A NOTE ON THE STABLE EQUIVALENCE PROBLEM

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Abstract. We provide counterexamples to the stable equivalence problem in every dimension $d \geq 2$. That means that we construct hypersurfaces $H_1, H_2 \subset \mathbb{C}^{d+1}$ whose cylinders $H_1 \times \mathbb{C}$ and $H_2 \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{d+2}$, although H_1 and H_2 themselves are not equivalent by an automorphism of $\mathbb{C}^{d+1}$. We also give, for every $d \geq 2$, examples of two non-isomorphic algebraic varieties of dimension d which are biholomorphic.

1. Introduction

The well known generalized cancellation problem asks the following question.

Generalized cancellation problem. Given two complex affine varieties V_1 and V_2 with the property that $V_1 \times \mathbb{C}^m$ and $V_2 \times \mathbb{C}^m$ are isomorphic for some $m \in \mathbb{N}$. Does this imply that V_1 and V_2 are isomorphic?

An affirmative answer was given by Abhyankar, Eakin and Heinzer [1] for the case of affine curves. The cancellation property holds also in the case where V_1 (or V_2) has nonnegative logarithmic Kodaira dimension. This was shown by Iitaka and Fujita in [10]. However, the answer to the generalized cancellation problem turns out to be negative in general. The first counterexamples are surfaces due to Danielewski [2] (see also [6]). Later on, Danielewski's construction was generalized by Dubouloz [4] to produce counterexamples of every dimension $d \geq 2$ (see also [7] and [5] for factorial and contractible 3-dimensional examples).

In 2004, Makar-Limanov, van Rossum, Shpilrain and Yu [15] considered the following analogous problem.

Stable equivalence problem. If two hypersurfaces in $\mathbb{C}^n$ are stably equivalent, are they equivalent?

Recall that two algebraic varieties V_1, V_2 in $\mathbb{C}^n$ are said to be *equivalent* if there exists a polynomial automorphism of $\mathbb{C}^n$ which maps V_1 onto V_2, and that they are said to be *stably equivalent* if there is an integer $m \in \mathbb{N}$ such that the cylinders $V_1 \times \mathbb{C}^m$ and $V_2 \times \mathbb{C}^m$ are equivalent varieties in $\mathbb{C}^{n+m}$. The stable equivalent problem has a positive answer for affine plane curves, as already shown by Makar-Limanov, van Rossum, Shpilrain and Yu in [15]. In the same vein of the result of Iitaka-Fujita, Drylo proved in [3] that two stably equivalent hypersurfaces in $\mathbb{C}^n$ are equivalent, if one of them is not $\mathbb{C}$-uniruled. The first counterexamples in $\mathbb{C}^3$, consisting in families of Danielewski hypersurfaces, were provided

2010 *Mathematics Subject Classification.* 14R10, 14L30.

Key words and phrases. equivalence of hypersurfaces, stable equivalence, locally nilpotent derivations, exotic models.

by Moser-Jauslin and the author [17]. Also, contractible 3-dimensional counterexamples appeared in [5].

In this note, we complete the analogy between the results on the generalized cancellation and stable equivalence problems. Indeed, we produce counterexamples to the stable equivalence problem for every $n \geq 3$. These new examples are easy generalizations of those of [17], inspired by the construction in [4].

We will actually give two kinds of counterexamples. On one hand, polynomials $P, Q \in \mathbb{C}[X_1, \ldots, X_n]$ whose zero-sets $V(P)$ and $V(Q)$ are non-isomorphic varieties, but such that the cylinders $V(P) \times \mathbb{C}$ and $V(Q) \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{n+1}$. On the other hand, polynomials $P, Q \in \mathbb{C}[X_1, \ldots, X_n]$ with the properties that $V(P) \times \mathbb{C}$ and $V(Q) \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{n+1}$ and that $V(P)$ and $V(Q)$ are non-equivalent hypersurfaces in $\mathbb{C}^n$, although the fibers $V(P-c)$ and $V(Q-c)$ of P and Q are pairwise isomorphic for all $c \in \mathbb{C}$. More precisely, we will prove the following result.

