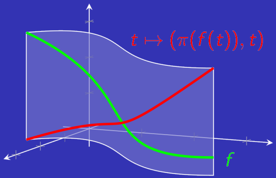
Embeddings and Automorphisms in Affine Algebraic Geometry



Immanuel van Santen

Universität Basel

21 September 2023

Outline

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Affine varieties - the geometric objects

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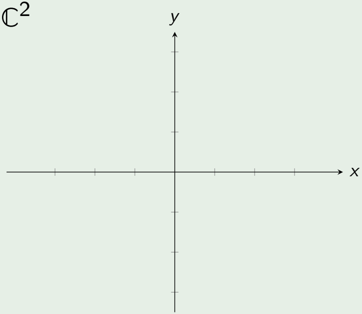

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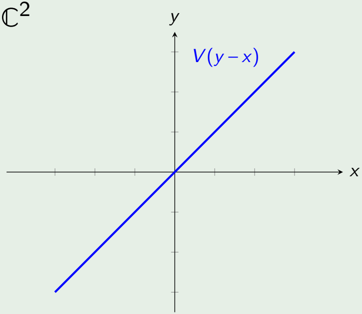

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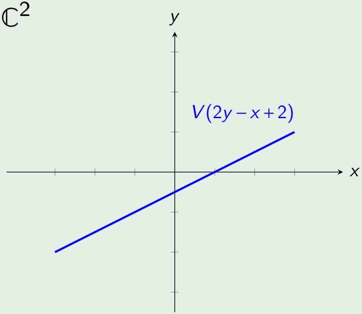

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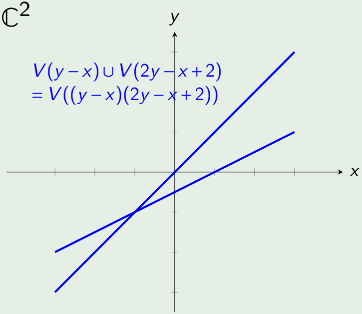

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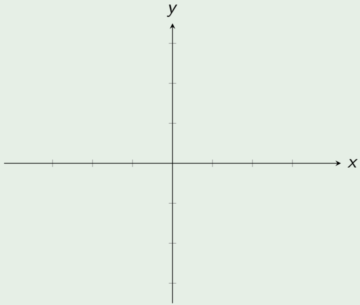

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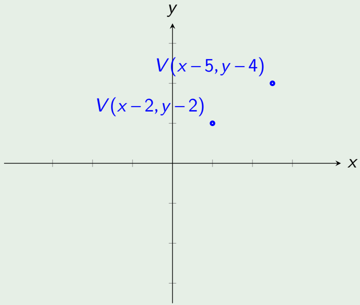

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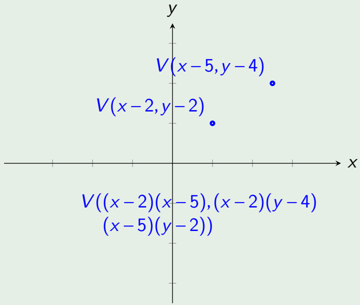

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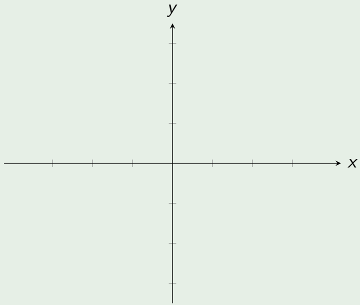

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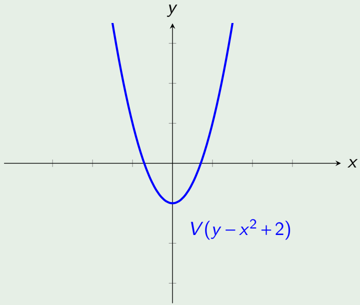

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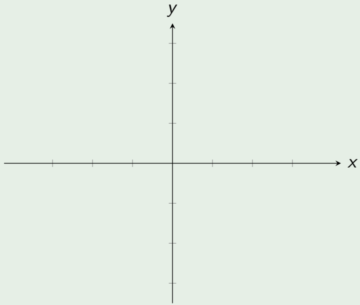

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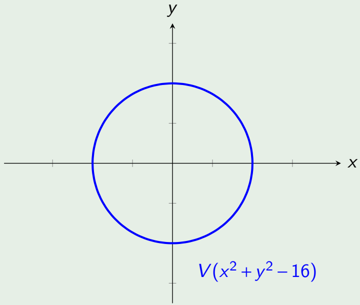

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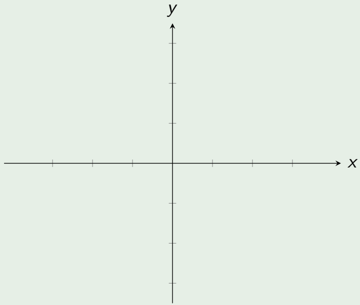

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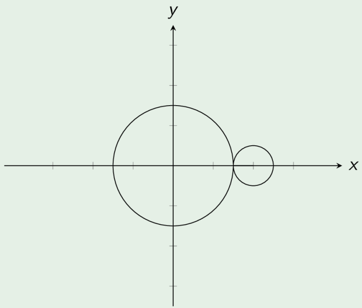

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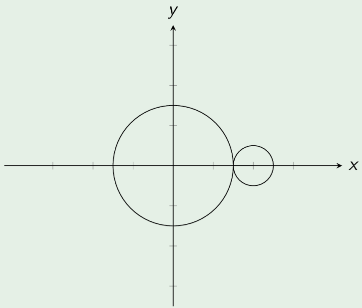

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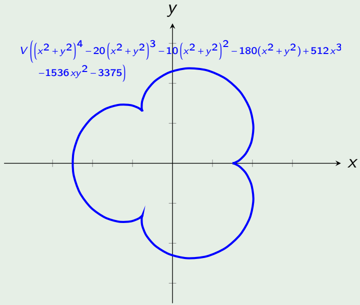

