On Automorphisms of the Affine Cremona Group

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Contents

1. Introduction

The affine Cremona group is the group of polynomial automorphisms of affine n-space with base field k. It is named after the Italian mathematician Luigi Cremona[1] and we denote it by $\mathcal{G}_n$. A natural and interesting question is, which kind of automorphisms of $\mathcal{G}_n$ there exist. If we look at it as an abstract group, two families are immediate:

(1) inner automorphisms
(2) automorphisms induced by an automorphism of the base field k.

In [KS11] Kraft and Stampfli show for the case $k = \mathbb{C}$ that any automorphism decomposes into an automorphisms of type (1) and an automorphism of type (2) if restricted to the subgroup of tame automorphisms $\mathcal{TG}_n \subset \mathcal{G}_n$ (see Section 2.1 for a definition):

Theorem 1.1 (Kraft-Stampfli)**.** *Let $k = \mathbb{C}$ and let $\theta\colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of groups. Then there is an element $g \in \mathcal{G}_n$ and an automorphism τ of $\mathbb{C}$ such that*

$$\theta(f) = \tau(g \circ f \circ g^{-1})$$

for all tame automorphisms $f \in \mathcal{TG}_n$.

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[1]Luigi Cremona, 1830 - 1903

Actually their methods work for any algebraically closed field of characteristic zero. Notice that this theorem yields a classification of all automorphisms of $\mathcal{G}_2$ because $\mathcal{T}\mathcal{G}_2 = \mathcal{G}_2$. This result was already obtained by Déserti, who proves Theorem 1.1 for $n = 2$ and an uncountable k of characteristic zero ([Dés06]).

In Section 3 we generalize Theorem 1.1 for algebraically closed fields of arbitrary characteristic. We follow the same strategy of proof as is used in [KS11]. However, in order to avoid some typical characteristic zero techniques such as the exponential map from the Lie Algebra to its group, we had to use more elementary tools and simplify certain steps.

While we look in Section 3 at all automorphisms of the affine Cremona group that preserve its structure as a mere group, we consider in Section 4 only those automorphisms that also respect its additional algebraic structure as an ind-group. In that case all automorphisms are inner if restricted to the subgroup of tame automorphisms. In characteristic zero the ind-group structure makes it possible to extend this result from the subgroup of tame automorphisms to the entire Cremona group. This will be done following an idea recently presented by Belov-Kanel and Yu [BKY12], which uses the observation from Anick that any automorphism of $\mathbb{A}^n$ can be approximated by a tame one up to any degree (see [Ani83]).

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2. Preliminaries and notation

We assume the reader to be familiar with basic concepts and results from affine algebraic geometry and the theory of linear algebraic groups as can be found for example in [Kra11] or [Hum75]. Certain results will be recalled when needed. The goal of this section is mainly to clarify some notation. We also state a few results on polynomial automorphisms and shortly introduce the concept of an ind-group.

If not stated otherwise, we work over an algebraically closed field k. Algebraic groups are always considered to be linear.

2.1. **The affine Cremona group.** We denote by $\mathbb{A}^n := k^n$ the affine n-space over the field k. Its coordinate ring $\mathcal{O}(\mathbb{A}^n) = k[x_1, \dots, x_n]$ is the polynomial ring in n variables. An endomorphism $g \in \mathrm{End}(\mathbb{A}^n)$ is given by an n-tuple $g = (g_1, \dots, g_n)$ with $g_i \in k[x_1, \dots, x_n]$, $i = 1, \dots, n$, such that $g(a) = (g_1(a), \dots, g_n(a))$ for $a \in \mathbb{A}^n$. The *degree* of an endomorphism g is the maximal degree of all the g_i. The composition of two endomorphisms $f, g \in \mathrm{End}(\mathbb{A}^n)$ is denoted by $f \circ g$. An endomorphism $g \in \mathrm{End}(\mathbb{A}^n)$ is an automorphism if g is bijective and g^{-1} a morphism. By

$$\mathcal{G}_n := \mathrm{Aut}(\mathbb{A}^n) \subset \mathrm{End}(\mathbb{A}^n)$$

we denote the group of automorphisms of $\mathbb{A}^n$, the *affine Cremona group*.

Let $\tau\colon k \to k$ be an automorphism of fields. Thus τ induces a bijective map $T\colon \mathbb{A}^n \to \mathbb{A}^n$ given by $T(a_1, \dots, a_n) := (\tau(a_1), \dots, \tau(a_n))$. Then $\mathcal{G}_n \to \mathcal{G}_n$, $g \mapsto T \circ g \circ T^{-1}$ is an automorphism of groups. Observe that we obtain the image of g by letting τ operate on the coefficients of g. By abuse of notation we denote this automorphism by τ as well.

We denote by $\left(\frac{\partial g_i}{\partial x_j}\right)$ the Jacobian of an endomorphism $g \in \mathrm{End}(\mathbb{A}^n)$ and by the polynomial

$$J(g) := \det\left(\frac{\partial g_i}{\partial x_j}\right)$$

its determinant. If g is an automorphism, then $J(g)$ is a nowhere vanishing polynomial and therefore a non-zero constant. The famous Jacobian Conjecture states that in characteristic zero an endomorphism is invertible if and only if its Jacobian determinant is a non-zero constant. In positive characteristics more care has to be taken. Indeed, for the endomorphism $g\colon x \mapsto x + x^p$, where $p = \operatorname{char}(k)$, one gets $J(g) = 1$ but g is clearly not invertible. A formulation of the Jacobian conjecture for positive characteristics can be found for example in [vdE00, Chapter 10.3].

There are several important subgroups of $\mathcal{G}_n$. By GL_n we denote the subgroup of linear automorphisms, by $\mathcal{T}_n$ the subgroup of translations, by $\mathrm{Aff}_n \coloneqq \mathrm{GL}_n \ltimes \mathcal{T}_n$ the affine group and by $D_n \subset \mathrm{GL}_n$ the subgroup of diagonal morphisms $(c_1x_1, \ldots, c_nx_n)$. The subgroup of monomial linear automorphisms is denoted by Mon_n and the center of GL_n by Z_n.

The *de Jonquiéres subgroup* $\mathcal{J}_n$ is the subgroup of *triangular automorphisms*, which are morphisms $g \in \mathcal{G}_n$ of the form $g = (g_1, \ldots, g_n)$ where $g_i \in k[x_i, \ldots, x_n]$. The subgroup of *tame automorphisms*

$$\mathcal{TG}_n \coloneqq \langle \mathrm{Aff}_n, \mathcal{J}_n \rangle$$

is the subgroup generated by Aff_n and $\mathcal{J}_n$. An automorphism $g = (g_1, \ldots, g_n)$ is called an *elementary transformation* if there is an i such that $g_i = x_i + f_i$ where $f_i \in k[x_1, \ldots, \widehat{x_i}, \ldots, x_n]$ and $g_j = x_j$ for $j \neq i$. It is well known that $\mathcal{TG}_n$ is generated by all elementary transformations and GL_n.

An important theorem of Jung and Van der Kulk states that in dimension two all automorphisms are tame (see for example [vdE00, Theorem 5.1.11]). However, in dimension three this is no longer the case. Shestakov and Umirbaev show in [SU03] that in characteristic zero the famous Nagata automorphism is *wild*, which means not contained in $\mathcal{TG}_n$.

2.2. **Ind-groups.** The concept of ind-groups was first introduced by Shafarevich. He called them infinite-dimensional groups.

Definition. An *ind-variety* is a set X together with an ascending filtration $X_1 \subset X_2 \subset \cdots \subset X$ such that:

(1) $X = \bigcup_{n\in\mathbb{N}} X_n$,
(2) X_k has the structure of an algebraic variety for all $k \in \mathbb{N}$,
(3) $X_k \subset X_{k+1}$ is closed.

Although the definition is more general, in this thesis we always consider the X_k to be affine algebraic varieties.

A subset $A \subset X$ of an ind-variety is closed if $A \cap X_k \subset X_k$ is closed for all $k \in \mathbb{N}$. This defines a topology on X. A closed subset $A \subset X$ naturally inherits the structure of an ind-variety; it is called a *closed ind-subvariety*. A map $\varphi\colon X \to Y$ between ind-varieties is a *morphism of ind-varieties*, if for all i there exists a j such that $\varphi(X_i) \subset Y_j$ and $\varphi|_{X_i}\colon X_i \to Y_j$ is a morphism of affine varieties. An *isomorphism of ind-varieties* is a bijective morphism whose inverse is again a morphism. Two filtrations $X = \bigcup_n X_n$ and $X = \bigcup_n Y_n$ are called *equivalent* if the identity map on X is an isomorphism.

If $X = \bigcup_n X_n$ and $Y = \bigcup_n Y_n$ are ind-varieties, we can give an ind-variety structure to $X \times Y$ by the filtration $X \times Y = \bigcup_n (X \times Y)_n$, where $(X \times Y)_n \coloneqq X_n \times Y_n$ with the product variety structure.

Definition. An *ind-group* is a group G with the structure of an ind-variety such that the maps $\iota\colon G \to G$, $g \mapsto g^{-1}$ and $m\colon G \times G \to G$, $(g, h) \mapsto gh$ are morphisms of ind-varieties.

A closed subgroup $H \subset G$ is called an *ind-subgroup*. A homomorphism of groups $\varphi\colon G \to G'$ between two ind-groups that is also a morphism of ind-varieties is called a *homomorphism of ind-groups*. Similarly we define *isomorphisms of ind-groups*.