Theorem. *The following assertions hold for every natural number $n \geq 1$.*

1. *The hypersurfaces $H_1, H_2 \subset \mathbb{C}^{n+2}$ defined by the equation $x_1^2 \cdots x_n^2 y + z^2 + x_1 \cdots x_n(z^2-1) = 1$ and $x_1^2 \cdots x_n^2 y + z^2 + x_1 \cdots x_n(z^2-2) = 1$, respectively, are non-isomorphic algebraic varieties such that $H_1 \times \mathbb{C}$ and $H_2 \times \mathbb{C}$ are equivalent hypersurfaces in $\mathbb{C}^{n+3}$.*
2. *The polynomials $Q_k = x_1^2 \cdots x_n^2 y + z^2 + x_1 \cdots x_n (z^2-1)^k \in \mathbb{C}[x_1, \ldots, x_n, y, z]$ are stably equivalent for all $k \geq 1$, whereas the hypersurfaces $V(Q_k) \subset \mathbb{C}^{n+2}$ are pairwise non-equivalent. However, the varieties $V(Q_k - c)$ and $V(Q_{k'} - c)$ are isomorphic for all $k, k' \geq 1$ and every $c \in \mathbb{C}$.*

It is worth mentioning that the special case of affine spaces is still open, for both cancellation and stable equivalence problems. The question to know whether an isomorphism $V \times \mathbb{C}^m \simeq \mathbb{C}^{n+m}$ implies $V \simeq \mathbb{C}^n$ is often referred to as the "Zariski cancellation problem", although Zariski's question was different. (In this form this question appeared first in a paper of Ramanujam [18].) It has a positive solution for $n = 1$ and for $n = 2$ by the results of Fujita and Miyanishi-Sugie ([9], [16]), whereas it is still an unsolved problem for $n \geq 3$.

Similarly, it was asked in [15] if every hypersurface in $\mathbb{C}^{n+1}$, which is stably equivalent to a (linear) hyperplane, is already equivalent to this hyperplane. This is true for $n = 1$ and for $n = 2$ by using the cancellation property of the affine plane and a result due to Sathaye [19] (see also [12]) saying that a polynomial whose fibers are all reduced and isomorphic to $\mathbb{C}^2$ is a variable. Moreover, as noticed in [15], a positive answer for an integer $n \geq 3$ would imply that the n-dimensional affine space has the cancellation property.

2. Four hypersurfaces in $\mathbb{C}^{n+2}$

We will work over the field $\mathbb{C}$ of complex numbers, even although all results of the present paper, exept Remark 2.6, would be valid over any algebraically closed field of characteristic zero. Let us fix some notations.

Notation 2.1. Given a ring R and an integer $m \in \mathbb{N}$, we denote by $R^{[m]}$ the polynomial ring in m variables over R. Throughout this paper, we fix a positive integer n and we denote by $\mathbb{C}[\underline{x}]$ the polynomial ring $\mathbb{C}[x_1, \ldots, x_n] \simeq \mathbb{C}^{[n]}$ in the variables $x_1, \ldots, x_n$.

For every integer $k \in \mathbb{N}$, we denote by $\underline{x}^{[k]}$ the element $\underline{x}^{[k]} = x_1^k \cdots x_n^k \in \mathbb{C}[\underline{x}]$ and, for every polynomial $q \in \mathbb{C}^{[1]}$, by P_q the polynomial of $\mathbb{C}[x_1, \ldots, x_n, y, z] = \mathbb{C}[\underline{x}][y, z]$ defined by

$$P_q = \underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} q(z^2).$$

The counterexamples to the stable equivalent problem mentioned in the introduction are realized as hypersurfaces in $\mathbb{C}^{n+2}$ given by the fibers $V(P_q - c)$ of some polynomials P_q. We will determine the isomorphism classes of these varieties. This will be done by using techniques mainly developed by Makar-Limanov in [14]. The idea is to exploit the fact that this kind of hypersurfaces admit additive group actions, but not too many. For instance, their Makar-Limanov invariants are non trivial.

It is in general very difficult and technical to compute such invariants. But we are in a good situation, since the method of Kaliman and Makar-Limanov ([13]) applies to the varieties that we are considering. Moreover, we can even use directly the results of Dubouloz [4], who already did the computation for the case where the polynomial q is constant. Remark that, thanks to the next lemma, it suffices to consider only this special case.