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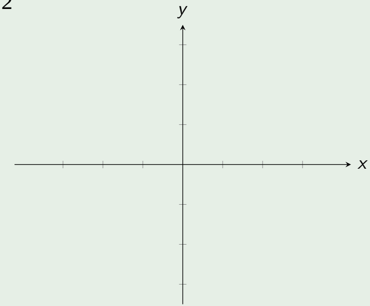

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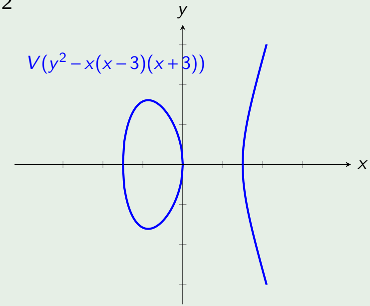

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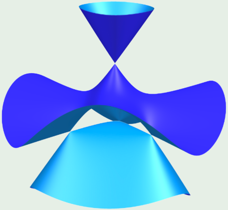

$$V\left(x^3 + y^3 + z^3 + 1 - \frac{1}{4} \cdot (x + y + z + 1)^3\right)$$

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$$V\left((x^2+y^2+z^2-\tfrac{9}{4})^2-\tfrac{23}{3}\cdot((1-z)^2-2x^2)\cdot((1+z)^2-2y^2)\right)$$

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The affine varieties in $\mathbb{C}^n$ form the closed subsets of a topology in $\mathbb{C}^n$, called **Zariski topology**. All affine varieties are endowed with the topology induced by the Zariski topology on $\mathbb{C}^n$.

Morphisms, isomorphisms, automorphisms and embeddings

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3. An **embedding** is a map $f\colon X \to Y$ such that the image $f(X)$ is an affine variety in $\mathbb{C}^m$ (i.e. it is closed in Y) and the restriction $f\colon X \to f(X)$ is an isomorphism.

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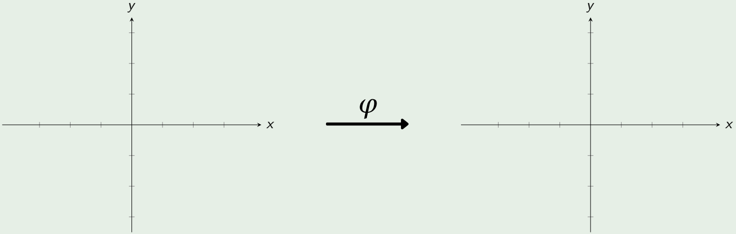

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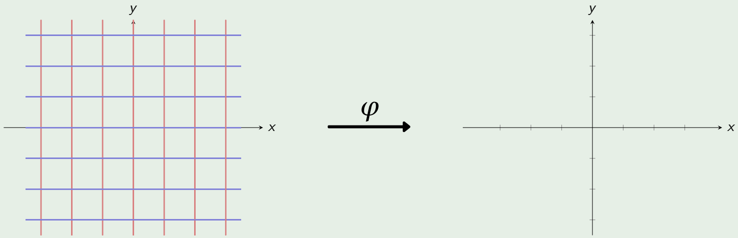

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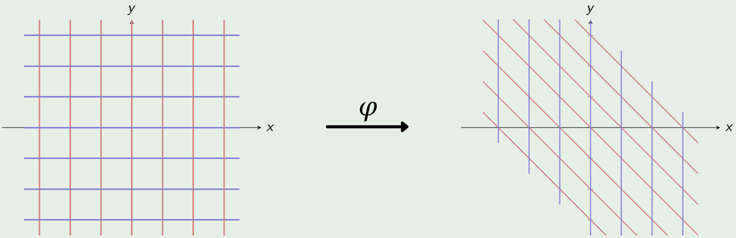

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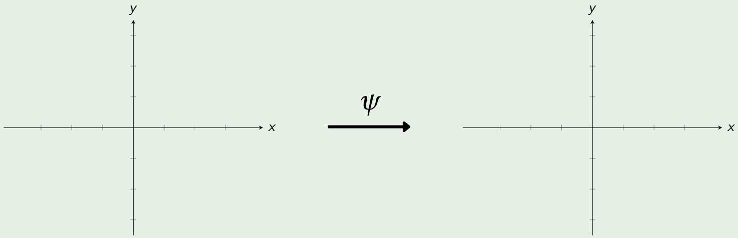

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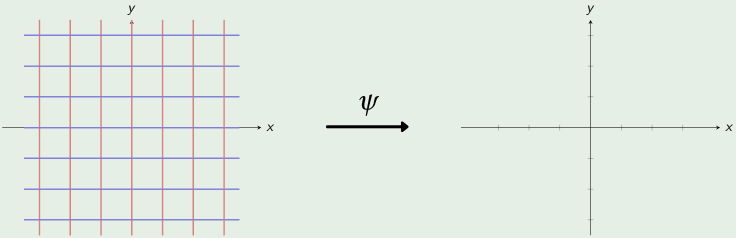

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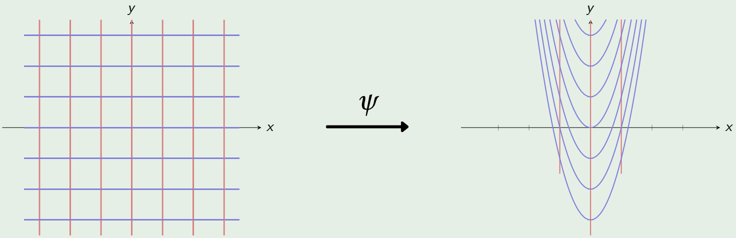

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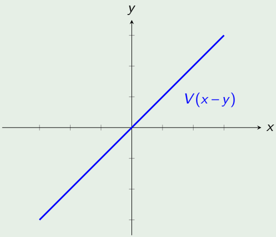

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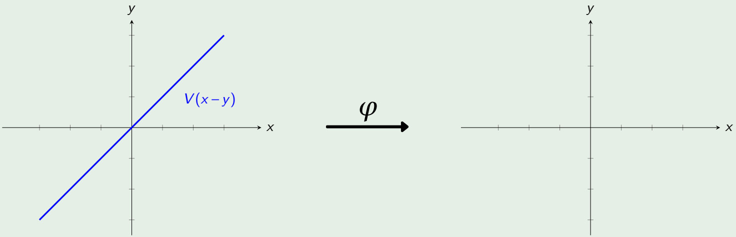

