 2$. *Let* $H \subseteq P$ *be a connected closed solvable subgroup such that the unipotent radical* $R_u(H)$ *is one-dimensional. Denote by* $Y = H \backslash P$ *the quotient space which is homogeneous under* P *via right-multiplication. Then, for every* $y \in Y$ *the fibers of the morphism*

$$\mathcal{U}_P \to Y\,, \quad u \mapsto yu$$

have codimension at least three in $\mathcal{U}_P$.

Proof of Lemma 10.10. We have to prove that $\mathcal{U}_P \cap qHp$ has codimension at least three in $\mathcal{U}_P$ for all $p, q \in P$. Since $\mathcal{U}_P$ is invariant under conjugation, this amounts to proving that the codimension of $\mathcal{U}_P \cap Hp$ is at least three in $\mathcal{U}_P$ for all $p \in P$. By the same argument we can replace H with a conjugate $p_0 H p_0^{-1}$ for some $p_0 \in P$, and thus we can assume that $H \subseteq TR_u(H)$.

In case the rank of G is at least three or G is of type G_2, it follows that

$$\dim \mathcal{U}_P - \dim H \geq \dim \mathcal{U}_P - \dim T - 1 \geq 3$$

by Lemma B.6, and thus the lemma is proved in these cases.

Assume that G is of type A_2. For every $p \in P$ the quotient $\eta\colon P \to T\backslash P$ restricts to a morphism $Hp \cap \mathcal{U}_P \to \eta(R_u(H)p)$. By Lemma 10.8 the fibers of this restriction are finite. Since $R_u(H)$ is one-dimensional, it follows that $Hp \cap \mathcal{U}_P$ is at most one-dimensional. By Lemma B.6, we have $\dim \mathcal{U}_P = 4$, which implies the lemma in this case.

Assume that G is of type B_2. Analogously, it follows from Lemma 10.8 and Lemma B.6 that $Hp \cap \mathcal{U}_P$ is at most two-dimensional and that $\dim \mathcal{U}_P = 5$, which proves the lemma in this case. □

Proof of Proposition 10.7. We start by observing that $K \cap P\backslash P$ is affine, since $K \cap P\backslash P \cong K\backslash KP$ is closed in $K\backslash G$ and since $K\backslash G$ is affine (K is reductive). In particular, every $\mathbb{C}^+$-orbit of a $\mathbb{C}^+$-action on $K \cap P\backslash P$ is closed.

We claim that for a generic $u \in \mathcal{U}_P$ the one-parameter unipotent subgroup $\mathbb{C}^+(u)$ of P acts without fixed point on $K \cap P\backslash P$. Every $\mathbb{C}^+(u)$-orbit in $K \cap P\backslash P$ is either a fixed point or isomorphic to $\mathbb{C}$. If $p \in P$ would map to a fixed point in $K \cap P\backslash P$ of the $\mathbb{C}^+(u)$-action, then $(K \cap P)p\mathbb{C}^+(u) = (K \cap P)p$. This would imply that $p\mathbb{C}^+(u)p^{-1} \subseteq K \cap P$. Since $K \cap P$ is solvable, $p\mathbb{C}^+(u)p^{-1}$ lies inside $R_u(K \cap P)$ and hence inside $R_u(P)$, by assumption. In particular, $\mathbb{C}^+(u)$ lies inside $R_u(P)$. However, generic $u \in \mathcal{U}_P$ are not contained in $R_u(P)$, since P is not a Borel subgroup of G. This proves our claim.

Denote by $\eta\colon KP \to K\backslash KP \cong K \cap P\backslash P$ the restriction of the canonical projection $G \to K\backslash G$. By Lemma B.6 we have $\dim \mathcal{U}_P \geq 4$ and hence there exists a one-parameter unipotent subgroup U of P such that $G \to G/U$ restricts to an embedding on X, by Remark 7.3. Moreover, we can assume by the previous paragraph that U acts without fixed point on $K \cap P\backslash P$. Thus we can apply Lemma C.1 to the U-equivariant morphism $XU \to \overline{\eta(XU)}$ to get a section X' of $XU \to XU/U$ that is mapped birationally via η onto its image. Hence, after applying an appropriate automorphism of G (that leaves KP invariant), we can assume that η maps X birationally onto its image; see Proposition 5.1. Let us denote this image inside $\overline{\eta(XU)}$ by Z. Note that Z is closed in $\overline{\eta(XU)}$, since X is isomorphic to $\mathbb{C}$. We apply Lemma 7.7 to the group P, the affine homogeneous P-space $K \cap P\backslash P$ and the curve Z in $K \cap P\backslash P$ (the codimension assumptions of Lemma 7.7 are guaranteed by Lemma 10.10). Thus we get a $u' \in \mathcal{U}_P \setminus \{e\}$ such that $G \to G/\mathbb{C}^+(u')$ restricts to an embedding on X (by Remark 7.3), $\mathbb{C}^+(u')$ acts without fixed point on $K \cap P\backslash P$, and $S_{u'} \to S_{u'}/\!\!/\mathbb{C}^+(u')$ restricts to a birational morphism on Z. Here $S_{u'}$ denotes the closure of all the $\mathbb{C}^+(u')$-orbits in $K \cap P\backslash P$ that pass through Z. Since X is mapped birationally onto $Z \subseteq S_{u'}$ and since Z is mapped birationally onto $S_{u'}/\!\!/\mathbb{C}^+(u')$ it follows that η restricts to a birational map $X\mathbb{C}^+(u') \to S_{u'}$. Hence we can apply Lemma C.2 to the $\mathbb{C}^+(u')$-equivariant morphism $X\mathbb{C}^+(u') \to S_{u'}$ and get a section X'' of $X\mathbb{C}^+(u') \to X\mathbb{C}^+(u')/\mathbb{C}^+(u')$ that is mapped isomorphically via η onto its image inside $S_{u'} \subseteq K \cap P\backslash P$. By Proposition 5.1 there exists an automorphism of G (that leaves KP invariant) and maps X to X'', and thus we can assume that η maps X isomorphically onto $K \cap P\backslash P$. Since η is the restriction of $G \to K\backslash G$ to KP, this finishes the proof. □

Appendix A. Principal bundles over the affine line

In [RR84] it is stated by referring to [Ste65] and [Ram83] that over an algebraically closed field every principal G-bundle over the affine line is trivial if G is a connected affine algebraic group. However, the connectedness assumption is in fact superfluous over an algebraically closed field of characteristic zero. For the sake of completeness we give a proof of this result.

Theorem A.1. *Let G be any affine algebraic group. Then every principal G-bundle over the affine line $\mathbb{C}$ is trivial.*

Before starting with the proof, let us recall a very important construction that associates a fiber bundle $P \times^G F \to X$ to a principal G-bundle $\pi\colon P \to X$ and a variety F with a left G-action (see [Ser58, Proposition 4]): the variety $P \times^G F$ is defined as the quotient of $P \times F$ by the right G-action

$$(p, f) \cdot g = (pg, g^{-1}f),$$

and the canonical map $P \times^G F \to X$ is a bundle with fiber F which becomes locally trivial after a finite étale base change; see [Ser58, §3.2, Example (c)].

Proof of Theorem A.1. Let $P \to \mathbb{C}$ be a principal G-bundle. Let G^0 be the connected component of the identity element in G. The principal G-bundle factorizes as

$$P \longrightarrow P \times^G G/G^0 \longrightarrow \mathbb{C}\,.$$

The first morphism is a principal G^0-bundle by [Ser58, Proposition 8]. The second morphism is a principal G/G^0-bundle and since G/G^0 is finite, it is a finite morphism; see [Ser58, Proposition 5 and §3.2, Example (a)]. Since the base is $\mathbb{C}$, this second principal bundle admits a section $s\colon \mathbb{C} \to P \times^G G/G^0$ (which follows from Hurwitz's Theorem [Har77, Ch. IV, Corollary 2.4]). Due to Theorem A.2, the principal G^0-bundle $P \to P \times^G G/G^0$ is trivial over $s(\mathbb{C})$, and thus $P \to \mathbb{C}$ admits a section, which proves the theorem. $\square$

The main step in the following Theorem is due to Steinberg [Ste65].

Theorem A.2. *Let G be a connected affine algebraic group. Then, every principal G-bundle over a smooth affine rational curve is trivial.*

Proof. Let X be a smooth affine rational curve and let $E \to X$ be a principal G-bundle.

First we prove that $E \to X$ admits a section that is defined over some open subset of X. By definition there exists a finite étale map from an affine curve U' onto an open subset U of the curve X such that the pullback $E_{U'} \to U'$ is a trivial principal G-bundle. Let K be the function field of U and let K' be the function field of U'. We can assume that the field extension K'/K is finite and Galois, by [Ser58, §1.5]. Let $\mathrm{Gal}(K'/K)$ denote the Galois group of this extension. We denote by $G(K')$ the K'-rational points of G, i.e., the group of rational maps $U' \dashrightarrow G$. By [Ser58, §2.3b)] it follows that the first Galois cohomology set

$$H^1(\mathrm{Gal}(K'/K), G(K'))$$

describes the isomorphism classes of principal G-bundles that are defined over some non-specified open subset of U such that their pullback via $U' \to U$ admit a section over some open $\mathrm{Gal}(K'/K)$-invariant subset of U'. Hence it is enough to prove that $H^1(\mathrm{Gal}(K'/K), G(K'))$ is trivial. Let $\bar{K}$ be an algebraic closure of K that contains K'. By [Ser94, §5.8, Ch. I], the natural map

$$H^1(\mathrm{Gal}(K'/K), G(K')) \to H^1(\mathrm{Gal}(\bar{K}/K), G(\bar{K}))$$

is injective. Note that $G(\bar{K})$ is an affine algebraic group over $\bar{K}$. Since K' has transcendence degree one over the ground field, the so-called (cohomological) dimension of K' is at most one by [Ser94, Ch. II, §3.3, Example (b)]. Now, by a result of Steinberg, $H^1(\mathrm{Gal}(\bar{K}/K), G(\bar{K}))$ is trivial; see [Ste65, Theorem 1.9]. Hence, $E \to X$ admits a section over some open subset of X.

Let $B \subseteq G$ be a Borel subgroup. The principal G-bundle $E \to X$ decomposes as

$$E \to E \times^G G/B \to X,$$

where the first morphism is a principal B-bundle and the second morphism is a G/B-bundle, locally trivial in the étale topology. Since E becomes trivial over some open subset V of X, it follows that $E \times^G G/B$ becomes also trivial over V, hence there exists a rational section $s\colon X \dashrightarrow E \times^G G/B$ that is defined over V. Since G/B is projective, to every point x in X there is a finite étale map f_x onto an open neighborhood of x such that the pullback of $E \times^G G/B \to X$ via f_x is projective. This implies that $E \times^G G/B \to X$ is universally closed and hence proper. Since X is a smooth curve, it follows by the Valuative Criterion of Properness that the section s is defined on the whole X; see [Har77, Ch. II, Theorem 4.7]. Thus the restriction of the principal B-bundle $E \to E \times^G G/B$ to $s(X)$ is trivial by Proposition A.3, since X has a trivial Picard group. Hence, we proved that $E \to X$ admits a section, which implies the statement of the theorem. □

Proposition A.3. *Let G be a connected, solvable affine algebraic group. Then, every principal G-bundle over any affine variety with vanishing Picard group is trivial.*

Proof. Let X be an affine variety. By [Ser58, Proposition 14] every principal G-bundle is locally trivial, since G is connected and solvable. Note that the first Čech cohomology

$$\check{H}^1(X, \underline{\mathrm{G}})$$

is a pointed set that corresponds to the isomorphism classes of locally trivial principal G-bundles over X, where $\underline{\mathrm{G}}$ denotes the sheaf of groups on X with sections over an open subset $U \subseteq X$ being the morphisms $U \to G$; see [Fre57, §3] and [Ser58, §3]. Since G is solvable and connected, there exists a semidirect product decomposition $G = U \rtimes T$ for a torus T and a unipotent group U. The short exact sequence corresponding to this decomposition yields an exact sequence in cohomology

$$\check{H}^1(X, \underline{\mathrm{U}}) \to \check{H}^1(X, \underline{\mathrm{G}}) \to \check{H}^1(X, \underline{\mathrm{T}})\,;$$

see [Fre57, Théorème I.2]. However, by using a decreasing chain of closed normal subgroups of U such that each factor is isomorphic to $\mathbb{C}^+$ and by using that $\check{H}^1(X, \underline{\mathbb{C}^+}) = H^1(X, \mathcal{O}_X)$ is trivial (since X is affine) it follows that $\check{H}^1(X, \underline{\mathrm{U}})$ is trivial. Since the Picard group $\check{H}^1(X, \underline{\mathbb{C}^*}) = H^1(X, \mathcal{O}_X^*)$ vanishes it follows analogously that $\check{H}^1(X, \underline{\mathrm{T}})$ is trivial, whence $\check{H}^1(X, \underline{\mathrm{G}})$ is trivial. This implies the proposition. □

Remark A.4. The proof of Proposition A.3 shows the following. If G is unipotent, then every principal G-bundle over any affine variety is trivial.

Appendix B. Generalities on the Weyl group, parabolic subgroups, and Schubert varieties

Throughout this section we fix the following notation. Let G be a connected reductive group, let B be a Borel subgroup, let T be a maximal torus in B, and let W be the Weyl group with respect to T. Moreover, with respect to (B,T), we denote by Ψ the set of all roots, by Ψ^+ the set of all positive roots, by Ψ^- the set of all negative roots, and by Δ the set of all simple roots.

B.1. The Weyl group. For any root $\alpha \in \Psi$ there is an associated reflection in W which we will denote by s_α. In case α is a simple root, we call s_α a *simple reflection*. Recall that W is generated by the simple reflections; see [Spr09, Theorem 8.2.8(i)]. Associated to W and the simple reflections there is a *length function*

$$\ell\colon W \to \mathbb{N}_0\,, \quad w \mapsto \ell(w) = \min\left\{ k \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} \exists \alpha_1, \ldots, \alpha_k \in \Delta \text{ with} \\ s_{\alpha_1}\cdots s_{\alpha_k} = w \end{array} \right\}$$

where $\mathbb{N}_0$ denotes the set of non-negative integers. A decomposition $w = s_{\alpha_1}\cdots s_{\alpha_k}$ into simple reflections is called *reduced* if $\ell(w) = k$.

Example B.1. In W there is a unique longest element w_0 with respect to ℓ; it satisfies $w_0(\Psi^+) = \Psi^-$; see [Spr09, §8.3.4]. In particular, we have $(w_0w_0)(\Psi^+) = \Psi^+$ and thus we get $w_0w_0 = e$ in W; see [Spr09, Proposition 8.2.4].

Remark B.2. Note that the action of W on Ψ satisfies the following relation: If $\alpha \in \Psi$, $w \in W$, and U_α denotes the root subgroup in G corresponding to α, we have

$$wU_\alpha w^{-1} = U_{w(\alpha)};$$

see [Hum75, Theorem 26.3(b)].

Associated to W and the simple reflections, there is a natural order $\leq$, called the *Bruhat order* on W. It is defined as follows. Let $w \in W$ and let $w = s_1 \cdots s_n$ be a reduced decomposition into simple reflections and let $v \in W$. Then

$$v \leq w \quad \text{if and only if} \quad \begin{array}{l} v = s_{i_1}\cdots s_{i_k} \text{ where } k \geq 0 \text{ and} \\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n. \end{array}$$

In other words, $v \leq w$ if and only if v is obtained from w by deleting some of the simple reflections s_i. This order relation is independent of the decomposition of w into simple reflections; see [Spr09, Corollary 8.5.6]. In Section B.4, we recall a more geometric interpretation of the Bruhat order in terms of Schubert varieties associated to elements in W; see Remark B.9.

B.2. The opposite parabolic subgroup. Let P be a parabolic subgroup that contains B; i.e., $P = BW_IB$ where I is a subset of Δ and W_I is the subgroup of W generated by the reflections corresponding to roots in I. There exists a unique parabolic subgroup P^- that contains T such that $P \cap P^-$ is a Levi factor of P and P^-; i.e., there are semidirect product decompositions

$$P = R_u(P) \rtimes (P \cap P^-) \quad \text{and} \quad P^- = R_u(P^-) \rtimes (P \cap P^-)\,.$$

See [Spr09, Corollary 8.4.4.] and [Bor91, Proposition 14.21]. We call P^- the *opposite parabolic subgroup* of P with respect to T.

Example B.3. The opposite Borel subgroup B^- of B with respect to T is the subgroup of G generated by all root subgroups corresponding to negative roots together with T. We have $B^- = w_0 B w_0$, since the unique longest element $w_0 \in W$ exchanges the positive and negative roots; see Example B.1 and Remark B.2.

In fact we can describe P^- as follows.

Lemma B.4. *We have $P^- = B^- W_I B^-$.*

For the lack of reference, we provide a proof.

Proof. Let Z be the connected component of the identity element in the group $\bigcap_{\gamma \in I} \ker \gamma$. By [Hum75, §30.2], the centralizer $\mathrm{C}_G(Z)$ is a Levi factor of P, i.e., $P = \mathrm{C}_G(Z) \ltimes R_u(P)$. Let Q be $B^- W_I B^-$. In fact, $Q = B^- W_{-I} B^-$ since $W_I = W_{-I}$. Moreover, Z is the connected component of the identity element in $\bigcap_{\gamma \in -I} \ker \gamma$ and thus it follows that $Q = \mathrm{C}_G(Z) \ltimes R_u(Q)$. Clearly, $R_u(Q) \cap P$ is a unipotent subgroup of $R_u(Q)$ that is invariant under conjugation by T. If $R_u(Q) \cap P$ is non-trivial, it contains a root subgroup U_β, by [Hum75, Proposition 28.1]) for a certain root β. Note that β is a negative root with respect to Δ which is not a $\mathbb{Z}$-linear combination of roots in I, by [Hum75, §30.2] applied to (B^-, Q). Since β is also a root of P with respect to T, we get a contradiction to [Hum75, Proposition 30.1] applied to (B, P). Hence $R_u(Q) \cap P$ is trivial. This implies that $P \cap Q = \mathrm{C}_G(Z)$ and thus P and Q are opposite parabolic subgroups. Since P and Q contain T, we get $Q = P^-$. ☐

By [Bor91, Proposition 14.21] we have that PP^- is open in G and the product map induces an isomorphism of varieties

$$R_u(P) \times (P \cap P^-) \times R_u(P^-) \xrightarrow{\cong} PP^-\,. \tag{B.1}$$

In particular, we get the following.

Lemma B.5. *We have* $\dim G = \dim R_u(P^-) + \dim P$.

B.3. Dimension of $\mathcal{U}_P$ and $R_u(P)$ of a parabolic subgroup P. We give here a result which estimates the dimension of $\mathcal{U}_P$ and $R_u(P)$ from below for a parabolic subgroup P. The proof is based on the following fact. Let α be a simple root and let β be a positive root which is a linear combination of simple roots different from α. If α and β are not perpendicular, then $\alpha+\beta$ is a positive root, by [Hum78, Lemma 9.4 and Lemma 10.1].

Lemma B.6. *Assume that G is a simple group and let P be a parabolic subgroup that contains B. Then the following hold:*

(i) *If* $\operatorname{rank}(G) \geq 3$ *and* $P \neq B$, *then* $\dim \mathcal{U}_P \geq 2\operatorname{rank}(G) + 1$.

(ii) *If* $\operatorname{rank}(G) = 2$ *and* $B \neq P \neq G$, *then*

$$\dim \mathcal{U}_P = \begin{cases} 4 & \text{if } G \text{ is of type } A_2, \\ 5 & \text{if } G \text{ is of type } B_2, \\ 7 & \text{if } G \text{ is of type } G_2. \end{cases}$$

(iii) *If* $\operatorname{rank}(G) \geq 2$ *and* $P \neq G$, *then* $\dim R_u(P) \geq 2$.

In particular, for a simple group with $\operatorname{rank}(G) \geq 2$, *we have* $\dim \mathcal{U}_G \geq 4$.

Proof. Assume that $P \neq B$. Since $\dim \mathcal{U}_P = \dim P - \operatorname{rank}(G)$ we get

$$\dim \mathcal{U}_P = \dim R_u(B) + (\dim R_u(B) - \dim R_u(P)).$$

Note that $\dim R_u(B)$ is equal to the number of positive roots. In a Dynkin diagram the vertices correspond to the simple roots and there is one (or more) edges between two simple roots if and only if they are not perpendicular. For each pair of non-perpendicular simple roots α, β, the sum $\alpha+\beta$ is again a (positive) root. Since any Dynkin diagram is a tree, the simple roots together with the above sums of pairs give $2\operatorname{rank}(G) - 1$ positive roots.

Assume that $\operatorname{rank}(G) \geq 3$ and $P \neq B$. Again, since any Dynkin diagram is a tree, one sees that there is a subgraph of the Dynkin diagram of G of the form

$$\alpha_1 \text{ ——— } \alpha_2 \text{ ——— } \alpha_3$$

and α_1, α_3 are not connected in the Dynkin diagram. Hence $\alpha_1+\alpha_2$ and α_3 are not perpendicular and thus the sum $\alpha_1+\alpha_2+\alpha_3$ is again a positive root. Thus we proved $\dim R_u(B) \geq 2\operatorname{rank}(G)$. Since P is not a Borel subgroup, we get $\dim R_u(B) - \dim R_u(P) \geq 1$. These two inequalities yield (i).

Assume that $\operatorname{rank}(G) = 2$ and $B \neq P \neq G$. Hence, we get $\dim R_u(B) - \dim R_u(P) = 1$, by [Hum75, §30.2]. Considering the classification of irreducible root systems of rank two and counting the number of positive roots in these root systems yield (ii).

Assume that $\mathrm{rank}(G) \geq 2$ and $P \neq G$. Hence, there exists a simple root α such that $-\alpha$ is not a root of P. Since $\mathrm{rank}(G) \geq 2$ and since the root system is irreducible, there exists a simple root $\beta \neq \alpha$ such that $\alpha+\beta$ is a positive root. By [Hum75, §30.2] it follows that α and $\alpha+\beta$ are distinct roots of $R_u(P)$, which proves (iii).

The "in particular"' follows from (i) and (ii) by choosing a parabolic subgroup P different from G and B and noting that $\mathcal{U}_G \supseteq \mathcal{U}_P$. $\square$

B.4. Basics from the theory of Schubert varieties. In this subsection, we recall the basics from the theory of Schubert varieties needed for this article. Our references are [BGG82, BL03, BK05, Spr09]. We use the notation of Subsection B.1.

Let P be a parabolic subgroup of G that contains B and let $I \subseteq \Delta$ such that $P = BW_IB$ where W_I is the subgroup of W generated by the simple reflections corresponding to elements from I.

Fix a $w \in W$. The *Schubert cell $X_P(w)$ associated to w* is defined as the B-orbit through the class of w in G/P; i.e., it is the image in G/P of BwP under the canonical projection $G \to G/P$. The *Schubert variety $S_P(w)$ associated to w* is defined as the closure of $X_P(w)$ in G/P. The *opposite Schubert cell $X_P^{\mathrm{op}}(w)$ associated to w* is $w_0X_P(w_0w)$ and the *opposite Schubert variety $S_P^{\mathrm{op}}(w)$ associated to w* is the subvariety $w_0S_P(w_0w)$ in G/P. Thus $X_P^{\mathrm{op}}(w)$ is the B^--orbit through the class of w in G/P and $S_P^{\mathrm{op}}(w)$ is its closure in G/P; see Example B.3.

Proposition B.7 (See [BGG82] and [BL03])**.** *Let $w_{0,I}$ be the longest element in W_I. Then the following statements hold:*

(i) *For every $w \in W$, the coset wW_I contains a unique element of minimal length. We denote by $W^I \subseteq W$ the subset of all such minimal representatives of cosets with respect to W_I.*

(ii) *We have a disjoint union*

$$G/P = S_P(w_0) = \bigcup_{w \in W^I} X_P(w)\,.$$

Moreover, for $w, w' \in W$ we have $X_P(w) = X_P(w')$ if and only if $w^{-1}w' \in W_I$.

(iii) *For every $w \in W^I$, the Schubert cell $X_P(w)$ is isomorphic to an affine space of dimension $\ell(w)$ and thus $S_P(w)$ is irreducible, rational, and of dimension $\ell(w)$. More precisely,*

$$S_P(w) = \bigcup_{v \in W^I,\, v \leq w} X_P(v),$$

where the union is disjoint.

(iv) *For every $w \in W^I$ we get $w_0ww_{0,I} \in W^I$. In particular,*

$$W^I \to W^I\,, \quad w \mapsto w_0ww_{0,I}$$

defines an involution on W^I. This involution reverses the Bruhat order $\leq$ on W^I. Moreover, for $w \in W^I$, the Schubert variety $S_P(w_0ww_{0,I})$ has codimension $\ell(w)$ in G/P.

Remark B.8. It follows from Proposition B.7(ii) that for all $w \in W$ we have $S_P(ww_{0,I}) = S_P(w)$.

Remark B.9. Proposition B.7(iii) implies the following geometric interpretation of the Bruhat order on W^I. For $v, w \in W^I$, one has $v \leq w$ if and only if $S_P(v) \subseteq S_P(w)$. In particular, if $P = B$ (i.e., $W^I = W$), one has for all $v, w \in W$ the relation $v \leq w$ if and only if $S_B(v) \subseteq S_B(w)$.

In the next corollary we show that there is a unique Schubert curve and a unique Schubert divisor in G/P provided that P is a maximal parabolic subgroup of G.

Corollary B.10. *Assume that $I = \Delta \setminus \{\alpha\}$ where α is a fixed simple root. Then the following hold:*

(a) *The variety $S_P(s_\alpha)$ is the unique Schubert variety of dimension one in G/P. Moreover, $S_P(s_\alpha)$ is the disjoint union of the two Schubert cells $X_P(s_\alpha)$ and $X_P(e)$.*
(b) *The variety $S_P(w_0s_\alpha)$ is the unique Schubert variety of codimension one in G/P. Moreover, G/P is the disjoint union of $S_P(w_0s_\alpha)$ and $X_P(w_0)$.*

Proof. We denote by $W^I \subseteq W$ the set of minimal representatives of W^I-cosets; see Proposition B.7(i).

(a) It follows from Proposition B.7(iii) that $S_P(s_\alpha)$ has dimension one and if $S_P(w)$ has dimension one for some $w \in W^I$, then w is a simple reflection. Since W_I contains all simple reflections except s_α, we get $w = s_\alpha$. This proves the first statement from (a). The second statement follows from Proposition B.7(iii).

(b) Analogously as in (a), it follows from Proposition B.7(iv) that $S_P(w_0s_\alpha)$ is the unique Schubert variety of codimension one in G/P. For the second statement of (b) it is enough to show that for all $w \in W^I$ we have

$$\operatorname{codim}_{G/P} S_P(w) \geq 1 \quad \Longrightarrow \quad S_P(w) \subseteq S_P(w_0s_\alpha)\,.$$

Let $w_{0,I}$ be the unique longest element in W_I. By Proposition B.7(iv) it follows that $\dim S_P(w_0ww_{0,I}) \geq 1$. Note that s_α occurs in any decomposition of $w_0ww_{0,I}$ into simple reflections, since otherwise $w_0ww_{0,I} \in W_I$ and thus $\dim S_P(w_0ww_{0,I}) = 0$ by Proposition B.7(ii). In particular, $s_\alpha \leq w_0ww_{0,I}$ in

W^I and thus $w \leq w_0 s_\alpha w_{0,I}$ by Proposition B.7(iv). Hence we get $S_P(w) \subseteq S_P(w_0 s_\alpha)$ by Proposition B.7(iii). □

Appendix C. Two results on $\mathbb{C}^+$-equivariant morphisms of surfaces

In this section we prove two results on $\mathbb{C}^+$-equivariant morphisms of surfaces that we use in the proof of Proposition 10.7. If S is an affine variety with a $\mathbb{C}^+$-action, then we denote by $S/\!/\mathbb{C}^+$ the spectrum of the ring of $\mathbb{C}^+$-invariant functions on S. In general $S/\!/\mathbb{C}^+$ is an affine scheme which is not a variety. If the quotient morphism $S \to S/\!/\mathbb{C}^+$ happens to be a principal $\mathbb{C}^+$-bundle, then we denote the algebraic quotient by $S/\mathbb{C}^+$. By Rentschler's Theorem, for a fixed point free action of $\mathbb{C}^+$ on the affine plane $\mathbb{C}^2$, the algebraic quotient of $\mathbb{C}^+$ is a trivial principal $\mathbb{C}^+$-bundle over the affine line $\mathbb{C} \cong \mathbb{C}^2/\mathbb{C}^+$; see [Ren68].

Lemma C.1. *Let S be an irreducible, quasi-affine surface and assume that $\mathbb{C}^+$ acts without fixed point on $\mathbb{C}^2$ and on S. If $f\colon \mathbb{C}^2 \to S$ is a dominant and $\mathbb{C}^+$-equivariant morphism, then there exists a section $X \subseteq \mathbb{C}^2$ of the algebraic quotient $\mathbb{C}^2 \to \mathbb{C}^2/\mathbb{C}^+$ such that f induces a birational morphism $X \to f(X)$.*

Proof. By [FM78, Lemma 1], there exists a $\mathbb{C}^+$-invariant open subset $V \subseteq S$ and a smooth affine curve C such that V and $C \times \mathbb{C}^+$ are $\mathbb{C}^+$-equivariantly isomorphic. Hence, f restricts on $f^{-1}(V)$ to a morphism of the form

$$(f^{-1}(V)/\mathbb{C}^+) \times \mathbb{C}^+ \longrightarrow C \times \mathbb{C}^+\,, \quad (x,t) \longmapsto (\bar{f}(x), t + q(x))$$

where q is a function defined on the curve $f^{-1}(V)/\mathbb{C}^+$ and $\bar{f}$ is the morphism $f^{-1}(V)/\mathbb{C}^+ \to C$ induced by f. Therefore, it suffices to find a function p on $\mathbb{C}^2/\mathbb{C}^+ \cong \mathbb{C}$ (which corresponds to a section of $\mathbb{C}^2 \to \mathbb{C}^2/\mathbb{C}^+$) such that the morphism

$$f^{-1}(V)/\mathbb{C}^+ \longrightarrow C \times \mathbb{C}^+\,, \quad x \longmapsto (\bar{f}(x), p(x) + q(x)) \tag{C.1}$$

is birational onto its image. After shrinking V, we can assume that $\bar{f}$ is finite and étale. Fix $c_0 \in C$. One can choose p such that the points

$$(c_0, p(x_1) + q(x_1))\,, \ldots, (c_0, p(x_k) + q(x_k))$$

are all distinct, where $x_1, \ldots, x_k$ denote the elements of the fiber of $\bar{f}$ over c_0. The same is still true for elements in a neighborhood of c_0 in C, as one can see by choosing an étale neighborhood of c_0 in C which trivializes $\bar{f}$ at c_0 with respect to the étale topology; see [Mil80, Ch. I, Corollary 3.12]. Hence (C.1) is injective on an open subset of $f^{-1}(V)/\mathbb{C}^+$; i.e., it is birational onto its image. □

Lemma C.2. *Let S be an irreducible, quasi-affine surface and assume that $\mathbb{C}^+$ acts without fixed point on $\mathbb{C}^2$ and on S. If $f\colon \mathbb{C}^2 \to S$ is a $\mathbb{C}^+$-equivariant birational morphism, then there exists a section $X \subseteq \mathbb{C}^2$ of $\mathbb{C}^2 \to \mathbb{C}^2/\mathbb{C}^+$ such that f restricts to an embedding on X.*

Proof. We identify $\mathbb{C}^2$ with $\mathbb{C} \times \mathbb{C}^+$ and consider it as a trivial principal $\mathbb{C}^+$-bundle over $\mathbb{C}$. For $\alpha \in \mathbb{C}^*$ let

$$Z_\alpha = \{\, (x, \alpha x) \mid x \in \mathbb{C} \,\} \subseteq \mathbb{C} \times \mathbb{C}^+ .$$

We claim that for generic $\alpha \in \mathbb{C}^*$ the map f restricts to an embedding on Z_α. In other words, we claim that f restricted to Z_α is injective and immersive for generic α (the properness is then automatically satisfied, since $Z_\alpha \cong \mathbb{C}$). The claim then implies the statement of the lemma.

Let us first prove injectivity. Since f is $\mathbb{C}^+$-equivariant and birational, there exists a $\mathbb{C}^+$-invariant open subset of $\mathbb{C} \times \mathbb{C}^+$ that is mapped isomorphically onto a $\mathbb{C}^+$-invariant open subset of S. Since $\mathbb{C}^+$ acts without fixed point, it follows that there are only finitely many $\mathbb{C}^+$-orbits F in S such that the inverse image $f^{-1}(F)$ consists of more than one $\mathbb{C}^+$-orbit. Thus, it is enough to show that f is injective on $f^{-1}(F) \cap Z_\alpha$ for fixed F and generic α in $\mathbb{C}^*$. So let $F \subseteq S$ be a $\mathbb{C}^+$-orbit such that there exist $k > 1$ and distinct $x_1, \ldots, x_k \in \mathbb{C}$ such that $f^{-1}(F)$ is the union of the lines $L_i = \{x_i\} \times \mathbb{C}^+$, $i = 1, \ldots, k$. Moreover, there exist $\beta_i \in \mathbb{C}^+$ such that $f|_{L_i}\colon L_i \to F$ is given by $t \mapsto t + \beta_i$, where we have identified the orbit F with $\mathbb{C}^+$. Injectivity of f on $f^{-1}(F) \cap Z_\alpha$ for generic α follows, since for generic α we have

$$\alpha x_i + \beta_i \neq \alpha x_j + \beta_j \quad \text{for all } i \neq j .$$

Let us prove immersivity. As already mentioned, there exists an open $\mathbb{C}^+$-invariant subset $U \subseteq \mathbb{C} \times \mathbb{C}^+$ such that f restricts to an open injective immersion on U. Let $x_0 \in \mathbb{C}$ such that $\{x_0\} \times \mathbb{C}^+$ lies in the complement of U in $\mathbb{C} \times \mathbb{C}^+$. Since there are only finitely many such $x_0 \in \mathbb{C}$, it is enough to show that for generic $\alpha \in \mathbb{C}^*$ the restriction $f|_{Z_\alpha}$ is immersive in the point $(x_0, \alpha x_0)$. Since $\mathbb{C}^+$ acts without fixed point on S and since f is $\mathbb{C}^+$-equivariant, the kernel of the differential of f is at most one-dimensional in every point of $\mathbb{C} \times \mathbb{C}^+$. Since the tangent direction of Z_α in the point $(x_0, \alpha x_0)$ is given by $(1, \alpha)$, we get that $f|_{Z_\alpha}$ is immersive in $(x_0, \alpha x_0)$ for generic α. This proves the immersivity. $\square$

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Department of Mathematics, ETH Zürich, Rämistr. 101, 8092 Zürich, Switzerland

Email address: peter.feller@math.ch

Fachbereich Mathematik, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany

Email address: immanuel.van.santen@math.ch

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EMBEDDINGS OF AFFINE SPACES INTO QUADRICS

JÉRÉMY BLANC AND IMMANUEL VAN SANTEN

ABSTRACT. This article provides, over any field, infinitely many algebraic embeddings of the affine spaces $\mathbb{A}^1$ and $\mathbb{A}^2$ into smooth quadrics of dimension two and three, respectively, which are pairwise non-equivalent under automorphisms of the smooth quadric. Our main tools are the study of the birational morphism $\mathrm{SL}_2 \to \mathbb{A}^3$ and the fibration $\mathrm{SL}_2 \to \mathbb{A}^3 \to \mathbb{A}^1$ obtained by projections, as well as degenerations of variables of polynomial rings, and families of $\mathbb{A}^1$-fibrations.

CONTENTS

1. INTRODUCTION

In what follows we denote by $\mathbf{k}$ the ground field of our algebraic varieties. Given two affine algebraic varieties X, Y, we say that two closed embeddings $\rho, \rho' \colon X \hookrightarrow Y$ are *equivalent* if there exists an automorphism $\varphi \in \mathrm{Aut}(Y)$ such that $\rho' = \varphi \circ \rho$. Similarly, we say that two closed subvarieties $X, X' \subset Y$ are *equivalent* if there exists an automorphism $\varphi \in \mathrm{Aut}(Y)$ such that $X' = \varphi(X)$. If two closed embeddings are equivalent, then their images are equivalent, but the converse is not always true and is related to the extension of automorphisms.

In the 1970s, Abhyankar and Sathaye conjectured that every closed embedding $\mathbb{A}^{n-1}_{\mathbf{k}} \hookrightarrow \mathbb{A}^n_{\mathbf{k}}$ is equivalent to a linear embedding (see for instance [vdE00, §3, p. 103]). This was the starting point for studying embedding problems in affine algebraic geometry. In the Bourbaki Seminar *Challenging problems on affine n-space* [Kra96], Kraft gives a list of eight fundamental problems related to the affine n-spaces. The third one is the following generalisation of the Abhyankar-Sathaye conjecture.

Embedding Problem. *Is every closed embedding $\mathbb{A}^m_{\mathbf{k}} \hookrightarrow \mathbb{A}^n_{\mathbf{k}}$ equivalent to the standard embedding $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$?*

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This question, asked over the ground field $\mathbf{k} = \mathbb{C}$ in [Kra96], has until now no negative answer. For $\mathbf{k} = \mathbb{R}$, it is easy to find counterexamples for $m = 1$ and $n = 3$, by taking embeddings which are not topologically trivial (non-trivial knots); see for instance the example of [Sha92], reproduced below in Example 6.1. In positive characteristic, there are counterexamples when $m = n - 1$ (see Proposition 3.17). The embedding problem has however a positive answer in the following cases:

(1) $m = 1$, $n = 2$, $\mathrm{char}(\mathbf{k}) = 0$ (Abhyankar-Moh-Suzuki Theorem) [AM75, Suz74], [vdE00, Theorem 2.3.5];
(2) $n \geq 2m + 2$, $\mathbf{k}$ infinite (Theorem of Kaliman, Nori and Srinivas [Kal91, Sri91]).

The case of hypersurfaces ($m = n-1$), corresponding to the Abyhankar-Sathaye conjecture, is of particular interest. In this case, the image is given by the zero set of an irreducible polynomial equation $f \in \mathbf{k}[\mathbb{A}^n]$. One necessary condition for an embedding to be equivalent to the standard embedding consists of asking that the other fibres of $f\colon \mathbb{A}^n_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ are affine spaces. In fact, for any field $\mathbf{k}$ and any $n \geq 1$, there is no known example of a hypersurface $X \subset \mathbb{A}^n_{\mathbf{k}}$ isomorphic to $\mathbb{A}^{n-1}_{\mathbf{k}}$ and given by $f = 0$, $f \in \mathbf{k}[\mathbb{A}^n]$ irreducible such that another fibre $f = \lambda$ is not isomorphic to $\mathbb{A}^{n-1}_{\mathbf{k}}$. The non-existence of such examples (at least when $\mathrm{char}(\mathbf{k}) = 0$), is the Sathaye conjecture [vdE00, §3, p. 103], which is implied by the Abhyankar-Sathaye conjecture. Even this weaker conjecture is quite strong and seems "unlikely" (as Arno van den Essen says in [vdE00, §3, p. 103]). Moreover, for $n = 3$ and $\mathrm{char}(\mathbf{k}) = 0$, the fact that infinitely many fibres $f = \lambda$ are isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ implies that the fibration is equivalent to the standard one, and in particular that all fibres are isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ [KZ01, Kal02, DK09]. The positive characteristic version of the Sathaye conjecture has until now no counterexample, and is open even in dimension $n = 2$ (this latter case corresponds to a question of Abhyankar; see [Gan11, Question 1.1]).

In this paper, we replace the affine space at the target by some analogue varieties, namely affine smooth quadrics. This simplifies the question in such a way that one can actually give an answer. Moreover, it also gives some idea on what kind of behaviour one could expect in a general situation.

In dimension $n = 2$, the most natural quadric is

$$Q_2 = \mathrm{Spec}(\mathbf{k}[x, y, z]/(xy - z(z + 1))) \subset \mathbb{A}^3_{\mathbf{k}}.$$

In fact, if $\mathbf{k}$ is an algebraically closed field, then every smooth quadric hypersurface $Q \subset \mathbb{A}^3_{\mathbf{k}}$ is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$, $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}) \times \mathbb{A}^1_{\mathbf{k}}$ or Q_2, as one can see using the classification of quadratic forms. As all embeddings of $\mathbb{A}^1_{\mathbf{k}}$ into $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}) \times \mathbb{A}^1_{\mathbf{k}}$ are constant on the first factor, they are all equivalent. Over any field, the group of automorphisms of Q_2 is similar to the one of $\mathbb{A}^2_{\mathbf{k}}$, as it is an amalgamated product of two factors, corresponding to affine maps and triangular maps (see [DG77] or [BD11, Theorem 5.4.5(7)(a)]). This is also the case for the affine surface $\mathbb{P}^2_{\mathbf{k}} \setminus \Gamma$, where $\Gamma \subset \mathbb{P}^2_{\mathbf{k}}$ is any smooth conic having a $\mathbf{k}$-point (see for instance [DD16, Theorem 2]). If $\mathrm{char}(\mathbf{k}) = 0$, there is exactly one (respectively, two) closed curve $C \subset \mathbb{A}^2_{\mathbf{k}}$ (respectively, $C \subset \mathbb{P}^2_{\mathbf{k}} \setminus \Gamma$) isomorphic to $\mathbb{A}^1_{\mathbf{k}}$, up to automorphism of the surface. This follows from the Abhyankar-Moh-Suzuki Theorem for $\mathbb{A}^2_{\mathbf{k}}$ and from [DD16] for $\mathbb{P}^2_{\mathbf{k}} \setminus \Gamma$. In particular, all automorphisms of the corresponding curves extend to automorphisms of $\mathbb{A}^2_{\mathbf{k}}$ or $\mathbb{P}^2_{\mathbf{k}} \setminus \Gamma$. Similarly, a complex toric affine surface admits only finitely many embeddings of $\mathbb{A}^1_{\mathbb{C}}$, up to equivalence [AZ13]. By contrast, we prove the following result.

Theorem 1. *For each field $\boldsymbol{k}$, there is an infinite set of closed curves*

$$C_i \subset Q_2 = \operatorname{Spec}(\boldsymbol{k}[x,y,z]/(xy - z(z+1))),\ i \in I,$$

which are pairwise non-equivalent up to automorphism of Q_2, such that each C_i is isomorphic to $\mathbb{A}^1_{\boldsymbol{k}}$ and such that the identity is the only automorphism of C_i that extends to an automorphism of Q_2. Moreover, if $\boldsymbol{k}$ is uncountable, then I can be chosen uncountable as well.

In dimension $n = 3$, the most natural quadric is

$$\mathrm{SL}_2 = \operatorname{Spec}(\mathbf{k}[t,u,x,y]/(xy - tu - 1)) \subset \mathbb{A}^4_{\mathbf{k}}\,.$$

Similarly as in dimension two, over an algebraically closed field $\mathbf{k}$, every quadric hypersurface in $\mathbb{A}^4_{\mathbf{k}}$ is isomorphic to $\mathbb{A}^3_{\mathbf{k}}$, $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}) \times \mathbb{A}^2_{\mathbf{k}}$, $Q_2 \times \mathbb{A}^1_{\mathbf{k}}$ or SL_2. Moreover, the quotient of SL_2 by its maximal torus yields a morphism $\mathrm{SL}_2 \to \mathrm{SL}_2 / T \simeq Q_2$, which is the “universal torsor” (also called the Cox quotient presentation or the characteristic space); see [ADHL15, Examples 4.5.13–4.5.14].

We consider the quadric hypersurface SL_2 more closely. Its automorphism group shares similar properties with the one of $\operatorname{Aut}(\mathbb{A}^3_{\mathbf{k}})$ (see [LV13, BFL14, Mar15]). Both are known to be complicated, as they contain “wild” automorphisms [LV13], and do not preserve any fibration, as is the case for other varieties being topologically closer to $\mathbb{A}^3_{\mathbf{k}}$, like the Koras-Russell threefold. However, in contrast to the quadric Q_2, the quadric SL_2 is closer to a contractible affine variety in the sense that the ring of regular functions on SL_2 is a unique factorisation domain (see [Pop74, Proposition 1] or Lemma 4.4). The first difference concerning embeddings of affine spaces with the surfaces $Q_2, \mathbb{A}^2_{\mathbf{k}}, \mathbb{P}^2_{\mathbf{k}} \setminus \Gamma$ and with $\mathbb{A}^3_{\mathbf{k}}$ is that the “simplest embedding” $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ is more rigid in the following sense.