In order to see that $\mathcal{G}_n$ is an ind-group, we need some preparation:

Proposition 2.1. *Let $f \in \mathcal{G}_n$. Then $\deg(f^{-1}) \leq \deg(f)^{n-1}$.*

Proof. See [vdE00, Proposition 2.3.1]. □

Proposition 2.2. *Let $g \in k[[x_1, \ldots, x_n]]^n$ such that $J(g) \in k^*$, where $k[[x_1, \ldots, x_n]]$ is the formal power series ring. If $g(0) = 0$, then g has a formal inverse g^{-1} in $k[[x_1, \ldots, x_n]]^n$ and the coefficients of g^{-1} are polynomial functions in the coefficients of g.*

Proof. See [Kam79, Lemma 0.1]. □

Let $\mathrm{End}(\mathbb{A}^n)_k \subset \mathrm{End}(\mathbb{A}^n)$ be the set of all endomorphisms of degree $\leq k$. We define

$$E \coloneqq \{g \in \mathrm{End}(\mathbb{A}^n) \mid J(g) \in k^*\} \text{ and } E_k \coloneqq E \cap \mathrm{End}(\mathbb{A}^n)_k.$$

So the E_k are affine varieties, and $E \subset \mathrm{End}(\mathbb{A}^n)$ is locally closed. Proposition 2.2 implies that there is a morphism $\rho\colon E_k \to E_k$ such that $\rho(g) = g^{-1}$ if $g^{-1} \in E_k$.

Let

$$A_k \coloneqq \{(f, g) \in \mathrm{End}(\mathbb{A}^n)_k \times \mathrm{End}(\mathbb{A}^n)_k \mid f \circ g = g \circ f = \mathrm{id}\}.$$

Then, for all $k \in \mathbb{N}_0$, the set A_k is an affine variety and $A_k \subset A_{k+1}$ is closed. Observe that

$$G_k \coloneqq \{g \in \mathcal{G}_n \mid g, g^{-1} \in \mathrm{End}(\mathbb{A}^n)_k\}$$

is isomorphic to A_k.

Define the morphism $\varphi\colon E_k \to \mathrm{End}(\mathbb{A}^n)_k \times \mathrm{End}(\mathbb{A}^n)_k$ by $g \mapsto (g, \rho(g))$. Since $G_k = \varphi^{-1}(A_k)$, the subset $G_k \subset E_k$ is closed. Let $(\mathcal{G}_n)_k \coloneqq \mathcal{G}_n \cap \mathrm{End}(\mathbb{A}^n)_k$. By Proposition 2.1, we get $(\mathcal{G}_n)_k = \mathrm{End}(\mathbb{A}^n) \cap G_{k^{n-1}}$, and therefore, that the $(\mathcal{G}_n)_k \subset \mathrm{End}(\mathbb{A}^n)_k$ are locally closed affine subvarieties.

So $\mathcal{G}_n = \bigcup_{k \in \mathbb{N}_0} (\mathcal{G}_n)_k$ is an ind-variety, and $\mathcal{G}_n \subset \mathrm{End}(\mathbb{A}^n)$ is locally closed. Since $m((\mathcal{G}_n)_k \times (\mathcal{G}_n)_l) \subset (\mathcal{G}_n)_{kl}$, and $\iota((\mathcal{G}_n)_k) \subset (\mathcal{G}_n)_{k^{n-1}}$, it follows that $\mathcal{G}_n = \bigcup_{k \in \mathbb{N}_0} (\mathcal{G}_n)_k$ is an ind-group.

It is easy to check that the subgroups GL_n, Aff_n and $\mathcal{J}_n$ introduced in Section 2.1 are closed ind-subroups of $\mathcal{G}_n$.

Remark. We followed the proof from [Kra12]. In that paper there is also a proof that $\mathrm{Aut}(X)$ is an ind-group for an arbitrary affine variety X. In general it is not known whether $\mathrm{Aut}(X) \subset \mathrm{End}(X)$ is always locally closed.

3. Automorphisms of $\mathcal{G}_n$ as an abstract group

As mentioned in the introduction, the main goal of this section is the proof of the following theorem for arbitrary algebraically closed fields k:

Theorem 3.1. *Let $\theta\colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of groups. Then there is an element $g \in \mathcal{G}_n$ and an automorphism of fields $\tau\colon k \to k$ such that for all elements $f \in \mathcal{TG}_n$*

$$\theta(f) = \tau(g \circ f \circ g^{-1}).$$

We proceed in several steps. First we show in 3.1 that the image of D_n is conjugated to D_n. In 3.2 we classify all one-dimensional connected unipotent subgroups of $\mathcal{G}_n$ that are normalized by the torus and examine their images. Since these groups generate $\mathcal{TG}_n$, our tools are sufficient to prove the theorem in 3.3.

3.1. **The images of the torus and the general linear group.** Our first cornerstone is Proposition 3.9. For this we follow the proof from [KS11], which works in any characteristic. In order to establish the result, we first introduce some technical utilities.

Let G be a group acting faithfully on an affine variety X, and let $x_0 \in X$ be a fixed point. Then the map $\rho\colon G \to \mathrm{GL}(T_{x_0}X)$, given by the differential $\rho(g) := d_{x_0}g$, is a representation of G. If g acts on X as the identity, we get $g \in \ker(\rho)$. Lemma 3.2 shows that in some cases the converse is true as well.

Lemma 3.2. *Let G be a group acting on an irreducible affine variety X, and let $x_0 \in X$ be a fixed point. Assume there exists a G-stable decomposition $\mathfrak{m}_{x_0} = V \oplus \mathfrak{m}_{x_0}^2$. If the induced action of an element $g \in G$ on $T_{x_0}X$ is the identity, then the action of g on X is also the identity. In particular, if G acts faithfully on X, then the induced representation on $T_{x_0}X$ is faithful.*

Proof. Let $g \in \ker(\rho)$. Since $V \simeq (T_{x_0}X)^*$, g acts as the identity on V and hence as the identity on all powers V^k. We have $\mathfrak{m}_{x_0} = V + V^2 + \cdots + V^k \oplus \mathfrak{m}_{x_0}^{k+1}$ for all k. As $\bigcap_k \mathfrak{m}_{x_0}^k = \{0\}$, we get that g acts as the identity on $\mathfrak{m}_{x_0}$ and thus also on $\mathfrak{m}_{x_0} \oplus k = \mathcal{O}(X)$; so g acts as the identity on X. $\square$

There always exists a G-stable decomposition as preconditioned in Lemma 3.2, if G is a linearly reductive group, so in particular if G is finite.

The action of a group G on an affine variety X is called *locally finite* if, for all $f \in \mathcal{O}(X)$, the linear span $\langle Gf \rangle$ is finite-dimensional as a vector space.

Lemma 3.3. *Let G be a group acting locally finitely on an affine variety X. Then the closure $\overline{G}$ of the image of G in the ind-group* $\mathrm{Aut}(X)$ *is an algebraic group that acts algebraically on X.*

Proof. See [KS11, Section 2] $\square$

Lemma 3.4. *Let X and Y be varieties. Then $\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$.*

Proof. The lemma is well known if both, X and Y, are affine. We first consider the case where X is affine. Let $\{h_i \mid i \in I\}$ be a basis of $\mathcal{O}(X)$ as a k-vector space. So for every $f \in \mathcal{O}(X \times Y)$ there are k-valued functions $f_i\colon Y \to k, i \in I$, such that $f(x,y) = \sum_i h_i(x) f_i(y)$ for all $(x,y) \in X \times Y$. Let $Y = \bigcup_j Y_j$ be a finite affine open covering. Then $f|_{X \times Y_j} \in \mathcal{O}(X) \otimes \mathcal{O}(Y_j)$, hence $f = \sum_i h_i \otimes f_{ij}$ for some $f_{ij} \in \mathcal{O}(Y_j)$ where the sum is finite. Since $f_i|_{Y_j} = f_{ij}$, it follows that $f_i \in \mathcal{O}(Y)$ and $f_{ij} = 0$ for all but finitely many $i \in I$.

For X an arbitrary variety, we can repeat the same argument, this time using the fact that $\mathcal{O}(X \times Y_j) = \mathcal{O}(X) \otimes \mathcal{O}(Y_j)$ for X arbitrary and Y_j affine. $\square$

Lemma 3.5. *Let G be an algebraic group acting regularly on a variety X. Then the action of G on X is locally finite.*

Proof. The action $G \times X \to X$ yields a homomorphism $\mathcal{O}(X) \to \mathcal{O}(G \times X), f \mapsto \tilde{f} = \sum_i h_i \otimes f_i$ for suitable $h_i \in \mathcal{O}(G), f_i \in \mathcal{O}(X)$, and the sum is finite. So,

$$\tilde{f}(g,x) := f(gx) = \sum_i h_i(g) f_i(x),$$

hence $gf = \sum h_i(g^-1) f_i$ for all $g \in G$, and thus $\langle Gf \rangle \subset \sum_i kf_i$. □

Lemma 3.6. *Let X and Y be varieties, X irreducible, and let $\varphi \colon X \to Y$ be a local isomorphism. Then φ is an open immersion.*

Proof. Let $X = \bigcup_i X_i$ be a finite open covering such that $Y_i := \varphi(X_i) \subset Y$ is open and $\varphi|_{X_i} \colon X_i \to Y_i$ is an isomorphism for all i. Define open immersions $\psi_i \colon Y_i \to X$ by $\psi_i(y) := (\varphi|_{X_i})^{-1}(y) \in X_i$. For any pair i, j, the open subset $U_{ij} := \varphi(X_i \cap X_j) \subset Y_i \cap Y_j$ is non-empty and $\psi_i|_{U_{ij}} = \psi_j|_{U_{ji}}$. Hence $\psi_i|_{Y_i \cap Y_j} = \psi_j|_{Y_i \cap Y_j}$, so we get a morphism $\psi \colon Y \to X$ such that $\psi|_{Y_i} = \psi_i$. Since $\psi \circ \varphi = id_X$ and $\varphi \circ \psi = id_Y$, we are done. □

Lemma 3.7. *Let $G \subset \mathcal{G}_n$ be a subgroup, let X be an irreducible affine variety of dimension n and let $\varphi \colon \mathbb{A}^n \to X$ be a dominant morphism. Assume that $\varphi^*(\mathcal{O}(X))$ is a G-stable subalgebra and that the induced action of G on X is locally finite. Then the action of G on $\mathbb{A}^n$ is locally finite.*

Proof. Our first goal is to reduce the claim to the case that X is normal. Let R be the integral closure of $\varphi^*(\mathcal{O}(X))$ in $k[x_1, \dots, x_n]$ and let $f \in R$. By definition, f satisfies an integral equation

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0$$

with $a_i \in \varphi^*(\mathcal{O}(X))$. The spaces $\langle Ga_i \rangle$ are finite-dimensional, so there is a $d \in \mathbb{N}$ such that $\deg(ga_i) < d$ for all a_i and all $g \in G$. From

$$(gf)^m + (ga_1)(gf)^{m-1} + \cdots + (ga_m) = 0$$

we get $\deg(gf) < d$ for all $g \in G$. Hence G acts locally finite on R. So without loss of generality we may assume that X is normal and φ birational.