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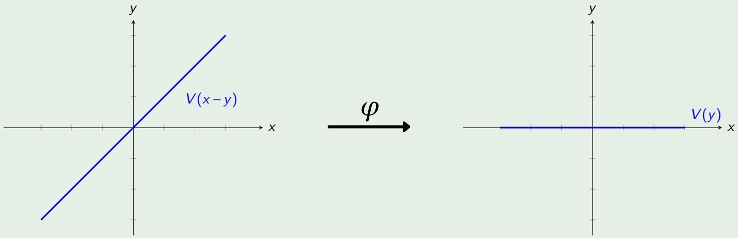

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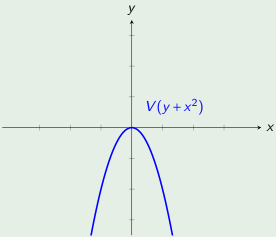

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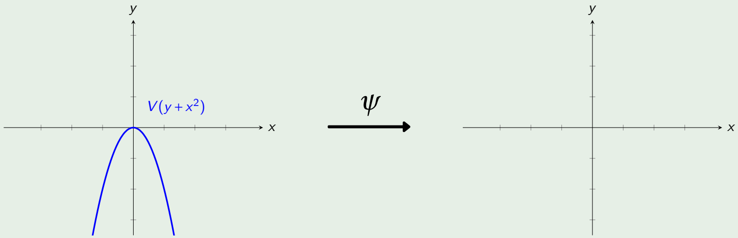

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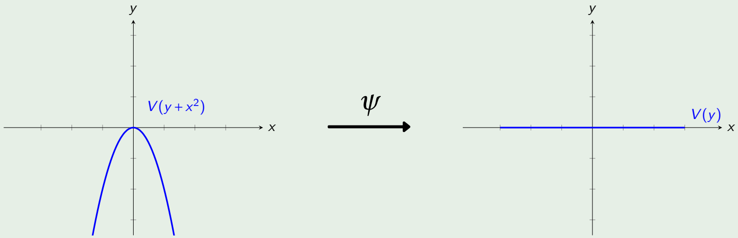

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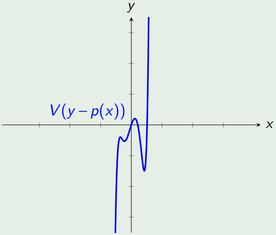

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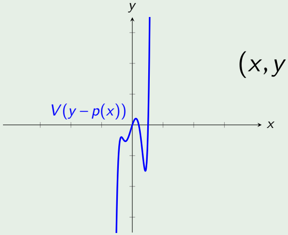

$$(x, y) \mapsto (x, y - p(x))$$

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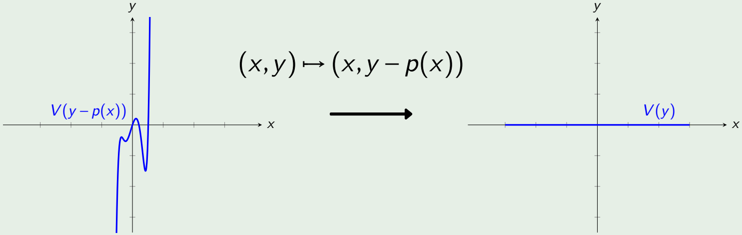

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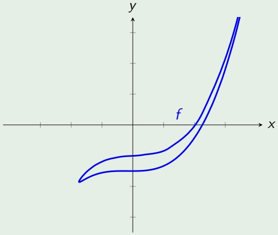

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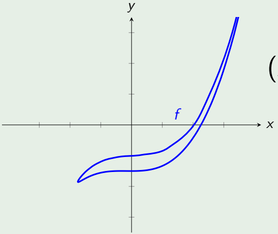

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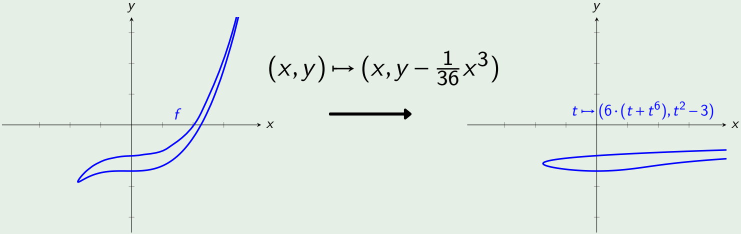

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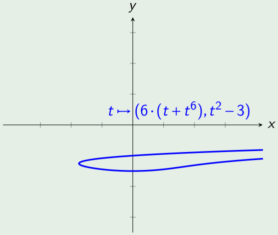

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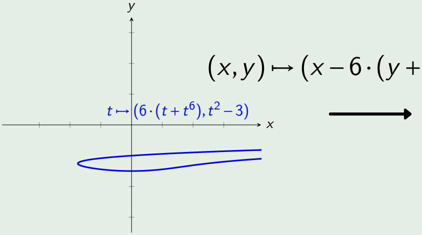

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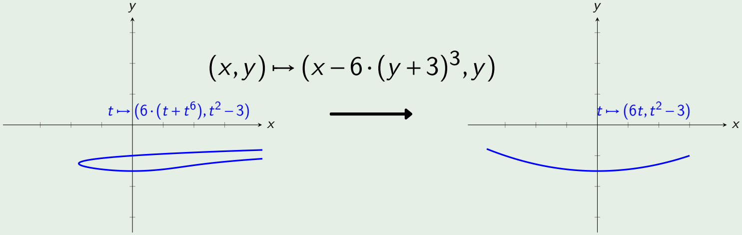

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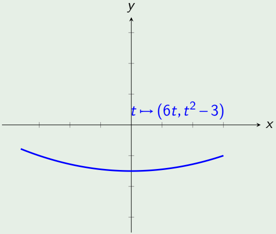