Lemma 2.2. *Let $R = \mathbb{C}[\underline{x}] \simeq \mathbb{C}^{[n]}$. Given $q \in \mathbb{C}^{[1]}$ and $c \in \mathbb{C}$, we let $g_c \in \mathbb{C}^{[1]}$ be the polynomial such that the equality $q(z^2) - q(c) = g_c(z^2)(z^2 - c)$ holds in $\mathbb{C}[z]$. Then, the endomorphism $\varphi_c \in \operatorname{End}_R R[y, z]$ of $R[y, z]$ fixing R and defined by*

$$\varphi_c(y) = \left(1 + \underline{x}^{[1]} g_c(z^2)\right) y + q(c) g_c(z^2) \quad \text{and} \quad \varphi_c(z) = z$$

induces an isomorphism between the rings $\mathbb{C}[\underline{x}, y, z]/(P_q - c)$ and $\mathbb{C}[\underline{x}, y, z]/(P_{q(c)} - c)$.

Proof. First, one checks that $\varphi_c\left(P_q - c\right) = \left(1 + \underline{x}^{[1]} g_c(z^2)\right)\left(P_{q(c)} - c\right)$. Thus, φ_c induces a morphism between $\mathbb{C}[\underline{x}, y, z]/(P_q - c)$ and $\mathbb{C}[\underline{x}, y, z]/(P_{q(c)} - c)$. The latter is invertible. To see this, one checks that the inverse morphism is induced by the endomorphism $\psi_c \in \operatorname{End}_R R[y, z]$ defined by $\psi_c(y) = \left(1 - \underline{x}^{[1]} g_c(z^2)\right) y - q(z^2) g_c(z^2)$ and $\varphi_c(z) = z$. $\square$

We will now compute, for all $q \in \mathbb{C}^{[1]}$ and all $c \in \mathbb{C}$, the set $\operatorname{LND}(B_{q,c})$ of locally nilpotent derivations on the coordinate ring $B_{q,c}$ of the varieties $V(P_q - c)$. Recall that a derivation δ of a $\mathbb{C}$-algebra B is called *locally nilpotent* if there exists, for every element $b \in B$, an integer $m = m(b) \geq 1$ such that $\delta^m(b) = 0$. Let Δ be the derivation of $\mathbb{C}[\underline{x}, y, z]$ defined by

$$\Delta = \underline{x}^{[2]} \frac{\partial}{\partial z} - 2z(1 + \underline{x}^{[1]} q'(z^2)) \frac{\partial}{\partial y},$$

where q' denotes the derivative of q. Note that Δ is locally nilpotent (it is a triangular derivation) and that it annihilates the polynomial $P_q - c$. Therefore, it induces a locally nilpotent derivation on $B_{q,c}$, which we still denote by Δ. It turns out that all other locally nilpotent derivations on $B_{q,c}$ are multiple of Δ by elements of $\mathbb{C}[\underline{x}]$.

Proposition 2.3. *Let $q \in \mathbb{C}^{[1]}$, $c \in \mathbb{C}$ and $B_{q,c} = \mathbb{C}[\underline{x}, y, z]/(P_q - c)$, where $P_q = \underline{x}^{[2]} y + z^2 + \underline{x}^{[1]} q(z^2) \in \mathbb{C}[\underline{x}, y, z]$. Then, the following hold for every nonzero locally nilpotent derivation δ of $B_{q,c}$.*

(1) $\operatorname{Ker}(\delta) = \mathbb{C}[\underline{x}]$ *and* $\operatorname{Ker}(\delta^2) = \mathbb{C}[\underline{x}] z + \mathbb{C}[\underline{x}]$.