Theorem 2. *Let $\boldsymbol{k}$ be any field and let*

$$\begin{array}{cccc} \rho_1\colon & \mathbb{A}^2_{\boldsymbol{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} 1 & t \\ s & 1+st \end{pmatrix} \end{array}$$

be the “standard” embedding. Then, an automorphism $(s,t) \mapsto (f(s,t), g(s,t))$ of $\mathbb{A}^2_{\boldsymbol{k}}$ extends to an automorphism of SL_2, via ρ_1, if and only if it has Jacobian determinant $\frac{\partial f}{\partial s}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial s} \in \boldsymbol{k}^*$ equal to ± 1. In particular, the following holds:

1. *every embedding $\mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$ with image $\rho_1(\mathbb{A}^2_{\boldsymbol{k}})$ is equivalent to an embedding*

$$\begin{array}{cccc} \rho_\lambda\colon & \mathbb{A}^2_{\boldsymbol{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} 1 & t \\ \lambda s & 1+\lambda st \end{pmatrix} \end{array}$$

   for a certain $\lambda \in \boldsymbol{k}^*$. Moreover, ρ_λ and $\rho_{\lambda'}$ are equivalent if and only if $\lambda' = \pm\lambda$;
2. *if $\boldsymbol{k}$ has at least four elements, then not all automorphisms of $\mathbb{A}^2_{\boldsymbol{k}}$ extend to SL_2 via ρ_1.*

Remark 1.1. Let us make some comments on Theorem 2:

1. Over the field of complex numbers $\mathbf{k} = \mathbb{C}$, we show that all algebraic automorphisms of $\mathbb{A}^2_{\mathbf{k}}$ extend via the standard embedding ρ_1 to holomorphic automorphisms of SL_2; see Remark 4.7.

(2) Let $f\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ be a closed embedding and denote by $\iota\colon \mathrm{SL}_2 \hookrightarrow \mathbb{A}^4_{\mathbf{k}}$ the standard embedding. If all coordinate functions of $\iota \circ f\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2 \subset \mathbb{A}^4_{\mathbf{k}}$ are polynomials of degree ≤ 2, then f is equivalent to ρ_λ for a certain $\lambda \in \mathbf{k}^*$ (Proposition 5.19).

Next, we focus on the closed embeddings $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ that are compatible with the simplest $\mathbb{A}^2$-fibration of SL_2. More precisely we have the following.

Definition 1.2. A closed embedding $\rho\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ is said to be a *fibred embedding* if it is of the form

$$(\diamondsuit) \qquad \begin{array}{rccc} \rho\colon & \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} p(s,t) & t \\ r(s,t) & q(s,t) \end{pmatrix} \end{array}$$

for some $p, q, r \in \mathbf{k}[s,t]$. This corresponds to the commutativity of the diagram

$$\begin{tikzcd} \mathbb{A}^2_{\mathbf{k}} \arrow[rr, hook, "\rho"] \arrow[dr, "\pi_1"'] & & \mathrm{SL}_2 \arrow[dl, "\pi_2"] \\ & \mathbb{A}^1_{\mathbf{k}}, & \end{tikzcd}$$

where $\pi_1\colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$, $\pi_2\colon \mathrm{SL}_2 \to \mathbb{A}^1_{\mathbf{k}}$ are, respectively, given by $(s,t) \mapsto t$ and $\left(\begin{smallmatrix} x & t \\ u & y \end{smallmatrix}\right) \mapsto t$.

As we will show, there are a lot of fibred embeddings (i.e., embeddings of the form $(\diamondsuit)$).

Theorem 3. *Let $\boldsymbol{k}$ be any field, let $P \in \boldsymbol{k}[t,x,y]$ be a polynomial that is a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$ (which means that P is the image of x by some automorphism of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$), and let*

$$H_P \subset \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t,u,x,y]/(xy - tu - 1))$$

and $Z_P \subset \mathbb{A}^3_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t,x,y])$ be the hypersurfaces given by $P = 0$.

(1) *The following conditions are equivalent:*
   (a) *The hypersurface $H_P \subset \mathrm{SL}_2$ is isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$.*
   (b) *The hypersurface $H_P \subset \mathrm{SL}_2$ is the image of a fibred embedding $\mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$.*
   (c) *The fibre of $Z_P \to \mathbb{A}^1_{\boldsymbol{k}}$, $(t,x,y) \mapsto t$ over every closed point of $\mathbb{A}^1_{\boldsymbol{k}} \setminus \{0\}$ is isomorphic to $\mathbb{A}^1$ and the polynomial $P(0,x,y) \in \boldsymbol{k}[x,y]$ is of the form $\mu x^m(x-\lambda)$ or $\mu y^m(y-\lambda)$ for some $\mu, \lambda \in \boldsymbol{k}^*$ and some $m \geq 0$.*

(2) *If $P, Q \in \boldsymbol{k}[t,x,y]$ are two polynomials of the above form satisfying the conditions* (a) – (b) – (c)*, such that $H_P, H_Q \subset \mathrm{SL}_2$ are equivalent under an automorphism of* SL_2*, then $Z_P, Z_Q \subset \mathbb{A}^3_{\boldsymbol{k}}$ are equivalent under an automorphism of $\mathbb{A}^3_{\boldsymbol{k}}$.*

(3) *There are infinitely many fibred embeddings $\mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$ having pairwise non-equivalent images in* SL_2*. If $\boldsymbol{k}$ is uncountable, we can moreover choose uncountably many such embeddings.*

Remark 1.3. Let us make some comments on Theorem 3:

(1) It is possible that H_P, H_Q are non-equivalent, even if Z_P, Z_Q are equivalent (Lemma 5.11).

(2) If $\mathrm{char}(\mathbf{k}) = 0$, then every image of a fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ is of the form H_P as above (Lemma 5.2(2)). This is false if $\mathrm{char}(\mathbf{k}) > 0$ (Lemma 5.3).

Let us make the following comment concerning embeddings of $\mathbb{A}^1_{\mathbf{k}}$ into the smooth quadric SL_2 over the field $\mathbf{k}=\mathbb{C}$. Although there are infinitely many non-equivalent embeddings of $\mathbb{A}^2_{\mathbb{C}}$ into SL_2, it is not known whether all embeddings of $\mathbb{A}^1_{\mathbb{C}}$ into SL_2 are equivalent under an algebraic automorphism. It seems that this question is as difficult as the question of non-equivalent embeddings $\mathbb{A}^1_{\mathbb{C}} \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$. However, up to holomorphic automorphisms, all embeddings of $\mathbb{A}^1_{\mathbb{C}}$ into $\mathbb{A}^3_{\mathbb{C}}$ and into SL_2 are equivalent; see [Kal92, Sta15].

In the last section (Lemma 6.2), we give an example of an embedding $\mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathrm{SL}_2$ which is non-equivalent to the standard embedding.

2. The smooth quadric of dimension two and the proof of Theorem 1

2.1. **The isomorphism with the complement of the diagonal in $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$.** In this section, we study the smooth quadric $Q_2 \subset \mathbb{A}^3_{\mathbf{k}}$ given by

$$Q_2 = \mathrm{Spec}(\mathbf{k}[x,y,z]/(xy - z(z+1))),$$

and more particularly closed embeddings $\mathbb{A}^1_{\mathbf{k}} \hookrightarrow Q_2$. Since the closure of Q_2 in $\mathbb{P}^3_{\mathbf{k}}$ is a smooth quadric, isomorphic to $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$, we get the following classical isomorphism.

Lemma 2.1. *The morphism*

$$\begin{array}{rccl} \rho\colon & Q_2 & \to & \mathbb{P}^1_{\boldsymbol{k}} \times \mathbb{P}^1_{\boldsymbol{k}} \\ & (x,y,z) & \mapsto & \begin{cases} ([y:z], [z:x]) & \text{if } z \neq 0, \\ ([z+1:x], [y:z+1]) & \text{if } z \neq -1, \end{cases} \end{array}$$

yields an isomorphism $Q_2 \xrightarrow{\simeq} (\mathbb{P}^1_{\boldsymbol{k}} \times \mathbb{P}^1_{\boldsymbol{k}}) \setminus \Delta$, where $\Delta \subset \mathbb{P}^1_{\boldsymbol{k}} \times \mathbb{P}^1_{\boldsymbol{k}}$ is the diagonal, with an inverse given by

$$\begin{array}{rccc} \psi\colon & (\mathbb{P}^1_{\boldsymbol{k}} \times \mathbb{P}^1_{\boldsymbol{k}}) \setminus \Delta & \to & Q_2 \\ & ([u_0:u_1],[v_0:v_1]) & \mapsto & \left(\frac{u_1 v_1}{u_0v_1 - u_1v_0}, \frac{u_0 v_0}{u_0v_1 - u_1v_0}, \frac{u_1 v_0}{u_0v_1 - u_1v_0} \right). \end{array}$$

Proof. We first check that $\rho((x,y,z)) \in (\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta$ for each $(x,y,z) \in Q_2$. If $z \neq 0$, then $[y:z] \neq [z:x]$, since $xy - z^2 = z \neq 0$. If $z=0$, then $xy=0$, whence $[z+1:x] = [1:x] \neq [y:1] = [y:z+1]$.

It remains then to check that $\rho \circ \psi = \mathrm{id}_{(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta}$ and $\psi \circ \rho = \mathrm{id}_{Q_2}$, which follows from a straightforward calculation. □

2.2. **Families of embeddings.** The following result is the key step in the proof of Theorem 1.

Lemma 2.2.

(1) *For each polynomial $p \in \boldsymbol{k}[t]$, the morphism $\nu_p\colon \mathbb{A}^1_{\boldsymbol{k}} \hookrightarrow Q_2$ given by*

$$\begin{array}{rccc} \nu_p\colon & \mathbb{A}^1_{\boldsymbol{k}} & \to & Q_2 \\ & t & \mapsto & (t(1+tp(t)), p(t), tp(t)) \end{array}$$

is a closed embedding.

(2) *If $p, q \in \boldsymbol{k}[t]$ are polynomials of degree ≥ 3 such that $\alpha\nu_p = \nu_q\beta$ for some $\beta \in \mathrm{Aut}(\mathbb{A}^1_{\boldsymbol{k}})$ and $\alpha \in \mathrm{Aut}(Q_2)$, then there exist $\mu \in \boldsymbol{k}$ and $\lambda \in \boldsymbol{k}^*$ such that*

$$p(t) = \lambda q(\lambda t + \mu)\,, \quad \beta(t) = \lambda t + \mu\,, \quad \alpha(x,y,z) = \left(\lambda x + \frac{\mu^2}{\lambda} y + 2\mu z + \mu, \frac{y}{\lambda}, z + \frac{\mu}{\lambda} y \right).$$

Proof. Using the isomorphism $\rho\colon Q_2 \xrightarrow{\simeq} (\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta$ of Lemma 2.1, we obtain that $\rho \circ \nu_p\colon \mathbb{A}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ is given by $t \mapsto ([1:t],[p(t):1+tp(t)])$, which is the restriction of the closed embedding

$$\hat{\nu}_p\colon \mathbb{P}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}, [u:v] \mapsto ([u:v],[uP(u,v):u^{d+1}+vP(u,v)]),$$

where $d = \deg(p)$ and $P(u,v) = p(\frac{v}{u})u^d$ is the homogenisation of p. This implies that $\Gamma_p = \hat{\nu}_p(\mathbb{P}^1_{\mathbf{k}}) \subset \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ is a smooth closed curve (isomorphic to $\mathbb{P}^1_{\mathbf{k}}$), and since $\Gamma_p \cap \Delta$ is given by $u(u^{d+1}+vP(u,v)) - vuP(u,v) = 0$, i.e., $u^{d+2} = 0$, this shows that ν_p is a closed embedding, and thus yields (1).

It remains to prove Assertion (2). We fix two polynomials $p, q \in \mathbf{k}[t]$ of degree ≥ 3 such that $\alpha\nu_p = \nu_q\beta$ for some $\beta \in \mathrm{Aut}(\mathbb{A}^1_{\mathbf{k}})$ and $\alpha \in \mathrm{Aut}(Q_2)$. This implies in particular that the automorphism $\alpha' = \rho^{-1}\alpha\rho \in \mathrm{Aut}((\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta)$ sends $\Gamma_p \setminus \Delta$ onto $\Gamma_q \setminus \Delta$.

We first prove that $\alpha' \in \mathrm{Aut}((\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta)$ extends to an automorphism $\hat{\alpha} \in \mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})$. Assume for contradiction that this is not the case. The map α' would then extend to a birational map $\hat{\alpha}\colon \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$, which is not an automorphism. We consider the minimal resolution of $\hat{\alpha}$, which is

$$\begin{tikzcd}
 & Z \arrow[ld, "\chi_1"'] \arrow[rd, "\chi_2"] & \\
\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \arrow[rr, dashed, "\hat{\alpha}"] & & \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \\
(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta \arrow[u, hook] \arrow[rr, "\simeq"] & & (\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta \arrow[u, hook]
\end{tikzcd}$$

where χ_1, χ_2 are birational morphisms. The resolution being minimal, every (-1)-curve $E \subset Z$ contracted by χ_2 is not contracted by χ_1, so $\chi_1(E) \subset \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ is contracted by $\hat{\alpha}$. There is thus a unique (-1)-curve contracted by χ_2, which is the strict transform $\tilde{\Delta}$ of Δ, and satisfies $\chi_1(\tilde{\Delta}) = \Delta$. As $\Delta^2 = 2$ and $(\tilde{\Delta})^2 = -1$, there are exactly three base-points of χ_1^{-1} that lie on the curve Δ (as proper point or infinitely near points). Since Γ_p is smooth of bidegree $(1, 1+\deg p)$, we get $\Gamma_p \cdot \Delta = 2 + \deg p \geq 5$, which implies that the strict transforms of Γ_p and Δ on Z satisfy $\tilde{\Gamma}_p \cdot \tilde{\Delta} \geq 2$ (as only three points belonging to Δ have been blown-up). As the curve $\tilde{\Delta}$ is contracted by χ_2, the curve $\chi_2(\tilde{\Gamma}_p)$ is singular. This contradicts the equality $\chi_2(\tilde{\Gamma}_p) = \Gamma_q$, which follows from the fact that $\hat{\alpha}(\Gamma_p \setminus \Delta) = \Gamma_q \setminus \Delta$.

We have shown that the extension of $\alpha' = \rho^{-1}\alpha\rho \in \mathrm{Aut}((\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta)$ is an automorphism $\hat{\alpha} \in \mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})$, which satisfies therefore $\hat{\alpha}(\Delta) = \Delta$ and $\hat{\alpha}(\Gamma_p) = \Gamma_q$. The curves Γ_p and Γ_q being of bidegree $(1, 1+\deg p)$ and $(1, 1+\deg q)$, we get $\deg p = \deg q$ and we obtain that $\hat{\alpha}$ does not exchange the two factors of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$. Moreover, as the point $([0:1],[0:1]) = \Delta \cap \Gamma_p = \Delta \cap \Gamma_q$ is fixed, and as the diagonal Δ is invariant, we can write $\hat{\alpha}$ as

$$\hat{\alpha}([u_0:u_1],[v_0:v_1]) = ([u_0:\lambda u_1 + \mu u_0],[v_0:\lambda v_1 + \mu v_0])$$

for some $\lambda \in \mathbf{k}^*$, $\mu \in \mathbf{k}$.

The equality $\alpha\nu_p = \nu_q\beta$ implies that $\hat{\alpha}\hat{\nu}_p = \hat{\nu}_q\hat{\beta}$, for some automorphism $\hat{\beta} \in \mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}})$, which is the extension of β and therefore it is of the form $[u:v] \mapsto [u:\lambda v + \mu u]$. We then compute

$$\begin{aligned}
\hat{\alpha}\hat{\nu}_p([u:v]) &= ([u:\lambda v+\mu u],[uP(u,v):\lambda u^{d+1}+\lambda vP(u,v)+\mu uP(u,v)]),\\
\hat{\nu}_q\hat{\beta}([u:v]) &= ([u:\lambda v+\mu u],[uQ(u,\lambda v+\mu u):u^{d+1}+(\lambda v+\mu u)Q(u,\lambda v+\mu u)]),
\end{aligned}$$

and obtain that $P(u,v) = \lambda Q(u, \lambda v + \mu u)$. Remembering that $P(u,v) = p(\frac{v}{u})u^d$ and $Q(u,v) = q(\frac{v}{u})u^d$ we obtain that $p(t) = \lambda q(\lambda t + \mu)$. We then compute the explicit form of α by conjugating $\hat{\alpha}$ with ρ^{-1}. $\square$

Example 2.3. For each $n \geq 1$, let $p_n(t) = t^n(t+1)^{n+1}$. The closed curve $C_n = \nu_{p_n}(\mathbb{A}^1_{\mathbf{k}}) \subset Q_2$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}}$, via

$$\begin{array}{cccc} \nu_{p_n}\colon & \mathbb{A}^1_{\mathbf{k}} & \hookrightarrow & Q_2 \\ & t & \mapsto & (t(1+tp_n(t)), p_n(t), tp_n(t)). \end{array}$$

Then Lemma 2.2(2) shows that all curves C_n are non-equivalent for different $n \geq 1$, and that the identity is the only automorphism of C_n that extends to Q_2.

The proof of Theorem 1 is now a consequence of Lemma 2.2.

Proof of Theorem 1. If $\mathbf{k}$ is the field with two elements, then we conclude by Example 2.3. Hence we can assume that $\mathbf{k}$ contains more than two elements. For each $n \geq 1$ and each $\lambda \in \mathbf{k}$, $\varepsilon \in \mathbf{k} \setminus \{0,1\}$, one defines $p_{n,\varepsilon}(t) = t^n(t+1)^{n+1}(t+\varepsilon)^{n+2} \in \mathbf{k}[t]$, and let $C_{n,\varepsilon} \subset Q_2$ be the closed curve given by $\nu_{p_{n,\varepsilon}}(\mathbb{A}^1_{\mathbf{k}})$, which is isomorphic to $\mathbb{A}^1_{\mathbf{k}}$ (Lemma 2.2(1)).

Lemma 2.2(2) implies that the identity is the only automorphism of $C_{n,\varepsilon}$ that extends to an automorphism of Q_2, since $\lambda p_{n,\varepsilon}(\lambda t + \mu) \neq p_{n,\varepsilon}(t)$ for $(\lambda, \mu) \in (\mathbf{k}^* \times \mathbf{k}) \setminus \{(1,0)\}$.

Similarly, Lemma 2.2(2) shows that $C_{n,\varepsilon}$ is equivalent to $C_{n',\varepsilon'}$ if and only if $n = n'$ and $\varepsilon = \varepsilon'$. $\square$

3. Variables of polynomial rings

In this section, we give some results on variables of polynomial rings. Most of them are classical or belong to the folklore. We include them for self-containedness and for lack of precise references.

Definition 3.1. Let S be a ring and let $R \subset S$ be a subring. We denote by $\mathrm{Aut}_R(S)$ the group of automorphisms of the R-algebra S. More precisely,

$$\mathrm{Aut}_R(S) = \{\text{ automorphism of rings } f\colon S \to S \text{ such that } f|_R = \mathrm{id}_R\}.$$

Definition 3.2. Let R be a domain and let S be a polynomial ring in $n \geq 1$ variables over R, i.e., $R \subset S$ and there exist $x_1, \ldots, x_n \in S$ such that each element of S can be written in a unique way as $f(x_1, \ldots, x_n)$, where f is a polynomial in the x_i with coefficients in R. An element $v \in S$ is called *variable of the R-algebra S* if there exists $f \in \mathrm{Aut}_R(S)$ such that $f(v) = x_1$.

In what follows, we often denote by $R[t]$ or $R[x]$ the polynomial ring in one variable over R, by $R[x,y]$ the polynomial ring in two variables over R, and by $R[x_1, \ldots, x_n]$ the polynomial ring in n variables over R.

Lemma 3.3. *Let R be a domain, let $S = R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over R, and let $v \in S$. The following conditions are equivalent:*

(1) *v is a variable of the R-algebra S;*
(2) *the $R[t]$-algebra $S[t]/(v-t)$ is isomorphic to a polynomial ring in $n-1$ variables over $R[t]$.*

Proof. If v is a variable, then there exists $f \in \mathrm{Aut}_R(S)$ such that $f(v) = x_1$. Using the natural inclusion $\mathrm{Aut}_R(S) \hookrightarrow \mathrm{Aut}_{R[t]}(S[t])$, we get isomorphisms of $R[t]$-algebras

$$S[t]/(v-t) \xrightarrow{\simeq} S[t]/(x_1-t) \xrightarrow{\simeq} R[x_2,\ldots,x_n,t] \xrightarrow{\simeq} R[t][x_2,\ldots,x_n].$$

Conversely, suppose that the $R[t]$-algebra $S[t]/(v-t)$ is isomorphic to a polynomial ring in $n-1$ variables over $R[t]$. This yields an $R[t]$-isomorphism $\psi\colon S[t]/(t-v) \xrightarrow{\simeq} R[t][x_2,\ldots,x_n]$. We then compose the isomorphisms of R-algebras

$$\begin{array}{ccccc} S = R[x_1,\ldots,x_n] & \xrightarrow{\simeq} & S[t]/(t-v) & \xrightarrow{\psi} & R[t][x_2,\ldots,x_n] \\ f & \mapsto & f + (t-v)\cdot S[t] & & \end{array}$$

and

$$\begin{array}{ccc} R[t][x_2,\ldots,x_n] & \xrightarrow{\simeq} & R[x_1,\ldots,x_n] \\ f(t,x_2,\ldots,x_n) & \mapsto & f(x_1,x_2,\ldots,x_n) \end{array}$$

and obtain an element of $\mathrm{Aut}_R(S)$ that sends v onto x_1. □

Lemma 3.4. *Let $\boldsymbol{k}$ be a field, let $\boldsymbol{k}[x_1,\ldots,x_n]$ be the polynomial ring in $n \geq 1$ variables over $\boldsymbol{k}$, and let $w \in \boldsymbol{k}[x_1,\ldots,x_n]$ be a variable of this $\boldsymbol{k}$-algebra.*

Then $\boldsymbol{k}[w]$ is factorially closed in $\boldsymbol{k}[x_1,\ldots,x_n]$, i.e., for all $f,g \in \boldsymbol{k}[x_1,\ldots,x_n] \setminus \{0\}$, and we have $fg \in \boldsymbol{k}[w] \Leftrightarrow f \in \boldsymbol{k}[w]$ and $g \in \boldsymbol{k}[w]$.

Proof. If $f \in \mathbf{k}[w]$ and $g \in \mathbf{k}[w]$, then $fg \in \mathbf{k}[w]$, since $\mathbf{k}[w]$ is a subring of $\mathbf{k}[x_1,\ldots,x_n]$.

Conversely, suppose that $fg \in \mathbf{k}[w]$. Choose $\psi \in \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[x_1,\ldots,x_n])$ such that $\psi(w) = x_1$. Then, $\psi(f), \psi(g) \in \mathbf{k}[x_1,\ldots,x_n]$ are two polynomials such that $\psi(f)\cdot\psi(g) \in \mathbf{k}[x_1]$. For each $i \geq 2$, the degree in x_i satisfies $\deg_{x_i}(\psi(f)) + \deg_{x_i}(\psi(f)) = \deg_{x_i}(\psi(f)\cdot\psi(g)) = 0$, so $\deg_{x_i}(\psi(f)) = \deg_{x_i}(\psi(f)) = 0$ since both elements are non-zero. Hence, $\psi(f), \psi(g) \in \mathbf{k}[x_1]$. Applying ψ^{-1}, we get $f \in \mathbf{k}[w]$ and $g \in \mathbf{k}[w]$. □

3.1. **Variables of polynomial rings in two variables.** We will need the following two technical lemmas.

Lemma 3.5. *Let $\boldsymbol{k}$ be a field, let w be a variable of the $\boldsymbol{k}$-algebra $\boldsymbol{k}[x,y]$, and let $v \in \boldsymbol{k}[x,y]$ be a polynomial. The following conditions are equivalent:*

1. *$v \in \boldsymbol{k}[w]$;*
2. *for each $u \in \boldsymbol{k}[w]$, the elements u and v are algebraically dependent over $\boldsymbol{k}$;*
3. *there exists $u \in \boldsymbol{k}[w] \setminus \boldsymbol{k}$ such that u and v are algebraically dependent over $\boldsymbol{k}$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) being clear, we only need to prove (3) ⇒ (1). Replacing v and w with $f(v)$ and $f(w)$, for some $f \in \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[x,y])$, we can assume that $w = x$. Denoting by $\overline{\mathbf{k}}$ the algebraic closure of $\mathbf{k}$, we have $\overline{\mathbf{k}}[x] \cap \mathbf{k}[x,y] = \mathbf{k}[x]$, so we can assume that $\mathbf{k} = \overline{\mathbf{k}}$.

We then consider the morphism $\tau\colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^2_{\mathbf{k}}$ given by $(x,y) \mapsto (u(x), v(x,y))$, which is dominant if and only if u, v are algebraically independent over $\mathbf{k}$. It remains then to see that τ is dominant if $u \in \mathbf{k}[x] \setminus \mathbf{k}$ and $v \notin \mathbf{k}[x]$. Let $v(x,y) = \sum_{i=0}^{d} v_i(x) y^i$, where $v_d \neq 0$ and $d > 0$. For a general $a \in \mathbf{k}$, $u(x) = a$ has a solution x_0 such that $v_d(x_0) \neq 0$, since $\mathbf{k}$ is algebraically closed. Hence $v(x_0, y) = b$ has a solution for all $b \in \mathbf{k}$. This proves that τ is dominant. □

Lemma 3.6. *Let $\boldsymbol{k}$ be a field, let $p \in \boldsymbol{k}[t]$ be an irreducible element, and let $\boldsymbol{k}_p = \boldsymbol{k}[t]/(p)$ be the corresponding residue field. Let $u, v \in \boldsymbol{k}[t][x,y]$ be elements such that $\boldsymbol{k}(t)[u,v] = \boldsymbol{k}(t)[x,y]$. Then, the classes $u_0, v_0 \in \boldsymbol{k}_p[x,y]$ of u, v satisfy one of the following properties, depending on the Jacobian determinant $\nu = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} \in \boldsymbol{k}[t] \setminus \{0\}$:*

(1) *If p divides ν, then u_0, v_0 are algebraically dependent over $\boldsymbol{k}_p$.*
(2) *If p does not divide ν, then $\boldsymbol{k}_p[u_0, v_0] = \boldsymbol{k}_p[x,y]$. In particular, both u_0 and v_0 are variables of the $\boldsymbol{k}_p$-algebra $\boldsymbol{k}_p[x,y]$.*

Proof. Since $\mathbf{k}(t)[u,v] = \mathbf{k}(t)[x,y]$, there are polynomials $P, Q \in \mathbf{k}(t)[X,Y]$ such that $P(u,v) = x$, $Q(u,v) = y$. Moreover, the polynomial $\nu = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} \in \mathbf{k}[t,x,y]$ belongs to $\mathbf{k}(t)^*$ and thus to $\mathbf{k}[t] \setminus \{0\}$. The element $\nu_0 = \frac{\partial u_0}{\partial x} \cdot \frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y} \cdot \frac{\partial v_0}{\partial x}$ is then the class of ν in $\mathbf{k}_p$.

We write $P = \frac{\tilde{P}}{\alpha}$, $Q = \frac{\tilde{Q}}{\beta}$, where $\tilde{P}, \tilde{Q} \in \mathbf{k}[t][X,Y]$, $\alpha, \beta \in \mathbf{k}[t] \setminus \{0\}$ and such that p does not divide both α and $\tilde{P}$ (and the same for β and $\tilde{Q}$). We then get

$$\tilde{P}_0(u_0, v_0) = \alpha_0 x\,, \quad \tilde{Q}_0(u_0, v_0) = \beta_0 y,$$

where $\tilde{P}_0, \tilde{Q}_0 \in \mathbf{k}_p[X,Y]$ are the classes of $\tilde{P}, \tilde{Q}$ and $\alpha_0, \beta_0 \in \mathbf{k}_p$ are the classes of α, β.

If α_0 and β_0 are not equal to zero, then $\mathbf{k}_p[u_0, v_0] = \mathbf{k}_p[x,y]$. In particular, u_0 and v_0 are variables of the $\mathbf{k}_p$-algebra $\mathbf{k}_p[x,y]$ and $\nu_0 \in \mathbf{k}_p^*$, so p does not divide ν.

If $\alpha_0 = 0$, then $\tilde{P}_0 \neq 0$ and $\tilde{P}_0(u_0, v_0) = 0$ implies that u_0 and v_0 are algebraically dependent over $\mathbf{k}_p$. The same conclusion holds when $\beta_0 = 0$. In both cases, the Jacobian determinant ν_0 is equal to zero, so p divides ν.

This yields (1) and (2). ☐

We recall the following classical result, essentially equivalent to the Jung-van der Kulk Theorem.

Lemma 3.7. *Let $\boldsymbol{k}$ be a field, let $\boldsymbol{k}[x,y]$ be the polynomial ring in two variables over $\boldsymbol{k}$, let $f \in \mathrm{Aut}_{\boldsymbol{k}}(\boldsymbol{k}[x,y])$ and $u = f(x), v = f(y) \in \boldsymbol{k}[x,y]$. If $\deg(u) \geq \deg(v) > 1$, then there exists a polynomial P with coefficients in $\boldsymbol{k}$ such that $\deg(u - P(v)) < \deg(u)$.*

Proof. By van der Kulk's Theorem all automorphisms of $\mathbf{k}[x,y]$ are tame [Jun42, vdK53]. The statement is then a direct consequence of [vdE00, Corollary 5.1.6]. ☐

The following result is needed in what follows. When the characteristic of $\mathbf{k}$ is zero, and $p = t$, it follows from [Fur02, Theorem 4]. We adapt here the proof of [Fur02] for our purpose.

Lemma 3.8. *Let $\boldsymbol{k}$ be a field, let $p \in \boldsymbol{k}[t]$ be an irreducible element, and let $\boldsymbol{k}_p = \boldsymbol{k}[t]/(p)$ be the corresponding residue field. If $v \in \boldsymbol{k}[t,x,y]$ is a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$, then its class in $\boldsymbol{k}_p[x,y]$ is an element which belongs to $\boldsymbol{k}_p[w] \subset \boldsymbol{k}_p[x,y]$ for some variable w of the $\boldsymbol{k}_p$-algebra $\boldsymbol{k}_p[x,y]$.*

Proof. Let $f \in \mathrm{Aut}_{\mathbf{k}(t)}(\mathbf{k}(t)[x,y])$ such that $f(x) = v$, and let us define $u = f(y)$. We denote by $v_0 \in \mathbf{k}_p[x,y]$ the class of v and will use the degree of polynomials in x, y with coefficients in $\mathbf{k}(t)$ or $\mathbf{k}_p$.

If $\deg(v) = 1$, then $\deg(v_0) \leq 1$. If $v_0 \in \mathbf{k}_p$ the result follows by taking any variable for w, for instance $w = x$. Otherwise, $v_0 = \alpha x + \beta y + \gamma$ for some $\alpha, \beta, \gamma \in \mathbf{k}_p$

with $(\alpha,\beta) \neq (0,0)$. This implies that $w = \alpha x + \beta y$ is a variable, as it is the component of an element of $\mathrm{GL}_2(\mathbf{k}_p)$, and the result follows.

We can thus assume that $\deg(v) > 1$ and prove the result by induction on the pair $(\deg(v), \deg(u))$, ordered lexicographically.

(i) If $\deg(u) \geq \deg(v)$, then there exists a polynomial $P \in \mathbf{k}(t)[X]$ such that $\deg(u - P(v)) < \deg(u)$ (Lemma 3.7). We can thus apply an induction hypothesis to $(u - P(v), v)$, since $\mathbf{k}(t)[u,v] = \mathbf{k}(t)[u - P(v), v]$, and obtain the result.

(ii) If $\deg(u) < \deg(v)$, we first replace u with $u-\lambda$ for some $\lambda \in \mathbf{k}(t)$ and assume that $u \in \mathbf{k}(t)[x,y]$ is a polynomial in x, y with no constant term. We then replace u with qu for some $q \in \mathbf{k}(t)^*$ and assume that $u \in \mathbf{k}[t][x,y]$ and the greatest common divisor in $\mathbf{k}[t]$ of the coefficients of u (as a polynomial in x, y) is equal to 1. One can then define the class $u_0 \in \mathbf{k}_p[x,y]$ of u, which is not equal to zero. Moreover, u_0 does not belong to $\mathbf{k}_p$, since u_0 has no constant term.

If v_0 is a variable of the $\mathbf{k}_p$-algebra $\mathbf{k}_p[x,y]$, then we are done. Otherwise, u_0, v_0 are algebraically dependent over $\mathbf{k}_p$ (Lemma 3.6).

Since the pair $(\deg(v), \deg(u))$ is smaller than $(\deg(u), \deg(v))$, we can apply an induction hypothesis and get a variable $w \in \mathbf{k}_p[x,y]$ such that $u_0 \in \mathbf{k}_p[w]$. The fact that u_0 and v_0 are algebraically dependent over $\mathbf{k}_p$ and that $u_0 \notin \mathbf{k}_p$ imply that $v_0 \in \mathbf{k}_p[w]$ (Lemma 3.5). ☐

We finish this section with several results relating variables and $\mathbb{A}^1$-bundles.

Lemma 3.9. *Let $\boldsymbol{k}$ be a field and let $P \in \boldsymbol{k}[x,y]$. Then, the following conditions are equivalent:*

(1) *The polynomial P is a variable of the $\boldsymbol{k}$-algebra $\boldsymbol{k}[x,y]$.*
(2) *The $\boldsymbol{k}[t]$-algebra $\boldsymbol{k}[t,x,y]/(P-t)$ is a polynomial ring in one variable over $\boldsymbol{k}[t]$.*
(3) *The $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]/(P-t)$ is a polynomial ring in one variable over $\boldsymbol{k}(t)$.*
(4) *The morphism $\mathbb{A}^2_{\boldsymbol{k}} \to \mathbb{A}^1_{\boldsymbol{k}}$ given by $(x,y) \mapsto P(x,y)$ is a trivial $\mathbb{A}^1$-bundle.*
(5) *The morphism $\mathbb{A}^2_{\boldsymbol{k}} \to \mathbb{A}^1_{\boldsymbol{k}}$ given by $(x,y) \mapsto P(x,y)$ is a trivial $\mathbb{A}^1$-bundle over some dense open subset $U \subset \mathbb{A}^1_{\boldsymbol{k}}$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 3.3.

(1) $\Leftrightarrow$ (4): By definition, (1) is equivalent to the existence of $f \in \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[x,y])$ such that $f(x) = P$. As $f = \varphi^*$ for some $\varphi \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$, this is equivalent to asking for $\varphi \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ that $\mathrm{pr}_x \circ \varphi$ is the map $(x,y) \mapsto P(x,y)$, where $\mathrm{pr}_x \colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ is given by $(x,y) \mapsto x$. This yields the equivalence (1) $\Leftrightarrow$ (4).

(2) $\Rightarrow$ (3) is trivially true.

(3) $\Rightarrow$ (5): Assertion (3) corresponds to say that the generic fibre of $(x,y) \mapsto P(x,y)$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$. This yields (5).

(5) $\Rightarrow$ (1) : Assume that the subset U given in (5) contains a $\mathbf{k}$-rational point. Replacing P with $P + \lambda$, $\lambda \in \mathbf{k}$, one can assume that 0 belongs to the open subset U. One then observes that the curve $\Gamma \subset \mathbb{A}^2_{\mathbf{k}}$ given by $P = 0$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}}$ and equivalent to a line by a birational map of $\mathbb{A}^2_{\mathbf{k}}$, hence can be contracted by a birational map of $\mathbb{A}^2_{\mathbf{k}}$. By [BFH16, Proposition 2.29], there exists an automorphism $\varphi \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ which sends Γ onto the line given by $x = 0$. This implies that P is a variable of the $\mathbf{k}$-algebra $\mathbf{k}[x,y]$.

If U contains no $\mathbf{k}$-rational point, then $\mathbf{k}$ is a finite field and thus it is perfect. For a finite Galois extension $\mathbf{k}' \supset \mathbf{k}$ the subset U contains a $\mathbf{k}'$-rational point. By the argument above, P is a variable of the $\mathbf{k}'$-algebra $\mathbf{k}'[x,y]$ and hence $\mathbf{k}'[x,y] = \mathbf{k}'[P,Q]$ for some $Q \in \mathbf{k}'[x,y]$. Since P is a polynomial with coefficients in $\mathbf{k}$, it is fixed under the action of the Galois group $G = \mathrm{Gal}(\mathbf{k}'/\mathbf{k})$ on $\mathbf{k}[x,y] = \mathbf{k}'[P,Q]$. For each $\sigma \in G$, there exists $(a_\sigma, b_\sigma) \in (\mathbf{k}')^* \ltimes \mathbf{k}'[T]$ with

$$\sigma(Q) = a_\sigma Q + b_\sigma(P).$$

We can then find $d \geq 0$ such that $\{b_\sigma \mid \sigma \in G\}$ is contained in the finite dimensional $\mathbf{k}'$-vector subspace $V_d = \{f \in \mathbf{k}'[T] \mid \deg(P) \leq d\} \subset \mathbf{k}'[T]$. Thus $\sigma \mapsto (a_\sigma, b_\sigma)$ defines an element of $H^1(G, (\mathbf{k}')^* \ltimes V_d)$. As $H^1(G, (\mathbf{k}')^*) = \{1\}$ and $H^1(G, V_d) = \{1\}$ [Ser68, Proposition 1, 2, Chp. X], we have $H^1(G, (\mathbf{k}')^* \ltimes V_d) = \{1\}$. The fact that (a_σ, b_σ) is a trivial cocycle corresponds to the existence of a polynomial $Q_0 \in \mathbf{k}[x,y]$ such that $\mathbf{k}[P, Q_0] = \mathbf{k}[x,y]$. This implies that P is a variable of the $\mathbf{k}$-algebra $\mathbf{k}[x,y]$. $\square$

We recall the following classical result.

Lemma 3.10. *Let $\boldsymbol{k}$ be a field, let Z be an affine variety over $\boldsymbol{k}$, all of its irreducible components being surfaces, let $U \subseteq \mathbb{A}^1_{\boldsymbol{k}}$ be a dense open subset and let $\pi\colon Z \to U$ be a dominant morphism. Then, the following conditions are equivalent:*

(1) *The morphism $\pi\colon Z \to U$ is a trivial $\mathbb{A}^1$-bundle.*
(2) *The morphism $\pi\colon Z \to U$ is a locally trivial $\mathbb{A}^1$-bundle.*
(3) *For each maximal ideal $\mathfrak{m} \subset \boldsymbol{k}[U]$, the fibre $\pi^{-1}(\mathfrak{m}) \subset Z$ is isomorphic to $\mathbb{A}^1_{\kappa(\mathfrak{m})}$ and the generic fibre of π is isomorphic to $\mathbb{A}^1_{\boldsymbol{k}(t)}$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. Assume (3) holds. Since each irreducible component of Z has dimension two, it follows that each of these irreducible components is mapped dominantly onto U via π. Thus π is flat. By [Asa87, Corollary 3.2] it follows now from (3) that $\mathbf{k}[Z]_\mathfrak{m}$ is a polynomial ring in one variable over $\mathbf{k}[U]_\mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subset \mathbf{k}[U]$. Hence, by [BCW77], the morphism π is a vector bundle with respect to the Zariski topology and since $\mathbf{k}[U]$ is a principal ideal domain, π is a trivial $\mathbb{A}^1$-bundle. $\square$

Lemma 3.11. *Let $\boldsymbol{k}$ be a field, let $P \in \boldsymbol{k}[t,x,y]$ be a polynomial which is a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$, let $U \subset \mathbb{A}^1_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t])$ be a dense open subset, let $Z \subset U \times \mathbb{A}^2_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[U][x,y])$ be the hypersurface given by $P = 0$, and let $\pi\colon Z \to U$ be the morphism $(t,x,y) \mapsto t$. Then, the following conditions are equivalent:*

(i) *P is a variable of the $\boldsymbol{k}[U]$-algebra $\boldsymbol{k}[U][x,y]$;*
(ii) *there is an isomorphism $\varphi\colon U \times \mathbb{A}^1_{\boldsymbol{k}} \xrightarrow{\simeq} Z$ such that $\pi\varphi$ is the projection $(t,x) \mapsto t$;*
(iii) *the morphism $\pi\colon Z \to U$ is a trivial $\mathbb{A}^1$-bundle;*
(iv) *the morphism $\pi\colon Z \to U$ is a locally trivial $\mathbb{A}^1$-bundle;*
(v) *for each maximal ideal $\mathfrak{m} \subset \boldsymbol{k}[U]$, the fibre $\pi^{-1}(\mathfrak{m}) \subset Z$ is isomorphic to $\mathbb{A}^1_{\kappa(\mathfrak{m})}$.*

Proof. (i) $\Rightarrow$ (ii): If P is a variable of the $\mathbf{k}[U]$-algebra $\mathbf{k}[U][x,y]$, there exists $f \in \mathrm{Aut}_{\mathbf{k}[U]}(\mathbf{k}[U][x,y])$ such that $f(x) = P$. The element f is then equal to ψ^* for some $\psi \in \mathrm{Aut}(U \times \mathbb{A}^2_{\mathbf{k}})$ such that $\pi\psi = \pi$. Hence, $\psi(Z)$ is the closed subset of $U \times \mathbb{A}^2_{\mathbf{k}}$ given by $x = 0$. Let $\theta\colon U \times \mathbb{A}^2_{\mathbf{k}} \to U \times \mathbb{A}^1_{\mathbf{k}}$ be the projection given by $(t,x,y) \mapsto (t,y)$. The composition $\theta \circ \psi$ restricts to an isomorphism $Z \xrightarrow{\simeq} U \times \mathbb{A}^1_{\mathbf{k}}$ that we denote by φ^{-1}. Thus $\mathrm{pr}_1 \circ \varphi^{-1} = \pi$, where $\mathrm{pr}_1\colon U \times \mathbb{A}^1_{\mathbf{k}} \to U$ is the projection on the first factor. This yields (ii).

(ii) $\Leftrightarrow$ (iii): is the definition of a trivial $\mathbb{A}^1$-bundle.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious.

(v) $\Rightarrow$ (i): This follows from the implication (3) $\Rightarrow$ (1) of Lemma 3.10 (we use here the fact that the generic fibre of π is isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$, which is provided by the assumption that P is a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$). $\square$

Remark 3.12. Lemma 3.11 is false if we do not assume P to be a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$. Let us choose for example $\mathbf{k}$ to be algebraically closed of characteristic $p > 0$, $P = x + x^{pq} + y^{p^2} - t$, and $U = \mathbb{A}^1$. Corollary 3.16 shows that (v) is satisfied but not (i). Moreover, the fact that P is not a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$ follows from Lemma 3.15.

Corollary 3.13. *Let $\boldsymbol{k}$ be a field, let $P \in \boldsymbol{k}[t,x,y]$ be a polynomial which is a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$, let $Z \subset \mathbb{A}^3_{\boldsymbol{k}} = \operatorname{Spec}(\boldsymbol{k}[t,x,y])$ be the hypersurface given by $P = 0$, and let $\pi\colon Z \to \mathbb{A}^1_{\boldsymbol{k}}$ be the morphism $(t,x,y) \mapsto t$. Then, the following conditions are equivalent:*

(i) *P is a variable of the $\boldsymbol{k}[t]$-algebra $\boldsymbol{k}[t][x,y]$;*
(ii) *there is an isomorphism $\varphi\colon \mathbb{A}^2_{\boldsymbol{k}} \xrightarrow{\simeq} Z$ such that $\pi\varphi$ is the projection $(t,x) \mapsto t$;*
(iii) *there is an isomorphism $\varphi\colon \mathbb{A}^2_{\boldsymbol{k}} \xrightarrow{\simeq} Z$;*
(iv) *the morphism $\pi\colon Z \to \mathbb{A}^1_{\boldsymbol{k}}$ is a trivial $\mathbb{A}^1$-bundle;*
(v) *the morphism $\pi\colon Z \to \mathbb{A}^1_{\boldsymbol{k}}$ is a locally trivial $\mathbb{A}^1$-bundle;*
(vi) *for each maximal ideal $\mathfrak{m} \subset \boldsymbol{k}[t]$, the fibre $\pi^{-1}(\mathfrak{m}) \subset Z$ is isomorphic to $\mathbb{A}^1_{\kappa(\mathfrak{m})}$.*

Proof. Applying Lemma 3.11 with $U = \mathbb{A}^1_{\mathbf{k}}$, we obtain the equivalence between (i)-(ii)-(iv)-(v)-(vi).