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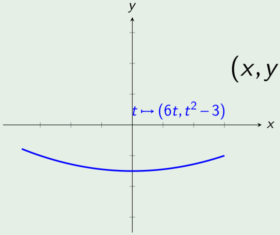

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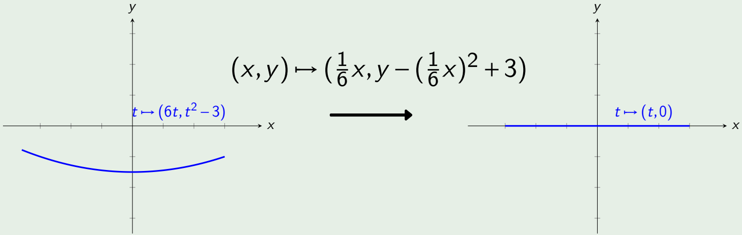

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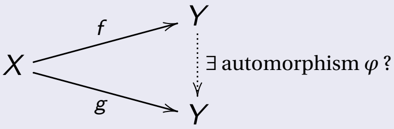

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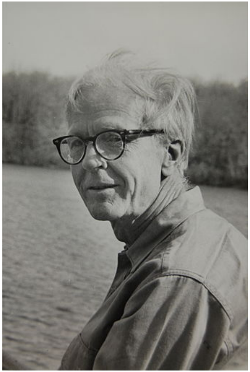

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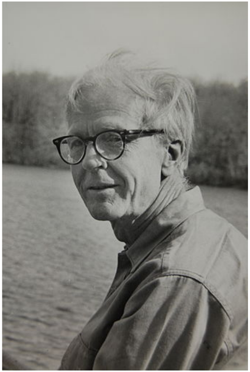
Theorem (Whitney, 36)

Let M be a smooth manifold of dimension $d \geq 0$. Then:



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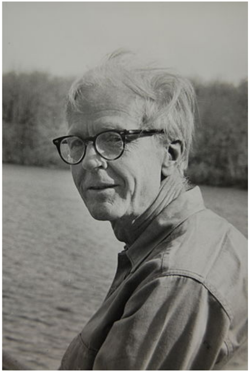

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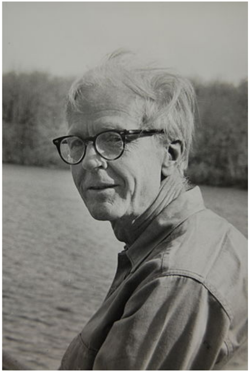

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S.S. Abhyankar

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M. Suzuki

Embeddings of $\mathbb{C}$ into $\mathbb{C}^2$

Theorem (Abhyankar-Moh, Suzuki, 74,75)

Two embeddings $\mathbb{C} \to \mathbb{C}^2$ are always equivalent.

S.S. Abhyankar

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Embeddings of $\mathbb{C}$ into $\mathbb{C}^3$

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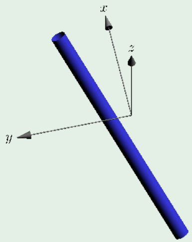

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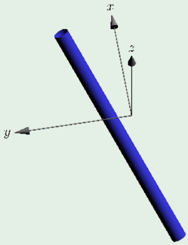

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$t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$

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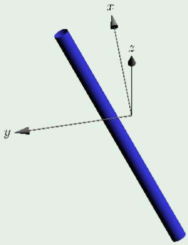

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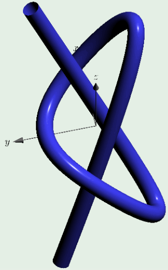

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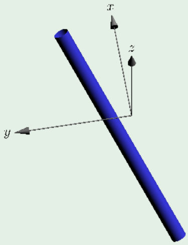

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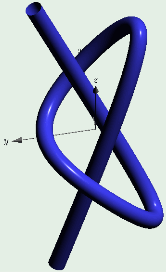

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Open question, Shastri, 92

Are these two embeddings equivalent?

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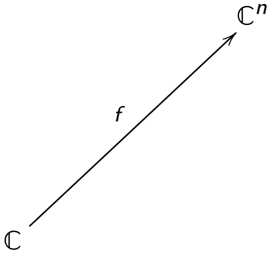
Embeddings of $\mathbb{C}$ into $\mathbb{C}^n$, $n \geq 4$, continued...



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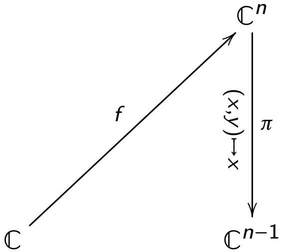

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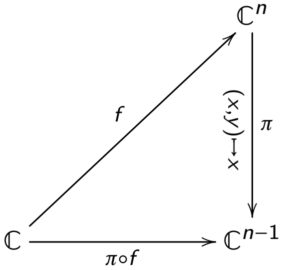

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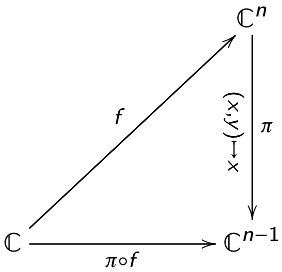

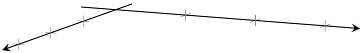

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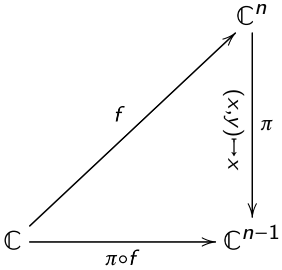

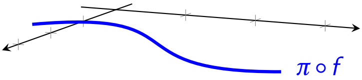

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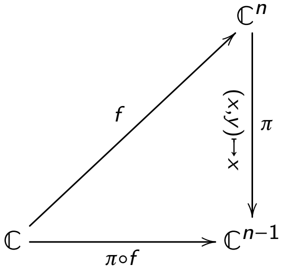

