Mini-course: Algebraic families of automorphisms and solvability

1.9.2026

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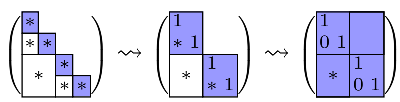
The ground field $\mathbf{k}$ is algebraically closed of arbitrary characteristic. We denote by X a variety over $\mathbf{k}$.

Exercise 1. Let G be an algebraic group. The following holds:

a) If $U, V \subseteq G$ are open and dense subsets, then $U \cdot V = G$.
b) If $H \subseteq G$ is a subgroup, then its closure $\overline{H}$ in G is also a subgroup.
c) If $H \subseteq G$ is a constructible subgroup, then H is closed in G.

Exercise 2. Let $B_n \subseteq \mathrm{GL}_n$ be the subgroup of lower triangular matrices. For $n = 2^k$ show that the derived length $\mathrm{dl}(B_n)$ is equal to $k+1$. In fact, for $n = 2^2$:



$$B_4 \quad \supset \quad B_4' \quad \supset \quad B_4''$$

In general, it holds that $\mathrm{dl}(B_n) = \lceil \log_2(n) \rceil + 1$.

Exercise 3. In this exercise you prove in several steps

> **Borel's fixed point theorem.** Let G be a connected solvable linear algebraic group acting on a complete variety X. Then G has a fixed point in X.

The proof proceeds by induction on the derived length of G.

a) Assume that an algebraic group H acts transitively on varieties Y, Z and let $f\colon Y \to Z$ be a bijective H-equivariant morphism. Then f is finite.
b) Show that $N := [G, G]$ is a normal, closed subgroup of G.
c) Show that there is a fixed point $x \in X$ for N such that $Gx \subseteq X$ is closed.
d) Prove that the stabilizer G_x of x is normal in G and that the orbit map $G/G_x \to Gx$ is a finite morphism.
   Hint. The quotient of a linear algebraic group by a closed normal subgroup is again a linear algebraic group.
e) Deduce that x is a fixed point for G.

Exercise 4. Consider the action of GL_2 on the 2×2-matrices Mat_2 by conjugation. Describe the biggest open subset U in Mat_2 such that the restriction of

$$\pi\colon \mathrm{Mat}_2 \to \mathbb{A}^2\,, \quad A \mapsto (\mathrm{tr}(A), \det(A))$$

to the open subset U is a geometric quotient.

Hint. Determine the dimension of the conjugacy classes and construct a section of $\pi\colon \mathrm{Mat}_2 \to \mathbb{A}^2$.

Exercise 5. Let $\mathcal{G} \subseteq \mathrm{Aut}(X)$ be a connected subgroup. Show that an orbit $\mathcal{G}x$ is locally closed in X. In particular, every $\mathcal{G}$-orbit is a subvariety of X.

Hint. There is a morphism $\rho\colon A \to \mathrm{Aut}(X)$ with image in $\mathcal{G}$ and $\overline{\rho(A)x} = \overline{\mathcal{G}x}$. Now, take an open dense subset in $\overline{\rho(A)x}$ that is contained in $\rho(A)x$.

Exercise 6. Show that $\mathrm{Aut}(\mathbb{A}^n)$ and the Cremona group $\mathrm{Bir}(\mathbb{A}^n)$ are connected.

Hint. If $f \in \mathrm{Aut}(\mathbb{A}^n)$ fixes the origin, then $t \mapsto t^{-1}f(tx_1, \ldots, tx_n)$ defines a morphism $\rho\colon \mathbb{A}^1 \to \mathrm{Aut}(\mathbb{A}^n)$ such that $\rho(0)$ is linear. One can argue similarly if $f \in \mathrm{Bir}(\mathbb{A}^n)$ fixes the origin and is a local isomorphism there.

Exercise 7. Suppose $\mathrm{char}(\mathbf{k}) = p > 0$. The following maps

$$\begin{array}{rcl} \rho_1\colon \mathbb{G}_a & \to & \mathrm{Aut}(\mathbb{A}^2) \\ t & \mapsto & (x, y + tx^2) \end{array} \quad \text{and} \quad \begin{array}{rcl} \rho_2\colon \mathbb{G}_a & \to & \mathrm{Aut}(\mathbb{A}^2) \\ t & \mapsto & (x, y + t^p x^2) \end{array}$$

are homomorphisms and morphisms, and they have the same image, denoted by G. Which one defines the algebraic group structure on G due to Ramanujam?

Exercise 8. Assume that the derived length of every connected solvable subgroup of $\mathrm{Aut}(X)$ is bounded by some constant. Show that in this case every connected solvable subgroup in $\mathrm{Aut}(X)$ is contained in a Borel subgroup.

Exercise 9. Let $d \geq 1$ and denote by $\mathbb{F}_d$ be the geometric quotient of $(\mathbb{A}^2 \setminus \{0\})^2$ by the following $\mathbb{G}_m^2$-action:

$$(\lambda_1, \lambda_2) \cdot (x_1, y_1, x_2, y_2) := (\lambda_1 x_1, \lambda_1 y_1, \lambda_2 \lambda_1^d x_2, \lambda_2 y_2) \tag{$*$}$$

($\mathbb{F}_d$ is the d-th Hirzebruch surface). Show the following:

a) The map $\pi\colon \mathbb{F}_d \to \mathbb{P}^1$ induced by the projection $(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1)$ is a $\mathbb{P}^1$-bundle.

b) The derived length of the following subgroup of $\mathcal{J}_2$ is 3:

$$\mathcal{J}_{2,\leq d} = \left\{ (a_1 x_1 + b_1, a_2 x_2 + b_2(x_1)) \;\middle|\; \begin{array}{l} a_1, a_2 \in \mathbf{k}^*, \\ b_1 \in \mathbf{k}, b_2 \in \mathbf{k}[x_1], \\ \deg(b_2) \leq d \end{array} \right\}.$$

c) The natural action of $\mathcal{J}_{2,\leq d}$ on $\mathbb{A}^2 \simeq \{ (x_1, 1, x_2, 1) \mid x_1, x_2 \in \mathbf{k} \}$ extends to an algebraic action on $(\mathbb{A}^2 \setminus \{0\})^2$ in such a way that it commutes with the $\mathbb{G}_m^2$-action given by $(*)$ and the induced action on $\mathbb{F}_d$ is faithful.

The above steps show that $\mathcal{J}_{2,\leq d}$ is a connected solvable algebraic subgroup of $\mathrm{Aut}(\mathbb{F}_d)$ of derived length 3.

Exercise 10. Assume $\mathrm{char}(\mathbf{k}) = 2$ and let $X = \mathbb{A}^2 \times \mathbb{A}^1$. Consider the morphism

$$\rho\colon \mathbb{A}^2 \to \mathrm{Aut}(X), \quad \rho(u, v)(x, y, a) = (x + u, y + v + f_a(xu)),$$

where $f_a(t) = t^4 + (1 + a + a^2)t^2 + a(1 + a)t$ for $a \in \mathbf{k}$. As the elements of $\rho(\mathbb{A}^2)$ commute and $\mathrm{id} \in \rho(\mathbb{A}^2)$, the subgroup $U := \langle \rho(\mathbb{A}^2) \rangle \subseteq \mathrm{Aut}(X)$ generated by $\rho(\mathbb{A}^2)$ is an algebraic group. Describe the algebraic group U.