Let $U \subset \mathbb{A}^n$ be an open set such that $\varphi(U)$ is open and $\varphi|_U \colon U \to \varphi(U)$ is an isomorphism. Define $Y := \bigcup_{g \in G} gU$. Then

$$\psi := \varphi|_Y \colon Y \to X$$

is a G-equivariant local isomorphism. By Lemma 3.6, ψ is an open immersion.

Since $k[x_1, \dots, x_n] \subset \mathcal{O}(Y)$, it suffices to show that the action of G on $\mathcal{O}(Y)$ is locally finite, or equivalently, that the action of G on $\mathcal{O}(\varphi(Y))$ is locally finite. In Lemma 3.3 we have seen that the algebraic group $\overline{G}$ acts regularly on X. The open subset $\varphi(Y) \subset X$ is G-stable and therefore also $\overline{G}$-stable. So the action of $\overline{G}$ on $\varphi(Y)$ is, by Lemma 3.5, locally finite. Hence the action of G is locally finite as well. □

In Lemma 3.8 we collect two results for whose proofs we refer to the literature:

Lemma 3.8. *Let G be a finite p-group acting on $\mathbb{A}^n$. Then $(\mathbb{A}^n)^G$, the fixed-point set of $\mathbb{A}^n$ under the action of G, is nonempty, connected and smooth. Moreover $T_{x_0}(\mathbb{A}^n)^G = (T_{x_0}\mathbb{A}^n)^G$ for all fixed points $x_0 \in (\mathbb{A}^n)^G$.*

Proof. By a theorem of SERRE ([Ser09, Theorem 7.5]), the fixed point set $(\mathbb{A}^n)^G$ is nonempty and connected. FOGARTY proves in ([Fog73, Theorem 5.2]), that $(\mathbb{A}^n)^G$ is smooth and that $T_{x_0}(\mathbb{A}^n)^G = (T_{x_0}\mathbb{A}^n)^G$ for all $x_0 \in (\mathbb{A}^n)^G$. □

For an $l \in \mathbb{N}$ such that $\mathrm{char}(k)$ does not divide l, we define the subgroup

$$\mu_l := \{g \in D_n \mid g^l = \mathrm{id}\}.$$

Clearly $\mu_l \simeq (\mathbb{Z}/l\mathbb{Z})^n$. We denote the l-th root of unity by ξ_l.

Proposition 3.9. *Let $\mu \subset \mathcal{G}_n$ be a subgroup isomorphic to μ_p where p is a prime $\neq \mathrm{char}(k)$. Then $\mathrm{Cent}(\mu) \subset \mathcal{G}_n$, the centralizer of μ, is a diagonalizable algebraic subgroup of dimension $\leq n$.*

Proof. $\mathrm{Cent}(\mu)$ is a closed ind-subgroup of $\mathcal{G}_n$. Indeed, let $\mathcal{G}_n = \bigcup_d (\mathcal{G}_n)^d$ be a filtration. Then $\mathrm{Cent}(\mu) \cap (\mathcal{G}_n)^d$ is closed for all d and clearly $\mathrm{Cent}(\mu)$ is a subgroup.

By Lemma 3.8 the fixed-point set $(\mathbb{A}^n)^\mu$ is nonempty. Let a be such a fixed point; thus $T_a(\mathbb{A}^n)^\mu = (T_a\mathbb{A}^n)^\mu$. According to Lemma 3.2, the tangent representation $\rho\colon \mu \to \mathrm{GL}(T_a\mathbb{A}^n)$ is faithful. Therefore, $\rho(\mu) \simeq \mu_p$, so $(T_a\mathbb{A}^n)^\mu = \{0\}$. As $(\mathbb{A}^n)^\mu$ is connected, a is an isolated fixed point.

Since $\rho(\mu) \subset \mathrm{GL}(T_a\mathbb{A}^n)$ is isomorphic to μ_p, it is conjugated to μ_p. Therefore, we can choose generators $\sigma_1, \ldots, \sigma_n$ of μ such that for all $i = 1, \ldots, n$ the linear map $\rho(\sigma_i) \in \mathrm{GL}(T_a\mathbb{A}^n)$ has a single eigenvalue ξ_p and eigenvalue 1 with multiplicity $n-1$. Set

$$H_i := (\mathbb{A}^n)^{\sigma_i}.$$

By Lemma 3.8, the closed subset $H_i \subset \mathbb{A}^n$ is smooth, connected and of dimension $n-1$; that is, a hypersurface. Denote by $f_i \in k[x_1, \ldots, x_n]$, $i = 1, \ldots, n$, the irreducible polynomials such that H_i is the zero locus of f_i. Note that $H_1 \cap \cdots \cap H_n = \{a\}$ and that the polynomials $f_1, \ldots, f_n$ are linearly independent.

We claim $\sigma_i^*(f_i) = \xi_p f_i$ and $\sigma_i^*(f_j) = f_j$ for $i \neq j$. For $x \in H_j$, we have

$$\sigma_j \circ \sigma_i(x) = \sigma_i \circ \sigma_j(x) = \sigma_i(x);$$

in other words, $\sigma_i(H_j) = H_j$ for all j. This implies that $\sigma_i(f_j)$ generates the vanishing ideal of H_j and thus $\sigma_i^*(f_j) = cf_j$ for a $c \in k^*$. From

$$d_a f_i \circ d_a \sigma_i = c d_a f_i$$

we get $c \in \{1, \xi_p\}$. Since $d_a f_i$ is zero if restricted to $T_a H_i = (T_a\mathbb{A}^n)^{\sigma_i}$, we get

$$d_a f_i \circ d_a \sigma_i = \xi_p d_a f_i,$$

so $\sigma_i^*(f_i) = \xi_p f_i$. Similarly it follows that $\sigma_i^*(f_j) = f_j$.

Let $g \in \mathrm{Cent}(\mu)$. As $g \circ \sigma_i \circ g^{-1} = \sigma_i$ for all i, the automorphism g fixes a. Moreover $\sigma_i^*(f \circ g) = \xi_p f \circ g$, so $f \circ g = cf$. We conclude that g fixes the n-dimensional linear subspace

$$V := kf_1 \oplus \cdots \oplus kf_n \subset k[x_1, \ldots, x_n].$$

In particular, the action of $\mathrm{Cent}(\mu)$ on the algebra $k[f_1, \ldots, f_n]$ is locally finite. The morphism $\varphi := (f_1, \ldots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^n$ is dominant since $\varphi^{-1}(0) = \{a\}$. So we apply Proposition 3.7 to conclude that $\mathrm{Cent}(\mu)$ acts locally finite on $\mathbb{A}^n$. By Lemma 3.3 the centralizer $\mathrm{Cent}(\mu)$ is contained in an algebraic group. Since $\mathrm{Cent}(\mu)$ is already an ind-subgroup of $\mathcal{G}_n$, it is therefore an algebraic subgroup.

As observed above, the image of $\mathrm{Cent}(\mu)$ in $\mathrm{GL}(V)$ is contained in a maximal torus. Being the image of a homomorphism of algebraic groups it is closed, hence a diagonalizable group. As $\mathfrak{m}_a = V \oplus \mathfrak{m}_a^2$, it follows from Lemma 3.2 that the homomorphism of algebraic groups $\mathrm{Cent}(\mu) \to \mathrm{GL}(T_a\mathbb{A}^n)$ and therefore the homomorphism of algebraic groups $G \to \mathrm{GL}(V)$ are injective. This concludes the proof. □

The following observation is easy, but a useful tool:

Lemma 3.10. *Let $\theta\colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of groups and $M \subset \mathcal{G}_n$ a subset. We denote by $\mathrm{Cent}(M)$ its centralizer and by $\mathrm{Norm}(M)$ its normalizer in $\mathcal{G}_n$. Then*

$$\theta(\mathrm{Cent}(M)) = \mathrm{Cent}(\theta(M)) \quad \textit{and} \quad \theta(\mathrm{Norm}(M)) = \mathrm{Norm}(\theta(M)).$$

If M is a subgroup we get in addition

$$\theta(\mathrm{Center}(M)) = \mathrm{Center}(\theta(M)).$$

Proof. We have

$$\begin{aligned}\theta(\mathrm{Cent}(M)) &= \theta(\{g \in \mathcal{G}_n \mid g \circ f \circ g^{-1} = f \text{ for all } f \in M\})\\ &= \{g \in \mathcal{G}_n \mid \theta^{-1}(g) \circ f \circ \theta^{-1}(g^{-1}) = f \text{ for all } f \in M\}\\ &= \{g \in \mathcal{G}_n \mid g \circ f \circ g^{-1} = f \text{ for all } f \in \theta(M)\}\\ &= \mathrm{Cent}(\theta(M)).\end{aligned}$$

The other claims can be proven analogously. □

Proposition 3.11. *In $\mathcal{G}_n$ we have the following identities:*

(1) $\mathrm{Cent}(\mu_l) = D_n$ *for* $\mathrm{char}(k) \neq l$,
(2) $\mathrm{Norm}(D_n) = \mathrm{Mon}_n$,
(3) $\mathrm{Center}(\mathrm{Mon}_n) = Z_n$,
(4) $\mathrm{Cent}(Z_n) = \mathrm{GL}_n$,
(5) $\mathrm{Norm}(\mathrm{Aff}_n) = \mathrm{Aff}_n$.