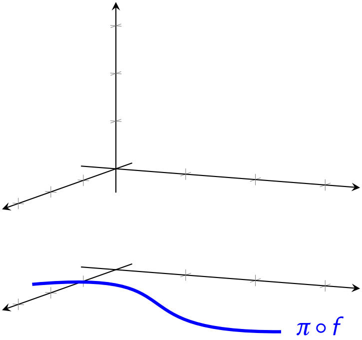

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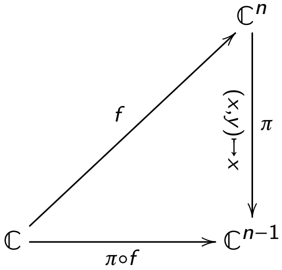

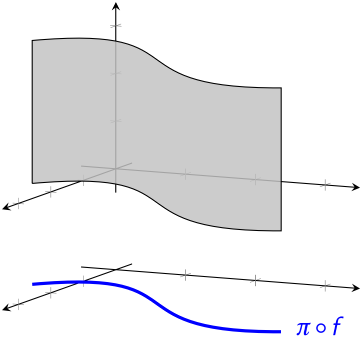

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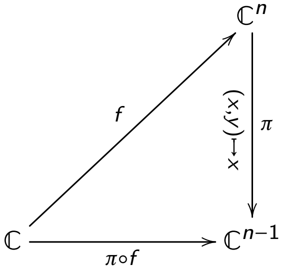

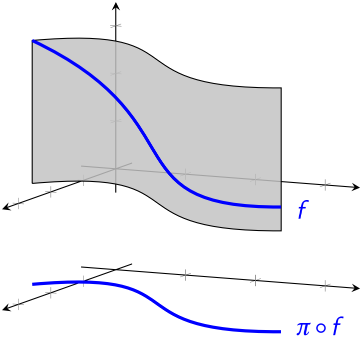

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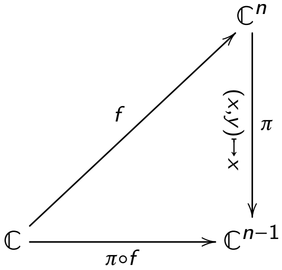

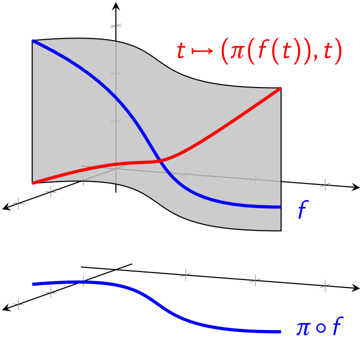

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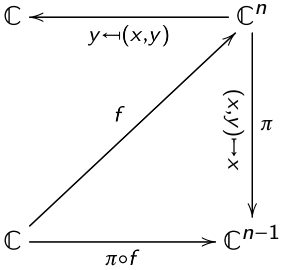

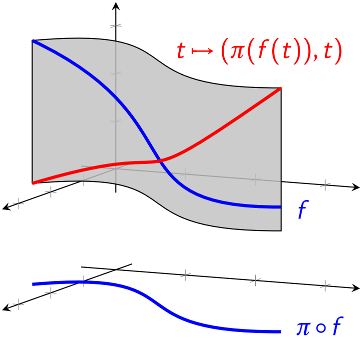

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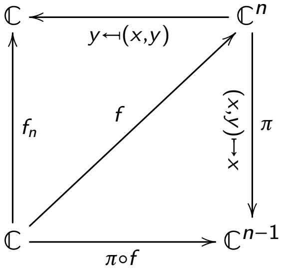

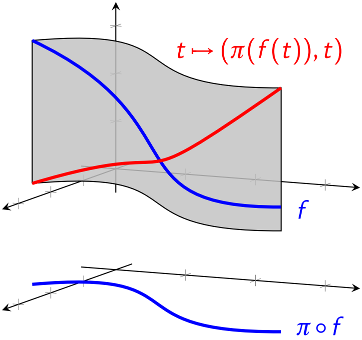

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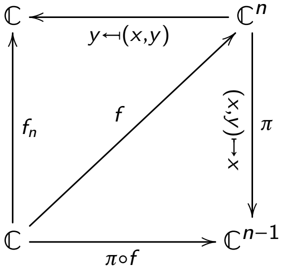

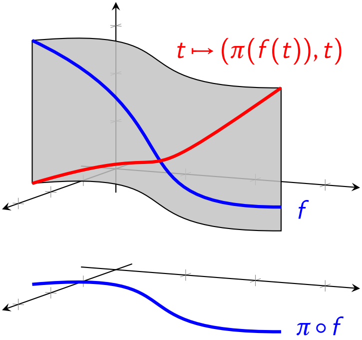

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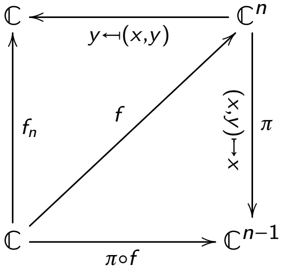

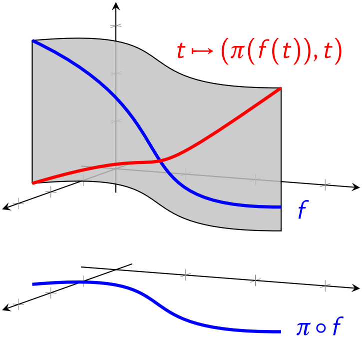

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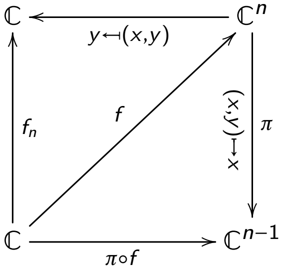

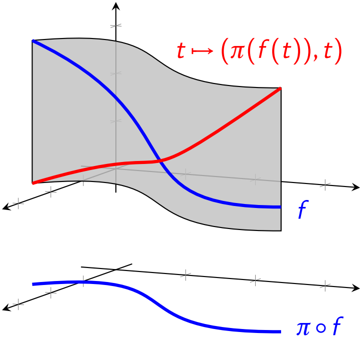

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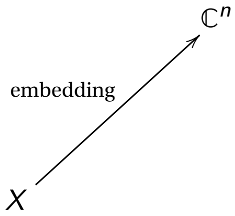