We then observe that (ii) implies (iii). It remains then to prove (iii) $\Rightarrow$ (iv). As P is a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$, the generic fibre of $\pi\colon Z \to \mathbb{A}^1_{\mathbf{k}}$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$, so π is a trivial $\mathbb{A}^1$-bundle over some dense open subset $U \subset \mathbb{A}^1_{\mathbf{k}}$. The fact that Z is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ implies then that π is a trivial $\mathbb{A}^1_{\mathbf{k}}$-bundle (Implication (5) $\Rightarrow$ (4) of Lemma 3.9). $\square$

3.2. **Non-trivial embeddings in positive characteristic.** In this paragraph, we recall the existence of non-trivial embeddings in positive characteristic. The family of examples that we give below seems classical (the case $\mathbb{A}^1_{\mathbf{k}} \hookrightarrow \mathbb{A}^2_{\mathbf{k}}$ with parameters equal to 1 corresponds in particular to [vdE00, Exercise 5(iii) in §5]). We give the (simple) proof here for a lack of a precise reference and for self-containedness.

Lemma 3.14. *For each field $\boldsymbol{k}$ of characteristic $p > 0$, each $a \in \boldsymbol{k}$, $b \in \boldsymbol{k}^*$ and each integer $q \geq 0$, the morphism*

$$\begin{array}{cccc} \rho\colon & \mathbb{A}^1_{\boldsymbol{k}} & \hookrightarrow & \mathbb{A}^2_{\boldsymbol{k}} \\ & u & \mapsto & (u^{p^2}, \frac{1}{b}(au^{pq} + u)) \end{array}$$

is a closed embedding, with image being the closed curve of $\mathbb{A}^2_{\boldsymbol{k}} = \operatorname{Spec}(\boldsymbol{k}[x,y])$ given by

$$x + a^{p^2}x^{pq} - b^{p^2}y^{p^2} = 0.$$

Proof. We first compute the equality

$$b^p\left(\frac{1}{b}(au^{pq} + u)\right)^p - a^p(u^{p^2})^q = (a^p u^{p^2q} + u^p) - a^p u^{p^2q} = u^p,$$

which shows that $\rho(\mathbb{A}^1_{\mathbf{k}})$ is contained in the closed curve $\Gamma \subset \mathbb{A}^2_{\mathbf{k}}$ given by $P = 0$, where $P = x - (b^p y^p - a^p x^q)^p = x - b^{p^2} y^{p^2} + a^{p^2} x^{qp} \in \mathbf{k}[x, y]$. The equality also yields $u^p \in \mathbf{k}[u^{p^2}, \frac{1}{b}(au^{pq} + u)]$ and thus yields $\mathbf{k}[u^{p^2}, \frac{1}{b}(au^{pq} + u)] = \mathbf{k}[u]$, which implies that ρ is a closed embedding. It remains to see that the degree of ρ (maximum of the degree of both components) is equal to the degree of P, to obtain that P is irreducible and that it defines the irreducible curve $\rho(\mathbb{A}^1_{\mathbf{k}})$. For $a = 0$, this follows, since $\deg(\rho) = p^2 = \deg(P)$. For $a \neq 0$, we have $\deg(\rho) = \max(p^2, pq) = \deg(P)$. $\square$

To show that the above embeddings are not equivalent to the standard one, when $q \geq 2$ is not a multiple of p and $a, b \neq 0$, one could make the argument on the degree of the components (no one divides the other) or can use the characterisation of variables given in Lemma 3.3 to show that $P = x + a^{p^2} x^{pq} - b^{p^2} y^{p^2} \in \mathbf{k}[x, y]$ is not a variable, by proving that $\mathbf{k}[x, y, t]/(P - t)$ is not a polynomial ring in one variable over $\mathbf{k}[t]$, as we do in Lemma 3.15 below. The second way has the advantage of giving examples in any dimension (see Proposition 3.17). This is related to the forms of the affine line over non-perfect fields (for more details on this subject, see [Rus70]).

Lemma 3.15. *For each field $\boldsymbol{k}$ of characteristic $p > 0$, each $a, b \in \boldsymbol{k}^*$ and each integer $q \geq 2$, not a multiple of p, the curve*

$$\Gamma = \operatorname{Spec}\left(\boldsymbol{k}(t)[x, y]/(x + a^{p^2} x^{pq} - b^{p^2} y^{p^2} - t)\right)$$

is not isomorphic to $\mathbb{A}^1_{\boldsymbol{k}(t)}$, but after the extension of scalars to $\boldsymbol{k}(t^{1/p})$ we have an isomorphism

$$\Gamma_{\boldsymbol{k}(t^{1/p})} \xrightarrow{\simeq} \mathbb{A}^1_{\boldsymbol{k}(t^{1/p})}.$$

Proof. After extending the scalars to $\mathbf{k}(t^{1/p^2})$, the curve Γ becomes

$$\begin{aligned} \Gamma_{\mathbf{k}(t^{1/p^2})} &= \operatorname{Spec}\left(\mathbf{k}(t^{1/p^2})[x, y]/(x + a^{p^2} x^{pq} - b^{p^2} y^{p^2} - t)\right) \\ &= \operatorname{Spec}\left(\mathbf{k}(t^{1/p^2})[x, y]/\left(x + a^{p^2} x^{pq} - b^{p^2}\left(y + \tfrac{t^{1/p^2}}{b}\right)^{p^2}\right)\right). \end{aligned}$$

Replacing y with $y + \frac{t^{1/p^2}}{b}$ and applying Lemma 3.14 we obtain an isomorphism

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbf{k}(t^{1/p^2})} & \xrightarrow{\simeq} & \Gamma_{\mathbf{k}(t^{1/p^2})} \\ u & \mapsto & \left(u^{p^2}, \frac{1}{b}\left(au^{pq} + u - t^{1/p^2}\right)\right). \end{array}$$

Replacing then u with $u + t^{1/p^2}$ we get an isomorphism defined over $\mathbf{k}(t^{1/p})$:

$$\begin{array}{cccc} \nu\colon & \mathbb{A}^1_{\mathbf{k}(t^{1/p})} & \xrightarrow{\simeq} & \Gamma_{\mathbf{k}(t^{1/p})} \\ & u & \mapsto & \left(u^{p^2} + t, \frac{1}{b}\left(a\left(u^p + t^{1/p}\right)^q + u\right)\right). \end{array}$$

It remains that no isomorphism $\hat{\nu}\colon \mathbb{A}^1_{\mathbf{k}(t)} \xrightarrow{\simeq} \Gamma_{\mathbf{k}(t)}$ exists. If $\hat{\nu}$ exists, then $\nu^{-1}\hat{\nu} \in \operatorname{Aut}(\mathbb{A}^1_{\mathbf{k}(t^{1/p})})$ would be given by $u \mapsto \alpha u + \beta$, with $\alpha, \beta \in \mathbf{k}(t^{1/p})$, $\alpha \neq 0$. The second coordinate of $\hat{\nu}(u)$ would then be

$$\frac{1}{b}\left(a\left((\alpha u + \beta)^p + t^{1/p}\right)^q + (\alpha u + \beta)\right) \in \mathbf{k}(t)[u].$$

The coefficient of u being $\frac{\alpha}{b}$, we get $\alpha \in \mathbf{k}(t)$. Remembering that $q \geq 2$, the coefficient of $u^{p(q-1)}$ is equal to $\frac{a}{b}q\alpha^{p(q-1)}(\beta^p + t^{1/p})$. As $\beta^p \in \mathbf{k}(t)$ we have $\beta^p + t^{1/p} \notin \mathbf{k}(t)$. Impossible, since q is not a multiple of p and $\alpha, a \neq 0$. □

Corollary 3.16. *For each field $\boldsymbol{k}$ of characteristic $p > 0$, each integer $q \geq 2$ which is not a multiple of p, each $\lambda, \mu \in \boldsymbol{k}^*$, and each integer $n \geq 2$, the polynomial*

$$f = x_1 + \lambda x_1^{pq} + \mu x_2^{p^2} \in \boldsymbol{k}[x_1, x_2] \subset \boldsymbol{k}[x_1, \ldots, x_n]$$

is not a variable of the $\boldsymbol{k}$-algebra $\boldsymbol{k}[x_1, \ldots, x_n]$.

However, when $\boldsymbol{k}$ is algebraically closed, the $\boldsymbol{k}$-algebra $\boldsymbol{k}[x_1, \ldots, x_n]/(f - t)$ is isomorphic to $\boldsymbol{k}[x_1, \ldots, x_{n-1}]$ for each $t \in \boldsymbol{k}$.

Proof. Showing that f is not a variable of $\mathbf{k}[x_1, \ldots, x_n]$ is equivalent to asking that the $\mathbf{k}[t]$-algebra $\mathbf{k}[x_1, x_2, \ldots, x_n, t]/(f-t)$ not be a polynomial ring in $n-1$ variables over $\mathbf{k}[t]$ (Lemma 3.3). It suffices then to show that $A_n = \mathbf{k}(t)[x_1, \ldots, x_n]/(f - t)$ is not a polynomial ring in $n - 1$ variables over $\mathbf{k}(t)$.

We first prove the result for $n = 2$. By extending the scalars, we can assume that $\lambda = a^{p^2}$ and $\mu = -b^{p^2}$ for some $a, b \in \mathbf{k}^*$. Lemma 3.15 then shows that $A_2 = \mathbf{k}(t)[x_1, x_2]/(f - t)$ is not a polynomial ring in one variable.

As $A_n = A_2[x_3, \ldots, x_n]$, the positive answer to the cancellation problem for curves [AHE72] implies that A_n is not a polynomial ring in $n - 1$ variables over $\mathbf{k}(t)$ for each $n \geq 2$.

It remains to assume that $\mathbf{k}$ is algebraically closed and to show that the $\mathbf{k}$-algebra $\mathbf{k}[x_1, \ldots, x_n]/(f - t)$ is isomorphic to $\mathbf{k}[x_1, \ldots, x_{n-1}]$ for each $t \in \mathbf{k}$. Replacing x_2 with $x_2 + \nu$ for some $\nu \in \mathbf{k}$, we only need to consider the case $t = 0$, which follows from Lemma 3.14. □

Proposition 3.17. *For each field $\boldsymbol{k}$ of characteristic $p > 0$, each integer $q \geq 2$ which is not a multiple of p, each $a \in \boldsymbol{k}^*$ and each $n \geq 1$, the morphism*

$$\begin{array}{cccc} \rho\colon & \mathbb{A}^n & \hookrightarrow & \mathbb{A}^{n+1} \\ & (x_1, \ldots, x_n) & \mapsto & (x_1^{p^2}, a x_1^{pq} + x_1, x_2, \ldots, x_n) \end{array}$$

is a closed embedding, which is not equivalent to the standard one.

Proof. It follows from Lemma 3.14 that ρ is a closed embedding and that its image is given by the hypersurface with equation $f = 0$, where

$$f = x_1 + a^{p^2} x_1^{pq} - x_2^{p^2} \in \mathbf{k}[x_1, x_2] \subset \mathbf{k}[x_1, \ldots, x_n].$$

It remains to show that f is not a variable of $\mathbf{k}[x_1, \ldots, x_n]$, which follows from Corollary 3.16. □

4. Liftings of automorphisms and the proof of Theorem 2

4.1. Lifting of automorphisms of $\mathbb{A}^3_{\mathbf{k}}$ to affine modifications.

Proposition 4.1. *Let $\boldsymbol{k}$ be a field and let*

$$R = \boldsymbol{k}[t, u, x, y]/(t^n u - h(t, x, y)),$$

where $n \geq 1$ and $h \in \boldsymbol{k}[t, x, y]$ is a polynomial such that $h_0 = h(0, x, y) \in \boldsymbol{k}[x, y]$ does not belong to $\boldsymbol{k}[w]$ for each variable $w \in \boldsymbol{k}[x, y]$.

(1) *Every element of* $R \setminus \boldsymbol{k}[t,x,y]$ *can be written as*

$$s + \sum_{i=1}^{m} f_i u^i$$

where $s \in \boldsymbol{k}[t,x,y]$, $m \geq 1$, $f_1, \ldots, f_m \in \boldsymbol{k}[t,x,y]$ *are polynomials of degree* $< n$ *in* t, *and* $f_m \neq 0$.

(2) *If* $f \in R \setminus \boldsymbol{k}[t,x,y]$ *is written as in* (1) *and* $d = \nu(f_m)$ *is the valuation of* f_m *in* t, *then* $0 \leq d < n$ *and* $t^{mn-d}f = g(t,x,y) \in \boldsymbol{k}[t,x,y]$ *satisfies* $g(0,x,y) \in h_0 \cdot \boldsymbol{k}[x,y] \setminus \{0\}$.

(3) *Writing* $I \subset \boldsymbol{k}[t,x,y]$ *for the ideal* (t^n, h), *we have* $t^n R \cap \boldsymbol{k}[t,x,y] = I$.

(4) *Every element of* $\mathrm{Aut}_{\boldsymbol{k}[t]}(R)$ *preserves the sets* $\boldsymbol{k}[t,x,y]$ *and* I.

Proof. (1): We first prove that every element of $R \setminus \mathbf{k}[t,x,y]$ has the desired form. Every element of R can be written as $\sum_{i=0}^{m} f_i u^i$ for some polynomials $f_i \in \mathbf{k}[t,x,y]$. We denote by r the largest integer such that $\deg_t(f_r) \geq n$. If $r = 0$ or if no such integer exists, we are done. Otherwise, we write $f_r = t^n A + B$ for some $A, B \in \mathbf{k}[t,x,y]$ with $\deg_t(B) < n$. Then, replacing $f_{r-1}u^{r-1} + f_r u^r = f_{r-1}u^{r-1} + (t^n A + B)u^r$ in $\sum_{i=0}^{m} f_i u^i$ with $(f_{r-1} + h(t,x,y)A)u^{r-1} + Bu^r$ decreases the integer r. After finitely many such steps, we obtain the desired form.

(2): We write $f = s + \sum_{i=1}^{m} f_i u^i = s + \sum_{i=1}^{m} f_i \frac{h^i}{t^{ni}}$ as in (1), we write $d = \nu(f_m)$, which satisfies $0 \leq d < n$ (since $f_m \neq 0$ and $\deg_t(f_m) < n$), and obtain $g = t^{mn-d}f = st^{mn-d} + \sum_{i=1}^{m} f_i h^i t^{mn-d-ni}$. In particular, $g \in \mathbf{k}[t,x,y]$ and it satisfies $g(0,x,y) = (h_0)^m \cdot r$, where $r \in \mathbf{k}[x,y]$ is obtained by replacing $t = 0$ in $\frac{f_m}{t^d} \in \mathbf{k}[t,x,y]$. From $\{r, h_0\} \subset \mathbf{k}[x,y] \setminus \{0\}$, we deduce $g(0,x,y) \neq 0$.

(3): The inclusion $I \subset t^n R$ follows from $\{t^n, h\} = \{t^n \cdot 1, t^n \cdot u\} \subset t^n R$. To show that $t^n R \cap \mathbf{k}[t,x,y] \subset I$, we take $f \in R$ such that $t^n f \in \mathbf{k}[t,x,y]$ and show that $t^n f \in I$. If $f \in \mathbf{k}[t,x,y]$, then $t^n f \in t^n \mathbf{k}[t,x,y] \subset I$. Otherwise, we write $f = s + \sum_{i=1}^{m} f_i u^i$ as in (1) and use (2) to obtain that $g = t^{mn-d}f \in \mathbf{k}[t,x,y]$ with $0 \leq d = \nu(f_m) < n$, and we get $g(0,x,y) \neq 0$. The fact that $t^n f \in \mathbf{k}[t,x,y]$ implies then that $n > mn - d$, whence $n > d > (m-1)n$, so $m = 1$. Hence $t^n f = t^n(s + f_1 u) = t^n s + hf_1 \in I$.

(4): Using (3) it suffices to show that every $\psi \in \mathrm{Aut}_{\mathbf{k}[t]}(R)$ preserves $\mathbf{k}[t,x,y]$. The algebra R is canonically isomorphic to $\mathbf{k}[t,x,y][\frac{h}{t^n}] \subset \mathbf{k}(t)[x,y]$. Since $\mathbf{k}(t)[x,y]$ is the localisation of $\mathbf{k}[t,x,y][\frac{h}{t^n}]$ in the multiplicative system $\mathbf{k}[t] \setminus \{0\}$, we get a natural inclusion $\mathrm{Aut}_{\mathbf{k}[t]}(R) \subset \mathrm{Aut}_K K[x,y]$, with $K = \mathbf{k}(t)$.

Suppose for contradiction that some $\psi \in \mathrm{Aut}_{\mathbf{k}[t]}(R)$ satisfies $\psi(\mathbf{k}[t,x,y]) \not\subset \mathbf{k}[t,x,y]$. This implies that $\psi(x) \notin \mathbf{k}[t,x,y]$ or $\psi(y) \notin \mathbf{k}[t,x,y]$. We assume that $\psi(x) \notin \mathbf{k}[t,x,y]$ (the case $\psi(y) \notin \mathbf{k}[t,x,y]$ being similar) and use (2) to obtain an integer $l > 0$ such that $g = t^l \psi(x) \in \mathbf{k}[t,x,y]$ satisfies $g(0,x,y) \in h_0 \cdot \mathbf{k}[x,y] \setminus \{0\}$. Since $\psi \in \mathrm{Aut}_{\mathbf{k}[t]}(R) \subset \mathrm{Aut}_K K[x,y]$, the element $\psi(x)$ is a variable of $K[x,y]$ and the same holds for $g(t,x,y) = t^l \psi(x)$. By Lemma 3.8, $g(0,x,y)$ belongs to $\mathbf{k}[w]$ for some variable $w \in \mathbf{k}[x,y]$. The fact that $g(0,x,y) \in h_0 \cdot \mathbf{k}[x,y] \setminus \{0\}$ implies then that $h_0 \in \mathbf{k}[w]$ (Lemma 3.4), contradicting the hypothesis. □

Corollary 4.2. *Let* $\boldsymbol{k}$ *be a field and*

$$R = \boldsymbol{k}[t,u,x,y]/(t^n u - h(t,x,y)),$$

where $n \geq 1$ *and* $h \in \boldsymbol{k}[t,x,y]$ *is a polynomial such that* $h_0 = h(0,x,y) \in \boldsymbol{k}[x,y]$ *does not belong to* $\boldsymbol{k}[w]$ *for each variable* $w \in \boldsymbol{k}[x,y]$. *Writing* I *the ideal* $(t^n, h) \subset$

$\boldsymbol{k}[t,x,y]$, we obtain a group isomorphism

$$\begin{array}{ccc} \mathrm{Aut}_{\boldsymbol{k}[t]}(R) & \xrightarrow{\simeq} & \{\psi \in \mathrm{Aut}_{\boldsymbol{k}[t]}(\boldsymbol{k}[t,x,y]) \mid \psi(I) = I\} \\ \varphi & \mapsto & \varphi|_{\boldsymbol{k}[t,x,y]}. \end{array}$$

Proof. According to Proposition 4.1(4), every element $\varphi \in \mathrm{Aut}_{\mathbf{k}[t]}(R)$ preserves $\mathbf{k}[t,x,y]$ and I, and thus restricts to an element $\psi \in \mathrm{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t,x,y])$ that preserves I.

Conversely, each automorphism $\psi \in \mathrm{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t,x,y])$ that preserves I induces an automorphism of $R = \mathbf{k}[t,x,y][\frac{I}{t^n}] = \mathbf{k}[t,x,y][\frac{h}{t^n}]$. This latter is uniquely determined by ψ, since the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbf{k}[t,x,y])$ given by the inclusion $\mathbf{k}[t,x,y] \hookrightarrow R$ is birational. ☐

Remark 4.3. According to [KZ99, Definition 1.1 and Proposition 1.1], $\mathrm{Spec}(R)$ is the affine modification of $\mathbb{A}^3_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[t,x,y])$ with locus (I, t^n). It is thus natural that every automorphism of $\mathbb{A}^3_{\mathbf{k}}$ fixing the ideal and the fibration $(t,x,y) \mapsto t$ lifts to an automorphism of $\mathrm{Spec}(R)$. In fact, this holds more generally for any affine modification; see [KZ99, Corollary 2.2 (a)]. The interesting part of Corollary 4.2 consists then in saying that all automorphisms of the $\mathbf{k}[t]$-algebra R are obtained by a lift of an automorphism of this form.

4.2. **Application of liftings to the case of SL_2.** We will apply Corollary 4.2 to the variety $\mathrm{SL}_2 \subset \mathbb{A}^4$ given by

$$\mathrm{SL}_2 = \left\{ \begin{pmatrix} x & t \\ u & y \end{pmatrix} \in \mathbb{A}^4 \,\middle|\, xy - tu = 1 \right\},$$

and obtain Proposition 4.5 below. Before we give a proof, let us recall the following basic facts on the coordinate ring of the variety SL_2.

Lemma 4.4. *Let R be the coordinate ring of SL_2, i.e., $R = \boldsymbol{k}[t,u,x,y]/(xy-tu-1)$. Then R is a unique factorisation domain and the units of R satisfy $R^* = \boldsymbol{k}^*$.*

Proof. Since the localisation $R_t = \mathbf{k}[t, \frac{1}{t}][x,y]$ is a unique factorisation domain, we only have to see that tR is a prime ideal of R, by [Mat89, Theorem 20.2]. This is the case, since $R/tR \simeq \mathbf{k}[u,x,y]/(xy-1)$ is an integral domain. Moreover, we have $R^* \subseteq (R_t)^* = \{\mu t^n \mid \mu \in \mathbf{k}^*, n \in \mathbb{Z}\}$. Since t^n is invertible in R if and only if $n = 0$, it follows that $R^* = \mathbf{k}^*$. ☐

In fact, by [Pop74, Proposition 1], the ring of regular functions of any simply connected algebraic group is a unique factorisation domain.

Proposition 4.5. *We consider the morphisms*

$$\begin{array}{ccccc} \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t,u,x,y]/(xy-tu-1)) & \xrightarrow{\eta} & \mathbb{A}^3_{\boldsymbol{k}} & \xrightarrow{\pi} & \mathbb{A}^1_{\boldsymbol{k}} \\ (t,u,x,y) & \mapsto & (t,x,y) & \mapsto & t \end{array}$$

and denote by $X \subset \mathrm{SL}_2$ the hypersurface given by $t = 1$ and by $\Gamma \subset \mathbb{A}^3_{\boldsymbol{k}}$ the closed curve given by $t = xy - 1 = 0$.

Then, the birational morphism $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\boldsymbol{k}}$ yields a group isomorphism

$$\begin{array}{ccc} \{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid g(X) = X\} & \xrightarrow{\simeq} & \{g \in \mathrm{Aut}(\mathbb{A}^3_{\boldsymbol{k}}) \mid \pi g = \pi, g(\Gamma) = \Gamma\} \\ g & \mapsto & \eta g \eta^{-1}. \end{array}$$

We moreover have

$$\begin{array}{rcl} \{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid g(X) = X\} & = & \{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid \pi\eta g = \pi\eta\} \\ & = & \{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid g^*(t) = t\}. \end{array}$$

Proof. Every automorphism g of $\mathbb{A}^3_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[t,x,y])$ yields an automorphism $g^* \in \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[t,x,y])$. Moreover, the condition $\pi g = g$ corresponds to $g^*(t) = t$, and the condition $g(\Gamma) = \Gamma$ to $g^*(I) = I$, where $I \subset \mathbf{k}[t,x,y]$ is the ideal of Γ, generated by t and $xy-1$. The isomorphism $\mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}}) \to \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[t,x,y])$ then yields a bijection

$$\begin{array}{ccc} \{g \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}}) \mid \pi g = \pi, g(\Gamma) = \Gamma\} & \xrightarrow{\simeq} & \{\psi \in \mathrm{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t,x,y]) \mid \psi(I) = I\} \\ g & \mapsto & g^*. \end{array}$$

We then want to apply Corollary 4.2 with $n = 1$ and $h = xy-1$. To check that it is possible, we need to see that h does not belong to $\mathbf{k}[w]$ for each variable $w \in \mathbf{k}[x,y]$. Indeed, $xy - 1 \in \mathbf{k}[w]$ would imply that $xy \in \mathbf{k}[w]$, and thus that $x, y \in \mathbf{k}[w]$, since $\mathbf{k}[w]$ is factorially closed (Lemma 3.4). This would yield $\mathbf{k}[w] = \mathbf{k}[x,y]$, a contradiction.

We then apply Corollary 4.2 and obtain a group isomorphism

$$\begin{array}{ccc} \mathrm{Aut}_{\mathbf{k}[t]}(R) & \xrightarrow{\simeq} & \{\psi \in \mathrm{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t,x,y]) \mid \psi(I) = I\} \\ \varphi & \mapsto & \varphi|_{\mathbf{k}[t,x,y]}, \end{array}$$

where $R = \mathbf{k}[t,u,x,y]/(tu - xy - 1)$. This yields then a group isomorphism

$$\begin{array}{ccc} \{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid \pi\eta g = \pi\eta\} & \xrightarrow{\simeq} & \{g \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}}) \mid \pi g = \pi, g(\Gamma) = \Gamma\} \\ g & \mapsto & \eta g \eta^{-1}. \end{array}$$

It remains then to show that

$$\{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid \pi\eta g = \pi\eta\} = \{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid g(X) = X\}.$$

The inclusion "$\subset$" follows from the equality $X = (\pi\eta)^{-1}(1)$. It remains then to show the inclusion "$\supset$".

To do this, we take $g \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $g(X) = X$ and prove that $\pi\eta g = \pi\eta$. The element g corresponds to an element $g^* \in \mathrm{Aut}_{\mathbf{k}}(R)$. The fact that $g(X) = X$ is then equivalent to asking if g^* sends the ideal generated by $t-1$ onto itself. Since $R^* = \mathbf{k}^*$ by Lemma 4.4, so $t-1$ is sent onto $\mu(t-1)$ for some $\mu \in \mathbf{k}^*$. This implies that the restriction of g^* yields an automorphism of $\mathbf{k}[t]$, corresponding to an automorphism $\hat{g} \in \mathrm{Aut}(\mathbb{A}^1_{\mathbf{k}})$ such that $\hat{g}\pi\eta = \pi\eta g$. As $(\pi\eta)^{-1}(0)$ is the only fibre of $\pi\eta$ that is not isomorphic to $\mathbb{A}^2_{\mathbf{k}}$, it has to be preserved under g. As the fibre $\pi^{-1}(1) = X$ is also preserved under g, we find that $\hat{g}$ is the identity, so $\pi\eta g = \pi\eta$, as desired. ☐

Corollary 4.6. *The closed embedding*

$$\begin{array}{cccc} \nu\colon & \mathbb{A}^2_{\boldsymbol{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (x,y) & \mapsto & \begin{pmatrix} x & 1 \\ xy-1 & y \end{pmatrix} \end{array}$$

has the following property: an automorphism of $\mathbb{A}^2_{\boldsymbol{k}}$ extends to an automorphism of SL_2, via ν, if and only if it has Jacobian determinant equal to ± 1.

Proof. We denote by $X = \nu(\mathbb{A}^2_{\mathbf{k}}) \subset \mathrm{SL}_2$ the closed hypersurface given by

$$X = \nu(\mathbb{A}^2_{\mathbf{k}}) = \left\{ \begin{pmatrix} x & 1 \\ xy-1 & y \end{pmatrix} \middle| (x,y) \in \mathbb{A}^2_{\mathbf{k}} \right\} = \left\{ \begin{pmatrix} x & t \\ u & y \end{pmatrix} \in \mathrm{SL}_2 \middle| t = 1 \right\},$$

write $G = \{g \in \mathrm{Aut}(\mathrm{SL}_2) \mid g(X) = X\}$, and denote by $\tau\colon G \to \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ the group homomorphism such that $g \circ \nu = \nu \circ \tau(g)$ for each $g \in G$.

We first prove that the subgroup $H = \{h \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \mid \mathrm{Jac}(h) \pm 1\}$ is contained in $\tau(G)$. The group H is generated by $(x,y) \mapsto (y,x)$, which is induced by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} y & t \\ u & x \end{pmatrix},$$

and by automorphisms of the form $(x,y) \mapsto (x, y + p(x))$, $p \in \mathbf{k}[x]$, induced by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ p(x) & 1 \end{pmatrix} \cdot \begin{pmatrix} x & t \\ u & y \end{pmatrix}.$$

It remains to take $g \in G$ and to prove that $\tau(g) \in H$. Proposition 4.5 implies that g can be written as

$$g \begin{pmatrix} x & t \\ u & y \end{pmatrix} = \begin{pmatrix} a(t,x,y) & t \\ s(t,u,x,y) & b(t,x,y) \end{pmatrix},$$

where $a, b \in \mathbf{k}[t,x,y]$, $s \in \mathbf{k}[t,u,x,y]$ and such that $\tilde{g}\colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^2_{\mathbf{k}}$ given by $\tilde{g}(t,x,y) = (t, a(t,x,y), b(t,x,y))$ is an automorphism of $\mathbb{A}^3_{\mathbf{k}}$ that preserves the curve Γ given by $t = xy - 1 = 0$. The Jacobian determinant of $\tilde{g}$ is $\mu = \frac{\partial a}{\partial x} \cdot \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \cdot \frac{\partial b}{\partial x} \in \mathbf{k}^*$. Replacing with $t = 0$, we obtain the automorphism of $\mathbb{A}^2_{\mathbf{k}}$ given by $(x,y) \mapsto (a(0,x,y), b(0,x,y))$, which preserves the curve with equation $xy = 1$ and is thus of Jacobian ± 1. Indeed, it is of the form $(x,y) \mapsto (\xi x, \xi^{-1} y)$ or $(x,y) \mapsto (\xi y, \xi^{-1} x)$ for some $\xi \in \mathbf{k}^*$ (see [BS15, Theorem 2 (*iii*)]). This shows that $\mu = \pm 1$. Replacing then $t = 1$ we get that the automorphism $\tau(g)$ which is given by $\tau(g)(x,y) = (a(1,x,y), b(1,x,y))$ has Jacobian ± 1. ☐

Proof of Theorem 2. We first observe that the embeddings $\rho_1, \bar{\nu}\colon \mathbb{A}^2_{\mathbf{k}} \to \mathrm{SL}_2$ given by

$$\begin{array}{rccc} \rho_1\colon & \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (a,b) & \mapsto & \begin{pmatrix} 1 & b \\ a & ab+1 \end{pmatrix}, \end{array} \qquad \begin{array}{rccc} \bar{\nu}\colon & \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (a,b) & \mapsto & \begin{pmatrix} a & 1 \\ -ab-1 & -b \end{pmatrix}, \end{array}$$

are equivalent, under the map $\left(\begin{smallmatrix} x & t \\ u & y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} u & x \\ -y & -t \end{smallmatrix}\right)$. The embedding

$$\begin{array}{rccc} \nu\colon & \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (x,y) & \mapsto & \begin{pmatrix} x & 1 \\ xy-1 & y \end{pmatrix} \end{array}$$

satisfies $\bar{\nu} = \nu\tau$, where $\tau \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ is the automorphism of Jacobian -1 given by $\tau\colon (x,y) \mapsto (x, -y)$. Corollary 4.6 then implies that there exists $\hat{\tau} \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\hat{\tau}\nu = \nu\tau = \overline{\nu}$, i.e., that the embeddings $\overline{\nu}$ and ν are equivalent, so ν and ρ_1 are equivalent. Corollary 4.6 implies then that an automorphism of $\mathbb{A}^2_{\mathbf{k}}$ extends to an automorphism of SL_2, via ρ_1, if and only if it has Jacobian determinant equal to ± 1. It remains to prove Assertions (1) and (2) of Theorem 2.

Assertion (2) follows from the fact that the group homomorphism

$$\mathrm{Jac}\colon \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \to \mathbf{k}^*$$

is surjective (taking for instance diagonal automorphisms), so there are automorphisms of Jacobian determinant in $\mathbf{k}^* \setminus \{\pm 1\}$ if and only if $\mathbf{k}$ contains at least four elements.

To obtain Assertion (1), we observe that every closed embedding $\mathbb{A}^2_{\mathbf{k}} \to \mathrm{SL}_2$ having image in $\rho_1(\mathbb{A}^2)$ is of the form $\rho_1\nu$ for some $\nu \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$. Writing $d_\lambda \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ for the automorphism given by $(s,t) \mapsto (\lambda s, t)$, $\lambda \in \mathbf{k}^*$, we can write

$\nu = d_\lambda \nu_1$ for some $\lambda \in \mathbf{k}^*$ and some $\nu_1 \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ of Jacobian determinant equal to 1. The result above implies that $\rho_1\nu$ is equivalent to $\rho_1 d_\lambda = \rho_\lambda$.

It remains to observe that $\rho_{\lambda'} = \rho_\lambda d_{\lambda'\lambda^{-1}}$, so ρ_λ and $\rho_{\lambda'}$ are equivalent if and only if $\lambda'\lambda^{-1} \in \{\pm 1\}$, which corresponds to $\lambda' = \pm\lambda$. □

Remark 4.7. Over the field $\mathbf{k} = \mathbb{C}$ of complex numbers, all algebraic embeddings of $\mathbb{C}^2$ into $\mathrm{SL}_2(\mathbb{C})$ with image equal to $\rho_1(\mathbb{C}^2)$ are equivalent under holomorphic automorphisms of $\mathrm{SL}_2(\mathbb{C})$. Indeed, according to Theorem 2(1) it is enough to show that the embeddings

$$\begin{array}{rccc} \rho_1\colon & \mathbb{C}^2 & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} 1 & t \\ s & st+1 \end{pmatrix}, \end{array} \qquad \begin{array}{rccc} \rho_\lambda\colon & \mathbb{C}^2 & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} 1 & t \\ \lambda s & \lambda st+1 \end{pmatrix}, \end{array}$$

are equivalent under a holomorphic automorphism for all $\lambda \in \mathbb{C}^*$. Such a holomorphic automorphism of $\mathrm{SL}_2(\mathbb{C})$ is given by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & t \\ \mu(x)u & y + \frac{\mu(x)-1}{x} tu \end{pmatrix},$$

where $\mu\colon \mathbb{C} \to \mathbb{C}^*$ is a holomorphic function with $\mu(1) = \lambda$ and $\mu(0) = 1$.

5. Fibred embeddings of $\mathbb{A}^2_{\mathbf{k}}$ into SL_2 and the proof of Theorem 3

In this section, we study fibred embeddings as in (♢) from Definition 1.2.

We will need the following simple description of the morphism $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\mathbf{k}}$ already studied in Proposition 4.5.

Lemma 5.1. *Let $\Gamma \subset \mathbb{A}^3_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t,x,y])$ be the curve given by $t = xy-1 = 0$ and let $\eta\colon \mathrm{B}\ell_\Gamma(\mathbb{A}^3_{\boldsymbol{k}}) \to \mathbb{A}^3_{\boldsymbol{k}}$ be the blow-up of Γ. We then have a natural open embedding $\mathrm{SL}_2 \hookrightarrow \mathrm{B}\ell_\Gamma(\mathbb{A}^3_{\boldsymbol{k}})$ such that the restriction of η corresponds to $(t,u,x,y) \mapsto (t,x,y)$.*

For completeness we insert a proof of this easy fact, which can in fact also be deduced from the more general statement [KZ99, Lemma 1.2], describing affine modifications as open subsets of blow-ups.

Proof. The blow-up of Γ can be seen as

$$\begin{array}{rcccc} \eta\colon & \mathrm{B}\ell_\Gamma(\mathbb{A}^3_{\mathbf{k}}) & = & \{((t,x,y),[u:v]) \in \mathbb{A}^3_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \mid tu = (xy-1)v\} & \to & \mathbb{A}^3_{\mathbf{k}} \\ & & & ((t,x,y),[u:v]) & \mapsto & (t,x,y). \end{array}$$

The open subset of $\mathrm{B}\ell_\Gamma(\mathbb{A}^3_{\mathbf{k}})$ given by $v \neq 0$ is then naturally isomorphic to SL_2, by identifying $((t,x,y),[u:1])$ with $(t,u,x,y) \in \mathrm{SL}_2$, and the birational morphism $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\mathbf{k}}$ sends $((t,x,y),[u:1])$ onto (t,x,y). □

5.1. **Polynomials associated to fibred embeddings.** The following result associates to every fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ a polynomial in $\mathbf{k}[t,x,y]$, and gives some basic properties of this polynomial (which will be studied in more detail after).

Lemma 5.2. *Let $\rho\colon \mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$ be a fibred embedding, and let $Z \subset \mathbb{A}^3_{\boldsymbol{k}}$ be the closure of $\eta(\rho(\mathbb{A}^2_{\boldsymbol{k}}))$, where*

$$\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\boldsymbol{k}}, \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto (t,x,y)\,.$$

Then, Z is given by $P(t,x,y)=0$, where $P \in \boldsymbol{k}[t,x,y]$ is a polynomial having the following properties:

(1) *The ring $\boldsymbol{k}[t,\frac{1}{t}][x,y]/(P)$ is a polynomial ring in one variable over $\boldsymbol{k}[t,\frac{1}{t}]$ (equivalently the morphism $\pi\colon Z \to \mathbb{A}^1_{\boldsymbol{k}}$ given by $(t,x,y)\mapsto t$ is a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\boldsymbol{k}} \setminus \{0\}$).*

(2) *If P is a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$ (which is always true if $\mathrm{char}(\boldsymbol{k})=0$ by (1) and the Abhyankar-Moh-Suzuki Theorem), then the polynomial $P(0,x,y)\in \boldsymbol{k}[x,y]$ is given by $\mu x^m(x-\lambda)$ or $\mu y^m(y-\lambda)$ for some $\mu,\lambda \in \boldsymbol{k}^*$ and some $m \geq 0$, and $\rho(\mathbb{A}^2_{\boldsymbol{k}}) \subset \mathrm{SL}_2$ is the hypersurface given by $P=0$.*

Proof. We consider the morphisms

$$\begin{array}{ccccc} \mathrm{SL}_2 = \mathrm{Spec}(\mathbf{k}[t,u,x,y]/(xy-tu-1)) & \xrightarrow{\eta} & \mathbb{A}^3_{\mathbf{k}} & \xrightarrow{\pi} & \mathbb{A}^1_{\mathbf{k}} \\ (t,u,x,y) & \mapsto & (t,x,y) & \mapsto & t \end{array}$$

and observe that η yields an isomorphism between the two open subsets $(\mathrm{SL}_2)_t \subset \mathrm{SL}_2$ and $(\mathbb{A}^3_{\mathbf{k}})_t \subset \mathbb{A}^3_{\mathbf{k}}$ given by $t \neq 0$. The morphism $\eta\rho\colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^3_{\mathbf{k}}$ restricts thus to a closed embedding $(\mathbb{A}^2_{\mathbf{k}})_t \hookrightarrow (\mathbb{A}^3_{\mathbf{k}})_t$, where $(\mathbb{A}^2_{\mathbf{k}})_t \subset \mathbb{A}^2_{\mathbf{k}}$ is the open subset where $t\neq 0$. This yields (1).

We now assume that $P \in \mathbf{k}[t,x,y]$ is a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$. Applying Lemma 3.8 we obtain that $P_0 = P(0,x,y) \in \mathbf{k}[w]$ for some variable $w \in \mathbf{k}[x,y]$. In particular, $xy-1$ does not divide P_0 (otherwise, by Lemma 3.4 we would have $xy-1 \in \mathbf{k}[w]$ and then $x,y\in\mathbf{k}[w]$, impossible). This implies that $Z \cap \Gamma$ is a 0-dimensional scheme (which is a priori not reduced), where $\Gamma \subset \mathbb{A}^3_{\mathbf{k}}$ is the closed curve given by $t = xy-1=0$. Recall that SL_2 is an open subset of $\mathrm{B}\ell_\Gamma(\mathbb{A}^3_{\mathbf{k}})$ (Lemma 5.1) and that the exceptional divisor $E \subset \mathrm{SL}_2$ is simply given by $t = xy-1=0$ and is a trivial $\mathbb{A}^1$-bundle over Γ. Since the pull-back $H \subset \mathrm{SL}_2$ of Z on SL_2, given by the equation $P=0$ has all its irreducible components of pure codimension one we get $H = \rho(\mathbb{A}^2_{\mathbf{k}}) \simeq \mathbb{A}^2_{\mathbf{k}}$. As ρ is a fibred embedding, the morphism $H \to \mathbb{A}^1_{\mathbf{k}}$ given by the projection on t is a trivial $\mathbb{A}^1$-bundle. This implies that $Z\cap\Gamma$ consists of a single reduced point, which is defined over $\mathbf{k}$ and thus of the form $q=(0,\lambda,\frac{1}{\lambda}) \in \Gamma$ for some $\lambda \in \mathbf{k}^*$.

We can thus write $P_0 \in \mathbf{k}[w]$ as $P_0 = ab$ where $a,b\in\mathbf{k}[w]$ are such that $b(q)\neq 0$, a is irreducible, and $a(q)=0$. This implies that a is a polynomial of degree 1 in w, so we can assume that $a=w$ (by replacing w with a).

We now show that $w = x-\lambda$ or $w = y-\frac{1}{\lambda}$ (after replacing w with μw, $\mu\in\mathbf{k}^*$), and can for this assume that $\mathbf{k}$ is algebraically closed. As w is a variable in $\mathbf{k}[x,y]$, the curve $C\subset \mathbb{A}^2_{\mathbf{k}}$ defined by $w=0$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}}$, and its closure in $\mathbb{P}^2_{\mathbf{k}}$ is a curve $\overline{C}$ passing through exactly one point q_0 of the line at infinity $L_\infty = \mathbb{P}^2_{\mathbf{k}} \setminus \mathbb{A}^2_{\mathbf{k}}$. Moreover, the tangent cone of $\overline{C}$ in q_0 is supported on only one line L. The closure of Γ is then the curve $\overline{\Gamma}\subset\mathbb{P}^2_{\mathbf{k}}$ given by $xy-z^2=0$, and $\overline{\Gamma}\setminus\Gamma = \{[1:0:0],[0:1:0]\}$. We apply Bézout's Theorem and find

$$2\deg(\overline{C}) = \overline{C}\cdot\overline{\Gamma} = (\overline{C}\cdot\overline{\Gamma})_q + (\overline{C}\cdot\overline{\Gamma})_{q_0} = 1 + (\overline{C}\cdot\overline{\Gamma})_{q_0}.$$

In particular, we get $q_0 \in \overline{\Gamma}$, so $q_0 \in \overline{\Gamma}\cap L_\infty = \{[1:0:0],[0:1:0]\}$. If $\overline{C}$ is a line, it has to be given by $x-\lambda z=0$ or $y-\frac{1}{\lambda}z=0$, as we already know that $\overline{C}$ passes through $[\lambda:\frac{1}{\lambda}:1]$. This gives $w=x-\lambda$ or $w=y-\frac{1}{\lambda}$, as desired. It remains to derive a contradiction from $\deg(\overline{C})>1$. We denote by m the multiplicity of $\overline{C}$ at q_0 and have $m<\deg(\overline{C})$, since $\overline{C}$ is irreducible and $\deg(\overline{C})>1$. We apply again Bézout's Theorem to obtain $\deg(\overline{C}) = \overline{C} \cdot L_\infty = (\overline{C} \cdot L_\infty)_{q_0}$. The inequality $(\overline{C} \cdot L_\infty)_{q_0} > m$ implies that $L_\infty = L$, so the tangent cone of $\overline{C}$ is supported on L_∞. In particular, the tangent cones of $\overline{C}$ and $\overline{\Gamma}$ at q_0 have distinct supports, because the conic $\overline{\Gamma}$ and the line L_∞ intersect transversally. This yields $m = (\overline{C} \cdot \overline{\Gamma})_{q_0}$, hence $2\deg(\overline{C}) = 1 + (\overline{C} \cdot \overline{\Gamma})_{q_0} = 1 + m < 1 + \deg(\overline{C})$, and contradicts the assumption $\deg(\overline{C}) > 1$.