Proof. (1) Clearly $D_n \subset \mathrm{Norm}(\mu_p)$. Conversely, let $g = (g_1, \dots, g_n) \in \mathrm{Cent}(\mu_p)$. Then

$$g_i(\xi_l^{r_1}x_1, \dots, x_i, \dots, \xi_l^{r_n}x_n) = g_i(x_1, x_2, \dots, x_n)$$

and

$$g_i(x_1, \dots, \xi_l^{r_i}x_i, \dots, x_n) = \xi_l^{r_i} g_i(x_1, x_2, \dots, x_n)$$

for all exponents $r_1, \dots, r_n \in \mathbb{N}$. It follows that each g_i is of the form

$$c_i x_i + x_i p_i(x_1^l, \dots, x_n^l)$$

for some $c_i \in k^*$ and a polynomial p_i with constant term 0. The same is true for g^{-1}. Let $g^{-1} = (f_1, \dots, f_n)$ and assume $f_i = b_i x_i + x_i q_i(x_1^l, \dots, x_n^l)$. By definition

$$\begin{aligned}x_i &= g_i(f_1, \dots, f_n)\\ &= c_i(b_i x_i + x_i q_i(x_1^l, \dots, x_n^l)) + (b_i x_i + x_i q_i(x_1^l, \dots, x_n^l)) p_i(f_1^l, \dots, f_n^l),\end{aligned}$$

so p_i and q_i have to be 0 and thus $g \in D_n$.

(2) Let $g = (g_1, \dots, g_n) \in \mathrm{Norm}(D_n)$, so $g \circ d \circ g^{-1} \in D_n$. Therefore, for all $c_1, \dots, c_n \in k^*$ we get that $g_i(c_1x_1, \dots, c_nx_n)$ is a multiple of g_i. This is only possible if g_i is a monomial. But since g is invertible, this monomial has to be linear. It is well known that $\mathrm{Norm}_{\mathrm{GL}_n}(D_n) = \mathrm{Mon}_n$ in the general linear group.

(3) Since $\mathrm{Mon}_n \subset \mathrm{GL}_n$, clearly $Z_n = \mathrm{Center}(\mathrm{GL}_n) \subset \mathrm{Center}(\mathrm{Mon}_n)$. On the other hand, every element in GL_n can be written as a sum of monomial matrices and thus $\mathrm{Center}(\mathrm{Mon}_n) \subset \mathrm{Center}(\mathrm{GL}_n) = Z_n$.

(4) It is well known that $\mathrm{GL}_n \subset \mathrm{Cent}(Z_n)$. Let now $g = (g_1, \dots, g_n) \in \mathrm{Cent}(Z_n)$, so for all $c \in k^*$ we get $g_i(cx_1, \dots, cx_n) = cg_i(x_1, \dots, x_n)$ and thus that g is linear.

(5) Let $f \in \mathrm{Norm}(\mathrm{Aff}_n)$. As $\mathrm{Aff}_n \subset \mathrm{Norm}(\mathrm{Aff}_n)$, we assume $f = h \circ g$ with $g \in \mathrm{Aff}_n$ and $h = \mathrm{id} + p_d + q$ for some $d \geq 2$, where $p_k \in (k[x_1, \dots, x_n]_d)^n$ and every monomial appearing in q is of degree $> d$. The inverse of h has the form $h^{-1} = \mathrm{id} - p_d + q'$ and every monomial in q' is of degree $> d$. So for all $\lambda \in k^*$:

$$h \circ (\lambda\,\mathrm{id}) \circ h^{-1} = \lambda\,\mathrm{id} + (\lambda^d - \lambda)p_d + r,$$

where every monomial appearing in r has degree $> d$. Since $h \in \mathrm{Norm}(\mathrm{Aff}_n)$, the automorphism $h \circ (\lambda\,\mathrm{id}) \circ h^{-1}$ has to be in Aff_n. Therefore, $h = \mathrm{id}$.

□

Lemma 3.12. *Let S and S' be diagonalizable algebraic groups that are isomorphic as groups. Then S and S' are isomorphic as algebraic groups.*

Proof. We may assume $S = D \times F$ and $S' = D' \times H$, with D and D' tori, F and G finite commutative groups. Let $\varphi \colon S \to S'$ be an isomorphism of groups.

Let $S_q \subset S$ be the subgroup of all elements of order q. Then for almost all primes q we have $S_q \subset D$ and hence $S_q \simeq (\mathbb{Z}/q\mathbb{Z})^n$ with $n = \mathrm{rank}(S_q) = \dim(D) = \dim(S)$. This yields $\dim(S) = \dim(S')$ and in particular $D \simeq D'$.

Let $n := |F| \cdot |H|$. Then $(S)^n = D$ and $(S')^n = D'$, so $\varphi(D) = D'$. It follows now that $\varphi(F) = H$, which concludes the proof. □

In positive characteristics this result can be improved:

Lemma 3.13. *Assume* $\mathrm{char}(k) = p > 0$. *Let S be a diagonalizable group and G an algebraic group isomorphic to S as a group. Then S and G are isomorphic as algebraic groups.*

Proof. We may assume $G = D \times U$ with D diagonalizable and U unipotent (Jordan decomposition). Assume that U is a nontrivial subgroup. Then $U^p \subsetneq U$, in particular there are infinitely many elements $u \in U$ such that $u^p = e$. But the number of elements $s \in S$ such that $s^p = e$ is finite, so $U = \{e\}$ and the claim follows from Lemma 3.12. □

The lemma is not true in characteristic zero. For example $\mathbb{C}^*$ is isomorphic as a group to $\mathbb{C}^* \times \mathbb{C}^+$. Indeed, $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^+$ are isomorphic as groups, since they are isomorphic as $\mathbb{Q}$-vector spaces. So we get

$$\mathbb{C}^* \simeq S^1 \times \mathbb{R}_{>0} \simeq S^1 \times \mathbb{R}^+ \simeq S^1 \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \simeq \mathbb{C}^* \times \mathbb{C}^+.$$

The following theorem is due to Bialynicki-Birula. For a proof we refer to the original paper [BB66].

Theorem 3.14 (Bialynicki-Birula)**.** *Let D_n act regularly and faithfully on $\mathbb{A}^n$. Then the action is conjugated to a linear one.*

From Theorem 3.14 it follows in particular, that every subgroup in $\mathcal{G}_n$ that is isomorphic as an algebraic group to D_n, is conjugated to D_n.

Theorem 3.15. *Let $\theta \colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of groups. There exists an automorphism $g \in \mathcal{G}_n$ such that $\theta(D_n) = g \circ D_n \circ g^{-1}$ and $\theta(\mathrm{GL}_n) = g \circ \mathrm{GL}_n \circ g^{-1}$.*

Proof. By Proposition 3.11, $\mathrm{Cent}(\mu_l) = D_n$ for some l. Lemma 3.10 yields therefore $\theta(D_n) = \mathrm{Cent}(\theta(\mu_l))$, which is by Proposition 3.9 a diagonalizable group. We conclude from Lemma 3.12, that $\theta(D_n)$ and D_n are isomorphic as algebraic groups, so by Theorem 3.14 they are conjugated. In other words, there is an automorphism $g \in \mathcal{G}_n$ such that $\theta(D_n) = g \circ D_n \circ g^{-1}$.

With the identities from Lemma 3.11 and again Lemma 3.10 it follows that $\theta(\mathrm{Mon}_n) = g \circ \mathrm{Mon}_n \circ g^{-1}$, $\theta(Z_n) = g \circ Z_n \circ g^{-1}$ and finally $\theta(\mathrm{GL}_n) = g \circ \mathrm{GL}_n \circ g^{-1}$. □

3.2. **On unipotent subgroups and their images.** Let $U \subset \mathcal{G}_n$ be a unipotent connected algebraic subgroup of dimension one. Then U is isomorphic to k^+ (see for example [Hum75, Theorem 20.5]). If, in addition, U is normalized by D_n under conjugation, there is an element $u \in U$ such that $U = \{d \circ u \circ d^{-1} \mid d \in D_n\} \cup \{e\}$. From $\mathrm{Aut}(k^+) = k^*$ we get the canonical identification $\mathrm{Aut}(U) = k^*$, which does not depend on the choice of isomorphism between k^+ and U. This yields a scalar product on U in a natural way. The algebraic action of D_n on U is thus given by a character $\chi\colon D_n \to k^*$ through $t \circ u \circ t^{-1} = \chi(t) \cdot u$. In this way we can assign a character to each one-dimensional connected unipotent subgroup normalized by D_n. Proposition 3.16 shows that this map is injective and describes its image.

We use multi-index notation, so, for example, $t_1^{\gamma_1} \cdots t_n^{\gamma_n}$ is denoted by $\mathbf{t}^\gamma$.

Proposition 3.16. *Let $U \subset \mathcal{G}_n$ be a unipotent connected algebraic subgroup of dimension one, normalized by D_n under conjugation. Then U has the form:*

$$\{(x_1, \dots, x_i + s\mathbf{x}^\gamma, \dots x_n) \mid s \in k^+\},$$

where $\mathbf{x}^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ with $\gamma_i = 0$ and $\gamma_j \geq 0$ for $j \neq i$.

In particular, the character corresponding to U has the form $\chi(t) = \mathbf{t}^{-\gamma + e_i}$ for $t = (t_1x_1, \dots, t_nx_n)$.

Proof. Let $\delta\colon k^+ \to U$ be an isomorphism of algebraic groups. Then δ has the form $\delta(s) = (\delta(s)_1, \dots, \delta(s)_n)$ with

$$\delta(s)_j = \sum_{k=0}^{k_j} s^k f_{jk}(x_1, \dots, x_n).$$

From $\delta(0) = \mathrm{id}$ we get $f_{j0} = x_j$ for $j = 1, \dots n$.