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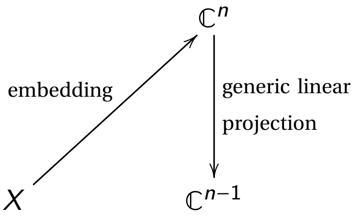

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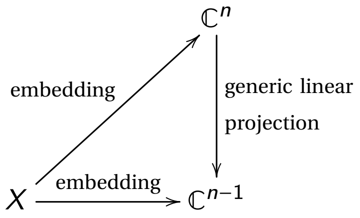

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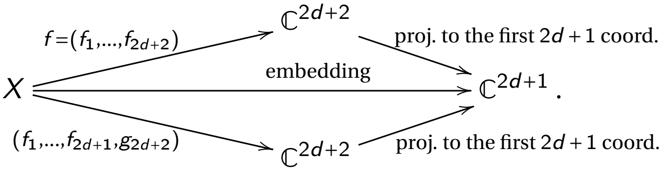

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Open question

Assume $1 \le d < n \le 2d+1$ are inetegers such that $(d,n) \neq (1,2)$. Are two embeddings $\mathbb{C}^d \to \mathbb{C}^n$ always equivalent?

Algebraic groups

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2. The special linear group $\mathrm{SL}_n(\mathbb{C}) = \{A \mid \det(A) = 1\} \subseteq \mathrm{Mat}_{n \times n}(\mathbb{C}) = \mathbb{C}^{n^2}$.

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2. $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{C})$ are algebraic subgroups of $\mathrm{SL}_n(\mathbb{C})$.

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An algebraic group G is called **characterless** if every homomorphism (i.e. morphism that is a group homo.) $G \to \mathbb{G}_m$ is constant.

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Every connected algebraic group G can be written as a semi-direct product $G^u \rtimes \mathbb{G}_m^r$, where G^u is the subgroup of G generated by subgroups isomorphic to $\mathbb{G}_a$ (and thus G^u is characterless) and $r \geq 0$.

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A bijective correspondence

$$\{\text{embeddings } X \to G\} / \sim \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{embeddings } X \to G^u\} / \sim$$

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Tool I: Moving embeddings

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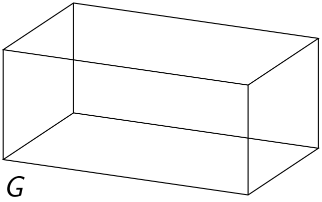

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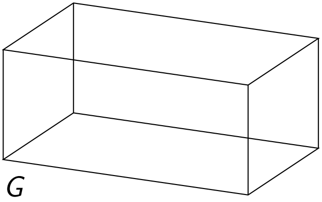

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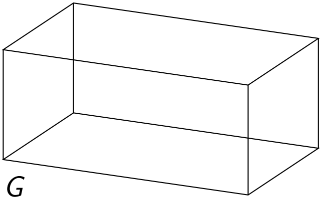

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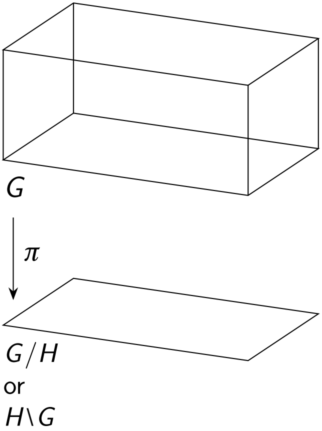

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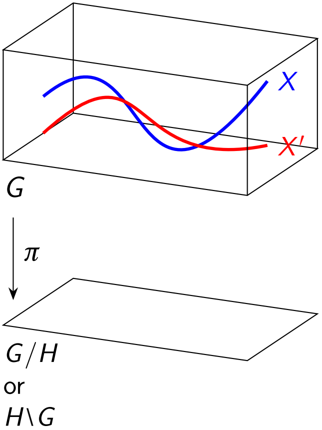

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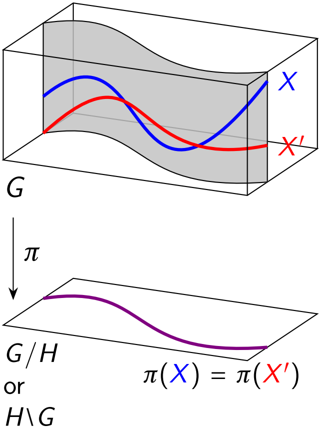

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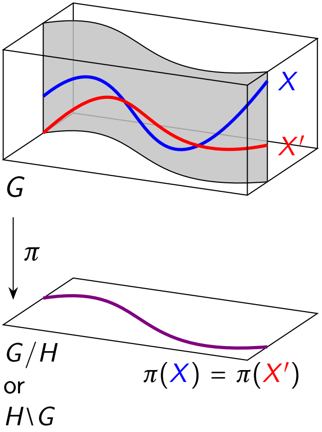

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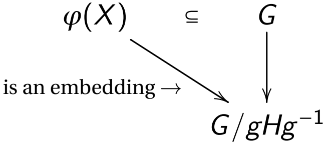

Embedding $\mathbb{C}$ into characterless algebraic groups

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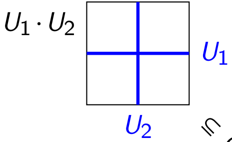

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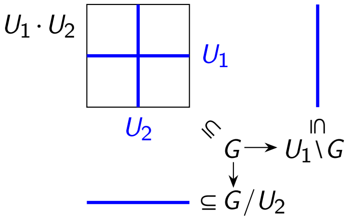

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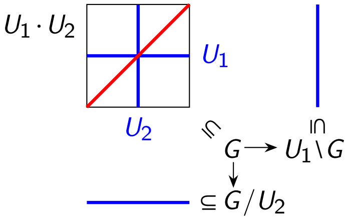

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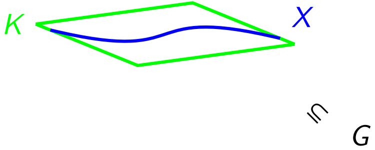

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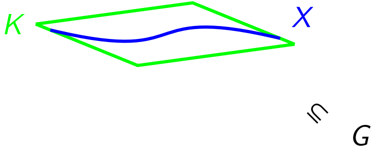