Now that $w = x - \lambda$ is proven (respectively, $w = y - \frac{1}{\lambda}$), we obtain $P_0 = wb$ for some $b \in \mathbf{k}[x]$ (respectively, $b \in \mathbf{k}[y]$) which does not vanish on any point of Γ. Hence, P_0 is equal to $x^m(x - \lambda)$ or $y^m(y - \frac{1}{\lambda})$ for some $m \geq 0$, after replacing P with μP, $\mu \in \mathbf{k}^*$. ☐

We now give an example which shows that the polynomial P given in Lemma 5.2 is not always a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x, y]$ (even if P is always such a variable when $\mathrm{char}(\mathbf{k}) = 0$).

Lemma 5.3. *Let $\boldsymbol{k}$ be a field of characteristic $p > 0$ and let $q \geq 2$ be an integer that does not divide p. Then, the polynomial*

$$P = (x - 1) - t^p(y^p - (x - 1)^q)^p \in \boldsymbol{k}[t, x, y]$$

has the following properties:

1. *P is not a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x, y]$.*
2. *The hypersurface $Z_P \subset \mathbb{A}^3_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t, x, y])$ given by $P = 0$ satisfies that $Z_P \to \mathbb{A}^1_{\boldsymbol{k}}$, $(t, x, y) \mapsto t$ is a trivial $\mathbb{A}^1$-bundle (in particular, Z_P is isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$).*
3. *The hypersurface $H_P \subset \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t, u, x, y]/(xy - tu - 1))$ given by $P = 0$ is the image of a fibred embedding $\mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$ (in particular, H_P is isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$).*

Proof. (1): Replacing x with $x + 1$, it suffices to show that $x - t^p(y^p - x^q)^p$ is not a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x, y]$. This follows from Corollary 3.16.

(2) We consider the morphisms

$$\begin{array}{rccc} \tau\colon & \mathbb{A}^2_{\mathbf{k}} & \to & Z_P \\ & (s, t) & \mapsto & (t, t^p s^{p^2} + 1, t^q s^{pq} + s) \\ \chi\colon & Z_P & \to & \mathbb{A}^2_{\mathbf{k}} \\ & (t, x, y) & \mapsto & (y - t^q(y^p - (x - 1)^q)^q, t) \end{array}$$

and check that $\tau \circ \chi = \mathrm{id}_{Z_P}$, $\chi \circ \tau = \mathrm{id}_{\mathbb{A}^2_{\mathbf{k}}}$.

(3): The morphism $\eta\colon H_P \to Z_P$, $(t, u, x, y) \mapsto (t, x, y)$ being an isomorphism on the subsets given by $t \neq 0$, the morphism $\pi \circ \eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$, $(t, u, x, y) \mapsto t$ is a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$. The zero fibre is moreover isomorphic to $\mathbb{A}^1_{\mathbf{k}}$ since $P(0, x, y) = x - 1$ and the line $\{x = 1\}$ intersects the conic $\{xy = 1\}$ transversally in one point (follows from Lemma 5.1). By Lemma 3.10 it follows that $\pi \circ \eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$ is a trivial $\mathbb{A}^1$-bundle. Hence H_P is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ and is the image of a fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$. ☐

We now start from a polynomial $P \in \mathbf{k}[t, x, y]$ that is a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x, y]$ and determine when this one comes from a fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$, by the process determined in Lemma 5.2. This yields the following result, which corresponds to Part (1) of Theorem 3.

Proposition 5.4. *Let $\boldsymbol{k}$ be any field, let $P \in \boldsymbol{k}[t,x,y]$ be a polynomial that is a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$, and let*

$$H_P \subset \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t,u,x,y]/(xy-tu-1))$$

and $Z_P \subset \mathbb{A}^3_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t,x,y])$ be the hypersurfaces given by $P=0$.

The following conditions are equivalent:

(a) *The hypersurface $H_P \subset \mathrm{SL}_2$ is isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$.*
(b) *The hypersurface $H_P \subset \mathrm{SL}_2$ is the image of a fibred embedding $\mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$.*
(c) *The fibre of $Z_P \to \mathbb{A}^1_{\boldsymbol{k}}$, $(t,x,y) \mapsto t$ over every closed point of $\mathbb{A}^1_{\boldsymbol{k}} \setminus \{0\}$ is isomorphic to $\mathbb{A}^1$ and the polynomial $P(0,x,y) \in \boldsymbol{k}[x,y]$ is of the form $\mu x^m(x-\lambda)$ or $\mu y^m(y-\lambda)$ for some $\mu, \lambda \in \boldsymbol{k}^*$ and some $m \geq 0$.*

Proof. We will use the morphisms

$$\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\mathbf{k}}, \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto (t,x,y), \quad \pi\colon \mathbb{A}^3_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}, \quad (t,x,y) \mapsto t\,.$$

(a) $\Rightarrow$ (b): Proving that H_P is the image of a fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ is equivalent to asking that $\pi \circ \eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$ be a trivial $\mathbb{A}^1$-bundle. Since P is a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$, it follows that the generic fibre of $\pi\colon Z_P \to \mathbb{A}^1_{\mathbf{k}}$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$. Moreover, $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\mathbf{k}}$ is an isomorphism over $\{t \neq 0\}$, so the generic fibre of $\pi \circ \eta$ is also isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$. The fact that H_P is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ (which is the hypothesis (a)) implies that $\pi \circ \eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$ is a trivial $\mathbb{A}^1$-bundle, by Lemma 3.9 ((3) $\Rightarrow$ (4)).

(b) $\Rightarrow$ (c): Follows from Lemma 5.2(1) and (2).

(c) $\Rightarrow$ (a): Since $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\mathbf{k}}$ is an isomorphism over the open subset $\{t \neq 0\}$, it follows that all fibres of $\pi \circ \eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$ over closed points of $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$ are isomorphic to $\mathbb{A}^1$. Moreover, the fibre of $\pi \circ \eta$ over 0 is isomorphic to $\mathbb{A}^1_{\mathbf{k}}$, since the restriction $\eta|_{\{t=0\}}\colon \{t=0\} \to \{t = xy-1 = 0\} \subset \{0\} \times \mathbb{A}^2_{\mathbf{k}}$ is a trivial $\mathbb{A}^1$-bundle over the curve $\{t = xy-1 = 0\}$ and since $\{P(0,x,y) = 0\}$ intersects $\{xy = 1\}$ in exactly one point, transversally. The generic fibre of $\pi \circ \eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$ being isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$, it follows from Lemma 3.10 that $\pi \circ \eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$ is a trivial $\mathbb{A}^1$-bundle and thus H_P is isomorphic to the affine plane $\mathbb{A}^2_{\mathbf{k}}$, which proves (a). $\square$

Example 5.5. For each $n \geq 1, m \geq 0$, $\mu \in \mathbf{k}^*$, and $q \in \mathbf{k}[t,x]$, the polynomial

$$P(t,x,y) = t^n y + \mu x^m(x-1) + tq(t,x) \in \mathbf{k}[t,x,y]$$

defines a hypersurface $H_P \subset \mathrm{SL}_2$ which is the image of a fibred embedding. Indeed, since P has degree 1 in y with coefficent t^n, it is a variable of $\mathbf{k}[t,t^{-1}][x,y]$. We can thus apply Proposition 5.4 and only need to check that $P(0,x,y) = \mu x^m(x-1)$ is of the desired form (as in Assertion (c)).

5.2. **Determining when two fibred embeddings are equivalent.** In this section, we consider embeddings satisfying the conditions of Proposition 5.4 (or equivalently of Theorem 3(1)) and determine when two of these are equivalent, by proving Theorem 3(2). We first characterise the case where the integer m of Proposition 5.4 (or equivalently of Lemma 5.2 or Theorem 3(1)) is equal to zero.

Lemma 5.6. *Let $\boldsymbol{k}$ be any field and let $P \in \boldsymbol{k}[t,x,y]$ be a polynomial that is a variable of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x,y]$, and let $H_P \subset \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t,u,x,y]/(xy - tu-1))$ and $Z_P \subset \mathbb{A}^3_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t,x,y])$ be the hypersurfaces given by $P=0$.*

Assume that H_P is isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$, which implies that $P(0,x,y) \in \boldsymbol{k}[x,y]$ is of the form $\mu x^m(x-\lambda)$ or $\mu y^m(y-\lambda)$ for some $\mu, \lambda \in \boldsymbol{k}^*$ and some $m \geq 0$. Then, the following conditions are equivalent:

(a) *$m = 0$;*
(b) *P is a variable of the $\boldsymbol{k}[t]$-algebra $\boldsymbol{k}[t][x,y]$;*
(c) *there is an isomorphism $\varphi\colon \mathbb{A}^2_{\boldsymbol{k}} \xrightarrow{\simeq} Z_P$ such that $\pi\varphi$ is the projection $(t,x) \mapsto t$;*
(d) *there is an isomorphism $\varphi\colon \mathbb{A}^2_{\boldsymbol{k}} \xrightarrow{\simeq} Z_P$;*
(e) *there exist $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_P) = \rho_1(\mathbb{A}^2_{\boldsymbol{k}})$, where ρ_1 is the standard embedding;*
(f) *there exist $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_P) = \rho_1(\mathbb{A}^2_{\boldsymbol{k}})$ and $\varphi^*(t) = t$.*

Proof. As before, we use the morphisms

$$\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\mathbf{k}}, \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto (t,x,y), \quad \pi\colon \mathbb{A}^3_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}, \quad (t,x,y) \mapsto t.$$

Proposition 5.4 says that $H_P \subset \mathrm{SL}_2$ is the image of a fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$, which corresponds to saying that $\pi\eta\colon H_P \to \mathbb{A}^1_{\mathbf{k}}$ is a trivial $\mathbb{A}^1$-bundle. Since $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\mathbf{k}}$ is an isomorphism over the open subset $\{t \neq 0\}$, we obtain that $\pi\colon Z_P \to \mathbb{A}^1_{\mathbf{k}}$ is a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$.

We first prove (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d), using Corollary 3.13. We observe that (b), (c), and (d) correspond, respectively, to the equivalent assertions (i), (ii), and (iii) of Corollary 3.13. Moreover, the condition $m = 0$ (which is (a)) corresponds to saying that the 0-fibre of $\pi\colon Z_P \to \mathbb{A}^1_{\mathbf{k}}$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}}$. Since $\pi\colon Z_P \to \mathbb{A}^1_{\mathbf{k}}$ is a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$, Assertion (a) corresponds to Assertion (vi) of Corollary 3.13. Thus Corollary 3.13 yields

$$\text{(a)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)}.$$

It remains to show that these are also equivalent to (e) and (f).

(b) $\Rightarrow$ (f): Applying an automorphism of the form

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} \mu^{-1}x & t \\ u & \mu y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} \mu^{-1}y & t \\ u & \mu x \end{pmatrix}$$

for some $\mu \in \mathbf{k}^*$, we can assume that $P(0,x,y) = x - 1$. Since P is a variable of the $\mathbf{k}[t]$-algebra $\mathbf{k}[t][x,y]$, there exists $f \in \mathrm{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t,x,y])$ such that $f(x-1) = P$. The element $\psi \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}})$ satisfying $\psi^* = f$ is then such that $\pi\psi = \pi$ and sends Z_P onto the hypersurface of $\mathbb{A}^3_{\mathbf{k}}$ given by $x = 1$. The restriction of ψ to the hypersurface given by $t = 0$ is an automorphism of the form $(0,x,y) \mapsto (0, \nu(x,y), \rho(x,y))$ which preserves the curve given by $x - 1 = 0$. Replacing ψ with its composition with the inverse of $(t,x,y) \mapsto (t, \nu(x,y), \rho(x,y))$, we can assume that the restriction of ψ to the hypersurface $t = 0$ is the identity, so $\psi(\Gamma) = \Gamma$, where Γ is the curve given by $t = xy - 1 = 0$. Proposition 4.5 implies then that ψ lifts to an automorphism φ of SL_2 sending H_P onto $\rho_1(\mathbb{A}^2_{\mathbf{k}})$. We moreover have $\varphi^*(t) = t$, since $\psi^*(t) = t$.

(f) $\Rightarrow$ (e) being clear, it remains to show (e) $\Rightarrow$ (a). For this implication, one can assume that $\mathbf{k}$ is algebraically closed. Assertion (e) yields an automorphism $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_P) = \rho_1(\mathbb{A}^2_{\mathbf{k}})$. Hence the automorphism $\varphi^* \in \mathrm{Aut}_{\mathbf{k}}(R)$, where $R = \mathrm{Spec}(\mathbf{k}[t,u,x,y]/(xy - tu - 1))$, sends the ideal $(x-1) \subset R$ onto the ideal $(P) \subset R$. It follows from Lemma 4.4 that $x - 1$ is sent onto μP, for some $\mu \in \mathbf{k}^*$.

In particular, for a general $a \in \mathbf{k}$, the variety $H_{P-a} \subset \mathrm{SL}_2$ given by $P - a = 0$ is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$. It remains to show that this implies that $m = 0$.

Since P is a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x, y]$, so is $P-a$. There exists then an open dense subset $U \subset \mathbb{A}^1_{\mathbf{k}}$ such that $U \times \mathbb{A}^2_{\mathbf{k}} \to U \times \mathbb{A}^1_{\mathbf{k}}$, $(t, x, y) \mapsto (t, P(t, x, y) - a)$ is a trivial $\mathbb{A}^1$-bundle. This implies that $q_a \colon H_{P-a} \to \mathbb{A}^1_{\mathbf{k}}$ $(t, u, x, y) \mapsto t$ is a trivial $\mathbb{A}^1$-bundle over U. By Lemma 3.9, q_a is a trivial $\mathbb{A}^1$-bundle (since $H_{P-a} \simeq \mathbb{A}^2_{\mathbf{k}}$), so the fibre $(q_a)^{-1}(\{0\})$ needs to be isomorphic to an affine line. Since $(q_a)^{-1}(\{0\})$ is given by the equations $xy - 1 = P(0, x, y) - a = t = 0$ in the affine 4-space $\mathbb{A}^4_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[t, u, x, y])$ and since $P(0, x, y) - a$ is equal to $\mu x^m(x - \lambda) - a$ or $\mu y^m(y - \lambda) - a$ and $a \in \mathbf{k}$ is general ($\mathbf{k}$ is algebraically closed), this implies that $m = 0$ and yields (a) as desired. ☐

Remark 5.7. Lemma 5.6 shows in particular that if $H_P, H_Q \subset \mathrm{SL}_2$ are two hypersurfaces given by two polynomials $P, Q \in \mathbf{k}[t, x, y]$ as in Theorem 3 (or as in the previous results), and if one of the two integers $m, m' \in \mathbb{N}$ associated to P, Q is equal to zero, then H_P, H_Q are equivalent if and only if $m = m' = 0$.

Proposition 5.8. *Let $\boldsymbol{k}$ be any field, let $P, Q \in \boldsymbol{k}[t, x, y]$ be polynomials that are variables of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x, y]$, and let*

$$H_P, H_Q \subset \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t, u, x, y]/(xy - tu - 1))$$

and $Z_P, Z_Q \subset \mathbb{A}^3_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t, x, y])$ be the hypersurfaces given by $P = 0$ and $Q = 0$, respectively.

Suppose that H_P is isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$ but that Z_P is not isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$, and that there exists $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ that sends H_P onto H_Q. Then, the following hold:

1. *There exists $\mu \in \boldsymbol{k}^*$ such that $\varphi^*(t) = \mu t$.*
2. *The birational map $\psi = \eta\varphi\eta^{-1}$ is an automorphism of $\mathbb{A}^3_{\boldsymbol{k}}$ which sends Z_P onto Z_Q, where $\eta\colon \mathrm{SL}_2 \to \mathbb{A}^3_{\boldsymbol{k}}$ is as before given by $(t, u, x, y) \mapsto (t, x, y)$.*
3. *There exists $m \geq 1$ such that $P(0, x, y)$ and $Q(0, x, y)$ are of the form $\mu x^m(x - \lambda)$ or $\mu y^m(y - \lambda)$ for some $\mu, \lambda \in \boldsymbol{k}^*$ (the integer m is the same for P, Q but μ, λ and the choice between x and y depend on P, Q).*

Proof. Since H_P is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$, the same holds for H_Q. The hypersurfaces $H_P, H_Q \subset \mathrm{SL}_2$ are thus images of fibred embeddings $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ and there are thus integers $m, m' \geq 0$ and $\lambda, \lambda', \mu', \mu \in \mathbf{k}^*$ such that $P(0, x, y) \in \{\mu x^m(x-\lambda), \mu y^m(y-\lambda)\}$ and $Q(0, x, y) \in \{\mu' x^{m'}(x - \lambda'), \mu' y^{m'}(y - \lambda')\}$ (Proposition 5.4). Moreover, the fact that Z_P is not isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ is equivalent to $m > 0$ and to the fact that H_P is not equivalent to the image $\rho_1(\mathbb{A}^2)$ of the standard embedding (Lemma 5.6). As H_P and H_Q are equivalent, the same holds for H_Q, so $m' > 0$.

The main part of the proof consists in proving (1). To do this, one can extend the scalars and assume $\mathbf{k}$ to be algebraically closed. We moreover have $\varphi^*(Q) = \xi P$ for some $\xi \in \mathbf{k}^*$ (follows from Lemma 4.4). Replacing P with ξP, we can assume that $\varphi^*(Q) = P$. For each $a \in \mathbf{k}^*$, the element φ then sends H_{P-a} onto H_{Q-a}, where $H_{P-a}, H_{Q-a} \subset \mathrm{SL}_2$ are given by the polynomials $P - a, Q - a \in \mathbf{k}[t, x, y]$. Since P, Q are variables of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x, y]$ and because the t-projections $Z_P \to \mathbb{A}^1$ and $Z_Q \to \mathbb{A}^1$ are trivial $\mathbb{A}^1$-bundles over $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$ (Lemma 5.2(1)), the polynomials P, Q are also variables of the $\mathbf{k}[t, \frac{1}{t}]$-algebra $\mathbf{k}[t, \frac{1}{t}][x, y]$ (follows from Lemma 3.11 with $U = \mathbb{A}^1 \setminus \{0\}$). Hence, the same holds for $P - a$ and $Q - a$. The morphisms $H_{P-a}, H_{Q-a}, Z_{P-a}, Z_{Q-a} \to \mathbb{A}^1_{\mathbf{k}}$ given by the projection on t are therefore trivial $\mathbb{A}^1$-bundles over $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$. We show now that the surfaces

$H_{P-a}, H_{Q-a}, Z_{P-a}, Z_{Q-a}$ are smooth for general $a \in \mathbf{k}^*$. The hypersurfaces Z_{P-a}, Z_{Q-a} are in fact isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ for general $a \in \mathbf{k}^*$, as P, Q are variables of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$. Since H_{P-a} is an open subset of the strict transform of Z_{P-a} under the blow-up $\mathrm{B}\ell_\Gamma(\mathbb{A}^3_{\mathbf{k}}) \to \mathbb{A}^3_{\mathbf{k}}$ of Γ (follows from Lemma 5.1), it is enough to show that $\Gamma \cap Z_{P-a}$ and $\Gamma \cap Z_{Q-a}$ are reduced for general $a \in \mathbf{k}^*$. The corresponding ideal is given by $f(s) = \varepsilon_1 s^m(s-\varepsilon_2) - a$ in $\mathcal{O}(\Gamma) = \mathbf{k}[s, \frac{1}{s}]$ for $\varepsilon_1, \varepsilon_2 \in \mathbf{k}^*$, $m \geq 0$. This ideal is reduced for general $a \in \mathbf{k}^*$, since the derivative of f is a non-zero polynomial, not depending on a. We can thus see these varieties as open subsets of smooth projective surfaces $\overline{H_{P-a}}, \overline{H_{Q-a}}, \overline{Z_{P-a}}, \overline{Z_{Q-a}}$ obtained by blowing-up some Hirzebruch surfaces, so that the projection on t is the restriction of the morphism to $\mathbb{P}^1_{\mathbf{k}}$ given by a $\mathbb{P}^1$-bundle of the Hirzebruch surface and having only one singular fibre. We can moreover assume that the boundary is a union of smooth rational curves of self-intersection 0 or ≤ -2 (in particular the projectivisation is minimal). Indeed, if a component of the singular fibre has self-intersection -1 and is in the boundary, we can contract it, and if the section has self-intersection -1, then we blow-up a general point of the smooth fibre contained in the boundary and then contract the strict transform of this fibre to obtain a section of self-intersection 0. The 0-fibre of $Z_{P-a} \to \mathbb{A}^1_{\mathbf{k}}$ is given by $t = \mu x^m(x-\lambda) - a = 0$ or $t = \mu y^m(y-\lambda) - a = 0$ and is thus a disjoint union $C \simeq \coprod_{i=1}^{m+1} \mathbb{A}^1_{\mathbf{k}}$ of $m+1$ affine curves isomorphic to $\mathbb{A}^1_{\mathbf{k}}$. Similarly the 0-fibre of $Z_{Q-a} \to \mathbb{A}^1_{\mathbf{k}}$ is a disjoint union $C' \simeq \coprod_{i=1}^{m'+1} \mathbb{A}^1_{\mathbf{k}}$ of $m'+1$ affine curves isomorphic to $\mathbb{A}^1_{\mathbf{k}}$. The closure of C is contained in the singular fibre F_0 of $\overline{Z_{P-a}} \to \mathbb{P}^1_{\mathbf{k}}$, which is a tree of smooth rational curves of self-intersection ≤ -1, being an SNC divisor. Hence, the closure of each component of C is a smooth rational curve of self-intersection ≤ -1, which intersects the boundary into a component lying in F_0. A similar description holds for C'.

The curves C, C' meet transversally the conic Γ given by $xy = 1$ (because of the form of $P(0,x,y) - a$ and $Q(0,x,y) - a$). The surfaces $\overline{H_{P-a}}, \overline{H_{Q-a}}$ are then obtained by blowing-up some points in each of the components of C, C' and removing these components, so we can choose the a minimal projectivisations of H_{P-a}, H_{Q-a} to be blowing-ups of the above points in $\overline{Z_{P-a}}, \overline{Z_{Q-a}}$ and get a dual graph of the boundary of these surfaces which is not a chain (or which is not "linear" or not a "zigzag"). This implies that the $\mathbb{A}^1$-fibration given by the t-projection is unique up to automorphisms of the target (see [Giz71] or [Ber83, Théorème 1.8]). As the zero fibre of $H_{P-a}, H_{Q-a} \to \mathbb{A}^1_{\mathbf{k}}$ is the unique degenerate fibre, there exist $\mu_a \in \mathbf{k}^*$ and $q_a \in \mathbf{k}[t,u,x,y]$ such that $\varphi^*(t) = \mu_a t + q_a \cdot (P-a)$. Since this holds for a general a, we get $\varphi^*(t) = \mu t$ for some $\mu \in \mathbf{k}^*$. Indeed, replacing t with 0 in $\varphi^*(t)$ yields an element of $\mathbf{k}[u,x,y]/(xy-1)$ which is divisible by $P-a$ for infinitely many a. This element is thus equal to zero.

We now show how Assertion (1) implies the two others. We write $\varphi = \varphi_1\varphi_2$ where $(\varphi_1)^*(t) = t$ and φ_2 is given by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & \mu t \\ \mu^{-1}u & y \end{pmatrix}.$$

The fact that $(\varphi_1)^*(t) = t$ implies that $\psi_1 = \eta\varphi_1\eta^{-1}$ is an automorphism of $\mathbb{A}^3_{\mathbf{k}}$ (Proposition 4.5). Since $\psi_2 = \eta\varphi_2\eta^{-1}$ is a diagonal automorphism of $\mathbb{A}^3_{\mathbf{k}}$, the element $\psi = \psi_1\psi_2 = \eta\varphi\eta^{-1}$ is an automorphism of $\mathbb{A}^3_{\mathbf{k}}$. As φ sends H_P onto H_Q, the automorphism ψ sends Z_P onto Z_Q, which yields (2). As $\psi^*(t) = \mu t$, the hyperplane $W \subset \mathbb{A}^3_{\mathbf{k}}$ given by $t = 0$ is invariant, this implies that $m = m'$ and thus yields (3). $\square$

Lemma 5.6 and Proposition 5.8 yield then the following result, which yields in particular Assertion (2) of Theorem 3.

Corollary 5.9. *If $P, Q \in \boldsymbol{k}[t, x, y]$ are polynomials which are variables of the $\boldsymbol{k}(t)$-algebra $\boldsymbol{k}(t)[x, y]$ and if the corresponding hypersurfaces $H_P, H_Q \subset \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t, u, x, y]/(xy - tu - 1))$ are equivalent and isomorphic to $\mathbb{A}^2_{\boldsymbol{k}}$, the following hold:*

1. *H_P, H_Q are the image of fibred embeddings $\mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$.*
2. *There exists $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_P) = H_Q$ and $\varphi^*(t) = \mu t$ for some $\mu \in \boldsymbol{k}^*$. In particular, the element $\psi = \eta\varphi\eta^{-1} \in \mathrm{Aut}(\mathbb{A}^3_{\boldsymbol{k}})$ satisfies $\psi^*(t) = \mu t$, $\psi(Z_P) = Z_Q$ and $\psi(\Gamma) = \Gamma$, where $\eta \colon \mathrm{SL}_2 \to \mathbb{A}^3_{\boldsymbol{k}}$ is the morphism $(t, u, x, y) \mapsto (t, x, y)$, $Z_P, Z_Q \subset \mathbb{A}^3_{\boldsymbol{k}}$ are the two hypersurfaces given by $P = 0$, $Q = 0$, and $\Gamma \subset \mathbb{A}^3_{\boldsymbol{k}}$ is the conic given by $t = xy - 1 = 0$.*

Proof. Assertion (1) follows from Proposition 5.4. It remains then to show (2). We denote by $\varphi_0 \in \mathrm{Aut}(\mathrm{SL}_2)$ an element such that $\varphi_0(H_P) = H_Q$.

(i) If $\varphi_0^*(t) = \mu t$ for some $\mu \in \mathbf{k}^*$, we choose $\varphi = \varphi_0$ and denote by $\theta \in \mathrm{Aut}(\mathrm{SL}_2)$ the element

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & \mu^{-1}t \\ \mu u & y \end{pmatrix}$$

to obtain $(\varphi_0\theta)^*(t) = t$. Proposition 4.5 shows that $\hat{\psi} = \eta(\varphi_0\theta)\eta^{-1} \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}})$ and $\hat{\psi}(\Gamma) = \Gamma$. Since $\tilde{\theta} = \eta\theta\eta^{-1}$ is the automorphism of $\mathbb{A}^3_{\mathbf{k}}$ given by $(t, x, y) \mapsto (\mu^{-1}t, x, y)$, we have $\psi = \eta\varphi_0\eta^{-1} \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}})$ and $\psi(\Gamma) = \Gamma$. The fact that $\varphi_0^*(t) = \mu t$ and $\varphi_0(H_P) = H_Q$ yields then $\psi^*(t) = \mu t$ and $\psi(Z_P) = Z_Q$.

(ii) If $\varphi_0^*(t) \notin \{\mu t \mid \mu \in \mathbf{k}^*\}$, then Proposition 5.8(1) does not hold, so Z_P is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$. Applying the same argument to φ_0^{-1} shows that Z_Q is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$. Lemma 5.6((d) $\Rightarrow$ (f)) then shows that there exist $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi_1(H_P) = \varphi_2(H_Q) = \rho_1(\mathbb{A}^2_{\mathbf{k}})$ and $(\varphi_1)^*(t) = (\varphi_2)^*(t) = t$. We then choose $\varphi = (\varphi_2)^{-1}\varphi_1$ and apply case (i). $\square$

5.3. **Examples of non-equivalent embeddings.**

Lemma 5.10. *To each polynomial $r \in \boldsymbol{k}[t]$, we associate the polynomial*

$$P_r = ty - (x - t)(x - 1 - t^2 r(t)) \in \boldsymbol{k}[t, x, y]$$

and denote by $H_{P_r} \subset \mathrm{SL}_2 = \mathrm{Spec}(\boldsymbol{k}[t, u, x, y]/(xy - tu - 1))$ and $Z_{P_r} \subset \mathbb{A}^3_{\boldsymbol{k}} = \mathrm{Spec}(\boldsymbol{k}[t, x, y])$ the hypersurfaces given by $P_r = 0$. Then,

1. *For each $r \in \boldsymbol{k}[t]$, the surface H_{P_r} is the image of a fibred embedding $\mathbb{A}^2_{\boldsymbol{k}} \hookrightarrow \mathrm{SL}_2$.*
2. *For each $r, s \in \boldsymbol{k}[t]$, the following are equivalent:*
   (i) *There exists $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_{P_r}) = H_{P_s}$.*
   (ii) *There exists $\varphi \in \mathrm{Aut}(\mathbb{A}^3_{\boldsymbol{k}})$ such that $\varphi(Z_{P_r}) = Z_{P_s}$.*
   (iii) *The surfaces Z_{P_r} and Z_{P_s} are isomorphic.*
   (iv) *$r = s$.*

Proof. For each $r \in \mathbf{k}[t]$, we write $S_r(t, x) = (x - t)(x - 1 - t^2 r(t)) \in \mathbf{k}[t, x]$ and observe that $P_r(t, x, y) = ty - S_r(t, x)$.

(1): Since P_r is of degree 1 in y, it is a variable of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$. Moreover, $P_r(0,x,y) = S_r(0,x) = x(x-1)$ is of the form $\mu x^m(x-\lambda)$ (with $\mu,\lambda \in \mathbf{k}^*$ and $m \geq 0$). The coefficient of y in P_r being t, the morphism $Z_{P_r} \to \mathbb{A}^1_{\mathbf{k}}$, $(t,x,y) \mapsto t$ is a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$. Proposition 5.4 ((c) $\Rightarrow$ (b)) then implies that $H_{P_r} \subset \mathrm{SL}_2$ is the image of a fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$.

It remains to show that the assertions (i) – (ii) – (iii) – (iv) of (2) are equivalent.

The implications (iv) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are trivial.

Lemma 5.6 implies that Z_{P_r} and Z_{P_s} are not isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ (the integer m being here equal to 1). We can thus apply Proposition 5.8(2), which yields (i) $\Rightarrow$ (ii).

It remains then to show (iii) $\Rightarrow$ (iv). According to [DP09, Proposition 3.6], the surface Z_{P_r} and Z_{P_s} are isomorphic if and only if there exist $a,\mu \in \mathbf{k}^*$, $\tau \in \mathbf{k}[t]$, such that

$$S_r(at,x) = \mu^2 S_s(t,\mu^{-1}x+\tau(t)) \quad \text{inside } \mathbf{k}[t,x].$$

This corresponds to

$$(x-at)(x-1-a^2t^2r(at)) = (x+\mu(\tau(t)-t))(x+\mu(\tau(t)-1-t^2s(t)))$$

and thus gives two possibilities:

(I): $at = \mu(t-\tau(t))$ and $1+a^2t^2r(at) = \mu(1+t^2s(t)-\tau(t))$. The first equation yields $\tau(t) = (1-\frac{a}{\mu})t$ and the second yields $\mu\tau(t) \equiv \mu - 1 \pmod{t^2}$, which gives $\tau = 0$ and then $\mu = 1$ and $a = 1$. The second equation thus yields $r(t) = s(t)$.

(II): $at = \mu(1+t^2s(t)-\tau(t))$ and $\mu(t-\tau(t)) = 1+a^2t^2r(at)$. This yields

$$1+a^2t^2r(at)-\mu t = -\mu\tau(t) = at - \mu(1+t^2s(t))$$

and thus $1-\mu t \equiv -\mu + at \pmod{t^2}$, whence $\mu = -1$ and $a = 1$. Replacing in the equation above, we find $r(t) = s(t)$. ☐

The proof of Theorem 3 is now clear.

Proof of Theorem 3. Assertion (1) corresponds to Proposition 5.4.

Assertion (2) follows from Corollary 5.9.

Assertion (3) follows from Lemma 5.10, which yields hypersurfaces $H_{P_r} \subset \mathrm{SL}_2$ that are parametrised by $r \in \mathbf{k}[t]$, which are all images of fibred embeddings and are pairwise non-equivalent. ☐

We finish this subsection with two explicit examples.

Lemma 5.11. *Let us denote by $P, Q \in \boldsymbol{k}[t,x,y]$ the polynomials*

$$P = t^2y - x(x+1) \quad \textit{and} \quad Q = t^2y - x(x+1-t^2)\,.$$

Then, the following hold:

1. *The hypersurfaces $Z_P, Z_Q \subset \mathbb{A}^3_{\boldsymbol{k}}$ given by $P = 0$ and $Q = 0$ are equivalent.*
2. *The hypersurfaces $H_P, H_Q \subset \mathrm{SL}_2$ given by $P = 0$ and $Q = 0$ are both images of fibred embeddings but are not equivalent.*

Proof. To get (1), it suffices to observe that the linear automorphism $\theta \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}})$ given by $(t,x,y) \mapsto (t,x,y-x)$ satisfies $\theta^*(Q) = P$, so $\theta(Z_P) = Z_Q$.

Since P, Q are of degree 1 in y, both are variables of the $\mathbf{k}(t)$-algebra $\mathbf{k}(t)[x,y]$. Moreover, $P(0,x,y) = Q(0,x,y) = -x(x+1)$ is of the form $\mu x^m(x-\lambda)$ (with $\mu,\lambda \in \mathbf{k}^*$ and $m = 1 \geq 0$). Since the coefficient of y in P and Q is t^2, the morphisms $Z_P, Z_Q \to \mathbb{A}^1_{\mathbf{k}}$, $(t,x,y) \mapsto t$ are trivial $\mathbb{A}^1$-bundles over $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$. Proposition 5.4

((c) $\Rightarrow$ (b)) then implies that $H_P, H_Q \subset \mathrm{SL}_2$ are images of fibred embeddings $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$.

To get (2), we suppose that there is $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_P) = H_Q$ and derive a contradiction. Corollary 5.9 yields an automorphism $\psi \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}})$ such that $\psi^*(t) = \mu t$ for some $\mu \in \mathbf{k}^*$ and such that $\psi(Z_P) = Z_Q$ and $\psi(\Gamma) = \Gamma$, where $\Gamma \subset \mathbb{A}^3_{\mathbf{k}}$ is the conic given by $t = xy - 1 = 0$. The restriction of ψ to the hyperplane $H \subset \mathbb{A}^3_{\mathbf{k}}$ given by $t = 0$ then preserves Γ and also the curve $C = H \cap Z_P = H \cap Z_Q$, given by $t = x(x+1) = 0$ (which is isomorphic to two copies of $\mathbb{A}^1$). The fact that C is preserved implies that $\psi_{|H}$ is of the form $(x, y) \mapsto (x, ay + p(x))$ or $(x, y) \mapsto (-1 - x, ay + p(x))$ for some $a \in \mathbf{k}^*$ and $p \in \mathbf{k}[x]$. The fact that Γ is preserved implies that $\psi_{|H} = \mathrm{id}$.

The element $\xi = \theta^{-1}\psi \in \mathrm{Aut}(\mathbb{A}^3_{\mathbf{k}})$ then satisfies $\xi(Z_P) = Z_P$, $\xi^*(t) = \mu t$ and $\xi_{|H}$ is the automorphism $(x, y) \mapsto (x, x + y)$. To show that this is impossible, we use [DP09, Theorem 3.11] to see that every automorphism of Z_P preserves C and its action on C corresponds to an element of the subgroup $G = G_0 \cup G_1 \simeq G_0 \rtimes (\mathbb{Z}/2\mathbb{Z})$ of $\mathrm{Aut}(C)$ given by

$$\begin{aligned} G_0 &= \{(x, y) \mapsto (x, \alpha y + (2x+1)\beta) \mid \alpha \in \mathbf{k}^*, \beta \in \mathbf{k}\} \\ G_1 &= \{(x, y) \mapsto (-1 - x, \alpha y + (2x+1)\beta) \mid \alpha \in \mathbf{k}^*, \beta \in \mathbf{k}\}. \end{aligned} \qquad \square$$

We then study an explicit example of a fibred embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ whose image is not equivalent to the standard embedding.

Example 5.12. According to the above study, the "simplest" example of a hypersurface $E \subset \mathrm{SL}_2$ being the image of a fibred embedding but not being equivalent to the image of the standard embedding is given by

$$E = \{(t, u, x, y) \in \mathbb{A}^4_{\mathbf{k}} \mid xy - tu = 1, ty = x(x-1)\}.$$

Indeed, using the polynomial $P = ty - x(x-1)$, which yields $P(0, x, y) = -x(x-1)$, the surface E is the image of a fibred embedding $\rho\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ (Example 5.5) but is not equivalent to $\rho_1(\mathbb{A}^2_{\mathbf{k}})$ (Lemma 5.6).

One can construct an explicit embedding $\rho\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ having image E in the following way. First, denoting by $E_t \subset E$ and $(\mathbb{A}^2_{\mathbf{k}})_t \subset \mathbb{A}^2_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[x, t])$ the open subsets given by $t \neq 0$, we get isomorphisms

$$\begin{array}{ccc} (\mathbb{A}^2_{\mathbf{k}})_t & \xrightarrow{\simeq} & E_t \\ (x, t) & \mapsto & \begin{pmatrix} x & t \\ \frac{x^2(x-1)-t}{t^2} & \frac{x(x-1)}{t} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} E_t & \xrightarrow{\simeq} & (\mathbb{A}^2_{\mathbf{k}})_t \\ \begin{pmatrix} x & t \\ u & y \end{pmatrix} & \mapsto & (x, t). \end{array}$$

To obtain a fibred embedding $\rho\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ having image equal to E, we need to remove the denominators of the isomorphism $(\mathbb{A}^2_{\mathbf{k}})_t \xrightarrow{\simeq} E_t$. We then compose with the automorphism of $(\mathbb{A}^2_{\mathbf{k}})_t$ given by $(x, t) \mapsto (t^2 x + t + 1, t)$ and get isomorphisms

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbf{k}} & \xrightarrow{\rho} & E \\ (x, t) & \mapsto & \begin{pmatrix} 1 + t + t^2 x & t \\ \frac{(1+t+t^2x)^2(t+t^2x)-t}{t^2} & \frac{(1+t+t^2x)(t+t^2x)}{t} \end{pmatrix} \end{array}$$

and

$$\begin{array}{ccc} E & \xrightarrow{\rho^{-1}} & \mathbb{A}^2_{\mathbf{k}} \\ \begin{pmatrix} x & t \\ u & y \end{pmatrix} & \mapsto & \left(\frac{x-t-1}{t^2}, t\right). \end{array}$$

We can observe that all components of ρ are indeed polynomials, and that $\frac{x-t-1}{t^2} \in \mathbf{k}[E]$. To show the latter, we compute $y = \frac{x(x-1)}{t} \in \mathbf{k}[E]$, $u = \frac{xy-1}{t} = \frac{x^2(x-1)-t}{t^2} \in \mathbf{k}[E]$, $y^2 - ux + u = \frac{x-1}{t} \in \mathbf{k}[E]$, which yields $\frac{x-t-1}{t^2} = u - (x+1)\left(\frac{x-1}{t}\right)^2 \in \mathbf{k}[E]$.

Writing

$$\tilde{\rho}(x,t) = A\rho(x,t)A \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathrm{SL}_2\,,$$

we get an equivalent closed embedding $\tilde{\rho}\colon \mathbb{A}^2_{\mathbf{k}} \to \mathrm{SL}_2$, which is an isomorphism

$$\begin{array}{cccc} \tilde{\rho}\colon & \mathbb{A}^2_{\mathbf{k}} & \xrightarrow{\simeq} & \tilde{E} \\ & (x,t) & \mapsto & \begin{pmatrix} 1+t^2x & t \\ x+3tx+2t^2x^2+2t^3x^2+t^4x^3 & 1+tx+2t^2x+t^3x^2 \end{pmatrix}, \end{array}$$

where $\tilde{E} = \{(t,u,x,y) \in \mathbb{A}^4_{\mathbf{k}} \mid xy - tu = 1, t(y+t) = (x+t)(x+t-1)\}$. The morphism $\tilde{\rho}$ corresponds to the closed embedding $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathbb{A}^4_{\mathbf{k}}$

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathbb{A}^4_{\mathbf{k}} \\ (x,t) & \mapsto & (t, 1+t^2x, 1+tx+2t^2x+t^3x^2, x+3tx+2t^2x^2+2t^3x^2+t^4x^3) \end{array}$$

that we can simplify using elementary automorphisms of $\mathbb{A}^4_{\mathbf{k}}$ to the embedding

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathbb{A}^4_{\mathbf{k}} \\ (x,t) & \mapsto & (t, t^2x, tx+t^3x^2, x+2t^2x^2-t^3x^2+t^4x^3) \\ & = & (t, t^2x, tx(1+t^2x), x+t^2x^2(2-t+t^2x)). \end{array}$$

Question 5.13. Is the closed embedding

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathbb{A}^4_{\mathbf{k}} \\ (x,t) & \mapsto & (t, t^2x, tx(1+t^2x), x+t^2x^2(2-t+t^2x)) \end{array}$$

equivalent to the standard one?

5.4. **Embeddings of $\mathbb{A}^2_{\mathbf{k}}$ into SL_2 of small degree.** Let $\iota\colon \mathrm{SL}_2 \hookrightarrow \mathbb{A}^4_{\mathbf{k}}$ be the standard embedding and let $f\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ be a closed embedding. This last subsection consists in showing the second part of Remark 1.1, which claims that if all coordinate functions $\iota \circ f\colon \mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2 \subset \mathbb{A}^4_{\mathbf{k}}$ are polynomials of degree ≤ 2, then f is equivalent to ρ_λ for a certain $\lambda \in \mathbf{k}^*$. This will be done in Proposition 5.19 below, after a few lemmas.

We first make the following easy observation.

Lemma 5.14. *For each fibred embedding*

$$\begin{array}{cccc} \rho\colon & \mathbb{A}^2_{\boldsymbol{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} a(s,t) & t \\ c(s,t) & b(s,t) \end{pmatrix} \end{array}$$

(with $a,b,c \in \boldsymbol{k}[s,t]$) there is an automorphism $g \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $g\rho$ is a fibred embedding given by

$$\begin{array}{cccc} g\rho\colon & \mathbb{A}^2_{\boldsymbol{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} 1+stp(s,t) & t \\ s(p(s,t)+q(s,t)+stp(s,t)q(s,t)) & 1+stq(s,t) \end{pmatrix} \end{array}$$

for some $p,q \in \boldsymbol{k}[s,t]$ such that $p(s,0)+q(s,0) \in \boldsymbol{k}^*$ and such that $\deg(1+stp(s,t)) \le \deg(a)$, $\deg(1+stq(s,t)) \le \deg(b)$ (where the degree is here the degree of polynomials in s,t).

Remark 5.15. The standard embedding ρ_1 is of the above form with $p = 0$ and $q = 1$. More generally, the embeddings $\{\rho_\lambda\}_{\lambda\in\mathbf{k}^*}$ of Theorem 2 are given by $p = 0$ and $q = \lambda$.

Proof. Replacing t with 0 yields two elements $a(s,0), b(s,0) \in \mathbf{k}[s]$ such that $a(s,0)\cdot b(s,0) = 1$. This implies that $a(s,0), b(s,0) \in \mathbf{k}^*$. Applying the automorphism

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} \mu x & t \\ u & \mu^{-1} y \end{pmatrix}$$

for some $\mu \in \mathbf{k}^*$, we can assume that $a(s,0) = b(s,0) = 1$. We then apply

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & t \\ u & y \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ d(t) & 1 \end{pmatrix}$$

for some $d \in \mathbf{k}[t]$ and replace $a(s,t)$ with $a(s,t) + td(t)$, so can assume that $a(0,t) = 1$. Applying similarly an automorphism of the form

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ e(t) & 1 \end{pmatrix} \cdot \begin{pmatrix} x & t \\ u & y \end{pmatrix},$$

we can assume that $b(0,t) = 1$. This yields $p, q \in \mathbf{k}[s,t]$ such that $a = 1 + stp$ and $b = 1 + stq$, which yields $c = s(p + q + stpq)$. Replacing t with 0 yields a closed embedding

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbf{k}} & \hookrightarrow & \mathrm{SL}_2 \\ s & \mapsto & \begin{pmatrix} 1 & 0 \\ s(p(s,0) + q(s,0)) & 1 \end{pmatrix} \end{array},$$

whence $p(s,0) + q(s,0) \in \mathbf{k}^*$. □

Corollary 5.16. *Each fibred embedding*

$$\begin{array}{rccc} \rho\colon & \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} a(s,t) & t \\ c(s,t) & b(s,t) \end{pmatrix} \end{array}$$

where $a, b, c \in \mathbf{k}[s,t]$ are such that $\deg a + \deg b \leq 4$ is equivalent to the embedding

$$\begin{array}{rccc} \rho_\lambda\colon & \mathbb{A}^2_{\mathbf{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} 1 & t \\ \lambda s & 1 + \lambda st \end{pmatrix} \end{array}$$

for some $\lambda \in \mathbf{k}^*$.