Since δ is an isomorphism, there exists an i such that $f_{i1} \neq 0$. We write

$$\delta(s)_i = x_i + \sum_r (\sum_{k=0}^{k_i} c_{rk} s^k) \mathbf{x}^r,$$

with $c_{rk} \in k$ and the sum is taken over all monomials $\mathbf{x}^r = x_1^{r_1} \cdots x_n^{r_n}$ that appear as summands in $\delta(s)_i$.

Let $t = (t_1x_1, \dots, t_nx_n) \in D_n$. We have $t \circ \delta(s) \circ t^{-1} = \delta(\chi(t)s)$. This yields

$$t_i \sum_r (t_1^{-r_1} \cdots t_n^{-r_n} \sum_{k=0}^{k_i} c_{rk} s^k) \mathbf{x}^r = \sum_r (\sum_{k=0}^{k_i} c_{rk} s^k \chi(t)^k) \mathbf{x}^r$$

and hence for all multi-indices r, such that $\mathbf{x}^r$ appears as a summand in $\delta(s)_i$,

$$t_i t_1^{-r_1} \cdots t_n^{-r_n} \sum_{k=0}^{k_i} c_{rk} s^k = \sum_{k=0}^{k_i} c_{rk} s^k \chi(t)^k.$$

Therefore, for each r, all but one of the coefficients c_{rk} have to be zero. Let $m_d = x_1^{d_1} \cdots x_n^{d_n}$ be a monomial that appears as a summand in f_{i1}. So in particular, $t_i t_1^{-d_1} \cdots t_n^{-d_n} c_{d1} s = c_{d1} s \chi(t)$, hence $\chi(t) = \mathbf{t}^{-d+e_i}$ and also $f_{i1} = c_{d1} m_d$.

Our next goal is to show that $d_i = 0$. If $f_{ik} = 0$ for $k > 1$, we are done. Indeed, $\delta(s) \circ \delta(s) = \delta(2s)$ yields $\deg(\delta(s) \circ \delta(s)) \leq \deg(\delta(s))$ (with inequality if and only if $\mathrm{char}(k) = 2$), hence either $m_d = x_i$ or $d_i = 0$. Clearly $m_d = x_i$ is not possible. Assume now $f_{ik} \neq 0$ for a $k > 1$. Let m_l be a monomial appearing as a summand in f_{ik}. So

$$t_i t_1^{-l_1} \cdots t_n^{-l_n} c_{lk} s^k = c_{lk} s^k \chi(t)^k,$$

which yields $m_l = \mathbf{x}^{kd-(k-1)e_i}$ and $f_{ik} = c_{lk} m_l$. As all $f_{ik} \neq 0$ for $k > 1$ have this form, we use again $\deg(\delta(s) \circ \delta(s)) \leq \deg(\delta(s))$ to get $d_i = 0$. Therefore, $\chi(t)$ has the claimed form.

It follows now easily that $f_{ik} = 0$ for $k > 1$ as well as $f_{jk} = 0$ for $j \neq i$ and we are done. □

We denote the characters described in Proposition 3.16 above by $X_u(D_n)$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the standard basis of the space of all characters of D_n, that is, $\varepsilon_i(t) = t_i$ for $t = (t_1 x_1, \ldots, t_n x_n) \in D_n$. Then

$$X_u(D_n) := \{\lambda = \sum_j \lambda_j \varepsilon_j \mid \text{there exists an } i \text{ such that } \lambda_i = 1 \text{ and } \lambda_j \leq 0 \text{ for } j \neq i\}.$$

So there is a bijection between X_u and the set of one-dimensional connected unipotent subgroups normalized by D_n. The unipotent subgroup corresponding to $\lambda \in X_u$ is denoted by U_λ.

In characteristic zero a non-trivial algebraic group G is unipotent if and only if all elements except e have infinite order. Indeed, unipotent elements different from e are of infinite order, so the condition is necessary. On the other hand, assume G not to be unipotent. Then it follows from the Jordan decomposition that G contains a semisimple element and therefore elements of finite order.

In positive characteristics the situation is different. Let $\mathrm{char}(k) = p$ and G an algebraic group. An element $g \in G$ is unipotent if and only if it has order p^l for some $l \geq 0$. To see this, assume $G \subset \mathrm{GL}(V)$ for a finite dimensional k-vector space V. Then $g \in G$ is nilpotent if and only if $N := g - e$ is nilpotent, that is, for l large enough $N^{p^l} = 0$. This is equivalent to $g^{p^l} = (e + N)^{p^l} = e + N^{p^l} = e$.

This characterization of unipotent groups by the order of their elements is useful to show that the images of certain unipotent subgroups of $\mathcal{G}_n$ are unipotent again.

Proposition 3.17. *Let $\theta \colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of groups such that $\theta(D_n) = D_n$. Then for any character $\alpha \in X_u(D_n)$ there exists a character $\gamma \in X_u(D_n)$ such that $\theta(U_\alpha) = U_\gamma$, in other words θ induces a permutation on $X_u(D_n)$.*

Proof. Let $u \in U_\alpha \setminus \{e\}$. We can decompose U_α into two D_n-orbits:

$$U_\alpha = \{e\} \cup \{t \circ u \circ t^{-1} \mid t \in D_n\}.$$

As $\theta(D_n) = D_n$, it follows that

$$\theta(U_\alpha) = \{e\} \cup \{t \circ \theta(u) \circ t^{-1} \mid t \in D_n\}.$$

As a a union of two torus orbits, $\theta(U_\alpha)$ is a constructible subset of some filter set of $\mathcal{G}_n$. Furthermore it is a group, so $\overline{\theta(U_\alpha)}$ is an algebraic group. There exists a dense open subset $U \subset \overline{\theta(U_\alpha)}$ that is contained in $\theta(U_\alpha)$ and because $U \circ U = \overline{\theta(U_\alpha)}$, the group $\theta(U_\alpha)$ is algebraic.

If $\mathrm{char}(k) = 0$, all elements of U_α and therefore all elements of $\theta(U_\alpha)$ have infinite order. If $\mathrm{char}(k) = p > 0$, all elements of U_α and therefore all elements of $\theta(U_\alpha)$ have order p^r for some r. We thus conclude that $\theta(U_\alpha)$ is unipotent.

We denote by $\varphi\colon D_n \to \mathrm{Aut}(U_\alpha)$ the action of D_n on U_α and by $\psi\colon D_n \to \mathrm{Aut}(\theta(U_\alpha))$ the action of D_n on $\theta(U_\alpha)$. Then $n - \dim(\ker(\varphi)) = \dim(U_\alpha) = 1$. Clearly $\theta(\ker(\varphi)) = \ker(\psi)$, so by Lemma 3.12, $\dim(\ker(\varphi)) = \dim(\ker(\psi))$ and as a result $\dim(\theta(U_\alpha)) = 1$. Theorem 3.16 yields now $\theta(U_\alpha) = U_\gamma$ for a $\gamma \in X_u$. □

Another way to show $\dim(\theta(U_\alpha)) = 1$ could be to use the fact that an orbit of a torus action is always affine.

Proposition 3.18. *Let $\theta\colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of groups such that $\theta(D_n) = D_n$. Then $\theta(\mathcal{T}_n) = \mathcal{T}_n$.*

Proof. The translations $\mathcal{T}_n$ are generated by the unipotent subgroups U_{ε_i}, $i = 1, \dots, n$. It follows that $\theta(\mathcal{T}_n)$ is generated by the unipotent subgroups $\theta(U_{\varepsilon_i}) = U_{\delta_i}$ for $\delta_i \in X_u(D_n)$ (Proposition 3.17). Therefore, it is sufficient to show that for all i there is a j such that $\theta(U_{\varepsilon_i}) = U_{\varepsilon_j}$.

Now $\mathcal{T}_n$ is commutative and consequently so is $\theta(\mathcal{T}_n)$. In particular all elements in U_{δ_i} commute with all elements in U_{δ_j} for $i, j = 1, \dots, n$. An automorphism $(x_1, \dots, x_r + \mathbf{x}^a, \dots, x_n) \in U_{\delta_i}$ commutes with an automorphism of the form $(x_1, \dots, x_k + \mathbf{x}^b, \dots, x_n) \in U_{\delta_j}$ (where $a_r = b_k = 0$ and we assume $i \neq j$) if and only if $r = k$ or $a_k = b_r = 0$. However, $r = k$ is impossible since $\mathcal{T}_n$ and therefore also $\theta(\mathcal{T}_n)$ are stable under the action of GL_n. So we conclude $a = 0$, that is, $\theta(U_{\varepsilon_i}) = U_{\varepsilon_r}$ and we are done. □

3.3. **Proof of Theorem 3.1.** We start with a classification of the automorphisms of Aff_n:

Proposition 3.19. *Let $\theta\colon \mathrm{Aff}_n \to \mathrm{Aff}_n$ be an automorphism of groups. Then there is an automorphism of fields $\tau\colon k \to k$ and an affine transformation $g \in \mathrm{Aff}_n$ such that $\theta(f) = \tau(g \circ f \circ g^{-1})$ for all $f \in \mathrm{Aff}_n$.*

Proof. There is an element $g_1 \in \mathcal{G}_n$ such that $\theta(\mathrm{GL}_n) = g_1 \circ \mathrm{GL}_n \circ g_1^{-1}$ and $\theta(\mathcal{T}_n) = g_1 \circ \mathcal{T}_n \circ g_1^{-1}$ (cf. Theorem 3.15 and Proposition 3.18). Since $\mathrm{Norm}(\mathrm{Aff}_n) = \mathrm{Aff}_n$ (Proposition 3.11), the automorphism g_1 is in Aff_n. So we may assume that $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$ and $\theta(\mathcal{T}_n) = \mathcal{T}_n$.