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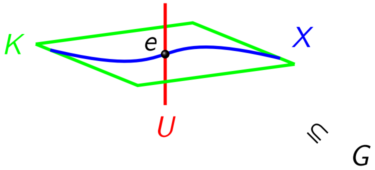

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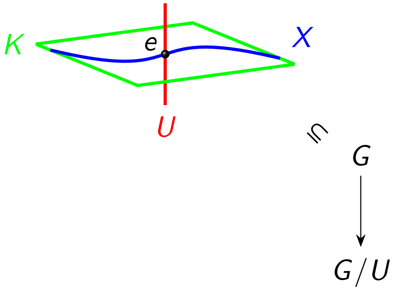

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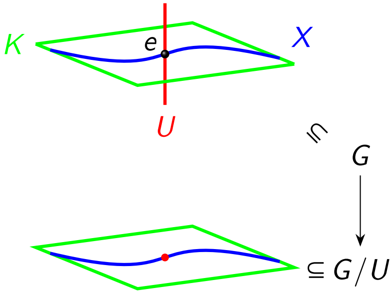

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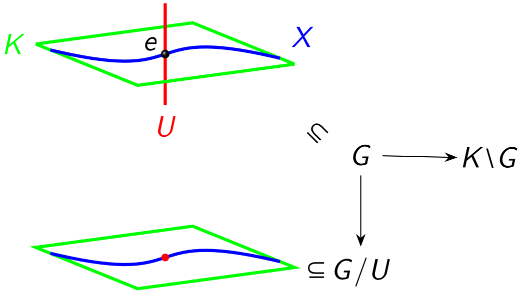

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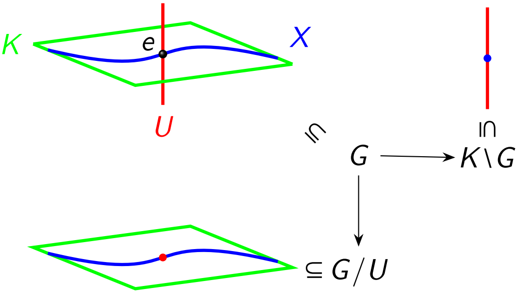

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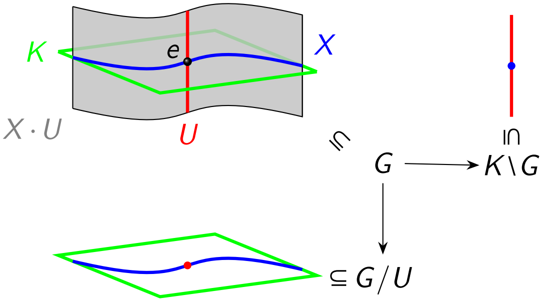

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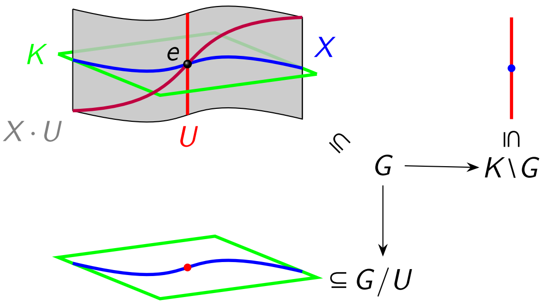

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Embedding $\mathbb{C}$ into characterless algebraic groups

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Embedding $\mathbb{C}$ into characterless algebraic groups

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The natural B-action on $\mathbb{P}^2 = \mathrm{SL}_3(\mathbb{C})/P$ has a unique orbit of dimension 1.

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is a locally trivial $\mathbb{C}$-bundle. Indeed: For $(v_2, v_3) \in \mathrm{Mat}_{3\times 2}(\mathbb{C}) \setminus W$:

$$(\pi|_E)^{-1}(v_2|v_3) = \{(v_1|v_2|v_3) \mid a_1 v_{11} - a_2 v_{12} = 1, v_{13} = 0\} \simeq \mathbb{C},$$

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Example

The natural B-action on $\mathbb{P}^2 = \mathrm{SL}_3(\mathbb{C})/P$ has a unique orbit of dimension 1. Its closure C are the elements with vanishing last entry. Hence,

$$E := \rho^{-1}(C) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \right\} \subseteq \mathrm{SL}_3(\mathbb{C}).$$

Upshot:

$$\pi|_E \colon E \to \mathrm{Mat}_{3\times 2}(\mathbb{C}) \setminus W = \mathrm{SL}_3(\mathbb{C})/R_u(P^-), \ (v_1|v_2|v_3) \mapsto (v_2|v_3)$$

is a locally trivial $\mathbb{C}$-bundle. Indeed: For $(v_2, v_3) \in \mathrm{Mat}_{3\times 2}(\mathbb{C}) \setminus W$:

$$(\pi|_E)^{-1}(v_2|v_3) = \{(v_1|v_2|v_3) \mid a_1 v_{11} - a_2 v_{12} = 1, v_{13} = 0\} \simeq \mathbb{C},$$

where a_1, a_2, a_3 are the 2×2-minors of $(v_2|v_3)$.

Embedding $\mathbb{C}$ into characterless algebraic groups

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Moreover, one can show that $\pi(E)$ is a **big open** subset of $G/R_u(P^-)$ (i.e. the complement has codimension ≥ 2) and that

$$\pi|_E\colon E \to \pi(E)$$

is a locally trivial $\mathbb{C}$-bundle.