Proof. Applying Lemma 5.14, one can assume that $a = 1 + stp$, $b = 1 + stq$, $c = s(p + q + stpq)$ for some $p, q \in \mathbf{k}[s,t]$ with $p(s,0) + q(s,0) \in \mathbf{k}^*$. If $p = 0$, then $\rho(\mathbb{A}^2)$ is equal to $\rho_1(\mathbb{A}^2)$, so the result follows from Theorem 2(1). The same holds if $q = 0$ by applying the automorphism

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} y & t \\ u & x \end{pmatrix}.$$

To finish the proof, we assume that $pq \neq 0$ and derive a contradiction. The fact that $\deg a + \deg b \leq 4$ implies that $p, q \in \mathbf{k}^*$. Hence we have $\mathbf{k}[s,t] = \mathbf{k}[a,b,c,t] = \mathbf{k}[t, st, s(p+q+stpq)] = \mathbf{k}[t, st, s(st+\xi)]$ with $\xi = \frac{p+q}{pq} \neq 0$, and thus the morphism

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbf{k}} & \mapsto & \mathbb{A}^3_{\mathbf{k}} \\ (s,t) & \mapsto & (t, st, s(st+\xi)) \end{array}$$

would be a closed embedding. This is false, since the image is properly contained in the irreducible hypersurface given by $\{(x,y,z) \in \mathbb{A}^3_{\mathbf{k}} \mid xz = y(y+\xi)\}$ (the line given by $x = y+\xi = 0$ is missing). $\square$

It remains to generalise Corollary 5.16 to the case of embeddings $\mathbb{A}^2_{\mathbf{k}} \hookrightarrow \mathrm{SL}_2$ of small degree (which are fibred or not).

In the sequel we will use the following subgroups of $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$.

Definition 5.17.

$$\begin{aligned} \mathrm{Aff}_2(\mathbf{k}) &= \left\{ (s,t) \mapsto (as+bt+e, cs+dt+f) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{k}), e,f \in \mathbf{k} \right\}, \\ \mathrm{GL}_2(\mathbf{k}) &= \{(s,t) \mapsto (as+bt, cs+dt) \,| a,b,c,d \in \mathbf{k},\ ad-bc \neq 0\}. \end{aligned}$$

Lemma 5.18. *Let $\boldsymbol{k}$ be an algebraically closed field and let $\rho\colon \mathbb{A}^2_{\boldsymbol{k}} \to \mathbb{A}^2_{\boldsymbol{k}} \setminus \{0\}$ be a morphism of the form*

$$(s,t) \mapsto (f(s,t), g(s,t))$$

such that f, g have degree 2 and that the homogeneous parts f_2 and g_2 of f, g of degree 2 are linearly independent. Then, there exist $\alpha \in \mathrm{Aff}_2$ and $\beta \in \mathrm{GL}_2$ such that

$$\beta\rho\alpha = (s,t) \mapsto (s^2, st+1).$$

Proof. We first observe that replacing ρ with $\beta\rho\alpha$, where $\alpha \in \mathrm{Aff}_2$ and $\beta \in \mathrm{GL}_2$, does not change the degree of f, g or the fact that f_2 and g_2 are linearly independent. We then observe that we can assume that $f_2 = s^2$. If f_2 is a square, it suffices to replace f with $\rho\alpha$ for some $\alpha \in \mathrm{GL}_2$. If f_2 is not a square, we choose $\xi \in \mathbf{k}$ such that $g_2 + \xi f_2$ is a square (this is possible since the discriminant of $g_2 + \xi f_2$ is a polynomial of degree 2 in ξ and $\mathbf{k}$ is algebraically closed). We then apply an element of GL_2 at the target to replace f_2, g_2 with $g_2 + \xi f_2, f_2$,and then apply as before an element of GL_2 at the source, to obtain $f_2 = s^2$.

For each irreducible factor P of f, we denote by $C_P \subset \mathbb{A}^2_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[s,t])$ the irreducible curve given by $P = 0$, and observe that g yields an invertible function on C_P.

(a) If f is a product of factors of degree 1, all belong to $\mathbf{k}[s]$, since $f_2 = s^2$. We can then write $f = \prod_{i=1}^2 (s - \lambda_i)$, for some $\lambda_i \in \mathbf{k}$. If $\lambda_1 = \lambda_2$, we replace s with $s - \lambda_1$ and get $f = s^2$, which yields $g = s(as+bt+c)+d$ for some $a,b,c \in \mathbf{k}, d \in \mathbf{k}^*$. The parts of degree 2 of f and g being linearly independent, we get $b \neq 0$. Replacing t with $\frac{t-as-c}{b}$, we replace g with $st+d$. We then apply diagonal elements of the form $(s,t) \mapsto (s, \mu t)$, $\mu \in k^*$ at the source and target and replace d with 1, which yields the desired form. To finish case (a), it remains to see that $\lambda_1 \neq \lambda_2$ is impossible. To derive this contradiction, we apply an element of Aff_2 at the source and get $f = s(s-1)$ This yields $g = sp(s,t) + \mu$, where $\mu \in \mathbf{k}^*$ and $p \in \mathbf{k}[s,t]$ is of degree 1. We moreover obtain $p(1,t) \in \mathbf{k} \setminus \{-\mu\}$, so $p(s,t) = (s-1)\xi + \nu$ for some $\xi, \nu \in \mathbf{k}$. This yields $g \in \mathbf{k}[s]$, which is impossible since g_2 is not a multiple of $f_2 = s^2$.

(b) We can now assume that f is not a product of factors of degree 1, i.e., f is irreducible, and derive a contradiction. We observe that the curve $C_f \subset \mathbb{A}^2$ given by $f = 0$ is isomorphic to $\mathbb{A}^1$. Indeed, the closure of C_f in $\mathbb{P}^2_{\mathbf{k}}$ is an irreducible and thus a smooth conic with one point at infinity since $f_2 = s^2$ (recall that $\mathbf{k}$ is assumed to be algebraically closed). This implies that the restriction $g|_{C_f}$ is a non-zero constant and so $g = \mu + \xi f$ for some $\mu \in \mathbf{k}^*, \xi \in \mathbf{k}$. This contradicts the fact that f_2 and g_2 are linearly independent. $\square$

Proposition 5.19. *Each closed embedding*

$$\begin{array}{rccc} \rho\colon & \mathbb{A}^2_{\boldsymbol{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} f_{11}(s,t) & f_{12}(s,t) \\ f_{21}(s,t) & f_{22}(s,t) \end{pmatrix} \end{array}$$

where $f_{11}, f_{12}, f_{21}, f_{22} \in \boldsymbol{k}[s,t]$ have at most degree 2 is equivalent to the embedding

$$\begin{array}{rccc} \rho_\lambda\colon & \mathbb{A}^2_{\boldsymbol{k}} & \hookrightarrow & \mathrm{SL}_2 \\ & (s,t) & \mapsto & \begin{pmatrix} 1 & t \\ \lambda s & 1+\lambda st \end{pmatrix} \end{array}$$

for some $\lambda \in \boldsymbol{k}^*$.

Proof. Applying Theorem 2, one only needs to show the existence of an automorphism of SL_2 that sends $\rho(\mathbb{A}^2_{\mathbf{k}})$ onto $\rho_1(\mathbb{A}^2_{\mathbf{k}})$. We distinguish the following cases:

(a) Suppose first that one of the polynomials f_{ij} is constant. One can assume that it is f_{11} by using permutation of coordinates (with signs). The case $f_{11} = 0$ is impossible, since the image would then be contained in

$$\left\{ \begin{pmatrix} x & t \\ u & y \end{pmatrix} \in \mathrm{SL}_2 \,\middle|\, x = 0 \right\} \simeq (\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}) \times \mathbb{A}^1_{\mathbf{k}}.$$

We then have $f_{11} \neq 0$ and apply a diagonal automorphism of SL_2 to get $f_{11} = 1$, which corresponds to $\rho(\mathbb{A}^2_{\mathbf{k}}) = \rho_1(\mathbb{A}^2_{\mathbf{k}})$.

(b) Suppose then that one of the f_{ij} has degree 1. Applying permutations one can assume that f_{12} has degree 1. Applying an element of Aff_2 at the source (see Definition 5.17), we do not change the degree of the $f_{ij}'s$ and can assume that $f_{12} = t$. Since $\deg(f_{11}f_{22}) = \deg(f_{12}f_{21}) \leq 4$, the result follows from Corollary 5.16.

(c) It remains to study the case where $\deg(f_{ij}) = 2$ for each $i, j \in \{1,2\}$. If the homogeneous parts of f_{11} and f_{12} of degree 2 are collinear, we apply

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & t \\ u & y \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & t+\mu x \\ u & y+\mu u \end{pmatrix}$$

for some $\mu \in \mathbf{k}$ and obtain $\deg(f_{12}) \leq 1$, which reduces to the cases (a), (b). To achieve the proof of (c), we now assume that the homogeneous parts of f_{11} and f_{12} of degree 2 are linearly independant and prove that this implies that $\mathbf{k}[f_{11}, f_{12}, f_{21}, f_{22}] \subsetneq \mathbf{k}[s,t]$ (which contradicts the fact that ρ is a closed embedding). To show this, one can extend the scalars and assume that $\mathbf{k}$ is algebraically closed. We then apply Lemma 5.18 to the morphism $\nu\colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^2_{\mathbf{k}} \setminus \{0\}$ given by $(s,t) \mapsto (f_{11}(s,t), f_{12}(s,t))$, and find $\alpha \in \mathrm{Aff}_2(\mathbf{k})$, $\beta \in \mathrm{GL}_2(\mathbf{k})$ such that $\beta\nu\alpha = (s,t) \mapsto (s^2, st+1)$. We write $\mu = \det(\beta) \in \mathbf{k}^*$ and replace ρ with $\hat{\beta}\rho\alpha$, where $\hat{\beta} \in \mathrm{Aut}(\mathrm{SL}_2)$ is of the form

$$\hat{\beta}\colon \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix} \cdot \beta \cdot \begin{pmatrix} x & t \\ u & y \end{pmatrix}.$$

This change being made, we obtain $f_{11} = s^2$, $f_{12} = st+1$. Since $1 = f_{11}f_{22} - f_{12}f_{21} = s^2 f_{22} - (st+1)f_{21}$, we obtain $f_{21} = st - 1 + g(s,t)s^2$ for some $g \in \mathbf{k}[s,t]$. This implies that $f_{11}, f_{12}-1, f_{21}+1$ all belong to the maximal ideal $(s,t)^2 \subset \mathbf{k}[s,t]$, which yields the desired contradiction

$$\mathbf{k}[f_{11}, f_{12}, f_{21}, f_{22}] = \mathbf{k}[f_{11}, f_{12}-1, f_{21}+1, f_{22}] \subsetneq \mathbf{k}[s,t].$$

$\square$

6. A non-trivial embedding of $\mathbb{A}^1$ into SL_2, over the reals

In this section, we provide over the field $\mathbf{k} = \mathbb{R}$ an explicit example of an algebraic embedding $\mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathrm{SL}_2$ which is not equivalent to the standard embedding

$$\begin{array}{rccc} \tau_1\colon & \mathbb{A}^1_{\mathbb{R}} & \hookrightarrow & \mathrm{SL}_2 \\ & t & \mapsto & \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \end{array}$$

Example 6.1. In [Sha92] the closed embedding

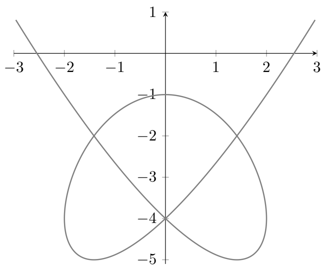
$$\begin{array}{rccc} \gamma\colon & \mathbb{A}^1 & \hookrightarrow & \mathbb{A}^3 \\ & t & \mapsto & (t^3 - 3t, t^4 - 4t^2 - 1, t^5 - 10t) \end{array}$$

is given. This one is not equivalent to the standard embedding $\mathbb{A}^1 \hookrightarrow \mathbb{A}^3$, $t \mapsto (t, 0, 0)$, over the field $\mathbb{R}$ of real numbers. The reason is that it corresponds, as an embedding $\mathbb{R} \hookrightarrow \mathbb{R}^3$, to the (open) trefoil knot.

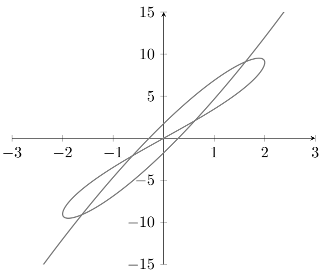
The fact that γ is a closed embedding, over any field $\mathbf{k}$, can be shown as follows. Writing $\gamma_1 = t^3 - 3t, \gamma_2 = t^4 - 4t^2 - 1, \gamma_3 = t^5 - 10t \in \mathbf{k}[t]$, we get

$$t = 3\gamma_3 - 12\gamma_1 - 5\gamma_1\gamma_2 + \gamma_2\gamma_3 - \gamma_1^3.$$

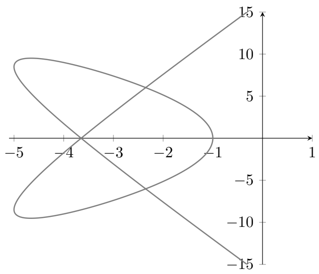
The fact that $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ corresponds to the open trefoil knot can be seen by looking at the three projections:



$t \mapsto (t^3 - 3t, t^4 - 4t^2 - 1)$



$t \mapsto (t^3 - 3t, t^5 - 10t)$



$t \mapsto (t^4 - 4t^2 - 1, t^5 - 10t)$

We now use Example 6.1 to provide a similar example in SL_2.

Lemma 6.2.

(1) *For each field* $\mathbf{k}$ *of characteristic* $\neq 2$, *the morphism*

$$\begin{array}{rccc} \tau\colon & \mathbb{A}^1 & \hookrightarrow & \mathrm{SL}_2 \\ & t & \mapsto & \begin{pmatrix} t^3 - 3t & t^4 - 4t^2 - 1 \\ 1 + \frac{t^2(17t^6 - 56t^4 - 137t^2 + 452)}{16} & \frac{t(17t^8 - 73t^6 - 149t^4 + 609t^2 + 172)}{16} \end{pmatrix} \end{array}$$

is a closed embedding.

(2) *If* $\mathbf{k} = \mathbb{R}$, *then* τ *is not equivalent to the standard embedding, because the fundamental group* $\pi_1(\mathrm{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R}))$ *is not isomorphic to the free group* $\pi_1(\mathrm{SL}_2(\mathbb{R}) \setminus \tau_1(\mathbb{R}))$.

Proof. (1): The fact that τ is a closed embedding can be done explicitly by giving a formula for t, but can also be shown by using the $\mathbb{A}^1$-bundle

$$\begin{array}{rccc} p\colon & \mathrm{SL}_2 & \to & \mathbb{A}^2 \setminus \{(0,0)\} \\ & \begin{pmatrix} x & t \\ u & y \end{pmatrix} & \mapsto & (x, t). \end{array}$$

Writing $\gamma_1 = t^3 - 3t, \gamma_2 = t^4 - 4t^2 - 1, \gamma_3 = t^5 - 10t \in \mathbf{k}[t]$ as in Example 6.1, we get $\gamma_1^2(\gamma_1^2 - 4) - \gamma_2(\gamma_2^2 + 9\gamma_2 + 24) = 16$ and thus get a birational morphism

$$\begin{array}{ccc} \mathbb{A}^1 & \to & \Gamma = \{(x,t) \in \mathbb{A}^2 \mid x^2(x^2-4) - t(t^2+9t+24) = 16\} \\ t & \mapsto & (\gamma_1(t), \gamma_2(t)) \end{array}$$

from $\mathbb{A}^1$ to the singular affine quartic curve $\Gamma \subset \mathbb{A}^2$. We then get a morphism

$$\begin{array}{cccc} f\colon & \Gamma & \to & \mathrm{SL}_2 \\ & (x,t) & \to & \begin{pmatrix} x & t \\ \frac{t^2+9t+24}{16} & \frac{x(x^2-4)}{16} \end{pmatrix} \end{array}$$

which satisfies $p \circ f = \mathrm{id}_\Gamma$ and is thus a section of p over Γ. This implies that

$$\begin{array}{ccc} \Gamma \times \mathbb{A}^1 & \hookrightarrow & \mathrm{SL}_2 \\ ((x,t),a) & \mapsto & \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} f(x,t) \end{array}$$

is a closed embedding. Since $\gamma = (\gamma_1, \gamma_2, \gamma_3)\colon \mathbb{A}^1 \to \Gamma \times \mathbb{A}^1 \subset \mathbb{A}^3$ is a closed embedding, the morphism

$$\begin{array}{cccc} \tau\colon & \mathbb{A}^1 & \hookrightarrow & \mathrm{SL}_2 \\ & t & \mapsto & \begin{pmatrix} 1 & 0 \\ t^5 - 10t & 1 \end{pmatrix} f(\gamma_1(t), \gamma_2(t)) \end{array}$$

is a closed embedding. Replacing γ_1 and γ_2 in the above formula yields the explicit form of the morphism given in the statement of the lemma.

(2): In the remaining part of the proof, we work over $\mathbf{k} = \mathbb{R}$ and use the Euclidean topology. The $\mathbb{R}$-bundle $p\colon \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}^2 \setminus \{(0,0)\}$ is trivial, as it admits a (rational) continuous section given by

$$\begin{array}{cccc} \xi\colon & \mathbb{R}^2 \setminus \{(0,0)\} & \to & \mathrm{SL}_2(\mathbb{R}) \\ & (x,t) & \mapsto & \begin{pmatrix} x & t \\ -\frac{t}{x^2+t^2} & \frac{x}{x^2+t^2} \end{pmatrix}. \end{array}$$

This yields a birational diffeomorphism

$$\begin{array}{cccc} \varphi\colon & \mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R} & \to & \mathrm{SL}_2(\mathbb{R}) \\ & ((x,t),a) & \mapsto & \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \xi(x,t)\,. \end{array}$$

In particular, $\mathrm{SL}_2(\mathbb{R}) \setminus \tau_1(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2 \setminus \{(0,0),(0,1)\} \times \mathbb{R}$, which implies that the fundamental group $\pi_1(\mathrm{SL}_2(\mathbb{R}) \setminus \tau_1(\mathbb{R}))$ is a free group (over two generators). It remains to show that $\pi_1(\mathrm{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R}))$ is not a free group. This will imply that no diffeomorphism of $\mathrm{SL}_2(\mathbb{R})$ sends $\tau(\mathbb{R})$ onto $\tau_1(\mathbb{R})$, and in particular no algebraic automorphism defined over $\mathbb{R}$.

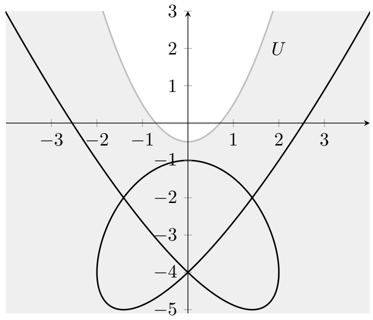
We extend $f\colon \Gamma(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ to a global continuous section $\hat{f}\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathrm{SL}_2(\mathbb{R})$ of p (which exists, since p is a trivial $\mathbb{R}$-bundle). This yields a rational diffeomorphism

$$\begin{array}{ccc} g\colon \mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R} & \xrightarrow{\simeq} & \mathrm{SL}_2(\mathbb{R}) \\ ((x,t),a) & \mapsto & \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \hat{f}(x,t) \end{array}$$

which maps $\gamma(\mathbb{R})$ onto $\tau(\mathbb{R})$.

We take an open subset $U \subset \mathbb{R}^2 \setminus \{(0,0)\}$ (for the Euclidean topology) that contains the singular curve $\Gamma(\mathbb{R})$ and a homeomorphism $h\colon U \xrightarrow{\simeq} \mathbb{R}^2$ which fixes

$\Gamma(\mathbb{R})$ pointwise, and which is homotopic to the inclusion $U \hookrightarrow \mathbb{R}^2$, via a homotopy that fixes $\Gamma(\mathbb{R})$ pointwise. We can for instance take $U = \{(x,t) \in \mathbb{R}^2 \mid t \leq x^2 - \frac{1}{2}\}$ and construct a homeomorphism and a homotopy which preserve the fibres of the projection $(x,t) \mapsto x$ as follows:



The homeomorphisms

$$(\star) \qquad p^{-1}(U) \setminus \tau(\mathbb{R}) \xrightarrow{g^{-1}} (U \times \mathbb{R}) \setminus \gamma(\mathbb{R}) \xrightarrow{h\times \mathrm{id}} \mathbb{R}^3 \setminus \gamma(\mathbb{R})$$

yield isomorphisms of the fundamental groups

$$\pi_1(p^{-1}(U) \setminus \tau(\mathbb{R})) \xrightarrow{\simeq} \pi_1((U \times \mathbb{R}) \setminus \gamma(\mathbb{R})) \xrightarrow{\simeq} \pi_1(\mathbb{R}^3 \setminus \gamma(\mathbb{R}))\,.$$

Since $\gamma\colon \mathbb{R} \hookrightarrow \mathbb{R}^3$ is the (open) trefoil knot, it follows that $\pi_1(\mathbb{R}^3 \setminus \gamma(\mathbb{R}))$ is the braid group with three strands and thus $\pi_1(p^{-1}(U) \setminus \tau(\mathbb{R}))$ is not a free group. It remains then to see that the group homomorphism

$$\iota\colon \pi_1(p^{-1}(U) \setminus \tau(\mathbb{R})) \to \pi_1(\mathrm{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R}))$$

induced by the inclusion $p^{-1}(U) \setminus \tau(\mathbb{R}) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R})$ is injective (as a subgroup of a free group is free). Every element $\alpha \in \mathrm{Ker}(\iota)$ lies in the kernel of the map $\iota'\colon \pi_1(p^{-1}(U) \setminus \tau(\mathbb{R})) \to \pi_1(\mathbb{R}^3 \setminus \gamma(\mathbb{R}))$ induced by the composition

$$p^{-1}(U) \setminus \tau(\mathbb{R}) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R}) \xrightarrow{g^{-1}} ((\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}) \setminus \gamma(\mathbb{R}) \hookrightarrow \mathbb{R}^3 \setminus \gamma(\mathbb{R}),$$

which corresponds simply to the composition

$$(\star\star) \qquad p^{-1}(U) \setminus \tau(\mathbb{R}) \xrightarrow{g^{-1}} (U \times \mathbb{R}) \setminus \gamma(\mathbb{R}) \hookrightarrow \mathbb{R}^3 \setminus \gamma(\mathbb{R}).$$

Since $h\colon U \to \mathbb{R}^2$ is homotopic to the inclusion $U \hookrightarrow \mathbb{R}^2$ via a homotopy that fixes $\Gamma(\mathbb{R})$ pointwise, the two compositions $p^{-1}(U) \setminus \tau(\mathbb{R}) \to \mathbb{R}^3 \setminus \gamma(\mathbb{R})$ of $(\star)$ and $(\star\star)$ are homotopic, so ι' is an isomorphism. This implies that ι is injective and achieves the proof. $\square$

Question 6.3. Working over the field of complex numbers $\mathbb{C}$, is the algebraic embedding $\tau\colon \mathbb{A}^1 \to \mathrm{SL}_2$ of Lemma 6.2(1) equivalent to the standard embedding $\tau_1\colon \mathbb{A}^1 \hookrightarrow \mathrm{SL}_2$?

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Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, CH-4051 Basel, Switzerland

Email address: jeremy.blanc@unibas.ch

Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, DE-20146 Hamburg, Germany

Email address: immanuel.van.santen@math.ch

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On maximal subalgebras

Stefan Maubach, Immanuel Stampfli *

Jacobs University Bremen gGmbH, School of Engineering and Science, Department of Mathematics, Campus Ring 1, 28759 Bremen, Germany

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ABSTRACT

Let $\mathbf{k}$ be an algebraically closed field. We classify all maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, t^{-1}, y]$. To the authors' knowledge, this is the first such classification result for a commutative algebra of dimension > 1. Moreover, we classify all maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that contain a coordinate of $\mathbf{k}[t, y]$. Furthermore, we give plenty examples of maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that do not contain a coordinate.

1. Introduction

All rings in this article are commutative an have a unity. A *minimal ring extension* is a non-trivial ring extension that does not allow a proper intermediate ring. A good overview of minimal ring extensions can be found in [10]. A first general treatment of minimal ring extensions was done by Ferrand and Olivier in [4]. They came up with the following important property of minimal ring extensions.

* Corresponding author.
E-mail addresses: stefan.maubach@gmail.com (S. Maubach), immanuel.e.stampfli@gmail.com (I. Stampfli).

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Theorem 1.0.1 *(see [4, Théorème 2.2]). Let $A \subsetneq R$ be a minimal ring extension and let $\varphi\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ be the induced morphism on spectra. Then there exists a unique maximal ideal $\mathfrak{m}$ of A such that φ induces an isomorphism*

$$\operatorname{Spec}(R) \setminus \varphi^{-1}(\mathfrak{m}) \xrightarrow{\simeq} \operatorname{Spec}(A) \setminus \{\mathfrak{m}\} .$$

Moreover, the following statements are equivalent:

i) The morphism $\varphi\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$ is surjective;
ii) The ring R is a finite A-module;
iii) We have $\mathfrak{m} = \mathfrak{m}R$.

Let $A \subsetneq R$ be a minimal ring extension. Then A is called a *maximal subring* of R. In the case where $\operatorname{Spec}(R) \to \operatorname{Spec}(A)$ is non-surjective, we call A an *extending*[1] maximal subring of R and otherwise, we call it a *non-extending*[2] maximal subring. Moreover, the unique maximal ideal $\mathfrak{m}$ of A (from the theorem above) is called the *crucial maximal ideal*.

In the non-extending case, Dobbs, Mullins, Picavet and Picavet-L'Hermitte gave in [3] a classification of all finite minimal ring extensions $A \subsetneq R$ based on the classification of all minimal ring extensions $A \subsetneq R$ where A is a field, due to Ferrand and Olivier [4]. Therefore, to some extent, the non-extending case is solved.

Our guiding problem is the following.

Problem. Classify all maximal subalgebras of a given affine $\mathbf{k}$-domain where $\mathbf{k}$ is an algebraically closed field.

Let $\mathbf{k}$ be an algebraically closed field and let R be an affine $\mathbf{k}$-domain. If R is one-dimensional and if $\operatorname{Spec}(R)$ contains more than one "smooth point at infinity", then the extending maximal subalgebras of R correspond bijectively to the "smooth points at infinity" of $\operatorname{Spec}(R)$. In fact, to every such point p at infinity, the subalgebra of functions in R that are defined in p is an extending maximal subalgebra of R and every extending maximal subalgebra of R is of this form. This is proven in Section 3 by using the Krull–Akizuki-Theorem.

In dimension two, the most natural algebra to study is the polynomial algebra in two variables $\mathbf{k}[t, y]$. Using the classification of extending maximal subalgebras of a one-dimensional affine $\mathbf{k}$-domain, we give in Section 4 plenty examples of extending maximal subalgebras of $\mathbf{k}[t, y]$ that do not contain a coordinate of $\mathbf{k}[t, y]$, i.e. they do not contain a polynomial in $\mathbf{k}[t, y]$ which is the component of an automorphism of $\mathbb{A}^2_{\mathbf{k}}$. These examples indicate that it is difficult to classify *all* extending maximal subalgebras of $\mathbf{k}[t, y]$.

[1] Since [4] proves that in this case f is a flat epimorphism, the literature calls this sometimes the "flat epimorphism case".
[2] The literature calls this sometimes the "finite case".

Therefore, we impose more structure in the problem. Namely, we search for all extending maximal subalgebras of $\mathbf{k}[t,y]$ that contain a coordinate of $\mathbf{k}[t,y]$.

Another natural 2-dimensional affine $\mathbf{k}$-domain beside the polynomial algebra $\mathbf{k}[t,y]$ is the localization of it in t, i.e. the 2-dimensional domain $\mathbf{k}[t,t^{-1},y]$. This algebra is directly related to our former problem, as it is isomorphic to the localization of $\mathbf{k}[t,y]$ in any coordinate of $\mathbf{k}[t,y]$. In fact, in this article we classify all extending maximal subalgebras of $\mathbf{k}[t,t^{-1},y]$ and get in the course of this classification all extending maximal subalgebras of $\mathbf{k}[t,y]$ that contain a coordinate. This is the bulk of this article. To the authors' knowledge, this is the first such classification result for an algebra of dimension > 1.

Let us give an instructive example, before we give more details on our results.

Example 1.0.1 *(see Lemma 2.0.7).* Let $\mathbf{k}$ be an algebraically closed field and let $R = \mathbf{k}[t,t^{-1},y]$. The ring

$$A = \mathbf{k}[t] + y\,\mathbf{k}[t,t^{-1},y] = \mathbf{k}[t,y,y/t,y/t^2,y/t^3,\ldots]$$

is an extending maximal subalgebra of $\mathbf{k}[t,t^{-1},y]$. The crucial maximal ideal of A is given by

$$\mathfrak{m} = (t,y,y/t,y/t^2,\ldots)\,.$$

Thus $A \subseteq \mathbf{k}[t,t^{-1},y]$ induces an open immersion $\mathbb{A}^*_{\mathbf{k}} \times \mathbb{A}^1_{\mathbf{k}} \to \mathrm{Spec}(A)$ and the complement of the image is just $\{\mathfrak{m}\}$. Moreover, the morphism $\mathrm{Spec}(A) \to \mathbb{A}^2_{\mathbf{k}}$ induced by $\mathbf{k}[t,y] \subseteq A$, sends the crucial maximal ideal $\mathfrak{m}$ to the origin $(0,0)$. So in some sense we "added" to $\mathbb{A}^*_{\mathbf{k}} \times \mathbb{A}^1_{\mathbf{k}}$ the point $(0,0) \in \{0\} \times \mathbb{A}^1_{\mathbf{k}}$.

Another description of the affine scheme $\mathrm{Spec}(A)$ is the following: It is the inverse limit of $\ldots \longrightarrow \mathbb{A}^2_{\mathbf{k}} \xrightarrow{\varphi} \mathbb{A}^2_{\mathbf{k}} \xrightarrow{\varphi} \mathbb{A}^2_{\mathbf{k}}$ inside the category of affine schemes, where $\varphi(t,x) = (t,tx)$.

A little more general, for any $\alpha \in \mathbf{k}[t]$, the ring $\mathbf{k}[t] + (y-\alpha)\,\mathbf{k}[t,t^{-1},y]$ is also an extending maximal subalgebra of $\mathbf{k}[t,t^{-1},y]$.

Towards the classification of all extending maximal subalgebras of $\mathbf{k}[t,t^{-1},y]$, we describe in Section 5 all extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t,t^{-1},y]$ that contain $\mathbf{k}[t,y]$. To formulate our results we introduce some notation. Let $\mathbf{k}[[t^{\mathbb{Q}}]]$ be the Hahn field over $\mathbf{k}$ with rational exponents, i.e. the field of formal power series

$$\alpha = \sum_{s\in\mathbb{Q}} a_s t^s \quad \text{such that } \mathrm{supp}(\alpha) = \{\, s \in \mathbb{Q} \mid a_s \neq 0 \,\} \text{ is well ordered.}$$

Moreover, we denote by $\mathbf{k}[[t^{\mathbb{Q}}]]^+$ the subring of elements $\alpha \in \mathbf{k}[[t^{\mathbb{Q}}]]$ that satisfy $\mathrm{supp}(\alpha) \subseteq [0,\infty)$. By extending the scalars $\mathbf{k}[t,t^{-1}]$ to the Hahn field $\mathbf{k}[[t^{\mathbb{Q}}]]$ one has a simple classification:

Theorem 1.0.2 *(see Corollary 5.3.8 and Remark 5.3.6). We have a bijection*

$$\boldsymbol{k}[[t^{\mathbb{Q}}]]^{+} \longrightarrow \left\{ \begin{array}{c} \textit{extending maximal} \\ \boldsymbol{k}\textit{-subalgebras of } \boldsymbol{k}[[t^{\mathbb{Q}}]][y] \\ \textit{that contain } \boldsymbol{k}[[t^{\mathbb{Q}}]]^{+}[y] \end{array} \right\}, \quad \alpha \longmapsto \boldsymbol{k}[[t^{\mathbb{Q}}]]^{+} + (y-\alpha)\,\boldsymbol{k}[[t^{\mathbb{Q}}]][y]$$

With the aid of this theorem, we are able to classify all extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, t^{-1}, y]$ that contain $\mathbf{k}[t, y]$.

Theorem 1.0.3 *(see Theorem 5.5.1). Let $\mathscr{S}$ be the set of $\alpha \in \boldsymbol{k}[[t^{\mathbb{Q}}]]^{+}$ such that* $\operatorname{supp}(\alpha)$ *is contained in a strictly increasing sequence of $\mathbb{Q}$. Then we have a surjection*

$$\mathscr{S} \longrightarrow \left\{ \begin{array}{c} \textit{extending maximal} \\ \boldsymbol{k}\textit{-subalgebras of } \boldsymbol{k}[t, t^{-1}, y] \\ \textit{that contain } \boldsymbol{k}[t, y] \end{array} \right\}, \quad \alpha \longmapsto A_{\alpha} \cap \boldsymbol{k}[t, t^{-1}, y]$$

where

$$A_{\alpha} = \boldsymbol{k}[[t^{\mathbb{Q}}]]^{+} + (y-\alpha)\,\boldsymbol{k}[[t^{\mathbb{Q}}]][y]\,.$$

Moreover, two elements of $\mathscr{S}$ are sent to the same $\boldsymbol{k}$-subalgebra, if and only if they lie in the same orbit under the natural action of $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \boldsymbol{k}^{*})$ *on $\mathscr{S}$.*

In Section 6 we start with the description of *all* maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, t^{-1}, y]$. Our main result of that section is the following.

Theorem 1.0.4 *(see Proposition 6.0.4). Let $A \subseteq \boldsymbol{k}[t, t^{-1}, y]$ be an extending maximal $\boldsymbol{k}$-subalgebra. Then, exactly one of the following cases occurs:*

i) There exists an automorphism σ of $\boldsymbol{k}[t, t^{-1}, y]$ such that $\sigma(A)$ contains $\boldsymbol{k}[t, y]$;
ii) A contains $\boldsymbol{k}[t, t^{-1}]$.

The maximal $\mathbf{k}$-subalgebras of case i) are described by Theorem 1.0.3. Thus, we are left with the description of the extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, t^{-1}, y]$ that contain $\mathbf{k}[t, t^{-1}]$. This will be done in Section 7. In order to state our result let us introduce some notation. Let $\mathscr{M}$ be the set of extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that contain $\mathbf{k}[t]$. Moreover, let $\mathscr{N}$ be the set of extending maximal $\mathbf{k}$-subalgebras A of $\mathbf{k}[t, y, y^{-1}]$ that contain $\mathbf{k}[t, y^{-1}]$ and such that

$$A \longrightarrow \mathbf{k}[t, y, y^{-1}]/(t-\lambda)$$

is surjective, where λ is the unique element in $\mathbf{k}$ such that the crucial maximal ideal of A contains $t-\lambda$ (this λ exists by Remark 7.0.1). The set $\mathscr{N}$ is described by Theorem 1.0.3.

Then, the maximal $\mathbf{k}$-subalgebras of case ii) in Theorem 1.0.4 are described by the following result.

Theorem 1.0.5 *(see Theorem 7.0.6 and Proposition 6.0.5). With the definitions of $\mathscr{M}$ and of $\mathscr{N}$ from above, we have bijections Θ and Φ*

$$\mathscr{N} \xrightarrow{\Theta} \mathscr{M} \supseteq \left\{ \begin{array}{c} B \text{ in } \mathscr{M} \text{ s.t. the crucial} \\ \text{maximal ideal of } B \\ \text{does not contain } t \end{array} \right\} \xleftarrow{\Phi} \left\{ \begin{array}{c} \text{extending maximal } \mathbf{k}\text{-} \\ \text{subalgebras of } \mathbf{k}[t, t^{-1}, y] \\ \text{that contain } \mathbf{k}[t, t^{-1}] \end{array} \right\}$$

given by $\Theta(A) = A \cap \mathbf{k}[t, y]$ and $\Phi(A') = A' \cap \mathbf{k}[t, y]$.

In particular, with the aid of the bijection $\Theta \colon \mathscr{N} \to \mathscr{M}$ in Theorem 1.0.5 we get a description of the extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that contain a coordinate of $\mathbf{k}[t, y]$.

2. Some general considerations about maximal subrings

In this section we gather some general properties of maximal subrings, that we will constantly use in the course of this article.

The first lemma says that maximal subrings behave well under localization.

Lemma 2.0.6 *(see [4, Lemme 1.3]). Let $A \subseteq R$ be a maximal subring and let S be a multiplicatively closed subset of A. Then the localization A_S is either a maximal subring of the localization R_S or $A_S = R_S$.*

The second lemma gives us the possibility for certain cases to reduce to quotient rings, while searching for maximal subrings. It is a direct consequence of [4, Lemme 1.4].

Lemma 2.0.7. *Let $A \subseteq R$ be a ring extension and let $I \subseteq A$ be an ideal such that $I = IR$. Then A is a maximal subring of R if and only if A/I is a maximal subring of R/I.*

In particular, for every ring extension $A \subseteq R$, the conductor ideal

$$I = \{\, a \in A \mid aR \subseteq A \,\}$$

satisfied $I = IR$. Note that every ideal J of A with $J = JR$ is contained in the conductor ideal I.

Lemma 2.0.8 *(see [4, Lemme 3.2]). Let $A \subsetneq R$ be an extending maximal subring. Then the conductor ideal of A in R is a prime ideal of R.*

Samuel introduced in [11] the P_2-property for ring extensions. This property will be crucial for our classification result.

Definition 2.0.2. Let $A \subseteq R$ be a subring. We say that A satisfies the *property* P_2 *in* R, if for all $r, q \in R$ with $rq \in A$ we have either $r \in A$ or $q \in A$.

Lemma 2.0.9 *(see [4, Proposition 3.1]). Let $A \subsetneq R$ be an extending maximal subring. Then A satisfies the property P_2 in R.*

The next lemma shows, that the extending maximal subrings of a field have a well known characterization. It is a direct consequence of [4, Proposition 3.3].

Lemma 2.0.10. *Let K be a field and let $R \subsetneq K$ be a subring. Then, R is an extending maximal subring of K if and only if R is a one-dimensional valuation ring of K.*

Let us state and prove the following rather technical lemma for future use.

Lemma 2.0.11. *Let C be a Noetherian domain such that the quotient field $Q(C)$ is not a finitely generated C-algebra. Let $A \subsetneq C[y]$ be an extending maximal subring that contains C and denote by $\mathfrak{m}$ the crucial maximal ideal of A. Then $\mathfrak{m} \cap C \neq 0$.*

Proof. Assume that $\mathfrak{m} \cap C = 0$. Then we have the following commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\pi"] & C[y] \\ Q(C) \arrow[r] & A/\mathfrak{m}\,. \end{tikzcd}$$

As A is a maximal subring of $C[y]$, we have $A \nsubseteq C$ and thus there exists $f \in A$ with $\deg_y(f) > 0$. Let $f = f_n y^n + \ldots + f_1 y + f_0$ where $f_i \in C$, $f_n \neq 0$. We have

$$y(f_n y^{n-1} + \ldots + f_1) = f - f_0 \in A\,.$$

Since A satisfies the property P_2 in $C[y]$ and since $y \notin A$ we get $f_n y^{n-1} + \ldots + f_1 \in A$. Proceeding in this way it follows that there exists $0 \neq c \in C$ such that $cy \in A$. Let us define the C-algebra homomorphism σ by

$$\sigma\colon C[y] \longrightarrow A/\mathfrak{m}\,, \quad y \mapsto \frac{\pi(cy)}{\pi(c)}\,.$$

We claim that σ and π coincide on A. We proceed by induction on the y-degree of the elements in A. By definition, σ and π coincide on C, i.e. they coincide on the elements of y-degree equal to zero. Let $g = g_n y^n + \ldots + g_1 y + g_0 \in A$ and assume that $g_n \neq 0$, $n > 0$. As before, we get $y(g_n y^{n-1} + \ldots + g_1) \in A$ and $g_n y^{n-1} + \ldots + g_1 \in A$. Thus we have

$$\begin{aligned}\pi(g) &= \pi(y(g_n y^{n-1} + \ldots + g_1)) + \pi(g_0) \\ &= \frac{\pi(cy(g_n y^{n-1} + \ldots + g_1))}{\pi(c)} + \pi(g_0) \\ &= \frac{\pi(cy)}{\pi(c)}\pi(g_n y^{n-1} + \ldots + g_1) + \pi(g_0) \\ &= \sigma(y)\sigma(g_n y^{n-1} + \ldots + g_1) + \sigma(g_0) \\ &= \sigma(g)\,,\end{aligned}$$

where we used in the second last equality the induction hypothesis. This proves the claim. Since π is surjective, σ is surjective too. Hence there exists $a \in A/\mathfrak{m}$ which is algebraic over $Q(C)$ such that $A/\mathfrak{m}$ is generated by a as a C-algebra. Let $h_0 + h_1 x + \ldots + h_m x^m + x^{m+1}$ be the minimal polynomial of a over $Q(C)$ and let

$$C_0 = C[h_0, \ldots, h_m] \subseteq Q(C)\,.$$

As C is Noetherian, C_0 is Noetherian. Moreover, $A/\mathfrak{m}$ is generated by $1, a, \ldots, a^m$ as a C_0-module. Hence, $Q(C)$ is a finitely generated C_0-module. Thus $Q(C)$ is a finitely generated C-algebra, a contradiction. □

3. The one-dimensional case

Let $\mathbf{k}$ be an algebraically closed field. The purpose of this section is to classify all extending maximal $\mathbf{k}$-subalgebras of a given one-dimensional affine $\mathbf{k}$-domain R. The key ingredient is the following lemma.

Lemma 3.0.12. *Let A be a $\mathbf{k}$-subalgebra of the one-dimensional affine $\mathbf{k}$-domain R. Then either $A = \mathbf{k}$ or A is a one-dimensional affine $\mathbf{k}$-domain.*

Proof. We can assume that $A \neq \mathbf{k}$. Then there exists $a \in A \setminus \mathbf{k}$, which is transcendental over $\mathbf{k}$. By the Krull–Akizuki-Theorem applied to $\mathbf{k}[a] \subseteq A$, it follows that A is Noetherian, see for example [8, Theorem 33.2]. By [9, Corollary 1.2], we have

$$\dim A = \mathrm{tr.deg}_{\mathbf{k}} A = 1\,.$$

Let A' be the integral closure of A in its quotient field. By [6, Theorem 9.3] it follows that $\dim A' = 1$. In particular, A' is equidimensional. [9, Theorem 3.2] implies now, that A is an affine $\mathbf{k}$-domain. □

Theorem 3.0.13. *Let R be a one-dimensional affine $\mathbf{k}$-domain. Take a projective closure $\overline{X}$ of the affine curve $X = \mathrm{Spec}(R)$ such that $\overline{X}$ is non-singular at every point of $\overline{X} \setminus X$. If $\overline{X} \setminus X$ contains just a single point, then R has no extending maximal $\mathbf{k}$-subalgebra. Otherwise, for any point $p \in \overline{X} \setminus X$,*

$$\{ f \in R \mid f \text{ is defined at } p \}$$

is an extending maximal $\mathbf{k}$*-subalgebra of* R *and every extending maximal* $\mathbf{k}$*-subalgebra of* R *is of this form.*

Proof. The first statement follows from Lemma 3.0.12. Thus we can assume that $\overline{X} \setminus X$ consists of more than one point. Let $p \in \overline{X} \setminus X$ and set $U = X \cup \{p\} \subsetneq \overline{X}$. Note that U is an affine curve, see [5, Chp. IV, Ex. 1.4]. Let $A \subseteq R$ be the image of $\Gamma(U, \mathcal{O}_U) \to R$ and consider an intermediate ring $A \subseteq B \subsetneq R$. By Lemma 3.0.12, B is a one-dimensional affine $\mathbf{k}$-domain. Consider the induced maps

$$X \xrightarrow{f} \operatorname{Spec}(B) \longrightarrow U \,.$$

As this composition is an open immersion, the first map is an open immersion. As f is not an isomorphism, the complement $\operatorname{Spec}(B) \setminus f(X)$ is non-empty. As $U \setminus X$ is a single point, this implies that $\operatorname{Spec}(B) \to U$ is surjective. In fact, since U is non-singular in $U \setminus X$, this map is an isomorphism and thus we get $A = B$. This proves that A is an extending maximal $\mathbf{k}$-subalgebra of R.