Observe that θ is determined by its restriction to $\mathcal{T}_n$. Indeed, asssume $\theta|_{\mathcal{T}_n} = \mathrm{id}_{\mathcal{T}_n}$. Then, for a linear automorphism $g \in \mathrm{GL}_n$ and a translation $v \in \mathcal{T}_n$, the map $g \circ v \circ g^{-1}$ is a translation. As a result, for all $v \in \mathcal{T}_n$:

$$g \circ v \circ g^{-1} = \theta(g \circ v \circ g^{-1}) = \theta(g) \circ v \circ \theta(g)^{-1}.$$

This implies $\theta(g) = g$.

Note that $\theta(Z_n) = Z_n$. There is a natural isomorphism of groups $k^* \simeq Z_n$ given by $c \mapsto c\,\mathrm{id}$. The group of translations has the structure of a k-vector space. Let $v \in \mathcal{T}_n$. Then scalar multiplication with $c \in k^*$ is given by $c \cdot v = c\,\mathrm{id} \circ v \circ (c\,\mathrm{id})^{-1}$. We define $\tau\colon k \to k$, such that $\tau|_{k^*}$ is the map corresponding to $\theta|_{Z_n}$ and $\tau(0) := 0$. Clearly $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in k^*$. In addition we get for all $c \in k^*$ and all $v \in \mathcal{T}_n$:

(1) $\theta(c\cdot v)=\theta(c\,\mathrm{id}\circ v\circ(c\,\mathrm{id})^{-1})=\theta(c\,\mathrm{id})\circ\theta(v)\circ\theta(c\,\mathrm{id})^{-1}=\tau(c)\cdot\theta(v),$

and therefore $\tau(a+b)=\tau(a)+\tau(b)$ for all $a,b\in k^*$ such that $a+b\neq 0$. The element $(-\mathrm{id})$ is the only one in Z_n of order 2, hence $\theta(-\mathrm{id})=-\mathrm{id}$ and consequently $\tau(-a)=-\tau(a)$ for all $a\in k^*$. So τ is an automorphism of the field k.

Consider the map $(\tau^{-1}\circ\theta)|_{\mathcal{T}_n}\colon\mathcal{T}_n\to\mathcal{T}_n$. Equation (1) shows that it is linear. So there is an element $g\in\mathrm{GL}_n$ such that $(\tau^{-1}\circ\theta)(v)=g\circ v\circ g^{-1}$. This shows $\theta(v)=\tau(g\circ v\circ g^{-1})$ for all translations v and thus completes the proof. □

Lemma 3.20. *Let $\chi_1,\chi_2\colon D_n\to k^*$ be two characters with the same kernel. Then $\chi_1=\pm p^r\chi_2$ for some $r\in\mathbb{Z}$ where $p=\mathrm{char}(k)$ for positive characteristics, $p=1$ if $\mathrm{char}(k)=0$.*

Proof. Let $n=1$. Then $\chi_1=x^n$ and $\chi_2=x^m$ and we may assume that $m,n\in\mathbb{N}$. Since $\ker(\chi_1)=\ker(\chi_2)$ the polynomials x^n-1 and x^m-1 have the same roots and the claim follows. We proceed by induction on n. Assume $n>1$ and $\chi_1(t)=t_1^{l_1}\cdots t_n^{l_n}$ and $\chi_2(t)=t_1^{k_1}\cdots t_n^{k_n}$ for $t=(t_1x_1,\ldots,t_nx_n)\in D_n$. Setting $t_n=1$ we get $(l_1,\ldots,l_{n-1})=z_1p^r(k_1,\ldots,k_{n-1})$, setting $t_1=\cdots=t_{n-1}=1$ we get $l_n=z_2p^sk_n$ where $z_1,z_2\in\{\pm1\}$ and $r,s\in\mathbb{Z}$. If $l_1=\cdots=l_{n-1}=0$ we are done. Otherwise, assume that $z_1p^r\neq z_2p^s$. Then there exists an element $a\in k^*$ such that $(a^{k_n})^{z_1p^r}\neq(a^{k_n})^{z_2p^s}$. Let $(b_1,\ldots,b_{n-1})\in k^{n-1}$ such that $b_1^{l_1}\cdots b_{n-1}^{l_{n-1}}=a^{-l_n}$. As a result

$$b_1^{l_1}\cdots b_{n-1}^{l_{n-1}}a^{l_n}=1=b_1^{k_1}\cdots b_{n-1}^{k_{n-1}}a^{k_n}=(b_1^{k_1}\cdots b_{n-1}^{k_{n-1}}a^{k_n})^{z_1p^r}$$

and therefore

$$(b_1^{k_1}\cdots b_{n-1}^{k_{n-1}})^{z_1p^r}(a^{k_n})^{z_2p^s}=(b_1^{k_1}\cdots b_{n-1}^{k_{n-1}})^{z_1p^r}(a^{k_n})^{z_1p^r}.$$

It follows that $(a^{k_n})^{z_1p^r}=(a^{k_n})^{z_2p^s}$, which is a contradiction. □

Lemma 3.21. *Let $\theta\colon\mathcal{G}_n\to\mathcal{G}_n$ be an automorphism of groups. Let $\alpha\in X_u$ and $U_\alpha\subset\mathcal{G}_n$ be the corresponding one-dimensional unipotent subgroup. Assume $\theta(U_\alpha)=U_\gamma$, where $\gamma\in X_u$. If $\theta|_{D_n}=\mathrm{id}_{D_n}$, then $\alpha=\pm\gamma$.*

Proof. For all $t\in D_n$, $u\in U_\alpha$ we have:

(2) $$\theta(\alpha(t)u)=\theta(t\circ u\circ t^{-1})=t\circ\theta(u)\circ t^{-1}=\gamma(t)\theta(u).$$

So α and γ have the same kernel. In characteristic zero this implies $\alpha=\pm\gamma$. In characteristic p we get $\alpha=\pm p^r\gamma$ for some integer r (Lemma 3.20). However, since $\alpha,\gamma\in X_u$, both must contain a coefficient $\alpha_i=1$ respectively $\gamma_j=1$. We see that $r=0$ and so $\alpha=\pm\gamma$. □

Remark. If $\mathrm{char}(k)$ is neither 2 nor 3, Lemma 3.21 can be improved. In this case, $2^{-1}\neq 2$. We choose $t\in D_n$ such that $\alpha(t)=2$. Therefore,

$$\gamma(t)\theta(u)=\theta(\alpha(t)u)=\theta(u\circ u)=\theta(u)\circ\theta(u)=2\theta(u)\neq-\alpha(t)\theta(u)$$

and thus $\alpha=\gamma$. Equation (2) shows, that in that case θ restricted to U_α is linear.

With this we are ready to show that the restriction of an automorphism of $\mathcal{G}_n$ to $\mathcal{TG}_n$ is already determined by its restriction to Aff_n.

Proposition 3.22. *Let $\theta\colon\mathcal{G}_n\to\mathcal{G}_n$ be an automorphism of groups such that θ if restricted to Aff_n is the identity. Then θ is also the identity if restricted to the group of tame automorphisms $\mathcal{TG}_n$.*

Proof. The tame automorphisms are generated by Aff_n and the elementary transformations. The latter ones are contained in one-dimensional connected unipotent subgroups normalized by D_n, so it is enough to show that θ is the identity if restricted to U_λ for all $\lambda \in X_u$.

Let U_α be a one-dimensional connected unipotent subgroup normalized by D_n and $\gamma \in X_u(D_n)$ such that $\theta(U_\alpha) = U_\gamma$ (cf. Proposition 3.17). Lemma 3.21 yields $\alpha = \pm\gamma$. If $\alpha = -\gamma$ we get $\alpha = \varepsilon_i - \varepsilon_j$, in other words $U_\alpha \subset \mathrm{GL}_n$, which contradicts $\theta|_{\mathrm{Aff}_n} = \mathrm{id}_{\mathrm{Aff}_n}$. As a result $\alpha = \gamma$.

So θ restricted to U_α is linear, that is, for all $\alpha \in X_u$ there is a constant $c_\alpha \in k^*$ such that $\theta(u) = c_\alpha u$ for all $u \in U_\alpha$. We claim $c_\alpha = 1$. Let $f = (x_1, \dots, x_i + \mathbf{x}^\gamma, \dots, x_n) \in U_\alpha$, where $\gamma_i := 0$ if $\alpha_i = 1$ and $\gamma_j := -\alpha_j$ else. Consider the translation

$$t\colon x \mapsto x - \sum_{j \neq i} e_j.$$

Then $t \circ f \circ t^{-1} = (x_1, \dots, x_i + h_\gamma, \dots, x_n)$ with $h_\gamma := (x_1+1)^{\gamma_1} \cdots (x_n+1)^{\gamma_n}$. So various monomials $x^{\gamma'}$ with corresponding $\alpha' \in X_u$ appear in h_γ, including the constant monomial 1 with corresponding character ε_i. From $\theta(t \circ f \circ t^{-1}) = t \circ \theta(f) \circ t^{-1}$ we get that all the constants $c_{\alpha'}$ are equal. By assumption $c_{\varepsilon_i} = 1$, since U_{ε_i} consists of translations. This yields $c_\alpha = 1$ and we are done. □

The proof of Theorem 3.1 follows easily:

Proof of Theorem 3.1. By Theorem 3.15 and Proposition 3.18, there exists an element $g_1 \in \mathcal{G}_n$ such that $g_1^{-1} \circ \theta(\mathrm{Aff}_n) \circ g_1 = \mathrm{Aff}_n$. Proposition 3.19 above shows that therefore there exists an automorphism τ of the base field k and an element $g_2 \in \mathrm{GL}_n$ such that for all elements $f \in \mathrm{Aff}_n$,

$$\theta(f) = g_1 \circ \tau(g_2 \circ f \circ g_2^{-1}) \circ g_1^{-1} = \tau(g \circ f \circ g^{-1})$$

for a suitable $g \in \mathcal{G}_n$. With Proposition 3.22 we conclude that the same is true for all automorphisms $f \in \mathcal{TG}_n$. □

4. Automorphisms of $\mathcal{G}_n$ as an ind-group

The main result of this section is:

Theorem 4.1. *Let the base field k be algebraically closed and of characteristic 0. Then every automorphism of $\mathcal{G}_n$ as an ind-group is inner.*

Theorem 4.1 was already formulated by Bodnarchuk [Bod01]. But his proof relies on results of Shafarevich that turned out to be wrong. In our proof we follow an idea from Belov-Kanel and Yu [BKY12].