(2) *There exists* $h(\underline{x}) \in \mathbb{C}[\underline{x}]$ *such that* $\delta = h(\underline{x})\Delta$, *where* Δ *is the locally nilpotent derivation on* $B_{q,c}$ *defined above.*

Proof. (1) First of all, remark that we can suppose that q is a constant polynomial. Indeed, take the isomorphism $\phi : B_{q,c} \to B_{q(c),c}$ given by Lemma 2.2 and let $\delta \in \mathrm{LND}(B_{q,c}) \setminus \{0\}$ be a nonzero locally nilpotent derivation. Then, $\widetilde{\delta} = \phi \circ \delta \circ \phi^{-1} \in \mathrm{LND}(B_{q(c),c}) \setminus \{0\}$ and we have $\mathrm{Ker}(\delta) = \phi^{-1}(\mathrm{Ker}(\widetilde{\delta}))$ and $\mathrm{Ker}(\delta^2) = \phi^{-1}(\mathrm{Ker}(\widetilde{\delta}^2))$. Since ϕ^{-1} maps $\mathbb{C}[\underline{x}]$ onto $\mathbb{C}[\underline{x}]$ and $\mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$ onto $\mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$, it suffices to prove $\mathrm{Ker}(\widetilde{\delta}) = \mathbb{C}[\underline{x}]$ and $\mathrm{Ker}(\widetilde{\delta}^2) = \mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$.

So, let $q, c \in \mathbb{C}$ be two constants and let δ be a nonzero locally nilpotent derivation on $B_{q,c}$. We are now in the case considered by Dubouloz in [4], where he proved (see paragraph 2.7 in [4]) that $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}]$ and $\mathrm{Ker}(\delta^2) \subset \mathbb{C}[\underline{x}, z]$ hold. This implies easily $\mathrm{Ker}(\delta^2) = \mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$.

Indeed, let $a \in \mathrm{Ker}(\delta^2) \setminus \mathrm{Ker}(\delta)$ and write $a = \sum_{i=0}^{d} \alpha_i(\underline{x})z^i$ with $d \geq 1$ and $\alpha_i(\underline{x}) \in \mathbb{C}[\underline{x}]$. Then $\delta(a) = \delta(z)\sum_{i=1}^{d} i\alpha_i(\underline{x})z^{i-1}$ is a nonzero element of $\mathrm{Ker}(\delta)$. Since the kernel of a locally nilpotent derivation is factorially closed, it follows that $\delta(z)$ lies in $\mathrm{Ker}(\delta)$. Thus, $z \in \mathrm{Ker}(\delta^2)$ and so $\mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}] \subset \mathrm{Ker}(\delta^2)$. On the other hand, $\delta(a) \in \mathrm{Ker}(\delta)$ implies $d = 1$, since $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}]$. Therefore, $a \in \mathbb{C}[\underline{x}]z + \mathbb{C}[\underline{x}]$ and (1) is proved.

(2) Let $\delta \in \mathrm{LND}(B_{q,c}) \setminus \{0\}$. By (1), $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}]$ and there exists a polynomial $a(\underline{x}) \in \mathbb{C}[\underline{x}] \setminus \{0\}$ such that $\delta(z) = a(\underline{x})$. To prove (2), it suffices to find an element $h(\underline{x}) \in \mathbb{C}[\underline{x}]$ such that $a(\underline{x}) = \underline{x}^{[2]}h(\underline{x})$, since

$$0 = \delta(P_q - c) = \underline{x}^{[2]}\delta(y) + a(\underline{x})2z(1 + \underline{x}^{[1]}q'(z^2)).$$

The equality above means that there exist polynomials $F, R \in \mathbb{C}^{[n+2]}$ such that

$$\underline{X}^{[2]}F(\underline{X}, Y, Z) + a(\underline{X})2Z(1 + \underline{X}^{[1]}q'(Z^2)) = R(\underline{X}, Y, Z)(\underline{X}^{[2]}Y + Z^2 + \underline{X}^{[1]}q(Z^2) - c).$$

From this, it follows that $a(\underline{X})$ and $R(\underline{X}, Y, Z)$ are both divisible by $\underline{X}^{[1]}$. Setting $a(\underline{X}) = \underline{X}^{[1]}\widetilde{a}(\underline{X})$ and $R(\underline{X}, Y, Z) = \underline{X}^{[1]}\widetilde{R}(\underline{X}, Y, Z)$, we obtain the equality

$$\underline{X}^{[1]}F(\underline{X}, Y, Z) + \widetilde{a}(\underline{X})2Z(1 + \underline{X}^{[1]}q'(Z^2)) = \widetilde{R}(\underline{X}, Y, Z)(\underline{X}^{[2]}Y + Z^2 + \underline{X}^{[1]}q(Z^2) - c).$$

The latter implies that $\widetilde{a}(\underline{X})$ is divisible by $\underline{X}^{[1]}$. Thus, $a(\underline{x}) = \underline{x}^{[2]}h(\underline{x})$ for an element $h(\underline{x}) \in \mathbb{C}[\underline{x}]$. This completes the proof. □

We are now in position to classify all hypersurfaces in $\mathbb{C}^{n+2}$ given by an equation of the form $P_q = c$. They have exactly four isomorphism classes. Each of them is given by one of the following varieties.