Conversely, let $A \subsetneq R$ be an extending maximal $\mathbf{k}$-subalgebra. By Lemma 3.0.12, A is an affine $\mathbf{k}$-domain. Let $g\colon X \to \operatorname{Spec}(A)$ be the induced map on affine varieties. It is an open immersion and $\operatorname{Spec}(A) \setminus X$ consists only of the crucial maximal ideal $\mathfrak{m}$ of A. Consider the birational map

$$\operatorname{Spec}(A) \overset{g^{-1}}{\dashrightarrow} X \longrightarrow \overline{X} \,, \tag{1}$$

which is an open immersion on $g(X)$. We have to show, that this map is an open immersion on $\operatorname{Spec}(A)$. By [4, Proposition 3.3], the localization $A_{\mathfrak{m}}$ is a one-dimensional valuation ring. Since $A_{\mathfrak{m}}$ is Noetherian, it is a discrete valuation ring. Thus $\operatorname{Spec}(A)$ is non-singular at $\mathfrak{m}$ and therefore the birational map (1) is an injective morphism, which is an open immersion locally at $\mathfrak{m}$ (note that $\overline{X}$ is smooth at every point of $\overline{X} \setminus X$). Thus the morphism (1) is an open immersion. □

4. Examples of extending maximal k-subalgebras of $\mathbf{k}[t, y]$ that do not contain a coordinate

It is thus natural to ask, whether all extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ contain a coordinate. In this section we construct plenty of examples, which give a non-affirmative answer to this question. These examples indicate that it is difficult to classify all maximal subalgebras of $\mathbf{k}[t, y]$.

For the construction of these examples we use techniques of birational geometry of surfaces and the classification of extending maximal subalgebras of one-dimensional affine $\mathbf{k}$-domains. As we fix the algebraically closed field $\mathbf{k}$, we write $\mathbb{P}^n$ for $\mathbb{P}^n_{\mathbf{k}}$ and $\mathbb{A}^n$ for $\mathbb{A}^n_{\mathbf{k}}$.

Definition 4.0.3. Let $L \subseteq \mathbb{P}^2$ be a line, let $p \in L$ be a point and let $\Gamma \subseteq \mathbb{P}^2$ be an irreducible curve with $\Gamma \neq L$ which passes through p. We say that Γ *is tangent to* L *at* p *of order at least* m, if there exists a sequence of blow-ups

$$S_m \xrightarrow{\pi_m} S_{m-1} \xrightarrow{\pi_{m-1}} \dots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2$$

such that π_1 is centered at p, π_i is centered at a point on the exceptional divisor of π_{i-1} for $i = 2, \dots, m$ and the strict transforms of L and of Γ under $\pi_1 \circ \dots \circ \pi_m$ have an intersection point on the exceptional divisor of π_m.

The following Lemma is crucial for our construction.

Lemma 4.0.14. *Let $\Gamma \subseteq \mathbb{P}^2$ be an irreducible curve and let $L \neq \Gamma$ be a line in $\mathbb{P}^2$. Fix some $p \in \Gamma \cap L$. If Γ is tangent to L at p of order at least 2 and if Γ is smooth at p, then there exists no coordinate $f\colon \mathbb{A}^2 = \mathbb{P}^2 \setminus L \to \mathbb{A}^1$ such that the rational map $f|_\Gamma \colon \Gamma \dashrightarrow \mathbb{A}^1$ is defined at p.*

Proof. Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map that restricts to an automorphism on $\mathbb{P}^2 \setminus L = \mathbb{A}^2$. Let $\mathrm{pr}\colon \mathbb{A}^2 \to \mathbb{A}^1$ be the projection given by $\mathrm{pr}(x, y) = x$. We have to prove that the rational map $\mathrm{pr} \circ \varphi|_\Gamma \colon \Gamma \dashrightarrow \mathbb{A}^1$ is not defined at p. Let $a \in \mathbb{P}^2$ be the image of p under the rational map $\varphi|_\Gamma \colon \Gamma \dashrightarrow \mathbb{P}^2$, which is defined at p since Γ is smooth at p. We have $a \in L$, since either φ contracts the line L to some point on L or φ maps L isomorphically onto itself. If $a \neq (0 : 1 : 0)$, then the map $\mathrm{pr} \circ \varphi|_\Gamma \colon \Gamma \dashrightarrow \mathbb{A}^1$ is not defined at p. Thus we can assume that $a = (0 : 1 : 0)$.

Let $\sigma\colon \mathrm{Bl}_a(\mathbb{P}^2) \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ centered at a. Then, $\mathrm{pr} \circ \varphi|_\Gamma \colon \Gamma \dashrightarrow \mathbb{A}^1$ is not defined at p if and only if

$$\Gamma \subseteq \mathbb{P}^2 \overset{\varphi}{\dashrightarrow} \mathbb{P}^2 \overset{\sigma^{-1}}{\dashrightarrow} \mathrm{Bl}_a(\mathbb{P}^2)$$

maps p to the intersection point of the exceptional divisor of σ and the strict transform of L under σ. In other words, we have to prove that $\varphi(\Gamma)$ is tangent to L at a of order at least 1.

If φ is an automorphism, then the result is obvious, so we can assume that there exist base-points of φ. By [1, Lemma 2.2] there exist birational morphisms $\varepsilon\colon Y \to \mathbb{P}^2$ and $\eta\colon Y \to \mathbb{P}^2$ such that the following is satisfied:

- we have $\eta = \varphi \circ \varepsilon$;
- no curve of self-intersection -1 of Y is contracted by both, ε and η;
- there are decompositions

$$\varepsilon = \varepsilon_1 \circ \dots \circ \varepsilon_n \colon Y \longrightarrow \mathbb{P}^2 \quad \text{and} \quad \eta = \eta_1 \circ \dots \circ \eta_n \colon Y \longrightarrow \mathbb{P}^2$$

where ε_1 (respectively η_1) is a blow-up centered at a point on L and ε_i (respectively η_i) is a blow-up centered at a point on the exceptional divisor of ε_{i-1} (respectively η_{i-1}) for $i > 1$;
- the integer n is greater than or equal to 3;
- the strict transform of L under ε (respectively η) has self-intersection -1.

Let q_{i-1} be the center of ε_i and let E_i be the exceptional divisor of ε_i for $i = 1, \ldots, n$. Moreover, we denote by L_i the strict transform of L under $\varepsilon_i \circ \cdots \circ \varepsilon_1$ for $i = 1, \ldots, n$. Since $(L_n)^2 = -1$, we see that L passes through q_0, that L_1 passes through q_1, but L_i passes not through q_i for $i > 1$.

By assumption, Γ is tangent to L at p of order at least 2, so there exists a sequence of blow-ups

$$S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2$$

such that π_1 is centered at p, π_2 is centered at some point on the exceptional divisor of π_1 and the strict transforms of L and of Γ under $\pi_1 \circ \pi_2$ intersect at one point of the exceptional divisor. Denote this intersection point on S_2 by p_2. Consider the birational map

$$\psi = \varepsilon_2^{-1} \circ \varepsilon_1^{-1} \circ \pi_1 \circ \pi_2 \,.$$

This map is defined at p_2 and we denote by p_2' its image under ψ. Since $p_2' \in L_2$ and since $q_2 \notin L_2$ there exists exactly one point $r \in L_n$ that is mapped onto p_2' via $\varepsilon_n \circ \cdots \circ \varepsilon_3$ (note that $n \geq 3$). Remark that the strict transform of Γ under ε passes through r.

Let r_{i-1} be the center of η_i and let F_i be the exceptional divisor of η_i for $i = 1, \ldots, n$. Since E_n and L_n are the only curves of self-intersection -1 lying in $Y \setminus \varepsilon^{-1}(\mathbb{P}^2 \setminus L)$, it follows that E_n is the strict transform of L under η and that η_n contracts L_n i.e. $F_n = L_n$. Hence we have for $i = 2, \ldots, n$

$$\eta_i \circ \cdots \circ \eta_n(r) = r_{i-1} \in F_{i-1} \,.$$

As $r \in L_n$, the curve L_n is contracted by η onto $\eta(r)$; this point being also the point where φ contracts L, we get $\eta(r) = a \in L$. Since the strict transform of L under η has self-intersection -1, it follows that r_1 is the intersection point of F_1 and the strict transform of L under η_1. As the strict transform of Γ under ε passes through r, its image passes through all the points r_i and thus also through r_1. So the curve $\varphi(\Gamma)$ is tangent to L at $a \in \mathbb{P}^2$ of order at least 1. $\square$

With this lemma we can construct plenty of examples of extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that do not contain a coordinate of $\mathbf{k}[t, y]$.

Let X be an irreducible curve of $\mathbb{A}^2$, which is defined by some polynomial f in $\mathbf{k}[t, y]$. Let Γ be the closure of X in $\mathbb{P}^2$. Assume that there exists a smooth point p on Γ that lies not in X and assume that $\Gamma \setminus X$ contains more than one point. Then the ring

$$A = \{ h \in \Gamma(X, \mathcal{O}_X) \mid h \text{ is defined at } p \}$$

is an extending maximal $\mathbf{k}$-subalgebra of $\Gamma(X, \mathcal{O}_X)$, which is finitely generated over $\mathbf{k}$, see Theorem 3.0.13 and Lemma 3.0.12. Let $a_1, \ldots, a_k \in A$ be a set of generators and let $r_1, \ldots, r_k \in \mathbf{k}[t, y]$ be elements such that $r_i|_X = a_i$. If Γ is tangent to $L = \mathbb{P}^2 \setminus \mathbb{A}^2$ at p of order at least 2, then

$$\mathbf{k}[r_1, \ldots, r_k] + f\,\mathbf{k}[t, y]$$

is an extending maximal $\mathbf{k}$-subalgebra of $\mathbf{k}[t, y]$ that does not contain a coordinate of $\mathbf{k}[t, y]$, see Lemma 2.0.7 and Lemma 4.0.14.

5. Classification of maximal subrings of $\mathbf{k}[t, t^{-1}, y]$ that contain $\mathbf{k}[t, y]$

The goal of this section is the classification of all maximal subrings of $\mathbf{k}[t, t^{-1}, y]$ that contain $\mathbf{k}[t, y]$. Let us start with a simple example.

Example 5.0.4. By using Lemma 2.0.7 one can see that

$$A = \mathbf{k}[t, y] + (y^2 - t)\,\mathbf{k}[t, t^{-1}, y]$$

is a maximal subring of $\mathbf{k}[t, t^{-1}, y]$, which contains $\mathbf{k}[t, y]$. Another description of this ring is the following

$$A = B \cap \mathbf{k}[t, t^{-1}, y]\,, \text{ where } B = \mathbf{k}[t^{1/2}, y] + (y - t^{1/2})\,\mathbf{k}[t^{1/2}, t^{-1/2}, y]\,.$$

By using Lemma 2.0.7, one can see that B is a maximal subring of $\mathbf{k}[t^{1/2}, t^{-1/2}, y]$, which contains $\mathbf{k}[t^{1/2}, y]$. However, the ring B is of a simpler form than A (we replaced $y^2 - t$ by a linear polynomial in y).

The general strategy works in a similar way. First we “enlarge” the coefficients $\mathbf{k}[t]$ to some ring F in such a way, that all maximal subrings of $F_t[y]$ that contain $F[y]$ have a simple form (in the example, we replaced $\mathbf{k}[t]$ by $F = \mathbf{k}[t^{1/2}]$). Then we prove that the intersection of such a simple maximal subring with $\mathbf{k}[t, t^{-1}, y]$ yields a maximal subring of $\mathbf{k}[t, t^{-1}, y]$ that contains $\mathbf{k}[t, y]$ and that we receive by this intersection-process every maximal subring that contains $\mathbf{k}[t, y]$.

For the “enlargement” of the coefficients we have to introduce some notation and terminology.

5.1. Notation and terminology

Let $\mathbf{k}$ be an algebraically closed field (of any characteristic). We denote by $\mathbf{k}[[t^{\mathbb{Q}}]]$ the Hahn field over $\mathbf{k}$ with exponents in $\mathbb{Q}$, i.e. the field of all formal power series

$$\alpha = \sum_{s \in \mathbb{Q}} a_s t^s$$

with coefficients $a_s \in \mathbf{k}$ and with the property that the support

$$\operatorname{supp}(\alpha) = \{\, s \in \mathbb{Q} \mid a_s \neq 0 \,\}$$

is a well ordered subset of $\mathbb{Q}$. There exists a natural valuation on $\mathbf{k}[[t^{\mathbb{Q}}]]$, namely

$$\nu \colon \; \mathbf{k}[[t^{\mathbb{Q}}]] \longrightarrow \mathbb{Q} \,, \quad \alpha \longmapsto \min \operatorname{supp}(\alpha) \,.$$

The valuation ring of ν we denote by $\mathbf{k}[[t^{\mathbb{Q}}]]^+$. More generally, for any subring $B \subseteq \mathbf{k}[[t^{\mathbb{Q}}]]$ we denote by B^+ the subring of elements with ν-valuation ≥ 0, i.e.

$$B^+ = \{\, b \in B \mid \nu(b) \geq 0 \,\} \,.$$

Finally, for any subring $A \subseteq \mathbf{k}[[t^{\mathbb{Q}}]][y]$ we denote by A_1 the subset of degree one elements, i.e.

$$A_1 = \{\, a \in A \mid \deg_y(a) = 1 \,\} \,.$$

5.2. Organization of the section

In Subsection 5.3, we classify all maximal subrings of $K[y]$ that contain $K^+[y]$ for any algebraically closed field $K \subseteq \mathbf{k}[[t^{\mathbb{Q}}]]$ that contains the field of rational functions $\mathbf{k}(t)$ and satisfies the so called *cutoff-property* (see Definition 5.3.2). For example, the Hahn field $\mathbf{k}[[t^{\mathbb{Q}}]]$, the Puiseux field $\bigcup_n \mathbf{k}((t^{1/n}))$ or the algebraic closure of $\mathbf{k}(t)$ enjoy the cutoff-property (see Example 5.3.3).

In Subsection 5.4, we prove that for any maximal subring $A \subsetneq K[y]$ containing $K^+[y]$, the intersection $A \cap \mathbf{k}[t, t^{-1}, y]$ is again a maximal subring of $\mathbf{k}[t, t^{-1}, y]$. Moreover, we prove that any maximal subring of $\mathbf{k}[t, t^{-1}, y]$ that contains $\mathbf{k}[t, y]$ can be constructed as an intersection like above.

Thus we are left with the question, which of the maximal subrings of $K[y]$ that contain $K^+[y]$ give the same ring, after intersection with $\mathbf{k}[t, t^{-1}, y]$. We give an answer to this question in Subsection 5.5.

5.3. Classification of maximal subrings of $K[y]$ that contain $K^+[y]$

Throughout this subsection, we fix an algebraically closed subfield $K \subseteq \mathbf{k}[[t^{\mathbb{Q}}]]$ that contains the field of rational functions $\mathbf{k}(t)$.

Proposition 5.3.1. *Let $K^+[y] \subseteq A \subsetneq K[y]$ be an intermediate ring and assume that $A \subsetneq K[y]$ satisfies the property P_2 in $K[y]$. Then $A = K^+[A_1]$.*

Proof. Let $a \in A$. After multiplying a with a unit of K^+, we can assume that

$$a = \frac{y^n}{t^s} + \text{lower degree terms in } y\,,$$

where $s \in \mathbb{Q}$ and $n \geq 0$ is an integer. We have to show that $a \in K^+[A_1]$. We proceed by induction on n. If $n = 0$, then $a \in A \cap K = K^+$. So let us assume $n > 0$. As K is algebraically closed and contains t, there exist $\alpha_1, \ldots, \alpha_n \in K$ with

$$a = \left(\frac{y - \alpha_1}{t^{s/n}}\right)\left(\frac{y - \alpha_2}{t^{s/n}}\right) \cdots \left(\frac{y - \alpha_n}{t^{s/n}}\right).$$

Since $A \subseteq K[y]$ satisfies the property P_2, we have $(y - \alpha_i)/(t^{s/n}) \in A$ for some i. This implies that

$$\frac{(y - \alpha_i)^n}{t^s} \in K^+[A_1]\,.$$

Thus $q = a - (y - \alpha_i)^n/t^s \in A$. By induction hypothesis we have $q \in K^+[A_1]$ and thus $a \in K^+[A_1]$. Hence $A \subseteq K^+[A_1]$, which implies the result. □

Lemma 5.3.2. *Let $K^+[y] \subseteq E \subsetneq K[y]$ be a proper subring. Then there exists a proper subring $E' \subsetneq K[y]$ that satisfies the property P_2 and contains E.*

Proof. Denote by $\tilde{E} \subseteq K[y]$ the integral closure of E in $K[y]$. As $E \neq K[y]$, it follows that tE is a proper ideal of E. In particular, $\varphi\colon \operatorname{Spec}(K[y]) \to \operatorname{Spec}(E)$ is nonsurjective. Since $\operatorname{Spec}(\tilde{E}) \to \operatorname{Spec}(E)$ is surjective (see [6, Theorem 9.3]), it follows that $\tilde{E} \neq K[y]$. Hence there exists an intermediate ring $\tilde{E} \subseteq E' \subsetneq K[y]$ that satisfies the property P_2 in $K[y]$, by [11, Théorème 8]. □

Now, we give an application of these two results to maximal subrings. Roughly speaking, the proposition says, that for rings which are generated by degree one elements, one can see the maximality already on the level of degree one elements.

Proposition 5.3.3. *Let $K^+[y] \subseteq A \subsetneq K[y]$ be a proper subring that satisfies $A = K^+[A_1]$. Then A is maximal in $K[y]$ if and only if*

$$\textit{for all } f \in K[y] \setminus A \textit{ of degree } 1 \textit{ we have } A[f] = K[y]\,. \tag{2}$$

Proof. Assume that A satisfies (2). Let $A \subseteq E \subsetneq K[y]$ be an intermediate ring. We want to prove $A = E$. By Lemma 5.3.2, there exists a proper subring $E' \subsetneq K[y]$ that satisfies the property P_2 and contains E. Now, if there would exist $f \in (E')_1 \setminus A_1$, then we would have by (2)

$$K[y] = A[f] \subseteq E' \subseteq K[y]\,.$$

This would imply that $E' = K[y]$, a contradiction. Thus we have $A_1 = (E')_1$. According to Proposition 5.3.1 we have $A = E'$ and therefore $A = E$.

The other implication is clear. □

Definition 5.3.1. Let $S = \{\, s_1 < s_2 < \dots \}$ be a strictly monotone sequence in $\mathbb{Q}_{\geq 0}$ and let $\Lambda = \{\, \alpha_1, \alpha_2, \dots \}$ be a sequence in K such that $\operatorname{supp}(\alpha_i) \subseteq [0, s_i)$ and $\operatorname{supp}(\alpha_{i+1} - \alpha_i) \subseteq [s_i, s_{i+1})$ for all $i > 0$. We call then (S, Λ) an *admissible pair of* K. If α is an element of K such that $\operatorname{supp}(\alpha - \alpha_i) \subseteq [s_i, \infty)$ for all $i > 0$, then we call α a *limit* of the admissible pair (S, Λ).

Lemma 5.3.4. *Let (S, Λ) be an admissible pair of $\mathbf{k}[[t^{\mathbb{Q}}]]$. Then there exists a limit in $\mathbf{k}[[t^{\mathbb{Q}}]]$. Moreover, if $\lim s_i = \infty$, then α is unique.*

Proof. Let $\alpha_i = \sum a_{is} t^s$. Now, we define $\alpha = \sum a_s t^s$, where $a_s = a_{is}$ for some i with $s_i > s$. One can easily check, that a_s is well defined. Moreover,

$$\operatorname{supp}(\alpha) = \bigcup_{i=1}^{\infty} \operatorname{supp}(\alpha_i) \quad \text{and} \quad \operatorname{supp}(\alpha) \cap [0, s_i) = \operatorname{supp}(\alpha_i) \text{ for } i = 1, 2, \dots$$

and thus $\operatorname{supp}(\alpha)$ is well ordered. It follows that $\alpha \in \mathbf{k}[[t^{\mathbb{Q}}]]^+$ and that $\operatorname{supp}(\alpha - \alpha_i) \subseteq [s_i, \infty)$ for all $i > 0$. The uniqueness statement is clear. □

Definition 5.3.2. The subfield $K \subseteq \mathbf{k}[[t^{\mathbb{Q}}]]$ satisfies the *cutoff property*, if for all $\alpha = \sum_s a_s t^s \in K$ and for all $u \in \mathbb{Q}$ we have $\sum_{s \geq u} a_s t^s \in K$.

Example 5.3.3. An important example of an algebraically closed field inside $\mathbf{k}[[t^{\mathbb{Q}}]]$ that contains $\mathbf{k}(t)$ and satisfies the cutoff property is the Puiseux field

$$\bigcup_{n=1}^{\infty} \mathbf{k}((t^{1/n}))\,.$$

Clearly, the Hahn field $\mathbf{k}[[t^{\mathbb{Q}}]]$ itself is an example. Another example is the algebraic closure of $\mathbf{k}(t)$ (inside the Puiseux field). This follows from the fact that $\sum_{i=-m}^{m} a_i t^{i/n}$ is algebraic over $\mathbf{k}(t)$ where $a_i \in \mathbf{k}$ and $n, m \in \mathbb{N}$ (it is the sum of algebraic elements).

In the next proposition we classify all P_2-subrings of $K[y]$ under the additional assumption, that K satisfies the cutoff property.

Proposition 5.3.5. *Assume that K satisfies the cutoff property. Let $K^+[y] \subseteq E \subsetneq K[y]$. Then E satisfies the property P_2 in $K[y]$ if and only if*

$$E = K^+ \left[\left\{ \frac{y - \alpha_i}{t^{s_i}} \mid i = 1, 2, \dots \right\} \right]$$

for an admissible pair (S, Λ) of K or $E = K^+[e]$ for some element $e \in K[y]$ of degree 1.

Proof. Assume that $K^+[y] \subseteq E \subsetneq K[y]$ is a subring that satisfies the property P_2 in $K[y]$. We consider the following subset of E_1:

$$N = \left\{ \frac{y - \alpha}{t^s} \in E \mid s \in \mathbb{Q}_{>0},\ \alpha \in K \text{ and } \operatorname{supp}(\alpha) \subseteq [0, s) \right\} .$$

Using the fact, that $E \cap K = K^+$ and that K satisfies the cutoff property, one can see that N has the following two properties:

i) If $(y - \alpha)/t^s, (y - \alpha')/t^{s'} \in N$ and $s \leq s'$, then $\operatorname{supp}(\alpha' - \alpha) \subseteq [s, s')$.
ii) If $(y - \alpha)/t^s \in E$, $s \in \mathbb{Q}_{>0}$ and $\alpha \in K$, then $\alpha \in K^+$ and there exists $n \in N$, such that $((y - \alpha)/t^s) - n \in K^+$.

Property ii) of N implies

$$K^+[N, y] = K^+[E_1] . \tag{3}$$

Let $U \subseteq \mathbb{Q}_{>0}$ be the set of all $s \in \mathbb{Q}_{>0}$ such that there exists $\alpha \in K^+$ with $(y - \alpha)/t^s \in N$. Property i) of N implies that for every $s \in U$ there exists a unique $\alpha_s \in K^+$ such that $(y - \alpha_s)/t^s \in N$. Now, we make the following distinction.

$\sup(U) \in U$: Let $u = \sup(U)$. It follows from property i) of N, that $(y - \alpha_s)/t^s \in K^+[(y - \alpha_u)/t^u]$ for all $s \in U$. This implies $K^+[N, y] \subseteq K^+[(y - \alpha_u)/t^u]$. Clearly, we have $K^+[(y - \alpha_u)/t^u] \subseteq K^+[N, y]$. With (3) and Proposition 5.3.1, we get the equality $E = K^+[(y - \alpha_u)/t^u]$.

$\sup(U) \notin U$: Let $S = \{ s_1 < s_2 < \dots \}$ be a sequence in U such that $\lim s_i = \sup(U)$. If we set $\alpha_i = \alpha_{s_i}$ and $\Lambda = \{ \alpha_1, \alpha_2, \dots \}$, then (S, Λ) is an admissible pair. Let $s \in U$. As $\sup(U) \notin U$, there exists i with $s_i > s$. With property i) of N, we get now $(y - \alpha_s)/t^s \in K^+[(y - \alpha_i)/t^{s_i}]$. Thus $K^+[N, y]$ is generated over K^+ by $(y - \alpha_i)/t^{s_i}$, $i = 1, 2, \dots$. By (3) and Proposition 5.3.1 we get $K^+[N, y] = K^+[E_1] = E$.

Thus E has the claimed form.

Now, we prove that $K^+[e]$ satisfies the property P_2 in $K[y]$, provided that $e \in K[y]$ has degree 1. By applying a K-algebra automorphism of $K[y]$, we can assume that $e = y$. Consider the following extension of the valuation $\nu|_K$ on K to $K[y]$

$$\mu\colon K[y] \longrightarrow \mathbb{Q}, \quad f_0 + \ldots + f_n y^n \longmapsto \min\{\nu(f_0), \ldots, \nu(f_n)\},$$

which extends (uniquely) to $K(y)$. Then $K^+[y]$ is exactly the set of elements in $K[y]$ with μ-valuation ≥ 0. From this it follows readily that $K^+[y]$ satisfies the property P_2 in $K[y]$.

Now, let (S, Λ) be an admissible pair of K. Then

$$K^+ \left[\left\{ \frac{y - \alpha_i}{t^{s_i}} \mid i = 1, 2, \ldots \right\} \right]$$

satisfies the property P_2 in $K[y]$ as it is the union of the increasing P_2-subrings

$$K^+ \left[\frac{y - \alpha_1}{t^{s_1}} \right] \subseteq K^+ \left[\frac{y - \alpha_2}{t^{s_2}} \right] \subseteq \cdots . \quad \square$$

With this classification result at hand, we can now achieve a classification of all maximal subrings of $K[y]$ that contain $K^+[y]$.

Proposition 5.3.6. *Assume that K satisfies the cutoff property. Let (S, Λ) be an admissible pair of K. Assume that either $\lim s_i = \infty$ or (S, Λ) has no limit in K. Then*

$$K^+ \left[\left\{ \frac{y - \alpha_i}{t^{s_i}} \mid i = 1, 2, \ldots \right\} \right] \tag{4}$$

is a maximal subring of $K[y]$ that contains $K^+[y]$. On the other hand, every maximal subring of $K[y]$ that contains $K^+[y]$ is of this form.

Proof. Let $B \subseteq K[y]$ be the ring of (4). We claim that $B \neq K[y]$. Otherwise, there exists i such that $1/t \in K^+[(y - \alpha_i)/t^{s_i}]$, as (S, Λ) is an admissible pair. This would imply $1/t \in K^+$, a contradiction.

Note, that we have $B = K^+[B_1]$. Thus, according to Proposition 5.3.3 it is enough to show, that $B[f] = K[y]$ for all $f \in K[y] \setminus B$ of degree 1. Up to multiplying f with a unit of K^+, we can assume that $f = (y - \alpha)/t^s$ for some $\alpha \in K$ and $s \in \mathbb{Q}$. First, assume that $s < \lim s_i$. Hence there exists i with $s < s_i$. Thus we have

$$\frac{\alpha_i - \alpha}{t^s} = \frac{y - \alpha}{t^s} - \frac{y - \alpha_i}{t^s} \in K[y] \setminus B .$$

So this last element lies in $K \setminus K^+$. Hence we have $B[f] = K[y]$. Now, assume $s \geq \lim s_i$ (and thus $\lim s_i$ is finite). As (S, Λ) has no limit in K, there exists i such that $\operatorname{supp}(\alpha - \alpha_i)$ is not contained in $[s_i, \infty)$. Thus,

$$\frac{y - \alpha}{t^{s_i}} - \frac{y - \alpha_i}{t^{s_i}} = \frac{\alpha_i - \alpha}{t^{s_i}} \in K \setminus K^+ ,$$

and hence we get $B[f] = K[y]$ again.

Now, let $A \subsetneq K[y]$ be a maximal subring that contains $K^+[y]$, which must be an extending maximal subring. By Lemma 2.0.9, $A \subseteq K[y]$ satisfies the property P_2. Since $K^+[e] \subseteq K[y]$ is not a maximal subring for all $e \in K[y]$ of degree 1, it follows from Proposition 5.3.5 that there exists an admissible pair (S, Λ), such that

$$A = K^+ \left[\left\{ \frac{y - \alpha_i}{t^{s_i}} \mid i = 1, 2, \dots \right\} \right].$$

It remains to prove that $\lim s_i = \infty$ or (S, Λ) has no limit in K. Assume towards a contradiction that $s = \lim s_i < \infty$ and $\alpha \in K$ is a limit of (S, Λ). Then, it follows that $A \subseteq K^+[(y - \alpha)/t^s]$. As $K^+[(y - \alpha)/t^s]$ is certainly not a maximal subring of $K[y]$, we get a contradiction. This finishes the proof. $\square$

Remark 5.3.4. Let A be the maximal subring (4) in the Proposition 5.3.6. We describe the crucial maximal ideal of A. Let $\mathfrak{n} \subseteq K^+$ be the unique maximal ideal. In fact, $\mathfrak{n} = \sum_{q \in \mathbb{Q}_{>0}} t^q K^+$. For $i \in \mathbb{N}$, let $\alpha_i' = \alpha_i + t^{s_i} a_{i+1}$ where $a_{i+1} \in \mathbf{k}$ denotes the coefficient of t^{s_i} in α_{i+1}. Thus A is generated over K^+ by the elements $(y - \alpha_i')/t^{s_i}$. We have the following inclusion of ideals in A

$$\mathfrak{n} + \sum_{i=0}^{\infty} \frac{y - \alpha_i'}{t^{s_i}} A \subseteq \sum_{q \in \mathbb{Q}_{>0}} t^q A\,.$$

As every element of A is an element of $\mathbf{k} \cdot 1$ modulo the left hand ideal and the right hand ideal is proper in A, these ideals are the same. It follows, that this ideal is maximal, has residue field $\mathbf{k}$ and it is the crucial maximal ideal.

Proposition 5.3.7. *For $\alpha \in K^+$, the ring $K^+ + (y - \alpha)K[y]$ is a maximal subring of $K[y]$ that contains $K^+[y]$, with non-zero conductor ideal $(y - \alpha)K[y]$. Moreover, all maximal subrings $K^+[y] \subseteq A \subsetneq K[y]$ with non-zero conductor are of this form.*

Proof. The first statement follows from Lemma 2.0.7. For the second statement, let $K^+[y] \subseteq A \subsetneq K[y]$ be a maximal subring and assume there exists $0 \neq f \in A$ such that $fK[y] \subseteq A$. We can assume that f is monic in y. Let $f = f_1 \cdots f_k$ be the decomposition of f into monic linear factors inside $K[y]$. As $A \subseteq K[y]$ satisfies the property P_2, for all $n \in \mathbb{N}$ there exists $i = i(n)$ such that $f_i/(t^{n/k}) \in A$. This implies that there exists i such that $f_i/t^n \in A$ for all $n \in \mathbb{N}$. Let $f_i = y - \alpha_i$. Hence, $K^+[y] + (y - \alpha_i)K[y] \subseteq A$. Since $A \subsetneq K[y]$ is a proper subring, we get $\alpha_i \in K^+$ and thus $A = K^+ + (y - \alpha_i)K[y]$. $\square$

Remark 5.3.5. Assume that K satisfies the cutoff property. Let $A \subseteq K[y]$ be a maximal subring that contains $K^+[y]$, $I \subseteq K[y]$ the conductor ideal of A in $K[y]$ (which could be zero), and let $\mathfrak{m} \subseteq A$ be the crucial maximal ideal. By [4, Proposition 3.3], the localization $(A/I)_{\mathfrak{m}}$ is a one-dimensional valuation ring. Let (S, Λ) be an admissible pair in K such that A is generated over K^+ by $(y - \alpha_i)/t^{s_i}$ for $i = 1, 2, \dots$, see Proposition 5.3.6. Let

$\alpha \in \mathbf{k}[[t^{\mathbb{Q}}]]^+$ be a limit of (S, Λ), which is not unique (however, it exists by Lemma 5.3.4). Using Proposition 5.3.6 and Proposition 5.3.7 one can check that the K-homomorphism

$$K[y]/I \longrightarrow \mathbf{k}[[t^{\mathbb{Q}}]]\,, \quad f \longmapsto f(\alpha)$$

is injective. Hence,

$$\omega\colon Q(K[y]/I) \longrightarrow \mathbb{Q}\,, \quad f \longmapsto \nu(f(\alpha))$$

is a valuation on the quotient field of $K[y]/I$. With the aid of Remark 5.3.4 one can see, that the valuation on $(A/I)_{\mathfrak{m}}$ is given by ω. In particular we have for $f \in K[y]$

$$f \in A \quad \Longleftrightarrow \quad \omega(\bar{f}) \geq 0\,,$$

where $\bar{f}$ denotes the residue class modulo I. Moreover, we get for the crucial maximal ideal

$$f \in \mathfrak{m} \quad \Longleftrightarrow \quad \omega(\bar{f}) > 0\,.$$

This characterization of A and $\mathfrak{m}$ will be very important for us.

As a consequence of Proposition 5.3.6 and Lemma 5.3.4 we can now classify all the maximal subrings of $\mathbf{k}[[t^{\mathbb{Q}}]][y]$ which contain $\mathbf{k}[[t^{\mathbb{Q}}]]^+[y]$.

Corollary 5.3.8. *If $K = \mathbf{k}[[t^{\mathbb{Q}}]]$, then for all $\alpha \in K^+$ the ring*

$$K^+\left[\left\{ \frac{y-\alpha}{t^s} \;\middle|\; s = 1, 2, \dots \right\}\right]$$

is maximal in $K[y]$ and contains $K^+[y]$. On the other hand, every maximal subring of $K[y]$ that contains $K^+[y]$ is of this form.

Remark 5.3.6. The maximal subring of $K[y]$ in Corollary 5.3.8 is the ring $K^+ + (y - \alpha)K[y]$. Its crucial maximal ideal is $\mathfrak{n} + (y-\alpha)K[y]$, where $\mathfrak{n} \subseteq K^+$ denotes the unique maximal ideal.

With Proposition 5.3.6 and Proposition 5.3.7 at hand, we can now give another description of the maximal subrings of $K[y]$ that contain $K^+[y]$ in the case where K is the algebraic closure of $\mathbf{k}(t)$. We just want to stress the following definition in advance.

Definition 5.3.7. A subset S of $\mathbb{Q}$ is called a *strictly increasing sequence* if there exists an isomorphism of the natural numbers to S that preserves the given orders.

Proposition 5.3.9. *Let K be the algebraic closure of $\boldsymbol{k}(t)$ inside $\boldsymbol{k}[[t^{\mathbb{Q}}]]$ and let $\mathscr{S}$ be the set of $\alpha \in \boldsymbol{k}[[t^{\mathbb{Q}}]]^+$ such that* $\operatorname{supp}(\alpha)$ *is contained in a strictly increasing sequence. Then we have bijections*

$$\Xi_1 \colon K^+ \longrightarrow \left\{ \begin{array}{c} \textit{maximal subrings of } K[y] \textit{ with} \\ \textit{non-zero conductor that contain } K^+[y] \end{array} \right\}$$

$$\Xi_2 \colon \mathscr{S} \setminus K^+ \longrightarrow \left\{ \begin{array}{c} \textit{maximal subrings of } K[y] \textit{ with} \\ \textit{zero conductor that contain } K^+[y] \end{array} \right\}$$

given by

$$\Xi_1(\alpha) = K^+ + (y-\alpha)K[y] \quad \textit{and} \quad \Xi_2(\beta) = K^+\left[\left\{ \frac{y-\beta_i}{t^{s_i}} \mid i \in \mathbb{N} \right\}\right]$$

where $\{ s_1 < s_2 < \dots \} = \operatorname{supp}(\beta)$ and β_i is the sum of the first $i-1$ non-zero terms of β.

Proof. Proposition 5.3.7 implies that Ξ_1 is bijective.

Let $\beta \in \mathscr{S} \setminus K^+$ and let $S = \{ s_1 < s_2 < \dots \}$, $\Lambda = \{ \beta_1, \beta_2, \dots \}$. Then (S, Λ) is an admissible pair and β is a limit of it. Since $\beta \notin K^+$ and since K satisfies the cutoff property, there exists no limit of (S, Λ) in K^+. Hence, by Proposition 5.3.6 the subring $\Xi_2(\beta)$ is maximal in $K[y]$. Thus Ξ_2 is well-defined.

Let $A \subseteq K[y]$ be a maximal subring with zero conductor that contains $K^+[y]$. By Proposition 5.3.6 there exists an admissible pair (S', Λ') in K such that

$$A = K^+\left[\left\{ \frac{y-\beta'_i}{t^{s'_i}} \mid i \in \mathbb{N} \right\}\right]$$

where $S' = \{ s'_1 < s'_2 < \dots \}$ and $\Lambda' = \{ \beta'_1, \beta'_2, \dots \}$, and either $\lim s'_i = \infty$ or (S', Λ') has no limit in K. If (S', Λ') has a limit in K, then the conductor of $A \subseteq K[y]$ is non-zero. Thus (S', Λ') has no limit in K. Since K is the algebraic closure of $\mathbf{k}(t)$, the support $\operatorname{supp}(\beta'_i)$ is finite for all i. Hence the pair (S', Λ') has a limit β' inside $\mathscr{S} \setminus K^+$. Moreover, this limit satisfies $\Xi_2(\beta') = A$, which proves the surjectivity of Ξ_2.

Let $\gamma_1, \gamma_2 \in \mathscr{S} \setminus K^+$ such that $\Xi_2(\gamma_1) = \Xi_2(\gamma_2)$ and denote this ring by D. For $k = 1, 2$, let $\{ s_{k1} < s_{k2} < \dots \} = \operatorname{supp}(\gamma_k)$ and let $\gamma_{ki} \in K$ be the sum of the first $i-1$ non-zero terms of γ_k. Let $i > 0$ be an integer. Without loss of generality we can assume that $s_{1i} \leq s_{2i}$. Since $(y - \gamma_{ki})/t^{s_{ki}} \in D$ for $k = 1, 2$, it follows that

$$\frac{\gamma_{2i} - \gamma_{1i}}{t^{s_{1i}}} = \frac{y - \gamma_{1i}}{t^{s_{1i}}} - \frac{y - \gamma_{2i}}{t^{s_{1i}}} \in D \cap K = K^+ .$$

Hence $\gamma_{2i} = \gamma_{1i} + t^{s_{1i}}\eta$ where $\eta \in K^+$. However, since $\operatorname{supp}(\gamma_{1i})$ and $\operatorname{supp}(\gamma_{2i})$ have the same number of elements, it follows that $\eta = 0$. Thus $\gamma_{1i} = \gamma_{2i}$ for all i. This implies that $\gamma_1 = \gamma_2$ and hence Ξ_2 is injective. $\square$

5.4. Description of all maximal subrings of $\mathbf{k}[t, t^{-1}, y]$ that contain $\mathbf{k}[t, y]$ by "intersection"

In this subsection we still fix an algebraically closed subfield $K \subseteq \mathbf{k}[[t^{\mathbb{Q}}]]$ that contains the field of rational functions $\mathbf{k}(t)$. Moreover, we fix a subring $L \subseteq K$ that contains $\mathbf{k}[t, t^{-1}]$. Recall that L^+ (respectively K^+) denotes the elements in L (respectively K) of ν-valuation ≥ 0.

Lemma 5.4.1. *The ring extension $L^+ \subseteq K^+$ is flat.*

Proof. Let $\mathfrak{n}$ be the unique maximal ideal of the valuation ring K^+. This ideal consists of all elements in K with ν-valuation > 0. Denote by L' the localization $(L^+)_{\mathfrak{n} \cap L^+}$. We show that K^+ is a flat L'-module, which implies then the result. Clearly, K^+ is a torsion-free L'-module. By [2, Chp. I, §2, no. 4, Proposition 3], it is thus enough to prove that L' is a valuation ring.

Let $g, h \in L^+$ and assume that $g \neq 0$, $\nu(h/g) \geq 0$. As the value group of ν is $\mathbb{Q}$, there exist integers $a \geq 0$, $b > 0$ such that $\nu(g) = a/b$. Thus we get

$$\nu\left(\frac{hg^{b-1}}{t^a}\right) \geq 0 \quad \text{and} \quad \nu\left(\frac{g^b}{t^a}\right) = 0$$

and therefore $hg^{b-1}/t^a \in L^+$, $g^b/t^a \in L^+ \setminus \mathfrak{n}$. This implies $h/g \in L'$. Hence, L' is a valuation ring (with valuation $\nu|_{Q(L^+)}$). $\square$

Our first result says that one can construct every maximal subring of $L[y]$ that contains $L^+[y]$ by intersecting $L[y]$ with some maximal subring of $K[y]$ that contains $K^+[y]$ under a certain assumption.

Proposition 5.4.2. *Assume that L^+ is a maximal subring of L. If $L^+[y] \subseteq B \subsetneq L[y]$ is a maximal subring, then there exists a maximal subring $K^+[y] \subseteq A \subsetneq K[y]$ such that $B = A \cap L[y]$.*

Remark 5.4.1. The assumption, that L^+ is a maximal subring of L is satisfied for example if $L = \mathbf{k}[t, t^{-1}]$ or $L = \mathbf{k}(t)$.

Proof. Let M be the L^+-module $L[y]/B$. By assumption, M is non-zero. In fact, since L^+ is a maximal subring of L, we have an injection

$$L^+/tL^+ \longrightarrow M\,, \quad \lambda \longmapsto \lambda t^{-1}\,.$$

Since K^+ is a flat L^+-module (see Lemma 5.4.1), we get an injection

$$K^+/tK^+ \simeq K^+ \otimes_{L^+} (L^+/tL^+) \longrightarrow K^+ \otimes_{L^+} M\,.$$

Thus $K^+ \otimes_{L^+} M$ is non-zero. Again, since K^+ is a flat L^+-module, this implies that

$$K^+ \otimes_{L^+} B \subsetneq K^+ \otimes_{L^+} L[y] \, .$$

Therefore, $K^+[B]$ is a proper subring of $K[y]$, which contains $K^+[y]$. Applying Zorn's Lemma to

$$\{ \, A \subseteq K[y] \mid A \supseteq K^+[B] \text{ and } t^{-1} \notin A \ \}$$

yields a maximal subring A in $K[y]$ that lies over $K^+[B]$. Thus $B = A \cap L[y]$. □

In the next proposition we prove that any maximal subring of $K[y]$ that lies over $K^+[y]$ gives a maximal subring of $L[y]$ after intersection with $L[y]$.

Proposition 5.4.3. *Assume that K satisfies the cutoff property. Let $K^+[y] \subseteq A \subsetneq K[y]$ be a maximal subring and let $B = A \cap L[y]$. Then*

i) If I denotes the conductor ideal of A in $K[y]$, then $I \cap B$ is the conductor ideal of B in $L[y]$.
ii) The subring $B \subsetneq L[y]$ is maximal. Moreover, if $\mathfrak{m}$ denotes the crucial maximal ideal of A, then $\mathfrak{m} \cap L[y]$ is the crucial maximal ideal of B.