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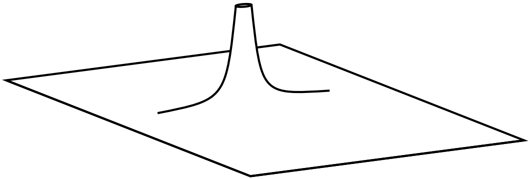

$$E \overset{\text{closed}}{\subseteq} G$$

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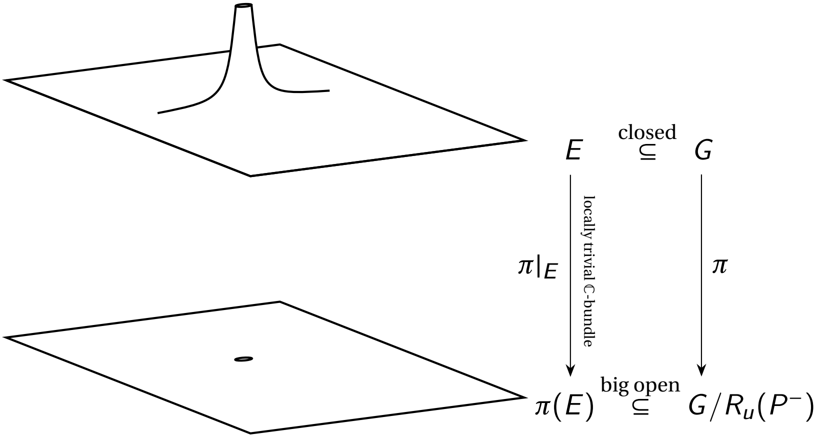

Embedding ℂ into characterless algebraic groups

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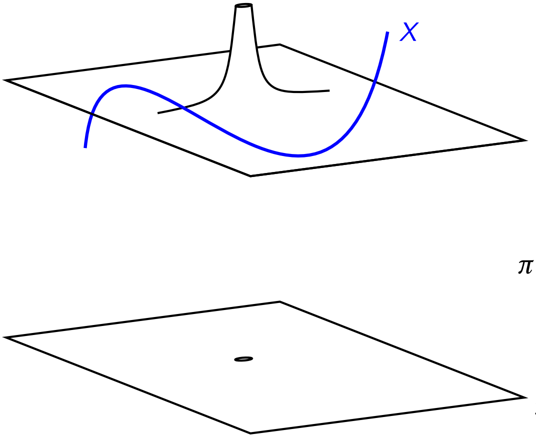

$$\begin{array}{ccc} E & \overset{\text{closed}}{\subseteq} & G \\ {\scriptstyle \pi|_E}\downarrow {\scriptstyle \text{locally trivial } \mathbb{C}\text{-bundle}} & & \downarrow {\scriptstyle \pi} \\ \pi(E) & \overset{\text{big open}}{\subseteq} & G/R_u(P^-) \end{array}$$

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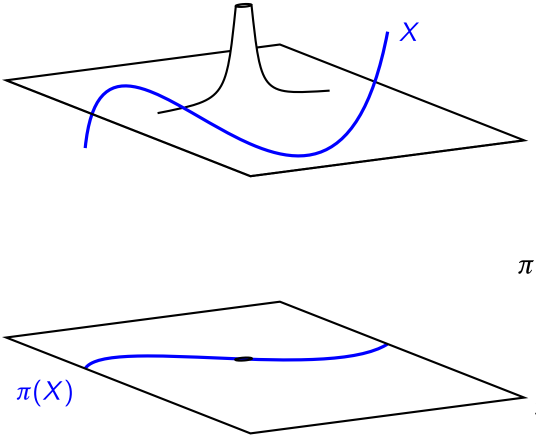

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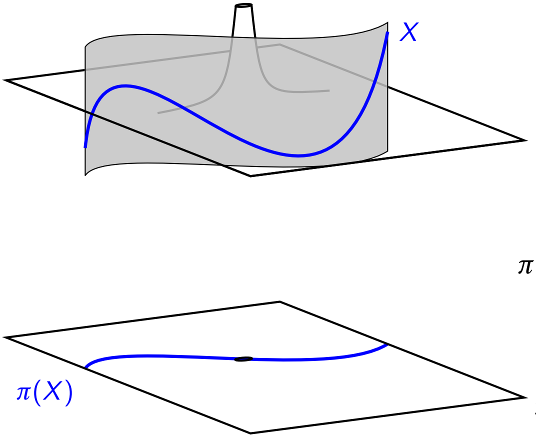

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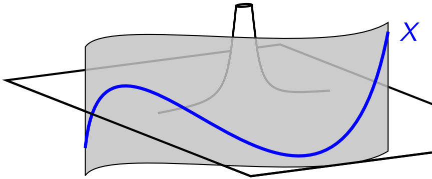
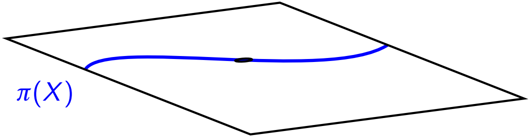


multiply with a generic g in G



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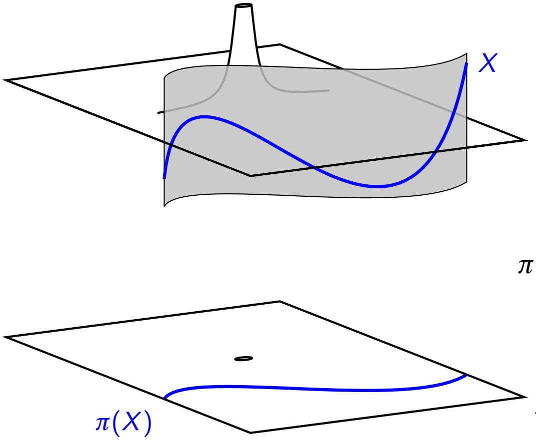

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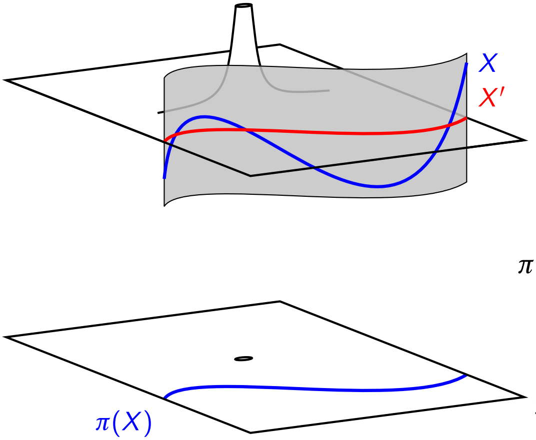

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Existence of embeddings into algebraic groups

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End

Thank you for your attention!