Notation 2.4. We denote by $V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1}$ the hypersurfaces in $\mathbb{C}^{n+2}$ defined by the equation $\underline{x}^{[2]}y + z^2 = 0$, $\underline{x}^{[2]}y + z^2 - 1 = 0$, $\underline{x}^{[2]}y + z^2 + \underline{x}^{[1]} = 0$ and $\underline{x}^{[2]}y + z^2 + \underline{x}^{[1]} - 1 = 0$, respectively.

These varieties are pairwise non-isomorphic and we have the following result, which was already proved in [17] for the case $n = 1$.

Proposition 2.5. *Let* $q \in \mathbb{C}^{[1]}$, $c \in \mathbb{C}$, *and let* $P_q = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}q(z^2) \in \mathbb{C}[\underline{x}, y, z]$ *as in Notation 2.1. Then, the variety* $V(P_q - c)$ *is isomorphic to:*

(1) $V_{0,0}$ *if an only if* $c = 0$ *and* $q(c) = 0$;

(2) $V_{1,0}$ *if an only if* $c = 0$ *and* $q(c) \neq 0$;
(3) $V_{0,1}$ *if an only if* $c \neq 0$ *and* $q(c) = 0$;
(4) $V_{1,1}$ *if an only if* $c \neq 0$ *and* $q(c) \neq 0$.

Proof. By Lemma 2.2, the variety $V(P_q - c)$ is isomorphic to the hypersurface of equation

$$\underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}q(c) - c = 0.$$

The "if parts" of the proposition follow then easily.

In order to prove that $V_{0,0}, V_{0,1}, V_{1,0}$ and $V_{1,1}$ are non-isomorphic, we consider two polynomials $q_1, q_2 \in \mathbb{C}^{[1]}$ and two constants $c_1, c_2 \in \mathbb{C}$. For $j = 1, 2$, let B_j denotes the ring $B_j = \mathbb{C}[\underline{x}, y, z]/(P_{q_j} - c_j)$ and let x_{i_j}, y_j, z_j denote the images of x_i, y, z in B_j. We also denote by $\mathbb{C}\left[\underline{x}_j\right]$ the ring $\mathbb{C}[x_{1_j}, \ldots, x_{n_j}]$. Suppose now that $\varphi : B_1 \to B_2$ is an isomorphism.

Let $\delta \in \mathrm{LND}(B_1) \setminus \{0\}$ be a nonzero locally derivation on B_1. Then, $\widetilde{\delta} = \varphi \circ \delta \circ \varphi^{-1}$ is a nonzero locally derivation on B_2 and we have $\mathrm{Ker}(\widetilde{\delta}) = \varphi(\mathrm{Ker}(\delta))$ and $\mathrm{Ker}((\widetilde{\delta})^2) = \varphi(\mathrm{Ker}(\delta^2))$. By Proposition 2.3, we have $\mathrm{Ker}(\delta) = \mathbb{C}[\underline{x}_1]$ and $\mathrm{Ker}(\widetilde{\delta}) = \mathbb{C}[\underline{x}_2]$. Thus, φ restricts to an isomorphism between $\mathbb{C}[\underline{x}_1]$ and $\mathbb{C}[\underline{x}_2]$. Moreover $\varphi(z_1) \in \mathrm{Ker}((\widetilde{\delta})^2) = \mathbb{C}[\underline{x}_2]z_2 + \mathbb{C}[\underline{x}_2]$. Therefore, $\varphi(z_1) = \alpha(\underline{x}_2)z_2 + \beta(\underline{x}_2)$ for some polynomials α and β. Repeating the same argument with φ^{-1}, we obtain that $\varphi^{-1}(z_2) = a(\underline{x}_1)z_1 + b(\underline{x}_1)$ for some polynomials a and b. From this, we get that the elements $\alpha(\underline{x}_2) \in \mathbb{C}[\underline{x}_2]$ and $a(\underline{x}_1) \in \mathbb{C}[\underline{x}_1]$ are in fact invertible, thus nonzero constants.