Proof.

i) Let $b \in I \cap B$. Then $bL[y] \subseteq A \cap L[y] = B$. Thus b lies in the conductor of B in $L[y]$. Now, let $f \in B$ be an element of the conductor of B in $L[y]$. Then we have $fL[y] \subseteq B$ and in particular, $f/t^n \in B \subseteq A$ for all $n \in \mathbb{N}$. As $K[y] = A_t$, this implies that $fK[y] \subseteq A$. Thus $f \in I \cap B$.
ii) Let $I \subseteq K[y]$ be the conductor ideal of A in $K[y]$. By i) the intersection $J = I \cap B$ is the conductor ideal of B in $K[y]$. Let $\mathfrak{m} \subseteq A$ be the crucial maximal ideal and let $\mathfrak{n} = \mathfrak{m} \cap B$. We divide the proof in several steps
    a) We claim that $(B/J)_\mathfrak{n}$ is a one-dimensional valuation ring. Since $(A/I)_\mathfrak{m}$ is a one-dimensional valuation ring (see [4, Proposition 3.3]), it is enough to prove that

$$(B/J)_\mathfrak{n} = (A/I)_\mathfrak{m} \cap Q(L[y]/J)$$

inside $Q(K[y]/I)$ (see also [6, Theorem 10.7]). Let $g, h \in L[y]/J$ be non-zero elements and assume that $h/g \in (A/I)_\mathfrak{m}$. Thus it follows for the valuation ω defined in Remark 5.3.5 that $\omega(h/g) \geq 0$. There exist integers a, b such that $\omega(g) = a/b$ and we can assume that $b > 0$. Thus we have $\omega(g^b/t^a) = 0$. Since

$\omega(h) \geq \omega(g)$ we get $\omega(hg^{b-1}/t^a) \geq 0$. Thus g^b/t^a and hg^{b-1}/t^a both lie inside A/I. Using the fact that

$$B/J = A/I \cap L[y]/J \subseteq K[y]/I$$

we get

$$\frac{h}{g} = \frac{h \cdot (g^{b-1}/t^a)}{g^b/t^a} \in (B/J)_{\mathfrak{n}} \,.$$

Thus we have $(A/I)_{\mathfrak{m}} \cap Q(L[y]/J) = (B/J)_{\mathfrak{n}}$. Note that the reasoning is similar to the proof of Lemma 5.4.1.

b) We claim that the complement of the image of $\operatorname{Spec} L[y] \to \operatorname{Spec} B$ is just the point $\mathfrak{n}$. By Remark 5.3.4, the residue field of the crucial maximal ideal $\mathfrak{m} \subseteq A$ is $\mathbf{k}$ and thus $\mathfrak{n}$ is a maximal ideal of B. Let $b \in \mathfrak{n}$. By Remark 5.3.4 we have $\mathfrak{m} = \operatorname{rad}(tA)$ and thus there exists an integer $q \geq 1$ such that $b^q \in tA$. Therefore $b^q/t \in A$. Since $b^q/t \in L[y]$, we get $b^q \in tB$. Thus we proved $\mathfrak{n} \subseteq \operatorname{rad}(tB)$. If $\mathfrak{p} \subseteq B$ is a prime ideal such that $\mathfrak{p}L[y] = L[y]$, then we get $t \in \mathfrak{p}$ (since $B_t = L[y]$). Thus we have $\mathfrak{n} \subseteq \operatorname{rad}(tB) \subseteq \mathfrak{p}$ and by the maximality of $\mathfrak{n}$ we get $\mathfrak{n} = \mathfrak{p}$.

c) Now, we prove that B/J is a maximal subring of $L[y]/J$. Let $C \subsetneq L[y]/J$ be a subring that lies over B/J. Using b), the fact that $J \subseteq \mathfrak{n}$ and that $(B/J)_t = L[y]/J$, we get the following commutative diagram

$$\begin{tikzcd}
 & & \operatorname{Spec} C \arrow[drr, "\varphi"] & & \\
\operatorname{Spec} L[y]/J \arrow[urr, "\not\simeq"] \arrow[r, "\simeq"] & (\operatorname{Spec} B/J) \setminus \{\mathfrak{n}\} \arrow[rrr, hook, "\text{open}"'] & & & \operatorname{Spec} B/J \,.
\end{tikzcd}$$

From this, one can easily deduce that φ is surjective. Let $\mathfrak{p} \in \operatorname{Spec} C$ with $\varphi(\mathfrak{p}) = \mathfrak{n}$. By a) and Lemma 2.0.10, $(B/J)_{\mathfrak{n}}$ is a maximal subring of $Q(L[y]/J)$. Since $t \in \mathfrak{p}$, this implies $(B/J)_{\mathfrak{n}} = C_{\mathfrak{p}}$. Hence we have $B/J = C$ by [6, Theorem 4.7].

From c) and from Lemma 2.0.7 it follows that B is a maximal subring of $L[y]$. From b) it follows that $\mathfrak{n} = \mathfrak{m} \cap L[y]$ is the crucial maximal ideal of B. $\square$

Remark 5.4.2. If the conductor ideal of A in $K[y]$ is non-zero, then there exists $\alpha \in K^+$ such that this ideal is $(y - \alpha)K[y]$. Now, if L is a field and $L \subseteq K$ an algebraic field extension, then the conductor of $B = A \cap L[y]$ is the ideal $m_\alpha L[y]$ where $m_\alpha \in L[y]$ is the minimal polynomial of α over L.

In the future we will need the following consequence of the last two propositions.

Corollary 5.4.4. *We have a bijective correspondence*

$$\varphi\colon \left\{\begin{array}{c}\textit{maximal subrings of}\\ \mathbf{k}[t,t^{-1},y] \textit{ that contain}\\ \mathbf{k}[t,y]\end{array}\right\} \xrightarrow{1:1} \left\{\begin{array}{c}\textit{maximal subrings of}\\ \mathbf{k}(t)[y] \textit{ that contain}\\ \mathbf{k}(t)^+[y]\end{array}\right\}$$

given by $\varphi(B) = B_S$ *and* $\varphi^{-1}(A) = A \cap \mathbf{k}[t,t^{-1},y]$*, where* S *denotes the multiplicative subset* $\mathbf{k}[t] \setminus (t)$ *of* $\mathbf{k}[t]$*.*

Proof. Let $B \subsetneq \mathbf{k}[t,t^{-1},y]$ be a maximal subring that contains $\mathbf{k}[t,y]$. By Lemma 2.0.6, the localization B_S is a maximal subring of $\mathbf{k}[t,t^{-1},y]_S$, since $S = \mathbf{k}[t] \setminus (t)$. Moreover, we have

$$B \subseteq B_S \cap \mathbf{k}[t,t^{-1},y] \subsetneq \mathbf{k}[t,t^{-1},y]$$

and thus by the maximality of B we get the equality $B = B_S \cap \mathbf{k}[t,t^{-1},y]$. This proves the injectivity of φ.

Let $A \subsetneq \mathbf{k}(t)[y]$ be a maximal subring that contains $\mathbf{k}(t)^+[y]$. By Proposition 5.4.2 there exists a maximal subring $K^+[y] \subseteq A' \subseteq K[y]$ such that $A' \cap \mathbf{k}(t)[y] = A$. By Proposition 5.4.3, it follows that $A \cap \mathbf{k}[t,t^{-1},y]$ is a maximal subring of $\mathbf{k}[t,t^{-1},y]$. Clearly, $A \cap \mathbf{k}[t,t^{-1},y]$ contains $\mathbf{k}[t,y]$. Moreover,

$$(A \cap \mathbf{k}[t,t^{-1},y])_S \subseteq A$$

and by the maximality of $(A \cap \mathbf{k}[t,t^{-1},y])_S$ we get equality. This proves the surjectivity of φ. □

5.5. Classification of the maximal subrings of $\mathbf{k}[t,t^{-1},y]$ *that contain* $\mathbf{k}[t,y]$

Throughout this subsection K denotes the algebraic closure of $\mathbf{k}(t)$ inside the Hahn field $\mathbf{k}[[t^{\mathbb{Q}}]]$. In this subsection we give a classification of all maximal subrings of $\mathbf{k}[t,t^{-1},y]$ that contain $\mathbf{k}[t,y]$.

Let α be in $\mathbf{k}[[t^{\mathbb{Q}}]]^+$. In this subsection we denote

$$A_\alpha = \mathbf{k}[[t^{\mathbb{Q}}]]^+ + (y-\alpha)\,\mathbf{k}[[t^{\mathbb{Q}}]][y]\,.$$

Thus $\alpha \mapsto A_\alpha$ is a bijective correspondence between $\mathbf{k}[[t^{\mathbb{Q}}]]^+$ and the maximal subrings of $\mathbf{k}[[t^{\mathbb{Q}}]][y]$ that contain $\mathbf{k}[[t^{\mathbb{Q}}]]^+[y]$ by Corollary 5.3.8 and Remark 5.3.6.

Let $(\mathbb{Q}/\mathbb{Z})^*$ be the group of group homomorphisms $\mathbb{Q}/\mathbb{Z} \to \mathbf{k}^*$. There exists a natural action of this group on the Hahn field, given by the homomorphism

$$(\mathbb{Q}/\mathbb{Z})^* \longrightarrow \operatorname{Aut}(\mathbf{k}[[t^{\mathbb{Q}}]]/\,\mathbf{k}((t)))\,, \quad \sigma \longmapsto \left(\sum_{s\in\mathbb{Q}} a_s t^s \mapsto \sum_{s\in\mathbb{Q}} a_s \sigma(s) t^s\right)\,, \qquad (5)$$

where $\mathrm{Aut}(\mathbf{k}[[t^{\mathbb{Q}}]]/\,\mathbf{k}((t)))$ denotes the group of field automorphisms of $\mathbf{k}[[t^{\mathbb{Q}}]]$ that fix the subfield $\mathbf{k}((t))$ pointwise (note that $\mathbf{k}((t)) \subseteq \mathbf{k}[[t^{\mathbb{Q}}]]$ is a Galois extension if and only if the characteristic of $\mathbf{k}$ is zero). The action (5) commutes with the valuation ν on $\mathbf{k}[[t^{\mathbb{Q}}]]$. In particular we have for all $\sigma \in (\mathbb{Q}/\mathbb{Z})^*$ and for all $\alpha \in \mathbf{k}[[t^{\mathbb{Q}}]]^+$

$$A_\alpha \cap \mathbf{k}[t, t^{-1}, y] = A_{\sigma(\alpha)} \cap \mathbf{k}[t, t^{-1}, y]\,.$$

The following result is the main theorem of this section.

Theorem 5.5.1. *Let $\mathscr{S}$ be the set of $\alpha \in \mathbf{k}[[t^{\mathbb{Q}}]]^+$ such that* $\mathrm{supp}(\alpha)$ *is contained in a strictly increasing sequence (see Definition 5.3.7). Then we have a bijection*

$$\Psi\colon \mathscr{S}/(\mathbb{Q}/\mathbb{Z})^* \longrightarrow \left\{ \begin{array}{c} \textit{maximal subrings of} \\ \mathbf{k}[t, t^{-1}, y] \textit{ that contain } \mathbf{k}[t, y] \end{array} \right\}, \quad \alpha \longmapsto A_\alpha \cap \mathbf{k}[t, t^{-1}, y]\,.$$

Moreover, $\Psi(\alpha)$ has non-zero conductor in $\mathbf{k}[t, t^{-1}, y]$ if and only if $\alpha \in K^+$ where K denotes the algebraic closure of $\mathbf{k}(t)$ inside the Hahn field $\mathbf{k}[[t^{\mathbb{Q}}]]$.

For the proof we need some preparation. First, we reformulate the action of $(\mathbb{Q}/\mathbb{Z})^*$ on the Hahn field. Let $\mathbf{k}(t^{\mathbb{Q}})$ be the subfield of the Hahn field generated by the ground field $\mathbf{k}$ and the elements t^s, $s \in \mathbb{Q}$. Then, $(\mathbb{Q}/\mathbb{Z})^*$ is isomorphic to the group $\mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\,\mathbf{k}(t))$ of field automorphisms of $\mathbf{k}(t^{\mathbb{Q}})$ that fix $\mathbf{k}(t)$ pointwise. An isomorphism is given by

$$(\mathbb{Q}/\mathbb{Z})^* \longrightarrow \mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\,\mathbf{k}(t))\,, \quad \sigma \longmapsto (t^s \mapsto \sigma(s) t^s)\ ,$$

and the homomorphism (5) identifies then under this isomorphism with

$$\mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\,\mathbf{k}(t)) \longrightarrow \mathrm{Aut}(\mathbf{k}[[t^{\mathbb{Q}}]]/\,\mathbf{k}((t)))\,, \quad \varphi \longmapsto \left(\sum_{s \in \mathbb{Q}} a_s t^s \mapsto \sum_{s \in \mathbb{Q}} a_s \varphi(t^s) \right)$$

(note that $\varphi(t^s)$ is a multiple of t^s with some element of $\mathbf{k}^*$). For proving the injectivity of the map Ψ in Theorem 5.5.1 we need two lemmas.

Lemma 5.5.2. *Let $q \in \mathbb{Q}_{\geq 0}$ and let α, $\alpha' \in \mathbf{k}[[t^{\mathbb{Q}}]]^+$. Assume that we have decompositions*

$$\alpha = \alpha_0 + \alpha_1\,, \quad \alpha' = \alpha_0 + ct^q + \alpha_1' \quad \textit{with} \quad \alpha_0,\, \alpha_1,\, \alpha_1' \in \mathbf{k}[[t^{\mathbb{Q}}]]\,, \quad c \in \mathbf{k}$$

such that

$$\mathrm{supp}(\alpha_0) \subseteq [0, q]\,, \quad \mathrm{supp}(\alpha_1)\,,\ \mathrm{supp}(\alpha_1') \subseteq (q, \infty)\,, \quad \mathrm{supp}(\alpha_0) \textit{ is finite}\,.$$

If $\nu(f(\alpha)) = \nu(f(\alpha'))$ for all $f \in \mathbf{k}(t)[y]$, then $\alpha_0 + ct^q = \sigma(\alpha_0)$ for some $\sigma \in \mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\,\mathbf{k}(t))$.

Proof. Let $m_0 \in \mathbf{k}(t)[y]$ be the minimal polynomial of α_0 over $\mathbf{k}(t)$. Note that α_0 is algebraic over $\mathbf{k}(t)$ since the support of α_0 is a finite set. Denote by $\alpha_0 = \beta_0, \ldots, \beta_r$ the different elements of the set

$$\{\, \sigma(\alpha_0) \mid \sigma \in \mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}}) / \mathbf{k}(t)) \,\} .$$

As the field extension $\mathbf{k}(t) \subseteq \mathbf{k}(t^{\mathbb{Q}})$ is normal, there exist integers $k_0 > 0$ and $k_1, \ldots, k_r \geq 0$ such that

$$m_0 = (y - \beta_0)^{k_0} (y - \beta_1)^{k_1} \cdots (y - \beta_r)^{k_r} ,$$

see [7, Theorem 3.20]. Assume towards a contradiction that $\alpha_0 + ct^q \neq \beta_j$ for all $0 \leq j \leq r$. Let i be an integer with $1 \leq i \leq r$. Since $\nu(\alpha_0 - \beta_i) \leq q$, we get

$$\begin{aligned} \nu(\alpha_1 + \alpha_0 - \beta_i) &= \nu(\alpha_0 - \beta_i) \\ &= \nu(ct^q + \alpha_0 - \beta_i) \\ &= \nu(\alpha_1' + ct^q + \alpha_0 - \beta_i) \end{aligned}$$

where we used in the second and third equality the fact that $ct^q + \alpha_0 \neq \beta_i$. Since $\alpha_0 = \beta_0 \neq \alpha_0 + ct^q$, the constant c is non-zero. Thus we have $\nu(\alpha_1) > \nu(\alpha_1' + ct^q)$. In summary we get

$$\begin{aligned} \nu(m_0(\alpha)) &= k_0 \nu(\alpha_1) + \sum_{i \neq 0} k_i \nu(\alpha_1 + \alpha_0 - \beta_i) \\ &> k_0 \nu(\alpha_1' + ct^q) + \sum_{i \neq 0} k_i \nu(\alpha_1' + ct^q + \alpha_0 - \beta_i) \quad = \quad \nu(m_0(\alpha')) \end{aligned}$$

and thus we arrive at a contradiction. $\square$

Lemma 5.5.3. *Let α, $\alpha' \in \boldsymbol{k}[[t^{\mathbb{Q}}]]^+$ and assume that* $\mathrm{supp}(\alpha)$, $\mathrm{supp}(\alpha')$ *are contained in strictly increasing sequences (see Definition 5.3.7). Then $\nu(f(\alpha)) = \nu(f(\alpha'))$ for all $f \in \boldsymbol{k}(t)[y]$ if and only if there exists $\sigma \in \mathrm{Aut}(\boldsymbol{k}(t^{\mathbb{Q}}) / \boldsymbol{k}(t))$ such that $\alpha' = \sigma(\alpha)$.*

Proof. Assume that $\nu(f(\alpha)) = \nu(f(\alpha'))$ for all $f \in \mathbf{k}(t)[y]$. By assumption, there exists a strictly increasing sequence $0 < s_1 < s_2 < \ldots$ in $\mathbb{Q}$ such that

$$\alpha = a_0 + \sum_{j=1}^{\infty} a_{s_j} t^{s_j} \quad \text{and} \quad \alpha' = a_0' + \sum_{j=1}^{\infty} a_{s_j}' t^{s_j} .$$

For $i \geq 1$ let

$$\alpha_i = a_0 + \sum_{j=1}^{i-1} a_{s_j} t^{s_j} \quad \text{and} \quad \alpha_i' = a_0' + \sum_{j=1}^{i-1} a_{s_j}' t^{s_j} \,.$$

We define inductively $\sigma_1, \sigma_2, \ldots \in \mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\,\mathbf{k}(t))$ such that $\sigma_i(\alpha_i) = \alpha_i'$.

Since $\nu(\alpha - a_0) = \nu(\alpha' - a_0)$ by assumption, it follows that $a_0' = a_0$. Hence $\sigma_1 = \mathrm{id}$ satisfies $\sigma_1(\alpha_1) = \alpha_1'$. Assume that $\sigma_i \in \mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\,\mathbf{k}(t))$ with $\sigma_i(\alpha_i) = \alpha_i'$ is already constructed. For all $f \in \mathbf{k}(t)[y]$ we have

$$\nu(f(\sigma_i(\alpha))) = \nu(\sigma_i(f(\alpha))) = \nu(f(\alpha)) = \nu(f(\alpha')) \,.$$

Since $\alpha_i' = \sigma_i(\alpha_i)$, Lemma 5.5.2 implies that there exists $\varphi \in \mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\,\mathbf{k}(t))$ such that $\alpha_{i+1}' = \varphi(\sigma_i(\alpha_{i+1}))$. Thus we can define $\sigma_{i+1} = \varphi \circ \sigma_i$.

By construction, σ_{i+1} and σ_i coincide on the field

$$K_i = \mathbf{k}\left(\{\, t^s \mid s \in \mathrm{supp}(\alpha_i)\,\}\right) \,.$$

Thus we get a well defined automorphism of the field $\bigcup_{i=0}^{\infty} K_i$ that restricts to σ_i on K_i. By the normality of the extension $\mathbf{k}(t) \subseteq \mathbf{k}(t^{\mathbb{Q}})$ we can extend this automorphism to an automorphism σ of $\mathbf{k}(t^{\mathbb{Q}})$ and we have $\sigma(\alpha) = \alpha'$ (see [7, Theorem 3.20]).

The converse of the statement is clear. $\square$

Proof of Theorem 5.5.1. Consider the bijections

$$\Xi_1 \colon \mathrm{K}^+ \longrightarrow \left\{ \begin{array}{c} \text{maximal subrings of } \mathrm{K}[y] \text{ with} \\ \text{non-zero conductor that contain } \mathrm{K}^+[y] \end{array} \right\}$$

$$\Xi_2 \colon \mathscr{S} \setminus \mathrm{K}^+ \longrightarrow \left\{ \begin{array}{c} \text{maximal subrings of } \mathrm{K}[y] \text{ with} \\ \text{zero conductor that contain } \mathrm{K}^+[y] \end{array} \right\},$$

of Proposition 5.3.9. For $\alpha \in \mathrm{K}^+$ and $\beta \in \mathscr{S} \setminus \mathrm{K}^+$ we have

$$\Xi_1(\alpha) \cap \mathbf{k}[t, t^{-1}, y] = \Psi(\alpha) \quad \text{and} \quad \Xi_2(\beta) \cap \mathbf{k}[t, t^{-1}, y] = \Psi(\beta) \,.$$

Using Proposition 5.4.3 and Remark 5.4.2 we see that $\Xi_1(\alpha) \cap \mathbf{k}[t, t^{-1}, y]$ is a maximal subring of $\mathbf{k}[t, t^{-1}, y]$ with non-zero conductor and $\Xi_2(\beta) \cap \mathbf{k}[t, t^{-1}, y]$ is a maximal subring of $\mathbf{k}[t, t^{-1}, y]$ with zero conductor. Thus Ψ is a well-defined map. Using Proposition 5.4.2, we see that Ψ is surjective.

For proving the injectivity, let $\alpha_1, \alpha_2 \in \mathscr{S}$ such that the rings $A_{\alpha_1} \cap \mathbf{k}[t, t^{-1}, y]$, $A_{\alpha_2} \cap \mathbf{k}[t, t^{-1}, y]$ are the same subsets of $\mathbf{k}[t, t^{-1}, y]$. For $i = 1, 2$, $A_{\alpha_i} \cap \mathbf{k}(t)[y]$ is a maximal subring of $\mathbf{k}(t)[y]$, see Proposition 5.4.3. By Corollary 5.4.4, we get the equality

$$A_{\alpha_1} \cap \mathbf{k}(t)[y] = A_{\alpha_2} \cap \mathbf{k}(t)[y] \,.$$

Let $B = A_{\alpha_1} \cap \mathbf{k}(t)[y] = A_{\alpha_2} \cap \mathbf{k}(t)[y]$. Let $\mathfrak{n}$ be the crucial maximal ideal of B and let J be the conductor ideal of B in $\mathbf{k}(t)[y]$. With Remark 5.3.5 we get for $i = 1, 2$

$$B = \{\, f \in \mathbf{k}(t)[y] \mid \omega_i(\bar{f}) \geq 0 \,\} \quad \text{and} \quad \mathfrak{n} = \{\, f \in \mathbf{k}(t)[y] \mid \omega_i(\bar{f}) > 0 \,\}\,,$$

where $\bar{f}$ denotes the residue class modulo J and ω_i denotes the valuation

$$\omega_i \colon Q(\mathbf{k}(t)[y]/J) \longrightarrow \mathbb{Q}\,, \quad g \longmapsto \nu(g(\alpha_i))\,.$$

By [4, Proposition 3.3], $(B/J)_{\mathfrak{n}}$ is a one-dimensional valuation ring of the field $Q(\mathbf{k}(t)[y]/J)$ and therefore it is a maximal subring of $Q(\mathbf{k}(t)[y]/J)$, see Lemma 2.0.10. The description above of B and $\mathfrak{n}$ implies that $(B/J)_{\mathfrak{n}}$ is the valuation ring with respect to ω_1 and with respect to ω_2. Therefore, the valuations ω_1, ω_2 are the same up to an order preserving isomorphism of $(\mathbb{Q}, +, <)$. However, since $\omega_1(t) = 1 = \omega_2(t)$, these valuations must then be the same. Thus by Lemma 5.5.3 there exists $\sigma \in \mathrm{Aut}(\mathbf{k}(t^{\mathbb{Q}})/\mathbf{k}(t))$ such that $\alpha_1 = \sigma(\alpha_2)$. This proves the injectivity of Ψ. $\square$

6. Classification of the maximal k-subalgebras of $\mathbf{k}[t, t^{-1}, y]$

The goal of this section is to classify all maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, t^{-1}, y]$. In fact, we reduce this problem in this section to another classification result, which we will solve then in the next section.

Proposition 6.0.4. *Let $A \subseteq \mathbf{k}[t, t^{-1}, y]$ be an extending maximal $\mathbf{k}$-subalgebra. Then, exactly one of the following cases occurs:*

i) There exists and automorphism σ of $\mathbf{k}[t, t^{-1}, y]$ such that $\sigma(A)$ contains $\mathbf{k}[t, y]$;
ii) A contains $\mathbf{k}[t, t^{-1}]$.

Proof of Proposition 6.0.4. Note that A satisfies the property P_2 in $\mathbf{k}[t, t^{-1}, y]$, see Lemma 2.0.9. Since $t \cdot t^{-1} = 1 \in A$, it follows that either $t \in A$ or $t^{-1} \in A$. Assume that we are not in case ii), i.e. assume that $\mathbf{k}[t, t^{-1}]$ is not contained in A. By applying an appropriate automorphism of $\mathbf{k}[t, t^{-1}, y]$, we can assume that $t \in A$ and hence $t^{-1} \notin A$. Therefore we get $A_t = A[t^{-1}] = \mathbf{k}[t, t^{-1}, y]$, since A is maximal. This implies that there exists an integer $k \geq 0$ such that $t^k y \in A$. Thus the $\mathbf{k}[t, t^{-1}]$-automorphism

$$\sigma \colon \mathbf{k}[t, t^{-1}, y] \longrightarrow \mathbf{k}[t, t^{-1}, y]\,, \quad y \longmapsto t^{-k} y$$

satisfies $y \in \sigma(A)$. Hence we get $\sigma(A) \supseteq \mathbf{k}[t, y]$ and therefore we are in case i). $\square$

The extending maximal $\mathbf{k}$-subalgebras in case i) of Proposition 6.0.4 are then described by Theorem 5.5.1. Thus we are left with the description of the extending maximal $\mathbf{k}$-subalgebras in case ii). In fact, they can be characterized in the following way:

Proposition 6.0.5. *There is a bijection*

$$\Phi\colon \left\{\begin{array}{c}\textit{extending maximal}\\ \boldsymbol{k}\textit{-subalgebras of } \boldsymbol{k}[t,t^{-1},y]\\ \textit{that contain } \boldsymbol{k}[t,t^{-1}]\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{extending maximal}\\ \boldsymbol{k}\textit{-subalgebras of } \boldsymbol{k}[t,y] \textit{ that}\\ \textit{contain } \boldsymbol{k}[t] \textit{ and } t \textit{ lies not in}\\ \textit{the crucial maximal ideal}\end{array}\right\}$$

given by $\Phi(A) = A \cap \mathbf{k}[t, y]$.

Proof. Let A be an extending maximal $\mathbf{k}$-subalgebra of $\mathbf{k}[t, t^{-1}, y]$ that contains $\mathbf{k}[t, t^{-1}]$. By Lemma 2.0.11, there exists $\lambda \in \mathbf{k}^*$ such that $t - \lambda$ lies in the crucial maximal ideal $\mathfrak{m}$ of A. Thus $A_{t-\lambda} = \mathbf{k}[t, t^{-1}, y]_{t-\lambda}$. Hence there exists $k \geq 1$ such that $(t - \lambda)^k y \in A$ and thus we get

$$\mathbf{k}[t, t^{-1}, (t - \lambda)^k y] \subseteq A \subsetneq \mathbf{k}[t, t^{-1}, y]\,.$$

This implies

$$\mathbf{k}[t, (t - \lambda)^k y] \subseteq A \cap \mathbf{k}[t, y] \subsetneq \mathbf{k}[t, y]\,.$$

We claim, that $A \cap \mathbf{k}[t, y]$ is a maximal subring of $\mathbf{k}[t, y]$. Let therefore $A \cap \mathbf{k}[t, y] \subseteq B \subseteq \mathbf{k}[t, y]$ be an intermediate ring. Thus we get

$$B = B_{t-\lambda} \cap B_t \supseteq \mathbf{k}[t, y]\ \cap B_t \supseteq B\,.$$

One can check that $A = (A \cap \mathbf{k}[t, y])_t$. Since A is maximal in $\mathbf{k}[t, t^{-1}, y]$, we get either $B_t = A$ or $B_t = \mathbf{k}[t, t^{-1}, y]$ and the claim follows. This proves that Φ is well-defined and injective.

Let A' be an extending maximal $\mathbf{k}$-subalgebra of $\mathbf{k}[t, y]$ that contains $\mathbf{k}[t]$ and the crucial maximal ideal does not contain t. By Lemma 2.0.6 it follows that A'_t is an extending maximal $\mathbf{k}$-subalgebra of $\mathbf{k}[t, t^{-1}, y]$ that contains $\mathbf{k}[t, t^{-1}]$. Moreover, we have $A'_t \cap \mathbf{k}[t, y] = A$ by the maximality of A in $\mathbf{k}[t, y]$. This proves the surjectivity of Φ. $\square$

After this proposition, one is now reduced to the problem of the description of all maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that contain $\mathbf{k}[t]$.

7. Classification of the maximal k-subalgebras of $\mathbf{k}[t, y]$ that contain $\mathbf{k}[t]$

Let $\mathscr{M}$ be the set of extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that contain $\mathbf{k}[t]$. The goal of this section is to describe the set $\mathscr{M}$ with the aid of the classification result Theorem 5.5.1. For this we introduce a subset $\mathscr{N}$ of the maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y, y^{-1}]$ that contain $\mathbf{k}[t, y^{-1}]$.

Remark 7.0.1. If A is an extending maximal $\mathbf{k}$-subalgebra of $\mathbf{k}[t, y, y^{-1}]$ that contains $\mathbf{k}[t, y^{-1}]$, then the residue field of the crucial maximal ideal is isomorphic to $\mathbf{k}$, by Remark 5.3.4 and Theorem 5.5.1. Hence there exists a unique $\lambda \in \mathbf{k}$ such that $t - \lambda$ lies in the crucial maximal ideal of A.

Define $\mathscr{N}$ to be the set of extending maximal $\mathbf{k}$-subalgebras A of $\mathbf{k}[t, y, y^{-1}]$ that contain $\mathbf{k}[t, y^{-1}]$ and such that

$$A \longrightarrow \mathbf{k}[t, y, y^{-1}]/(t - \lambda) \tag{6}$$

is surjective where λ denotes the unique element in $\mathbf{k}$ such that the crucial maximal ideal contains $t - \lambda$ (see Remark 7.0.1). Now, we can formulate the main result of this section.

Theorem 7.0.6. *The map* $\Theta\colon \mathscr{N} \to \mathscr{M}$, $A \mapsto A \cap \mathbf{k}[t, y]$ *is bijective.*

Remark 7.0.2. As we classified already all maximal subrings of $\mathbf{k}[t, y, y^{-1}]$ that contain $\mathbf{k}[t, y^{-1}]$ (see Theorem 5.5.1), Theorem 7.0.6 gives us a description of all extending maximal $\mathbf{k}$-subalgebras of $\mathbf{k}[t, y]$ that contain a coordinate of $\mathbf{k}[t, y]$ (up to automorphisms of $\mathbf{k}[t, y]$).

Remark 7.0.3. Lemma 2.0.11 implies the following: If A is an extending maximal $\mathbf{k}$-subalgebra of $\mathbf{k}[t, y]$ which contains $\mathbf{k}[t]$, then there exists a unique $\lambda \in \mathbf{k}$ such that $t - \lambda$ lies in the crucial maximal ideal of A. Thus $\mathscr{M}$ is the disjoint union of the sets

$$\mathscr{M}_\lambda = \{\, A \in \mathscr{M} \mid t - \lambda \text{ lies in the crucial maximal ideal of } A \,\}, \quad \lambda \in \mathbf{k}\,.$$

By Remark 7.0.1, $\mathscr{N}$ is the disjoint union of the sets

$$\mathscr{N}_\lambda = \{\, A \in \mathscr{N} \mid t - \lambda \text{ lies in the crucial maximal ideal of } A \,\}, \quad \lambda \in \mathbf{k}\,.$$

Note that we have canonical bijections

$$\mathscr{M}_0 \longmapsto \mathscr{M}_\lambda\,, \quad A \longmapsto \sigma_\lambda(A) \quad \text{and} \quad \mathscr{N}_0 \longrightarrow \mathscr{N}_\lambda\,, \quad A \longmapsto \sigma_\lambda(A)$$

where σ_λ is the automorphism of $\mathbf{k}[t, y, y^{-1}]$ given by $\sigma_\lambda(t) = t - \lambda$ and $\sigma_\lambda(y) = y$. Using the fact that for all $A \in \mathscr{M}_0$ we have

$$\sigma_\lambda(A) \cap \mathbf{k}[t, y] = \sigma_\lambda(A \cap \mathbf{k}[t, y])\,,$$

one is reduced for the proof of Theorem 7.0.6 to proving the following proposition.

Proposition 7.0.7. *The map* $\mathscr{N}_0 \to \mathscr{M}_0$, $A \mapsto A \cap \mathbf{k}[t, y]$ *is bijective.*

For the proof of Proposition 7.0.7 we need several (technical) lemmas.

Lemma 7.0.8. *Let $\mathbf{k}[t] \subseteq Q \subsetneq \mathbf{k}[t,y]$ be an intermediate ring that satisfies the P_2 property in $\mathbf{k}[t,y]$ and assume that*

$$Q \longrightarrow \mathbf{k}[t,y]/t\,\mathbf{k}[t,y]$$

is surjective. If $\mathfrak{p} \subseteq Q$ is an ideal that contains t and that does not contain $t\,\mathbf{k}[t,y] \cap Q$, then there exists $h \in Q \setminus \mathfrak{p}$ such that $y^{-1} \in Q_h$.

Proof. By assumption, there exists $g \in \mathbf{k}[t,y] \setminus y\,\mathbf{k}[t,y]$ and $n \geq 0$ such that

$$ty^n g \in Q \setminus \mathfrak{p}\,. \tag{7}$$

Let $g_0 \in \mathbf{k}[t]$, $g' \in \mathbf{k}[t,y] \setminus y\,\mathbf{k}[t,y]$ and $r \geq 1$ such that $g - g_0 = y^r g'$. If $n = 0$, we get

$$ty^r g' = tg - tg_0 \in Q \setminus \mathfrak{p}\,.$$

Thus we can and will assume that $n \geq 1$. Now, choose $g \in \mathbf{k}[t,y] \setminus y\,\mathbf{k}[t,y]$ of minimal y-degree such that (7) is satisfied for some $n \geq 1$. We claim that $g \in Q$. Otherwise, $\deg_y(g) > 0$ and $ty^n \in Q$, since Q satisfies the P_2 property in $\mathbf{k}[t,y]$. In fact, since g is of minimal y-degree, we get $ty^n \in \mathfrak{p}$. Thus we get a contradiction to the fact that

$$ty^{n+r}g' = ty^n g - ty^n g_0 \in Q \setminus \mathfrak{p} \quad \text{and} \quad \deg_y(g') < \deg_y(g)\,.$$

Let $h = ty^n g \in Q \setminus \mathfrak{p}$. Since $t, g \in Q$, it follows that $y^{-n} = tg/h \in Q_h$. Since Q satisfies the property P_2 in $\mathbf{k}[t,y]$, the localization Q_h satisfies the property P_2 in $\mathbf{k}[t,y]_h = \mathbf{k}[t,t^{-1},y,y^{-1}]_g$. Hence, we get $y^{-1} \in Q_h$. $\square$

Lemma 7.0.9. *Let $A \in \mathscr{N}_0$ and let $\mathfrak{m}$ be the crucial maximal ideal of A. Then the inclusion $A \cap \mathbf{k}[t,y] \subseteq \mathbf{k}[t,y]$ defines an open immersion*

$$\varphi\colon \mathbb{A}^2_{\mathbf{k}} \longrightarrow \operatorname{Spec} A \cap \mathbf{k}[t,y]$$

on spectra and the complement of the image of φ consists only of the maximal ideal $\mathfrak{m} \cap \mathbf{k}[t,y]$ of $A \cap \mathbf{k}[t,y]$.

Remark 7.0.4. The proof will show the following:

a) the maximal ideal $\mathfrak{m} \cap \mathbf{k}[t,y]$ of $A \cap \mathbf{k}[t,y]$ contains t and does not contain $t\,\mathbf{k}[t,y] \cap A \cap \mathbf{k}[t,y]$ (see iii) in the proof);
b) the homomorphism $A \cap \mathbf{k}[t,y] \to \mathbf{k}[t,y]/t\,\mathbf{k}[t,y]$ is surjective (see i) in the proof).

Proof of Lemma 7.0.9. Let $A' = A \cap \mathbf{k}[t,y]$ and let $\mathfrak{m}' = \mathfrak{m} \cap \mathbf{k}[t,y]$. Due to Remark 7.0.1, the residue field $A/\mathfrak{m}$ is isomorphic to $\mathbf{k}$. Hence, $\mathfrak{m}'$ is a maximal ideal of A'. We divide the proof in several steps.

i) We claim that φ induces a closed immersion $\{0\} \times \mathbb{A}^1_{\mathbf{k}} \to V_{\mathrm{Spec}(A')}(t)$. Due to the surjection (6), there exists $f \in \mathbf{k}[t, y, y^{-1}]$, $a \in A$ such that $y = a + tf$. Let $f = f^+ + f^-$ where $f^+ \in \mathbf{k}[t, y]$ and $\deg_y(f^-) < 0$. We have $a + tf^- \in A$ and $tf^+ \in t\,\mathbf{k}[t, y]$. Thus we get

$$a + tf^- = y - tf^+ \in A \cap \mathbf{k}[t, y] = A' .$$

This implies that

$$A'/tA' \longrightarrow \mathbf{k}[t, y]/t\,\mathbf{k}[t, y] = \mathbf{k}[y]$$

is surjective, which implies the claim.

ii) We claim that φ induces an isomorphism $\mathbb{A}^*_{\mathbf{k}} \times \mathbb{A}^1_{\mathbf{k}} \simeq \mathrm{Spec}(A') \setminus V_{\mathrm{Spec}(A')}(t)$. Since $t \in \mathfrak{m}$, we have $A_t = \mathbf{k}[t, t^{-1}, y, y^{-1}]$ and thus $t^k y \in A$ for some integer k. This implies $t^k y \in A'$ and thus $A'_t = \mathbf{k}[t, t^{-1}, y]$.

iii) We claim that $\mathrm{Spec}(A') \setminus \varphi(\mathbb{A}^2_{\mathbf{k}}) = \{\mathfrak{m}'\}$. Using i) and ii) this is equivalent to show that $\mathfrak{m}'$ is the only prime ideal of A' that contains t and does not contain $t\,\mathbf{k}[t, y] \cap A'$. Since $\mathfrak{m}$ contains t it follows that $\mathfrak{m}'$ contains t. Since there exists no prime ideal of $\mathbf{k}[t, y, y^{-1}]$ that lies over $\mathfrak{m}$, the surjection (6) implies that $\mathfrak{m}$ does not contain $t\,\mathbf{k}[t, y, y^{-1}] \cap A$. Hence there exists $f \in \mathbf{k}[t, y, y^{-1}]$ such that $tf \in A \setminus \mathfrak{m}$. Since $t\,\mathbf{k}[t, y^{-1}] \subseteq \mathfrak{m}$, we can even assume that $f \in \mathbf{k}[t, y]$. Hence $tf \in A' \setminus \mathfrak{m}'$ and therefore $\mathfrak{m}'$ does not contain $t\,\mathbf{k}[t, y] \cap A'$.
As $A \subseteq \mathbf{k}[t, y, y^{-1}]$ induces an isomorphism $\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^*_{\mathbf{k}} \simeq \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ and since $t \in \mathfrak{m}$, we have

$$\mathrm{rad}(tA) = t\,\mathbf{k}[t, y, y^{-1}] \cap A \cap \mathfrak{m} .$$

Intersecting with $\mathbf{k}[t, y]$ yields

$$\mathrm{rad}(tA') = t\,\mathbf{k}[t, y] \cap A' \cap \mathfrak{m}'.$$

Thus every prime ideal of A' that contains t and does not contain $t\,\mathbf{k}[t, y] \cap A'$ must be equal to $\mathfrak{m}'$ (note that $\mathfrak{m}'$ is a maximal ideal of A').

iv) We claim that φ is an open immersion. According to Theorem 5.5.1 and Remark 5.3.5, there exists $\alpha \in \mathbf{k}[[(y^{-1})^{\mathbb{Q}}]]^+$ such that

$$A = \{\, f \in \mathbf{k}[y^{-1}, y, t] \mid \nu(f(\alpha)) \geq 0 \,\} .$$

Note that y^{-1} corresponds to the t in Theorem 5.5.1 and t corresponds to the y in Theorem 5.5.1. In particular we have $\nu(y^{-1}) = 1$. If $\alpha = 0$, then $A = \mathbf{k}[y^{-1}] + t\,\mathbf{k}[y^{-1}, y, t]$ and thus (6) is not surjective. Hence $\alpha \neq 0$. Let $\nu(\alpha) = a/b$ for integers $a \geq 0$, $b > 0$ and let $\lambda \in \mathbf{k}^*$ be the coefficient of $y^{-a/b}$ of α. There exists $k \geq 1$ such that

$$y(\lambda^b - t^b y^a)^k \in A' \,. \tag{8}$$

Indeed, $\nu(\lambda^b - \alpha^b y^a) > 0$, since α is equal to $\lambda(y^{-1})^{a/b}$ plus higher oder terms in y^{-1}. Hence, there exists $k \geq 1$ such that

$$\nu(y(\lambda^b - \alpha^b y^a)^k) = -1 + k\nu(\lambda^b - \alpha^b y^a) \geq 0 \,,$$

which yields (8).

As A satisfies the property P_2 in $\mathbf{k}[y^{-1}, y, t]$ (see Lemma 2.0.9), it follows that A' satisfies the property P_2 in $\mathbf{k}[y, t]$. Since $y \notin A'$ we get thus $\lambda^b - t^b y^a \in A'$ by (8). Again by (8) we have $y \in A'_{\lambda^b - t^b y^a}$, which implies

$$A'_{\lambda^b - t^b y^a} = \mathbf{k}[t, y]_{\lambda^b - t^b y^a} \,.$$

As the zero set of $\lambda^b - t^b y^a$ and of t in $\mathbb{A}^2_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[t, y]$ are disjoint, it follows with ii) that $\varphi \colon \mathbb{A}^2_{\mathbf{k}} \to \operatorname{Spec}(A')$ is locally an open immersion. However, i) and ii) imply that φ is injective and thus φ is an open immersion. $\square$

Lemma 7.0.10. *Let $A \in \mathscr{N}_0$ and let $\mathfrak{m}$ be the crucial maximal ideal of A. Moreover, we denote $A' = A \cap \boldsymbol{k}[t, y]$ and $\mathfrak{m}' = \mathfrak{m} \cap \boldsymbol{k}[t, y]$. Then the following holds:*

a) A' is a maximal subring of $\boldsymbol{k}[t, y]$;
b) $\mathfrak{m}'$ is the crucial maximal ideal of A';
c) For all $h \in A' \setminus \mathfrak{m}'$ such that $y^{-1} \in A'_h$ we have

$$A = A'_h \cap \boldsymbol{k}[t, y, y^{-1}] \quad \textit{and} \quad A'_h = A_h \,.$$

Moreover, there exist $h \in A' \setminus \mathfrak{m}'$ with $y^{-1} \in A'_h$.

Proof of Lemma 7.0.10. As A satisfies the P_2 property in $\mathbf{k}[t, y, y^{-1}]$, A' satisfies the P_2 property in $\mathbf{k}[t, y]$. By Remark 7.0.4, $\mathfrak{m}'$ contains t and does not contain $t\,\mathbf{k}[t, y] \cap A'$. Moreover, the homomorphism $A' \to \mathbf{k}[t, y]/t\,\mathbf{k}[t, y]$ is surjective according to Remark 7.0.4. Let $h \in A' \setminus \mathfrak{m}'$ such that $y^{-1} \in A'_h$ (by Lemma 7.0.8 there exists such an h). We claim that

$$A'_h = A_h \,. \tag{9}$$

Indeed, if $a = a^+ + a^- \in A$ and $a^+ \in \mathbf{k}[t, y]$, $\deg_y(a^-) < 0$, then we get

$$a^+ = a - a^- \in A \cap \mathbf{k}[t, y] = A' \,.$$

However, $a^- \in \mathbf{k}[t, y^{-1}] \subseteq A'_h$ and thus $a = a^+ + a^- \in A'_h$, which implies the claim. Using Lemma 2.0.6 and the fact that $h \in A' \setminus \mathfrak{m}'$, the claim implies that

$$A'_h \subsetneq \mathbf{k}[t, y, y^{-1}]_h = \mathbf{k}[t, y]_h$$

is an extending maximal subring. Now, let $A' \subseteq B \subsetneq \mathbf{k}[t, y]$ be an intermediate ring. Since $\varphi\colon \mathbb{A}^2_{\mathbf{k}} \to \operatorname{Spec}(A')$ is an open immersion and $\operatorname{Spec}(A') \setminus \varphi(\mathbb{A}^2_{\mathbf{k}}) = \{\mathfrak{m}'\}$ (see Lemma 7.0.9), it follows that $\mathfrak{m}'$ lies in the image of the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A')$. Hence, there exists a prime ideal in B that lies over $\mathfrak{m}'$. Since $h \in A' \setminus \mathfrak{m}'$, it follows that there exists a prime ideal of B_h that lies over $\mathfrak{m}' A'_h$. In particular, $B_h \neq \mathbf{k}[t, y]_h$. By the maximality of A'_h in $\mathbf{k}[t, y]_h$ we get $A'_h = B_h$. Thus

$$B \subseteq B_h \cap \mathbf{k}[t, y] = A'_h \cap \mathbf{k}[t, y] = A_h \cap \mathbf{k}[t, y] = A'$$

where the last equality follows from the fact that

$$A_h \cap \mathbf{k}[t, y, y^{-1}] = A \tag{10}$$

(note that $y \notin A_h$, since otherwise $yh^k \in A \cap \mathbf{k}[t, y] = A'$ for a certain integer k and thus $y \in A'_h$, contradicting the maximality of A'_h in $\mathbf{k}[t, y]_h$). This proves the maximality of A' in $\mathbf{k}[t, y]$, which is a). Equations (9) and (10) say, that c) is satisfied. Statement b) is a consequence of statement a) and Lemma 7.0.9. □

Proof of Proposition 7.0.7. From Lemma 7.0.10 a), b) it follows that $\mathscr{N}_0 \to \mathscr{M}_0$ is well-defined. From Lemma 7.0.10 c) it follows that $\mathscr{N}_0 \to \mathscr{M}_0$ is injective.