4.1. **Automorphisms of $\mathcal{G}_n$ as an ind-group restricted to $\mathcal{TG}_n$.** First we show that every automorphism Φ of $\mathcal{G}_n$ as an ind-group is inner if restricted to the subgroup $\mathcal{TG}_n$. An automorphism of groups $\tau\colon \mathcal{G}_n \to \mathcal{G}_n$ induced by an automorphism τ of the base field k is never an automorphism of ind-groups. Therefore, we could deduce this result directly from Theorem 3.1. However, the additional algebraic structure of an automorphism of ind-groups makes some steps easier. For this reason we sketch the proof again. The results in this section still hold for any algebraically closed base field k.

Proposition 4.2. *Let $\Phi\colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of ind-groups. Then there is an element $g \in \mathcal{G}_n$ such that $\Phi(D_n) = g \circ D_n \circ g^{-1}$, $\Phi(\mathrm{GL}_n) = g \circ \mathrm{GL}_n \circ g^{-1}$ and $\Phi(Z_n) = g \circ Z_n \circ g^{-1}$.*

Proof. By Bialynicki-Birula's theorem (3.14), the image of D_n is conjugated to D_n. So $g^{-1} \circ \Phi \circ g$ leaves D_n stable; thus it also maps its normalizer Mon_n to itself. As a result, $\mathrm{Center}(\mathrm{Mon}_n) = Z_n$ is stable under $g^{-1} \circ \Phi \circ g$ and therefore also $\mathrm{GL}_n = \mathrm{Cent}(Z_n)$ (see Proposition 3.11 and Lemma 3.10). □

Consequently, we can assume that Φ maps D_n to D_n, GL_n to GL_n and Z_n to Z_n. In particular, $\Phi|_{Z_n}\colon Z_n \to Z_n$ is either the identity or the map $c\,\mathrm{id} \mapsto c^{-1}\,\mathrm{id}$.

The image $\Phi(U_\alpha)$ of a one-dimensional connected unipotent subgroup U_α with $\alpha \in X_u(D_n)$ is clearly again a one-dimensional connected unipotent subgroup normalized by D_n, so there exists a character $\gamma \in X_u(D_n)$ such that $\Phi(U_\alpha) = U_\gamma$. Moreover, by Proposition 3.18, $\Phi(\mathcal{T}_n) = \mathcal{T}_n$.

The equation

$$\Phi(c \cdot v) = \Phi(c\,\mathrm{id} \circ v \circ (c\,\mathrm{id})^{-1}) = \Phi(c\,\mathrm{id}) \circ \Phi(v) \circ \Phi(c\,\mathrm{id})^{-1} = \Phi(c\,\mathrm{id}) \cdot \Phi(v)$$

yields for $c, d \in k^*$, such that $c + d \neq 0$, that $\Phi((c+d)\,\mathrm{id}) = \Phi(c\,\mathrm{id}) + \Phi(d\,\mathrm{id})$ and thus $\Phi|_{Z_n} = \mathrm{id}_{Z_n}$. As a result $\Phi|_{\mathcal{T}_n}$ is linear; hence it is given by conjugation with an element in GL_n. So we may assume $\Phi|_{\mathcal{T}_n}$ to be the identity. As a consequence

$$g \circ v \circ g^{-1} = \Phi(g \circ v \circ g^{-1}) = \Phi(g) \circ v \circ \Phi(g)^{-1}$$

for all $v \in \mathcal{T}_n$ and $g \in \mathrm{GL}_n$ and therefore $\Phi|_{\mathrm{GL}_n} = \mathrm{id}_{\mathrm{GL}_n}$. As $\Phi|_{\mathcal{T}\mathcal{G}_n}$ is determined by its restriction to Aff_n (see Proposition 3.22), we have proven the following proposition:

Proposition 4.3. *Let $\Phi\colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of ind-groups. Then there is an element $g \in \mathcal{G}_n$ such that*

$$\Phi(f) = g \circ f \circ g^{-1}$$

for all tame automorphisms $f \in \mathcal{T}\mathcal{G}_n$.

An alternative way to establish Proposition 4.3 would be to use linearization results as can be found for example in [KS92].

4.2. **Tame approximation in characteristic zero.** In this section we assume the base field k to be of characteristic zero and as always algebraically closed.

Let $\mathfrak{m} := (x_1, \dots, x_n) \subset k[x_1, \dots, x_n]$ be the homogeneous maximal ideal. Let $f, g \in \mathrm{End}(\mathbb{A}^n)$. If f and g coincide in all homogeneous terms of degree $< d$, that is, $f_i - g_i \in \mathfrak{m}^d$ for $i = 1, \dots, n$, we write

$$f \equiv g \mod \mathfrak{m}^d.$$

We denote by $H_d \subset \mathcal{G}_n$ the subgroup of all automorphisms $f \in \mathcal{G}_n$ such that $f \equiv \mathrm{id} \mod \mathfrak{m}^{d+1}$. Note that $f \equiv g \mod \mathfrak{m}^{d+1}$ is equivalent to $f \circ g^{-1} \in H_d$.

The following theorem was first discovered by Anick [Ani83]. It states that every endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$ with $J(g) \in k^*$ – so in particular every automorphism – can be approximated by tame automorphisms up to any degree.

Theorem 4.4. *Let $f \in \mathrm{End}(\mathbb{A}^n)$ be an endomorphism with Jacobian determinant $J(g) \in k^*$. Then for any $d \in \mathbb{N}$ there exists a tame automorphism $g^{(d)} \in \mathcal{T}\mathcal{G}_n$ such that $f \equiv g^{(d)} \mod \mathfrak{m}^{d+1}$.*

We follow a proof as described in [Fur07] and [Kra12] that uses representation theory. For this we recall a classical statement concerning representations of GL_n, which is a special case of the well-known Littlewood-Richardson rule.

Define the set of *partitions* by

$$\mathcal{P} \coloneqq \{(\lambda_1, \lambda_2, \dots) \mid \lambda_i \in \mathbb{N}, \lambda_1 \geq \lambda_2 \geq \dots, \lambda_i = 0 \text{ for large } i\}.$$

The *height* of a partition λ is defined as $\max\{i \mid \lambda_i \neq 0\}$, the *length* as $|\lambda| \coloneqq \lambda_1 + \lambda_2 + \cdots$. For every partition $\lambda \in \mathcal{P}$ of height $\leq n$ there is an irreducible polynomial representation L_λ of GL_n. The L_λ represent all isomorphism classes of simple polynomial GL_n-modules. For a proof of this statement and further explanations we refer to [Pro07].

Proposition 4.5 (Pieri's formula)**.** *Let V be an n-dimensional GL_n-module. Then for any $\lambda \in \mathcal{P}$ of height $\leq n$ we have the following decomposition into irreducible GL_n-modules:*

$$L_\lambda \otimes \bigwedge^i V = \bigoplus_\mu L_\mu(V),$$

where the sum is taken over all partitions μ of height $\leq n$ such that $|\mu| = |\lambda| + i$ and $\lambda_j \leq \mu_j \leq \lambda_j + 1$ for all j.

Proof. See [Pro07, Theorem 9.10.2]. □

We denote by $k[x_1, \dots, x_n]_d$ the homogeneous polynomials of degree d. The k-vector space $(k[x_1, \dots, x_n]_d)^n \subset \mathrm{End}(\mathbb{A}^n)$ is a GL_n-module, where the GL_n-action is given by $gf(\mathbf{x}) \coloneqq g \circ f \circ g^{-1}(\mathbf{x})$ for $g \in \mathrm{GL}_n$ and $f \in (k[x_1, \dots, x_n])^n$.

Lemma 4.6. *The GL_n-module $(k[x_1, \dots, x_n]_d)^n$ is isomorphic to the direct sum of two irreducible GL_n-submodules. In fact, we have the following split exact sequence, which is GL_n-invariant:*

$$0 \longrightarrow \ker(\nabla) \longrightarrow (k[x_1, \dots, x_n]_d)^n \xrightarrow{\nabla} k[x_1, \dots, x_n]_{d-1} \longrightarrow 0,$$

where the linear map $\nabla \colon (k[x_1, \dots, x_n]_d)^n \to k[x_1, \dots, x_n]_{d-1}$ is defined by

$$\nabla(u) \coloneqq \sum_i \frac{\partial u_i}{\partial x_i}.$$

Proof. There is a natural GL_n-isomorphism

$$(k[x_1, \dots, x_n]_d)^n \simeq k^n \otimes k[x_1, \dots, x_n]_d,$$

where the GL_n-action on $k^n \otimes k[x_1, \dots, x_n]_d$ is given by $g(v \otimes p) \coloneqq gv \otimes gp$. Setting $V \coloneqq (k^n)^*$, we get

$$V^* \otimes S^d(V) \simeq \bigwedge^{n-1} V \otimes S^d(V),$$

where $S^d(V)$ denotes the symmetric algebra of degree d. Since $S^d(V) \simeq L_\lambda$, with $\lambda = (d)$, Pieri's formula yields

$$\bigwedge^{n-1} V \otimes S^d(V) = L_{\mu_1} \oplus L_{\mu_2},$$

where

$$\mu_1 = (d, \underbrace{1, \dots, 1}_{n-1}) \quad \text{and} \quad \mu_2 = (d+1, \underbrace{1, \dots, 1}_{n-2}).$$

In particular $(k[x_1, \dots, x_n]_d)^n$ is isomorphic to the direct sum of exactly two irreducible GL_n-submodules.