If we take the derivation $\delta = \Delta$ (see Proposition 2.3), one checks that $\widetilde{\delta}(z_2) = \varphi(\Delta(az_1 + b(\underline{x}_1))) = a\varphi(\underline{x}_1^{[2]})$. Consequently, there exists, again by Proposition 2.3, a polynomial h such that $a\varphi(\underline{x}_1^{[2]}) = h(\underline{x}_2)\underline{x}_2^{[2]}$. Since $\varphi : \mathbb{C}[\underline{x}_1] \to \mathbb{C}[\underline{x}_2]$ is an isomorphism, this implies that there exist a bijection σ of the set $\{1, \ldots, n\}$ and nonzero constants $\lambda_i \in \mathbb{C}^*$ such that $\varphi(x_i) = \lambda_i x_{\sigma(i)}$ for all $1 \leq i \leq n$.

Let $\lambda = \prod_{i=1}^{n} \lambda_i$ and suppose from now on that q_1 and q_2 are constant. Since $\lambda^2\underline{x}_2^{[2]}\varphi(y_1) + (\alpha z_2 + \beta(\underline{x}_2))^2 + \lambda\underline{x}_2^{[1]}q_1 - c_1 = \varphi(\underline{x}_1^{[2]}y_1 + z_1^2 + \underline{x}_1^{[1]}q_1 - c_1) = 0$ in B_2, there exist polynomials $F, A \in \mathbb{C}^{[n+2]}$ such that

$$\lambda^2\underline{x}^{[2]}F(\underline{x}, y, z) + (\alpha z + \beta(\underline{x}))^2 + \lambda q_1\underline{x}^{[1]} - c_1 = A(\underline{x}, y, z)(\underline{x}^{[2]}y + z^2 + q_2\underline{x}^{[1]} - c_2).$$

Looking at this equality modulo $(\underline{x}^{[2]})$, it follows that $\beta(\underline{x})$ lies in the ideal of $\mathbb{C}[\underline{x}]$ generated by $\underline{x}^{[2]}$, and that $c_1 = A(\underline{0}, 0, 0)c_2$ and $\lambda q_1 = A(\underline{0}, 0, 0)q_2$. This shows that $V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1}$ are pairwise non-isomorphic and proves the proposition. □

Remark 2.6. Even if they are non-isomorphic, the varieties $V_{0,1}$ and $V_{1,1}$ are biholomorphic. Indeed, the analytic automorphism Ψ of $\mathbb{C}[\underline{x}, y, z]$ defined by $\Psi(x_i) = x_i$ for all $1 \leq i \leq n$,

$$\Psi(y) = \exp(-\underline{x}^{[1]})y - \frac{\exp(-\underline{x}^{[1]}) - 1 + \underline{x}^{[1]}}{\underline{x}^{[2]}} \quad \text{and} \quad \Psi(z) = \exp(-\frac{1}{2}\underline{x}^{[1]})z,$$

satisfies $\Psi(\underline{x}^{[2]}y + z^2 + \underline{x}^{[1]} - 1) = \exp(-\underline{x}^{[1]})(\underline{x}^{[2]}y + z^2 - 1)$. The case $n = 1$ is due to Freudenburg and Moser-Jauslin [8]. Note that Jelonek [11] has recently constructed other examples, in every dimension $d \geq 2$, of algebraically non-isomorphic varieties that are holomorphically isomorphic.

3. Stable equivalence

In this paper, we will consider two notions of equivalence.

Definition 3.1.

(1) Two hypersurfaces $H_1, H_2 \subset \mathbb{C}^n$ are said to be *equivalent* if there exists a polynomial automorphism Φ of $\mathbb{C}^n$ such that $\Phi(H_1) = H_2$.
(2) Two polynomials $P, Q \in \mathbb{C}^{[n]}$ are said to be *equivalent* if there exists a polynomial automorphism Φ of $\mathbb{C}^n$ such that $\Phi^*(P) = Q$.