Now, we prove the surjectivity. Let $Q \in \mathscr{M}_0$. We have the following inclusion

$$Q/t\,\mathbf{k}[t, y] \cap Q \ \subseteq \mathbf{k}[t, y]/t\,\mathbf{k}[t, y] = \mathbf{k}[y]\,. \tag{11}$$

On spectra, this map yields an open immersion, since $\operatorname{Spec} \mathbf{k}[t, y] \to \operatorname{Spec} Q$ is an open immersion. Hence, (11) is a finite ring extension, and thus (11) must be an equality. This implies that the crucial maximal ideal $\mathfrak{p}$ of Q does not contain $t\,\mathbf{k}[t, y] \cap Q$ (note that $\operatorname{Spec} Q \setminus \operatorname{Spec} \mathbf{k}[t, y] = \{\mathfrak{p}\}$). By assumption, $t \in \mathfrak{p}$. Moreover, Q satisfies the P_2 property in $\mathbf{k}[t, y]$ by Lemma 2.0.9. By Lemma 7.0.8 there exists $h \in Q \setminus \mathfrak{p}$ such that $y^{-1} \in Q_h$. Thus, Lemma 2.0.6 implies that

$$Q_h \subsetneq \mathbf{k}[t, y]_h = \mathbf{k}[t, y, y^{-1}]_h$$

is an extending maximal subring. Since $y^{-1} \in Q_h$, the ring

$$Q' = Q_h \cap \mathbf{k}[t, y, y^{-1}] \subsetneq \mathbf{k}[t, y, y^{-1}]$$

contains $\mathbf{k}[t, y^{-1}]$. Now, we divide the proof in several steps.

i) We claim that Q' is a maximal subring of $\mathbf{k}[t, y, y^{-1}]$. Therefore, take an intermediate ring $Q' \subseteq B \subsetneq \mathbf{k}[t, y, y^{-1}]$. By the maximality of Q in $\mathbf{k}[t, y]$ we get $Q = B \cap \mathbf{k}[t, y]$ and hence

$$Q_h = (B \cap \mathbf{k}[t, y])_h \,.$$

If y would be in B_h, then y would be in $(B \cap \mathbf{k}[t, y])_h = Q_h$, a contradiction to the fact that $Q_h \neq \mathbf{k}[t, y, y^{-1}]_h$. Hence we have $B_h \neq \mathbf{k}[t, y, y^{-1}]_h$. The maximality of Q_h in $\mathbf{k}[t, y, y^{-1}]_h$ implies that $B_h = Q_h$. Hence, we have

$$B \subseteq B_h \cap \mathbf{k}[t, y, y^{-1}] = Q' \subseteq B \,,$$

which proves the maximality of Q' in $\mathbf{k}[t, y, y^{-1}]$.

ii) We claim that $\mathfrak{p}Q_h \cap \mathbf{k}[t, y, y^{-1}]$ is the crucial maximal ideal of Q'. Clearly, $\mathfrak{p}Q_h$ is the crucial maximal ideal of Q_h. If $\mathfrak{p}Q_h \cap \mathbf{k}[t, y, y^{-1}]$ would not be the crucial maximal ideal of Q', then $\operatorname{Spec} Q_h \to \operatorname{Spec} Q'$ would send $\mathfrak{p}Q_h$ to a point of the open subset $\operatorname{Spec} \mathbf{k}[t, y, y^{-1}]$ of $\operatorname{Spec} Q'$. This would imply that $\mathbf{k}[t, y, y^{-1}] \subseteq Q_h$, a contradiction.

iii) We claim that $Q' \in \mathscr{N}_0$. By ii), $\mathfrak{p}Q_h \cap \mathbf{k}[t, y, y^{-1}]$ is the crucial maximal ideal of Q' and it contains t. By the equality (11) we get $y = q + tf$ for some $q \in Q$, $f \in \mathbf{k}[t, y]$. Since $Q \subseteq Q'$ and $\mathbf{k}[t, y^{-1}] \subseteq Q'$, the homomorphism

$$Q' \longrightarrow \mathbf{k}[t, y, y^{-1}]/t\,\mathbf{k}[t, y, y^{-1}]$$

is surjective. With i) we get $Q' \in \mathscr{N}_0$.

iv) We claim that $Q' \cap \mathbf{k}[t, y] = Q$. This follows from the fact that $Q \subseteq Q' \cap \mathbf{k}[t, y] \subsetneq \mathbf{k}[t, y]$ and from the maximality of Q in $\mathbf{k}[t, y]$.

This proves the surjectivity. □

Let us interpret the map $\mathscr{N}_0 \to \mathscr{M}_0$, $A \mapsto A \cap \mathbf{k}[t, y]$ in geometric terms. For this we introduce the following terminology.

Definition 7.0.5. We call a dominant morphism $Y \to X$ of affine schemes an *(extending) minimal morphism*, if $\Gamma(X, \mathcal{O}_X)$ is an (extending) maximal subring of $\Gamma(Y, \mathcal{O}_Y)$. Moreover, the point in X which corresponds to the crucial maximal ideal of $\Gamma(X, \mathcal{O}_X)$ we call the *crucial point* of X.

Let us denote by $\mathrm{pr}\colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ the projection $(t, y) \mapsto t$. The set $\mathscr{M}_0$ corresponds to the extending minimal morphisms $\psi\colon \mathbb{A}^2_{\mathbf{k}} \to X$ such that $\mathrm{pr}\colon \mathbb{A}^2_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ factorizes as

$$\mathbb{A}^2_{\mathbf{k}} \xrightarrow{\psi} X \longrightarrow \mathbb{A}^1_{\mathbf{k}} \,,$$

and such that the crucial point of X is sent onto $0 \in \mathbb{A}^1_{\mathbf{k}}$ via $X \to \mathbb{A}^1_{\mathbf{k}}$. The set $\mathscr{N}_0$ corresponds to the extending minimal morphisms $\varphi\colon \mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^*_{\mathbf{k}} \to Y$ such that the open immersion $\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^*_{\mathbf{k}} \to \mathbb{A}^2_{\mathbf{k}}$, $(t, y) \mapsto (t, y^{-1})$ factorizes as

$$\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^*_{\mathbf{k}} \xrightarrow{\varphi} Y \longrightarrow \mathbb{A}^2_{\mathbf{k}}$$

and such that the image of $\{0\} \times \mathbb{A}^*_{\mathbf{k}}$ under φ is closed in Y.

Proposition 7.0.11. *Let $\varphi\colon \mathbb{A}^1_{\boldsymbol{k}} \times \mathbb{A}^*_{\boldsymbol{k}} \to Y$ be an extending minimal morphism corresponding to an element $A \in \mathcal{N}_0$. Then*

$$\operatorname{Spec} A \ \cap \boldsymbol{k}[t, y] = Y \cup_{\varphi} \mathbb{A}^2_{\boldsymbol{k}}$$

where $Y \cup_{\varphi} \mathbb{A}^2_{\boldsymbol{k}}$ denotes the glueing via $\mathbb{A}^2_{\boldsymbol{k}} \xleftarrow{\sigma} \mathbb{A}^1_{\boldsymbol{k}} \times \mathbb{A}^*_{\boldsymbol{k}} \xrightarrow{\varphi} Y$ where σ is the open immersion defined by $\sigma(t, y) = (t, y)$.

Proof. By Theorem 7.0.6 we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^*_{\mathbf{k}} \arrow[r, "\varphi", "\text{extend. minimal mor.}"'] \arrow[d, hook, "\sigma"'] & Y \arrow[d] \\
\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^1_{\mathbf{k}} \arrow[r, "\text{extend. minimal mor.}"'] & \operatorname{Spec} A \cap \mathbf{k}[t, y] \,.
\end{tikzcd}$$

By Lemma 7.0.10, there exists a regular function h on $\operatorname{Spec} A \cap \mathbf{k}[t, y]$ that does not vanish at the crucial point of $\operatorname{Spec} A \cap \mathbf{k}[t, y]$ and we have

$$A_h = (A \cap \mathbf{k}[t, y])_h \,.$$

Thus $Y \to \operatorname{Spec} A \cap \mathbf{k}[t, y]$ restricts to an open immersion on Y_h. By the commutativity of the diagram, it follows that $Y \to \operatorname{Spec} A \cap \mathbf{k}[t, y]$ restricts to an open immersion on $\varphi(\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^*_{\mathbf{k}})$. By Lemma 7.0.10, the morphism $Y \to \operatorname{Spec} A \cap \mathbf{k}[t, y]$ maps the crucial point of Y to that one of $\operatorname{Spec} A \cap \mathbf{k}[t, y]$. Hence, Y_h contains the crucial point of Y. In summary, we get that $Y \to \operatorname{Spec} A \cap \mathbf{k}[t, y]$ is an open immersion. Thus all morphisms in the diagram above are open immersions. Moreover, φ induces an isomorphism $\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^*_{\mathbf{k}} \to Y \cap \mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^1_{\mathbf{k}}$ where we consider $Y \cap \mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^1_{\mathbf{k}}$ as an open subset of $\operatorname{Spec} A \cap \mathbf{k}[t, y]$. Hence $\operatorname{Spec} A \cap \mathbf{k}[t, y]$ is the claimed glueing. □

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ALGEBRAIC EMBEDDINGS OF $\mathbb{C}$ INTO $\mathbf{SL}_n(\mathbb{C})$

IMMANUEL STAMPFLI*

Department of Mathematics
Jacobs University Bremen
Campus Ring 1
28759 Bremen, Germany

immanuel.e.stampfli@gmail.com

Abstract. We prove that any two algebraic embeddings $\mathbb{C} \to \mathrm{SL}_n(\mathbb{C})$ are the same up to an algebraic automorphism of $\mathrm{SL}_n(\mathbb{C})$, provided that n is at least 3. Moreover, we prove that two algebraic embeddings $\mathbb{C} \to \mathrm{SL}_2(\mathbb{C})$ are the same up to a holomorphic automorphism of $\mathrm{SL}_2(\mathbb{C})$.

1. Introduction

There are many results concerning algebraic embeddings of some variety into the affine space $\mathbb{C}^n$. Let me recall two of them. Any two algebraic embeddings of a smooth affine variety X into $\mathbb{C}^n$ are the same up to an algebraic automorphism of $\mathbb{C}^n$, provided that $n > 2\dim X + 1$. This result is due to Nori, Srinivas [Sri91], and Kaliman [Kal91]. If one relaxes the condition that the automorphism of $\mathbb{C}^n$ must be algebraic, Kaliman [Kal13] and independently, Feller and the author [FS14] proved the following improvement: Any two algebraic embeddings of a smooth affine variety X into $\mathbb{C}^n$ are the same up to a *holomorphic* automorphism of $\mathbb{C}^n$, provided that $n > 2\dim X$.

As a further development of these results, we study algebraic embeddings of $\mathbb{C}$ into SL_n. This article can be seen as a first example to understand algebraic embeddings of a curve into an arbitrary affine algebraic variety with a large automorphism group.

In dimension zero, Arzhantsev, Flenner, Kaliman, Kutzschebauch, and Zaidenberg proved that two embeddings of a finite set into any irreducible smooth affine flexible variety Z are the same up to an algebraic automorphism of Z, provided that $\dim Z > 1$ [AFK$^+$13]. Our main result is based on this work.

Main Theorem (cf. Theorems 4 and 7)**.** *Let $f, g\colon \mathbb{C} \to \mathrm{SL}_n$ be algebraic embeddings. If $n \geq 3$, then f and g are the same up to an algebraic automorphism of SL_n and if $n = 2$, then f and g are the same up to a holomorphic automorphism of SL_n.*

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Corresponding Author: I. Stampfli, e-mail: immanuel.e.stampfli@gmail.com.

To the author's knowledge it is not known whether all algebraic embeddings $\mathbb{C} \to \mathrm{SL}_2$ are the same up to an algebraic automorphism of SL_2. Also for algebraic embeddings $\mathbb{C} \to \mathbb{C}^3$, it is an open problem whether all these embeddings are the same up to an algebraic automorphism of $\mathbb{C}^3$; see [Sha92] for potential examples that are not equivalent to linear embeddings.

In fact, in a certain sense the class of algebraic embeddings $\mathbb{C} \to \mathrm{SL}_2$ is as big as the class of algebraic embeddings $\mathbb{C} \to \mathbb{C}^3$. More precisely, the following holds. If $g \colon \mathbb{C} \to \mathbb{C}^3$, $t \mapsto (g_1(t), g_2(t), g_3(t))$ is an algebraic embedding, then one can apply a (tame) algebraic automorphism of $\mathbb{C}^3$ such that afterwards the polynomial g_2 divides $g_1 g_3 - 1$ and thus the following map is an algebraic embedding

$$\mathbb{C} \to \mathrm{SL}_2, \quad t \mapsto \begin{pmatrix} g_1(t) & (g_1(t)g_3(t) - 1)/g_2(t) \\ g_2(t) & g_3(t) \end{pmatrix}.$$

The construction of the claimed (tame) algebraic automorphism of $\mathbb{C}^3$ can be seen as follows. First, one can apply a map of the form $(x, y, z) \mapsto (x, y + \lambda, z)$ such that afterwards the polynomial g_2 has only finitely many simple roots, say $t_1, \ldots, t_n$. Now, it is enough to apply some (tame) algebraic automorphism of the form $(x, y, z) \mapsto (\varphi_1(x, z), y, \varphi_3(x, z))$, which sends the points $g(t_1), \ldots, g(t_n)$ to the curve $\{xz = 1, y = 0\} \subseteq \mathbb{C}^3$; see [KZ99, Lemma 5.5].

The proof of the main theorem gives a method to construct the claimed automorphism. However, the proof does not produce a computer algorithm that would give such an automorphism. This is because the construction in the proof depends on certain zero sets of polynomials.

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2. Algebraic automorphisms of SL_n

Let us introduce first some notation. For i, j in $\{1, \ldots, n\}$, we denote the ijth entry of a matrix $X \in \mathrm{SL}_n$ by X_{ij}. The projection $\mathrm{SL}_n \to \mathbb{C}$, $X \to X_{ij}$ we denote by x_{ij}.

In the first lemma, we list algebraic automorphisms of SL_n that we use constantly. The proof is straight forward.

Lemma 1. *Let $n \geq 2$ and let $i \neq j$ be integers in $\{1, \ldots, n\}$. Then, for every polynomial p in the functions x_{kl}, $k \neq i$, the map*

$$\mathrm{SL}_n \to \mathrm{SL}_n, \quad X \mapsto E_{ij}(p(X)) \cdot X$$

is an automorphism, where $E_{ij}(a)$ denotes the elementary matrix with ijth entry equal to a. Similarly, for every polynomial q in the functions x_{kl}, $l \neq j$, the map

$$\mathrm{SL}_n \to \mathrm{SL}_n, \quad X \mapsto X \cdot E_{ij}(q(X))$$

is an automorphism.

Recall that the group of tame automorphisms of $\mathbb{C}^n$ is the subgroup of the automorphisms of $\mathbb{C}^n$ generated by the affine linear maps and the elementary automorphisms, i.e., the automorphisms of the form

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i + h_i(x_1, \ldots, \widehat{x_i}, \ldots, x_n), \ldots, x_n),$$

where h_i is a polynomial not depending on x_i. In the next result we list automorphisms of $\mathbb{C}^n$ that can be lifted to automorphisms of SL_n via the projection to the first column $\pi_1 \colon \mathrm{SL}_n \to \mathbb{C}^n$, i.e., automorphisms ψ of $\mathbb{C}^n$ such that there exists an automorphism Ψ of SL_n (depending on ψ) that makes the following diagram commutative:

$$\begin{tikzcd} \mathrm{SL}_n \arrow[r, "\Psi"] \arrow[d, "\pi_1"'] & \mathrm{SL}_n \arrow[d, "\pi_1"] \\ \mathbb{C}^n \arrow[r, "\psi"] & \mathbb{C}^n \end{tikzcd} .$$

Lemma 2. *Let $n \geq 2$. Every tame automorphism of $\mathbb{C}^n$ that preserves the origin can be lifted to some automorphism of SL_n via $\pi_1 \colon \mathrm{SL}_n \to \mathbb{C}^n$.*

Proof. First, remark that the group of tame automorphisms of $\mathbb{C}^n$ that preserve the origin is generated by the linear group GL_n and by the elementary automorphisms that preserve the origin. For every $A \in \mathrm{GL}_n$, the linear map $x \mapsto A \cdot x$ of $\mathbb{C}^n$ can be lifted to the automorphism

$$\mathrm{SL}_n \to \mathrm{SL}_n, \quad X \mapsto A \cdot X \cdot \mathrm{diag}\left(1, \ldots, 1, (\det A)^{-1}\right)$$

via π_1, where $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ denotes the $n \times n$-diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. Let ψ be an elementary automorphism of $\mathbb{C}^n$ that preserves the origin, i.e., there exist $i \in \{1, \ldots, n\}$ and polynomials $p_1, \ldots, \widehat{p_i}, \ldots, p_n$ in the variables $x_1, \ldots, \widehat{x_i}, \ldots, x_n$ such that

$$\psi(x_1, \ldots, x_n) = \left(x_1, \ldots, x_i + \sum_{j \neq i} x_j p_j(x_1, \ldots, \widehat{x_i}, \ldots, x_n), \ldots, x_n\right).$$

The automorphism ψ can be lifted to some automorphism of SL_n, e.g., to the automorphism

$$\mathrm{SL}_n \to \mathrm{SL}_n, \quad X \mapsto \left(\prod_{j \neq i} E_{ij}(p_j(X_{11}, \ldots, \widehat{X_{i1}}, \ldots, X_{n1}))\right) \cdot X;$$

cf. also Lemma 1. This finishes the proof. $\square$

3. A generic projection result

Let V be an algebraic variety. We say that a statement is true for *generic* $v \in V$ if there exists a Zariski dense open subset $U \subseteq V$ such that the statement is true for all $v \in U$.

Lemma 3. *Let $n \geq 3$. If $f\colon \mathbb{C} \to \mathrm{SL}_n$ is an algebraic embedding such that the matrices $f(0) - f(1)$ and $f'(0)$ have maximal rank, then, for generic $A \in \mathrm{M}_{n,n-1}$ the map*

$$\mathbb{C} \xrightarrow{f} \mathrm{SL}_n \xrightarrow{\pi_A} \mathrm{M}_{n,n-1}$$

is an algebraic embedding, where $\mathrm{M}_{n,n-1}$ denotes the space of $n \times (n-1)$-matrices and π_A is given by $X \mapsto X \cdot A$.

Proof. Let $\Delta \subseteq \mathbb{C}^2$ be the diagonal. Consider the following (Zariski) locally closed subsets of $\mathbb{C}^2 \setminus \Delta$:

$$C_i = \{\, (t, r) \in \mathbb{C}^2 \setminus \Delta \mid \operatorname{rank}(f(t) - f(r)) = i \,\}.$$

Consider for every $A \in \mathrm{M}_{n,n-1}$ the composition

$$C_i \to \mathbb{C}^2 \setminus \Delta \xrightarrow{(t,r) \mapsto f(t) - f(r)} \mathrm{M}_{n,n} \xrightarrow{\pi_A} \mathrm{M}_{n,n-1} \,. \tag{$*$}$$

This map is never zero for generic $A \in \mathrm{M}_{n,n-1}$; indeed:

- If $1 < i \leq n$, then ($*$) is never zero provided that $A \in \mathrm{M}_{n,n-1}$ has maximal rank.
- If $i = 1$, then $\dim C_1 \leq 1$, since $\dim C_n = 2$ (note that $f(0) - f(1)$ has maximal rank). For $(t, r) \in C_1$, let $Z_{(t,r)} = \ker(f(t) - f(r))$. Since $\dim C_1 \leq 1 < n - 1$, a generic $(n-1)$-dimensional subspace of $\mathbb{C}^n$ is different from $Z_{(t,r)}$ for all $(t, r) \in C_1$. Thus, for generic A the composition ($*$) is never zero.

Clearly, $C_0 = \varnothing$. Hence, we proved that the composition $\pi_A \circ f$ is injective for generic $A \in \mathrm{M}_{n,n-1}$. Clearly, $\pi_A \circ f$ is proper for generic $A \in \mathrm{M}_{n,n-1}$.

For the immersivity, we have to show for generic $A \in \mathrm{M}_{n,n-1}$ that

$$f'(t) \cdot A \neq 0 \tag{$\boxdot$}$$

for all $t \in \mathbb{C}$. Since $\operatorname{rank} f'(0) = n$, the set $U = \{\, t \in \mathbb{C} \mid \operatorname{rank} f'(t) = n \,\}$ is Zariski dense and open in $\mathbb{C}$. Thus ($\boxdot$) is satisfied for all $A \neq 0$ and for all $t \in U$. Since f is immersive, we have $f'(t) \neq 0$ for all $t \in \mathbb{C}$. This implies that for generic A we have $f'(t) \cdot A \neq 0$ for all $t \in \mathbb{C}$. $\square$

4. Algebraic embeddings of $\mathbb{C}$ into SL_n for $n \geq 3$

Theorem 4. *For $n \geq 3$, any two algebraic embeddings of $\mathbb{C}$ into SL_n are the same up to an algebraic automorphism of SL_n.*

Lemma 5. *Let $n \geq 2$. Assume that $f\colon \mathbb{C} \to \mathrm{SL}_n$ is an algebraic embedding such that*

$$\mathbb{C} \xrightarrow{f} \mathrm{SL}_n \xrightarrow{\pi_{n-1}} \mathrm{M}_{n,n-1}$$

is an algebraic embedding, where π_{n-1} denotes the projection to the first $n-1$ columns. Then there exists an algebraic automorphism φ of SL_n such that

$$\mathbb{C} \xrightarrow{f} \mathrm{SL}_n \xrightarrow{\varphi} \mathrm{SL}_n \xrightarrow{\pi_1} \mathbb{C}^n$$

is given by $t \mapsto (1, 0, \ldots, 0, t)^T$.

Proof of Lemma 5. Assume that $n = 2$. Since two algebraic embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ are the same up to an algebraic automorphism of $\mathbb{C}^2$ (Abhyankar–Moh–Suzuki Theorem; see [AM75], [Suz74]), one can see that there exists an algebraic automorphism of $\mathbb{C}^2$ that preserves the origin and changes the embedding $\pi_1 \circ f\colon \mathbb{C} \to \mathbb{C}^2$ to the embedding $\mathbb{C} \to \mathbb{C}^2$, $t \mapsto (1, t)$. Using the fact that every algebraic automorphism of $\mathbb{C}^2$ is tame (Jung's Theorem, see [Jun42]), it follows from Lemma 2 that there exists an algebraic automorphism φ of SL_2 such that $\pi_1 \circ \varphi \circ f(t) = (1, t)$.

Assume that $n \geq 3$. Let $A(t) = \pi_{n-1} \circ f(t)$. Since the kernel of $A(t)^T$ is one-dimensional for all t, the following affine variety

$$E = \{ (v, t) \in \mathbb{C}^n \times \mathbb{C} \mid A(t)^T \cdot v = 0 \}$$

defines the total space of a line bundle over $\mathbb{C}$ with projection map $(v, t) \mapsto t$. Since $n \geq 3 > \dim E$, this implies that there exists a vector $v \in \mathbb{C}^n$ such that $v^T \cdot A(t)$ is non-zero for all $t \in \mathbb{C}$. Now, complete v^T to a matrix $B \in \mathrm{SL}_n$ with last row equal to v^T. Since $n \geq 3$, there exists a permutation matrix $P \in \mathrm{SL}_n$, with first column equal to $(0, \ldots, 0, 1)^T$. After applying the automorphism $X \mapsto B \cdot X \cdot P$ of SL_n, we can assume that

i) the map $\mathbb{C} \to \mathrm{M}_{n,n-1}$ given by $t \mapsto (f_{ij}(t))_{1 \leq i \leq n, 2 \leq j \leq n}$ is an algebraic embedding and
ii) the vector $(f_{n2}(t), f_{n3}(t), \ldots, f_{nn}(t))$ is non-zero for all $t \in \mathbb{C}$,

where $f_{ij}(t)$ denotes the ijth entry of the matrix $f(t)$. By ii), there exist polynomials $\tilde{p}_k \in \mathbb{C}[t]$, $2 \leq k \leq n$ such that

$$\sum_{k=2}^{n} f_{nk}(t)\tilde{p}_k(t) = t - f_{n1}(t).$$

By i), there exist polynomials p_k in the functions x_{ij} with $1 \leq i \leq n$, $2 \leq j \leq n$ such that $\tilde{p}_k(t) = p_k(\ldots, f_{ij}(t), \ldots)$. Let $\varphi\colon \mathrm{SL}_n \to \mathrm{SL}_n$ be the automorphism

$$X \mapsto X \cdot \begin{pmatrix} 1 & & & \\ p_2(X) & 1 & & \\ \vdots & & \ddots & \\ p_n(X) & & & 1 \end{pmatrix}.$$

Clearly, the left down corner of the matrix $\varphi \circ f(t)$ is equal to t. Now, one can construct with the aid of Lemma 2 an automorphism ψ of SL_n such that the first column of $\psi \circ \varphi \circ f(t)$ is equal to $(1, 0, \ldots, 0, t)^T$. This proves the lemma. □

Lemma 6. *Let $n \geq 2$ and let $f\colon \mathbb{C} \to \mathrm{SL}_n$ be an algebraic embedding such that the first column of $f(t)$ is equal to $(1, 0, \ldots, 0, t)^T$. Then f is the same as*

$$\mathbb{C} \to \mathrm{SL}_n, \quad t \mapsto E_{n1}(t)$$

up to an algebraic automorphism of SL_n, where $E_{n1}(t)$ denotes the elementary matrix with left down corner equal to t.

Proof of Lemma 6. Let ψ be the automorphism of SL_n defined by

$$X \mapsto X \cdot f(X_{n1})^{-1} \cdot E_{n1}(X_{n1})$$

where X_{ij} denotes the ijth entry of the matrix X. Now, one can easily check that $\psi \circ f$ is the embedding $t \mapsto E_{n1}(t)$. $\square$

Proof of Theorem 4. Start with an algebraic embedding $f\colon \mathbb{C} \to \mathrm{SL}_n$. As SL_n is flexible, for any finite set Z in SL_n there exists an automorphism of SL_n which fixes Z and has prescribed volume preserving differentials in the points of Z; see [AFK$^+$13, Thm. 4.14 and Rem. 4.16]. Using the fact that $\mathrm{Aut}(\mathrm{SL}_n)$ acts 2-transitively on SL_n, see, e.g., [AFK$^+$13, Thm. 0.1], we can assume that

$$\det(f(0) - f(1)) \neq 0 \quad \text{and} \quad \det f'(0) \neq 0.$$

Since $n \geq 3$, by Lemma 3 there exists a matrix A in $\mathrm{M}_{n,n-1}$ of maximal rank, such that $t \mapsto f(t) \cdot A$ defines an algebraic embedding of $\mathbb{C}$ into $\mathrm{M}_{n,n-1}$. Extend A with an additional column $v \in \mathbb{C}^n$ to an $n \times n$-matrix $(A|v)$ of determinant one. After applying the algebraic automorphism $X \to X \cdot (A|v)$ of SL_n, we can assume that the composition

$$\mathbb{C} \xrightarrow{f} \mathrm{SL}_n \xrightarrow{\pi_{n-1}} \mathrm{M}_{n,n-1}$$

is an algebraic embedding. After an algebraic coordinate change of SL_n, we can assume that the first column of $f(t)$ is equal to $(1, 0, \ldots, 0, t)^T$ by Lemma 5. Thus, up to an algebraic automorphism of SL_n, f is the same as $t \mapsto E_{n1}(t)$ by Lemma 6. This finishes the proof. $\square$

5. Algebraic embeddings of $\mathbb{C}$ into SL_2

Theorem 7. *Any two algebraic embeddings* $\mathbb{C} \to \mathrm{SL}_2$ *are the same up to a holomorphic automorphism of* SL_2.

Remark 1. Since for all $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ the embeddings

$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad t \mapsto \begin{pmatrix} 1 & at + b \\ 0 & 1 \end{pmatrix}$$

are the same up to an algebraic automorphism of SL_2, it is enough to prove Theorem 7 up to an algebraic reparametrization of the embeddings $\mathbb{C} \to \mathrm{SL}_2$.

For the proof of Theorem 7, we need the following rather technical result, which enables us to bring an arbitrary algebraic embedding $\mathbb{C} \to \mathrm{SL}_2$ in a "nice" position.

Proposition 8. *Let* $f\colon \mathbb{C} \to \mathrm{SL}_2$ *be an algebraic embedding. Then there exists a holomorphic automorphism* φ *of* SL_2 *and a constant* $a \in \mathbb{C}$ *such that the embedding*

$$\mathbb{C} \to \mathrm{SL}_2, \quad t \mapsto \begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix} := (\varphi \circ f)(t + a)$$

satisfies:

(1) *for all* $t \in g_{11}^{-1}(0)$ *we have* $g_{12}(t) = t$;
(2) *the map* $t \mapsto (g_{11}(t), g_{21}(t))$ *is a proper, bimeromorphic immersion such that the image* Γ *has only simple normal crossing singularities;*
(3) *the singularities of* Γ *are distinguished by the first coordinate of* $\mathbb{C}^2$;
(4) *the line* $\{0\} \times \mathbb{C}$ *intersects* Γ *transversally; in particular,* Γ *is smooth in every point of* $\Gamma \cap \{0\} \times \mathbb{C}$;
(5) *the map* $t \mapsto g_{11}(t)$ *is polynomial.*

The proof of this proposition uses the following easy result which is a direct application of the Baire category theorem:

Lemma 9. *Let* $\mathcal{H}(\mathbb{C}^n)$ *be the Fréchet space of holomorphic functions on* $\mathbb{C}^n$ *with the compact-open topology. If* S *is the countable union of closed proper subspaces of* $\mathcal{H}(\mathbb{C}^n)$*, then* $\mathcal{H}(\mathbb{C}^n) \setminus S$ *is dense in* $\mathcal{H}(\mathbb{C}^n)$.

Let $p \in \mathbb{C}^n$ and let $i \in \{1, \dots, n\}$. In our proof of Proposition 8 we use the fact that the non-zero linear functionals on $\mathcal{H}(\mathbb{C}^n)$

$$h \mapsto h(p) \quad \text{and} \quad h \mapsto D_{x_i} h(p)$$

are continuous and thus their kernels are proper closed subspaces of $\mathcal{H}(\mathbb{C}^n)$.

Additionally, we use for the proof of Proposition 8 the following, again rather technical result:

Lemma 10. *Let* $f\colon \mathbb{C} \to \mathrm{SL}_2$ *be an algebraic embedding. Then there exists an algebraic automorphism* φ *of* SL_2 *such that the embedding*

$$\mathbb{C} \to \mathrm{SL}_2, \quad t \mapsto (\varphi \circ f)(t) = \begin{pmatrix} x(t) & y(t) \\ z(t) & w(t) \end{pmatrix}$$

satisfies:

a) *the maps* $t \mapsto x(t)$ *and* $t \mapsto w(t)$ *are non-constant polynomials;*
b) *the maps* $t \mapsto (x(t), z(t))$ *and* $t \mapsto (x(t), w(t))$ *are bimeromorphic and immersive;*
c) *the singularities of the image of* $t \mapsto (x(t), z(t))$ *lie inside* $(\mathbb{C}^*)^2$;
d) *the image of* $t \mapsto (x(t), z(t))$ *intersects* $\{0\} \times \mathbb{C}$ *transversally.*

Proof of Lemma 10. Clearly, we can assume that $f(0)$ is the identity matrix $E_2 \in \mathrm{SL}_2$. By [AFK+13, Thm. 4.14 and Rem. 4.16], there exists an algebraic automorphism of SL_2 which fixes E_2 and maps the tangent vector $f'(0) \in T_{E_2} \mathrm{SL}_2 = \operatorname{Lie} \mathrm{SL}_2$ to the matrix

$$F_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \operatorname{Lie} \mathrm{SL}_2 .$$

Thus we can assume that $f(0) = E_2$ and $f'(0) = F_2$. In particular, property a) is satisfied. Since $f'(t)$ is never zero and since $f'(t)$ is invertible for generic t (note that $f'(0) = F_2$ is invertible) it follows that $f'(t) \cdot v$ is non-zero for generic $v \in \mathbb{C}^2 \setminus \{(0,0)\}$. For generic $\mu \in \mathbb{C}$, this implies that the embedding

$$t \mapsto f(t) \cdot \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

still satisfies property a) and the projection to the first column gives an immersive map. Let us fix such a μ. For generic $\lambda \in \mathbb{C}$ the embedding

$$t \mapsto f(t) \cdot \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

still satisfies property a) and the projection to the first column and the projection to the diagonal give immersive maps. Since any immersive morphism of $\mathbb{C}$ to an irreducible affine curve is birational, we can assume that f satisfies properties a) and b). Now, for generic $a \in \mathbb{C}$ the embedding

$$t \mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot f(t) \tag{$\boxplus$}$$

satisfies properties a) and b) and the singularities of the image of the projection to the first column lie inside $\mathbb{C} \times \mathbb{C}^*$. Let us fix such an a. For generic $b \in \mathbb{C}$ the embedding

$$t \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot f(t) \tag{$\circledast$}$$

satisfies now the properties a) to c). Let $(p(t), q(t))^T$ be the first column of the embedding ($\boxplus$). Then the top left entry of the embedding ($\circledast$) is given by $p(t) + bq(t)$. Now, if ($\circledast$) satisfies properties a) to c), then ($\circledast$) satisfies property d) if and only if $p(t) + bq(t)$ has only simple roots. However, this last condition is satisfied for generic b, since $p(t) + bq(t)$ has only simple roots if and only if for all t the vector $(1, b)^T$ does not lie in the kernel of the matrix

$$\begin{pmatrix} p(t) & q(t) \\ p'(t) & q'(t) \end{pmatrix}$$

and since this last matrix is invertible for generic t and never vanishes. This finishes the proof. $\square$

Proof of Proposition 8. Using Lemma 10 we can assume that f satisfies the properties a) to d) of Lemma 10. As a consequence of b) and c) we get that the map $t \mapsto (x(t), z(t), w(t))$ is a proper holomorphic embedding.

Let $t_1, \ldots, t_n$ be the roots of $x(t) = 0$ (which are simple according to property d)). After a reparametrization of f of the form $t \mapsto t + a$ one can assume that $w(t_i) \neq w(t_j)$ for all $i \neq j$ and $t_i \neq 0$ for all i. Let $a_i \in \mathbb{C}$ such that $e^{-a_i} = -t_i z(t_i)$ and let $b\colon \mathbb{C} \to \mathbb{C}$ be a polynomial map such that $b(w(t_i)) = a_i$ and $b'(w(t)) = 0$ for all t with $x'(t) = 0$. After applying the holomorphic automorphism

$$\mathrm{SL}_2 \to \mathrm{SL}_2, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} x & e^{-b(w)} y \\ e^{b(w)} z & w \end{pmatrix}$$

we can assume that the embedding f satisfies $y(t_i) = t_i$ for all i, and f still satisfies the properties a) to d).

Let ρ be the embedding $t \mapsto (x(t), y(t), z(t))$. Fix $x_0 \neq 0$ such that

I) $z(s) \neq 0$ and $x'(s) \neq 0$ for all $s \in x^{-1}(x_0)$ and
II) the maps $t \mapsto z(t)$ and $t \mapsto w(t)$ are injective on $x^{-1}(x_0)$.

Let $\{s_1, \dots, s_n\} = x^{-1}(x_0)$. With the aid of Lemma 9 one can see that there exists a holomorphic function $c\colon \mathbb{C}^2 \to \mathbb{C}$ that satisfies the following:

i) for all $(x, z, w) \neq (x, z, w') \in \rho(\mathbb{C})$ we have $c(x, w) \neq c(x, w')$;
ii) for all t with $x'(t) = 0$, the partial derivative $D_w c$ vanishes in $(x(t), w(t))$;
iii) for all $i = 1, \dots, n$ we have $c(0, w(t_i)) = 0$;
iv) for all integers k, q and for all 2-element sets $\{i, j\} \neq \{l, p\}$ we have

$$\begin{aligned}\big[\log z(s_l) - \log z(s_p) + 2\pi i q\big] \cdot \big[c(x_0, w(s_j)) - c(x_0, w(s_i))\big]\\ \neq \big[\log z(s_i) - \log z(s_j) + 2\pi i k\big] \cdot \big[c(x_0, w(s_p)) - c(x_0, w(s_l))\big];\end{aligned}$$

v) for all integers k and for all $i \neq j$ we have

$$\begin{aligned}\big[\log z(s_i) - \log z(s_j) + 2\pi i k\big] \cdot \big[x'(s_i) c(x, w)'(s_j) - x'(s_j) c(x, w)'(s_i)\big]\\ \neq \big[z'(s_i) x'(s_j)/z(s_i) - z'(s_j) x'(s_i)/z(s_j)\big] \cdot \big[c(x_0, w(s_j)) - c(x_0, w(s_i))\big].\end{aligned}$$

Let $V \subseteq \mathbb{C}^*$ be the largest subset such that for all $x_0 \in V$ the properties I) and II) are satisfied. By property a), the complement $\mathbb{C} \setminus V$ is a closed discrete (countable) subset of $\mathbb{C}$. The inequalities in iv) and v) are locally holomorphic in $x_0 \in V$ after a local choice of sections $s_1, \dots, s_n$ of the covering $x^{-1}(V) \to V$ and a local choice of the branches of the logarithms. Since V is path-connected, one can now deduce that there exists a subset $U \subseteq V$ such that $\mathbb{C} \setminus U$ is countable and for all $x_0 \in U$ the properties iv) and v) are satisfied.

According to i) and c) there exists $\lambda \in \mathbb{C}^*$ such that for all $x_1 \in \mathbb{C} \setminus U$ we have the following: If $(x_1, z, w) \neq (x_1, z', w') \in \rho(\mathbb{C})$, then $e^{\lambda c(x_1, w)} z \neq e^{\lambda c(x_1, w')} z'$. Now, let φ be the following holomorphic automorphism

$$\mathrm{SL}_2 \to \mathrm{SL}_2, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} x & e^{-\lambda c(x,w)} y \\ e^{\lambda c(x,w)} z & w \end{pmatrix}$$

and let $g = \varphi \circ f$. According to iii), g satisfies property (1) of the proposition. Property ii) implies that $t \mapsto (g_{11}(t), g_{21}(t))$ is immersive. Clearly, $t \mapsto (g_{11}(t), g_{21}(t))$ is proper and g satisfies property (5) of the proposition. By iii), it follows that g satisfies property (4) of the proposition and thus $t \mapsto (g_{11}(t), g_{21}(t))$ is bimeromorphic. By the choice of λ, it follows for $x_1 \notin U$ that $g_{21}(t) \neq g_{21}(t')$ for all $t \neq t' \in x^{-1}(x_1)$. Since all $x_0 \in U$ satisfy iv) and v), the image of $t \mapsto (g_{11}(t), g_{21}(t))$ has only simple normal crossings, which have distinct first coordinates in $\mathbb{C}^2$. This implies properties (2) and (3) of the proposition. $\square$

Proof of Theorem 7. Let $f\colon \mathbb{C} \to \mathrm{SL}_2$ be an algebraic embedding. We will prove that up to a holomorphic automorphism of SL_2 and up to an algebraic reparametrization, f is the same as the standard embedding $t \mapsto E_{12}(t)$.

After applying a holomorphic automorphism of SL_2 and performing an algebraic reparametrization we can assume that f satisfies properties (1) to (5) of Proposition 8. We denote

$$f(t) = \begin{pmatrix} x(t) & y(t) \\ z(t) & w(t) \end{pmatrix}.$$

As usual, $\pi_1\colon \mathrm{SL}_2 \to \mathbb{C}^2$ denotes the projection onto the first column. Let S be the (countable) closed discrete set of points $s \in \mathbb{C}^2 \setminus \{0\}$ such that $(\pi_1 \circ f)^{-1}(s) = \{s_1, s_2\}$ with $s_1 \neq s_2$; see property (2). For every s in S, it holds that $y(s_1) \neq y(s_2)$, since f is an embedding and since all simple normal crossings of the image of $\pi_1 \circ f$ lie inside $\mathbb{C}^* \times \mathbb{C}$ due to property (4). Thus, we can choose $a_s \in \mathbb{C}$ such that

$$s_1 - e^{a_s} y(s_1) = s_2 - e^{a_s} y(s_2).$$

Let $\psi_1\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function with $\psi_1(0) = 0$ such that for all $s \in S$ we have $\psi_1(x(s_1)) = a_s$. This function exists, since $x(s_1) = x(s_2) \neq 0$ for all $s \in S$ (by property (4)), since $x((\pi_1 \circ f)^{-1}(S))$ is a closed analytic subset of $\mathbb{C}$ (by property (5)), and since $x(s_1) \neq x(s'_1)$ for distinct s, s' of S (by property (3)). Let α_1 be the holomorphic automorphism of SL_2 defined by

$$\alpha_1 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & e^{\psi_1(x)} y \\ e^{-\psi_1(x)} z & w \end{pmatrix}.$$

By composing f with α_1, we can assume that $s_1 - y(s_1) = s_2 - y(s_2)$ for all $s \in S$. The embedding f still satisfies the properties (1) to (5).

Let $\Gamma \subset \mathbb{C}^2$ be the image of $\pi_1 \circ f\colon \mathbb{C} \to \mathbb{C}^2$. By Remmert's proper mapping theorem [Rem57, Satz 23], Γ is a closed analytic subvariety of $\mathbb{C}^2$. Now, using that $\pi_1 \circ f$ is immersive and Γ has only simple normal crossings, we get a holomorphic factorization

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\pi_1 \circ f"] \arrow[dr, "t \mapsto t - y(t)"'] & \Gamma \arrow[d, dotted, "e"] \\ & \mathbb{C} \end{tikzcd}.$$

Using properties (1) and (4), it follows that the map

$$\tilde{e}\colon \Gamma \to \mathbb{C}, \quad (x, z) \mapsto \frac{e(x, z)}{x}$$

is holomorphic. Using Cartan's Theorem B [Car53, Thm. B], we can extend $\tilde{e}$ to a holomorphic map $\psi_2\colon \mathbb{C}^2 \to \mathbb{C}$. Let α_2 be the holomorphic automorphism of SL_2 defined by

$$\alpha_2 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y + x\psi_2(x, z) \\ z & w + z\psi_2(x, z) \end{pmatrix}.$$

After applying the automorphism α_2 to f we can assume that $y(t) = t$. This implies that $x(0)w(0) = 1$. Let p, q be the holomorphic functions such that $p(t)t = x(t) - x(0)$ and $q(t)t = w(t) - w(0)$. After applying the holomorphic automorphism

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} w(0) & 0 \\ 0 & x(0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q(y) & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p(y) & 1 \end{pmatrix}$$

we can additionally assume that $w(t) = x(t) = 1$, which implies $z(t) = 0$. The statement follows now from Remark 1. $\square$

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