Next, we observe that ∇ is GL_n-stable. Indeed, for $u \in (k[x_1, \dots, x_n]_d)^n$ and $g \in \mathrm{GL}_n$ we have

$$\nabla(gu) = \mathrm{tr}(d_x(g \circ u \circ g^{-1})) = \mathrm{tr}(g \circ d_{g^{-1}(x)} u \circ g^{-1}) = \mathrm{tr}(d_{g^{-1}(x)} u) = g\nabla(u).$$

It is also not hard to check that the linear map

$$\Delta\colon k[x_1, \dots, x_n]_{d-1} \to (k[x_1, \dots, x_n]_d)^n, \quad p \mapsto \frac{1}{n+d}\, p \cdot \mathrm{id}$$

is a GL_n-invariant section. The claim follows.

□

Proof of Theorem 4.4. For $d = 0$ and $d = 1$ the approximation can be obtained by translations and linear automorphisms, so the claim is certainly true.

We proceed now with induction on d. Let $d > 1$ and suppose there is a tame automorphism $g^{(d-1)}$ satisying $f \equiv g^{(d-1)} \mod \mathfrak{m}^d$. Therefore, we can assume $f \equiv \mathrm{id} \mod \mathfrak{m}^d$ and $J(f) = 1$. We show that there exists a tame automorphism $g^{(d)}$ such that $f \equiv g^{(d)} \mod \mathfrak{m}^{d+1}$. The homogeneous component of degree d of f is denoted by f_d; so $f_d \in (k[x_1, \dots, x_n]_d)^n$.

For $h, h' \in H_d$ and $g \in \mathrm{GL}_n$, straightforward calculations show that $(h \circ h')_d = h_d + h'_d$ and $(g \circ h \circ g^{-1})_d = gh_d$. The subset

$$E_d \coloneqq \{h_d \mid h \in \mathrm{End}(\mathbb{A}^n),\, J(h) = 1 \text{ and } h \equiv \mathrm{id} \mod \mathfrak{m}^d\} \subset (k[x_1, \dots, x_n]_d)^n$$

is thus a GL_n-submodule, as well as the subset

$$E_d^t \coloneqq \{h_d \mid h \in \mathcal{TG}_n,\, J(h) = 1 \text{ and } h \equiv \mathrm{id} \mod \mathfrak{m}^d\} \subset E_d.$$

It is easily seen, by the definition of the determinant, that the condition $J(f) = 1$ yields $\nabla(f_d) = 0$ for $f \equiv \mathrm{id} \mod \mathfrak{m}^d$. So it follows from Lemma 4.6 that $\ker(\nabla) = E_d = E_d^t$ and we are done.

□

We could also formulate Theorem 4.4 topologically. Define a metric on $\mathrm{End}(\mathbb{A}^n)$ by $d(f, g) \coloneqq \exp(-\mathrm{ht}(f - g))$ for $f, g \in \mathrm{End}(\mathbb{A}^n)$. Here $\mathrm{ht}(h)$ denotes the *height* of an endomorphism h, that is, the smallest degree of all homogeneous terms in h if $h \neq 0$, and $\mathrm{ht}(0) \coloneqq \infty$. Theorem 4.4 states then, that the tame automorphisms are dense in the set of endomorphisms $H \coloneqq \{f \in \mathrm{End}(\mathbb{A}^n) \mid J(f) \in k^*\}$. It can be shown that the set $H \subset \mathrm{End}(\mathbb{A}^n)$ is closed with respect to this topology (see [Ani83]). Therefore, the endomorphisms g with $J(g) \in k^*$ are the only ones that can be approximated by tame automorphisms.

4.3. **Proof of Theorem 4.1.** To be able to apply tame approximation to prove Theorem 4.1, we show $\Phi(H_d) = H_d$ for automorphisms Φ of $\mathcal{G}_n$ as an ind-group. For this we need a different characterization of the subgroups H_d.

Definition. Let X be an affine variety and $\lambda\colon k^* \to X$ a morphism. If there is a morphism $\widetilde{\lambda}\colon k \to X$ such that $\widetilde{\lambda}|_{k^*} = \lambda$, we define

$$\lim_{t \to 0} \lambda(t) \coloneqq \widetilde{\lambda}(0).$$

Note that the above defined limit, if it exists, is unique.

Definition. Let X be an affine variety and $\mu\colon k \to X$ be a non-constant morphism. Then $\mu^*(\mathfrak{m}_{\mu(0)}) \subset (t)$. We define $\sigma(\mu) \coloneqq \max\{d \in \mathbb{N} \mid \mu^*(\mathfrak{m}_{\mu(0)}) \subset (t^d)\}$ and call it the *speed* of μ.

Let $g \in \mathcal{G}_n$ and define the morphism $\lambda\colon k^* \to \mathcal{G}_n$, $\lambda_g(t) := (t^{-1}\,\mathrm{id}) \circ g \circ (t\,\mathrm{id})$. The image is contained in some filter set $(\mathcal{G}_n)^d$, so we may think of λ_g as a morphism of affine varieties. Define $\widetilde{\lambda}\colon k \to \mathcal{G}_n$ by $\widetilde{\lambda}_g|_{k^*} := \lambda_g$ and $\widetilde{\lambda}_g(0) := \mathrm{id}$. An automorphism g lies in H_d if and only if $\lim_{t\to 0}\lambda_g(t) = \mathrm{id}$ and $\sigma(\widetilde{\lambda}_g) \geq d$. Therefore,

$$H_d = \{g \in \mathcal{G}_n \mid \lim_{t\to 0}\lambda_g = \mathrm{id} \text{ and } \sigma(\widetilde{\lambda}_g) \geq d\}.$$

Proposition 4.7. *Let the base field k be of characteristic zero and $\Phi\colon \mathcal{G}_n \to \mathcal{G}_n$ an automorphism of ind-groups that is the identity if restricted to $\mathcal{TG}_n$. Then Φ is the identity on $\mathcal{G}_n$.*

Proof. Since Φ is the identity on Z_n, it follows for all $g \in \mathcal{G}_n$ that $\Phi \circ \lambda_g = \lambda_{\Phi(g)}$. As a result, if $g \in H_d$, then $\lim_{t\to 0}\lambda_{\Phi(g)} = \mathrm{id}$ and $\sigma(\widetilde{\lambda}_{\Phi(g)}) \geq d$, so $\Phi(g) \in H_d$.

Let $f \in \mathcal{G}_n$. By Theorem 4.4 there is for each $d \in \mathbb{N}$ a tame automorphism $g^{(d)}$ satisfying $f \equiv g^d \mod \mathfrak{m}^{d+1}$. This is equivalent to $f \circ (g^{(d)})^{-1} \in H_d$. As a result, $\Phi(f) \circ (g^{(d)})^{-1} = \Phi(f \circ (g^{(d)})^{-1}) \in H_d$. We conclude $\Phi(f) \equiv g^{(d)} \mod \mathfrak{m}^{d+1}$ for all d, therefore $\Phi(f) \equiv f \mod \mathfrak{m}^{d+1}$ and thus $\Phi(f) = f$. □

Proof of Theorem 4.1. By Theorem 4.3, every automorphism of ind-groups is inner if restricted to $\mathcal{TG}_n$. So the theorem follows directly from the above Proposition 4.7. □

Remark. Let $\Phi\colon \{g \in \operatorname{End}\mathbb{A}^n \mid J(g) \in k^*\} \to \{g \in \operatorname{End}\mathbb{A}^n \mid J(g) \in k^*\}$ be an automorphism of semigroups respecting the ind-structure. It is not hard to see that Proposition 4.7 can be generalized to this case, and therefore that such an automorphism of semigroups is always given by conjugation with an element of $\mathcal{G}_n$.

5. Outlook

A number of questions naturally arise from the results presented in this thesis. The most obvious is whether an automorphism of $\mathcal{G}_n$ as a group is already defined by its restriction to the subgroup of tame automorphisms $\mathcal{TG}_n$:

Question. Let $\theta\colon \mathcal{G}_n \to \mathcal{G}_n$ be an automorphism of groups, $n \geq 3$. Is θ inner up to field automorphisms?

Little is known about wild automorphisms, especially in higher dimensions. In particular, we do not know of a suitable set of automorphisms generating $\mathcal{G}_n$ to which we could reduce the problem. An alternative approach in characteristic zero could be to show that every automorphism θ that is the identity on $\mathcal{TG}_n$ leaves the subgroups H_d, as introduced in Section 4.2, stable and then to use again tame approximation. But it is unclear in which way this could be achieved. However, Stampfli recently showed that for $k = \mathbb{C}$ every automorphism of groups of $\mathcal{G}_3$ that fixes the subgroup of tame automorphisms, also fixes the Nagata automorphism (see [Sta12]).

We only looked at restrictions of automorphisms of all $\mathcal{G}_n$ to $\mathcal{TG}_n$. But we do not know yet, which automorphisms of $\mathcal{TG}_n$ exist:

Question. Let $\theta\colon \mathcal{TG}_n \to \mathcal{TG}_n$ be an automorphism of groups. Is θ inner up to field automorphisms?

Note that this is equivalent to the question of whether $\operatorname{Norm}(\mathcal{TG}_n) = \mathcal{TG}_n$.

An interesting question seems to be which endomorphisms in positive characteristics can be approximated by tame automorphisms:

Question. Let $\mathrm{char}(k) = p$. Can every automorphism $f \in \mathcal{G}_n$ be approximated by a tame one? Moreover, is every endomorphism that can be approximated by tame automorphisms an automorphism itself?

One of the main problems is that not every endomorphism $g \in \mathcal{G}_n$ with $J(g) \in k^*$ is invertible. It looks unlikely for various reasons, that $J(g) \in k^*$ is a sufficient condition for an endomorphism g to be approximable by tame automorphisms. We could therefore ask whether for an endomorphism the property of being approximable is equivalent to being an automorphism. In characteristic zero this question is equivalent to the Jacobian conjecture. For this reason it is stated as a new version of the Jacobian conjecture in positive characteristics in [BKY12]. It is not clear though, whether this version agrees with previous formulations of the Jacobian conjecture for positive characteristics.

Further research is also required to investigate, to which fields our results can be generalized.

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