These two notions are of course closely related, the zero-sets $V(P)$ and $V(Q)$ of irreducible polynomials $P, Q \in \mathbb{C}^{[n]}$ being equivalent hypersurfaces in $\mathbb{C}^n$ if and only if there exists a nonzero constant $\mu \in \mathbb{C}^*$ such that P and μQ are equivalent polynomials in $\mathbb{C}^{[n]}$.

The next proposition gives the classification, up to equivalence, of all polynomials P_q (see Notation 2.1) and of their fibers $V(P_q - c)$. It is an easy generalization of results of [17] to the case $n \geq 2$.

Proposition 3.2. *Let $q_1, q_2 \in \mathbb{C}^{[1]}$ be two polynomials and $c_1, c_2 \in \mathbb{C}$ be two constants. For $i = 1, 2$, let $P_{q_i} = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}q_i(z^2) \in \mathbb{C}[\underline{x}, y, z]$ as in Notation 2.1. Then, the following hold.*

(1) *The polynomials $P_{q_1} - c_1$ and $P_{q_2} - c_2$ of $\mathbb{C}^{[n+2]}$ are equivalent if and only if $c_1 = c_2$ and there exists a nonzero constant $\lambda \in \mathbb{C}^*$ such that $q_2 = \lambda q_1$.*
(2) *The hypersurfaces $H_1 = V(P_{q_1} - c_1), H_2 = V(P_{q_2} - c_2) \subset \mathbb{C}^{n+2}$ are equivalent if and only if there exist two nonzero constants $\lambda, \mu \in \mathbb{C}^*$ such that $c_2 = \mu^{-1}c_1$ and such that the equality $q_2(t) = \lambda q_1(\mu t)$ holds in $\mathbb{C}[t]$.*

Proof. (1) Suppose that $P_{q_1} - c_1$ and $P_{q_2} - c_2$ are equivalent polynomials of $\mathbb{C}[\underline{x}, y, z]$ and let Φ be an automorphism of $\mathbb{C}[\underline{x}, y, z]$ such that $\Phi(P_{q_1} - c_1) = P_{q_2} - c_2$. The key of the proof is to show that $\Phi(\underline{x}^{[1]}) = \lambda \underline{x}^{[1]}$ for some constant $\lambda \in \mathbb{C}^*$. Afterwards, we can conclude exactly as in [17].

Remark that Φ induces, for every $c \in \mathbb{C}$, an isomorphism Φ_c between the rings $B_1 = \mathbb{C}[\underline{x}, y, z]/(P_{q_1} - c_1 - c)$ and $B_2 = \mathbb{C}[\underline{x}, y, z]/(P_{q_2} - c_2 - c)$. Therefore, as we have seen in the proof of Proposition 2.5, the element $\Phi_c(\underline{x}^{[1]})$ lies in the ideal $\underline{x}^{[1]}B_2$. Thus,

$$\Phi(\underline{x}^{[1]}) \in \bigcap_{c \in \mathbb{C}} \left(\underline{x}^{[1]}, P_{q_2} - c_2 - c\right) = \bigcap_{c \in \mathbb{C}} \left(\underline{x}^{[1]}, z^2 - c_2 - c\right) = \left(\underline{x}^{[1]}\right).$$

Since Φ is an automorphism, this implies that there exists a nonzero constant $\lambda \in \mathbb{C}^*$ such that $\Phi(\underline{x}^{[1]}) = \lambda \underline{x}^{[1]}$, as desired.

Now, since $\Phi(P_{q_1} - c_1 + \alpha \underline{x}^{[1]} - c) = P_{q_2} - c_2 + \alpha\lambda \underline{x}^{[1]} - c$, the varieties $V(P_{q_1+\alpha} - c_1 - c)$ and $V(P_{q_2+\alpha\lambda} - c_2 - c)$ are isomorphic for all $\alpha, c \in \mathbb{C}$. By Proposition 2.5, this implies that $c_1 = c_2$ and then that the zeros of the polynomials $q_1 + \alpha$ and $q_2 + \alpha\lambda$ are the same for all $\alpha \in \mathbb{C}$. Thus, $q_2 = \lambda q_1$.

Conversely, if $q_2 = \lambda q_1$ for some $\lambda \in \mathbb{C}^*$, it suffices to check that $\Phi(P_{q_1}) = P_{q_2}$, where Φ is the automorphism of $\mathbb{C}[\underline{x}, y, z]$ defined by $\Phi(x_1) = \lambda x_1$, $\Phi(x_i) = x_i$ for all $2 \leq i \leq n$, $\Phi(y) = \lambda^{-2}y$ and $\Phi(z) = z$. This proves the assertion (1).

(2) The hypersurfaces $H_1 = V(P_{q_1} - c_1)$ and $H_2 = V(P_{q_2} - c_2)$ are equivalent if and only if there exists a nonzero constant $\mu \in \mathbb{C}^*$ such that the polynomials $P_{q_1} - c_1$ and $\mu(P_{q_2} - c_2)$ are equivalent. Then, Assertion (2) follows from Assertion (1), noting that

$\mu(P_{q_2} - c_2)$ is equivalent to the polynomial $P_{\widetilde{q_2}} - \mu c_2$, where $\widetilde{q_2}$ denotes the element of $\mathbb{C}[t]$ defined by $\widetilde{q_2}(t) = q_2(\mu^{-1}t)$. Indeed, one checks that this equivalence is realized by the automorphism of $\mathbb{C}^{n+2}$ defined by $(x_1, x_2, \ldots, x_n, y, z) \mapsto (\mu^{-1}x_1, x_2, \ldots, x_n, \mu y, \epsilon z)$, where ϵ is any complex number such that $\epsilon^2 = \mu^{-1}$. □

Before we state the next result, let us recall the notion of *stable equivalence*.

Definition 3.3.

(1) Two hypersurfaces $H_1, H_2 \subset \mathbb{C}^n$ are said to be *stably equivalent* if there exists a $m \in \mathbb{N}$ such that $H_1 \times \mathbb{C}^m$ and $H_2 \times \mathbb{C}^m$ are equivalent hypersurfaces in $\mathbb{C}^{n+m}$.

(2) Two polynomials $P, Q \in \mathbb{C}^{[n]}$ are said to be *stably equivalent* if there exists a $m \in \mathbb{N}$ such that P and Q are equivalent polynomials of $\mathbb{C}^{[n+m]}$.

In this context, we have the following obvious generalization of Theorem 2.5' of [17].

Lemma 3.4. *For every $q \in \mathbb{C}^{[1]}$, the polynomials P_q and $P_{q(0)}$ are stably equivalent.*

Proof. The case $n = 1$ was proved in [17], where an explicit automorphism Φ of $\mathbb{C}[x, y, z, w]$, fixing x and satisfying $\Phi(x^2y + z^2 + xq(z^2)) = x^2y + z^2 + xq(0)$, is constructed. Since this automorphism fixes x, it suffices to replace formally x by $\underline{x}^{[1]}$ to get an automorphism of $\mathbb{C}[\underline{x}, y, z, w]$ which maps P_q onto $P_{q(0)}$. For the sake of completeness, let us give the formula.

Let $r \in \mathbb{C}[t]$ be the polynomial such that the equality $q(t) - q(0) = 2tr(t)$ holds. We let $\Phi(x_i) = x_i$ for all $1 \leq i \leq n$, $\Phi(z) = (1 - \underline{x}^{[1]}r(P_{q(0)}))z + \underline{x}^{[2]}w$ and $\Phi(w) = (1 + \underline{x}^{[1]}r(P_{q(0)}))w - (r(P_{q(0)}))^2z$. Note that $\Phi(z^2 + \underline{x}^{[1]}q(z^2)) \equiv z^2 + \underline{x}^{[1]}q(0) \mod (\underline{x}^{[2]})$. Therefore, we can choose $\Phi(y) \in \mathbb{C}[\underline{x}, y, z, w]$ such that $\Phi(P_q) = P_{q(0)}$. Doing so, we get an endomorphism (we will show that it is in fact an automorphism) Φ of $\mathbb{C}[\underline{x}, y, z, w]$ which maps P_q onto $P_{q(0)}$.

Similarly, we define an endomorphism Ψ of $\mathbb{C}[\underline{x}, y, z, w]$ such that $\Psi(P_{q(0)}) = P_q$ by posing $\Psi(x_i) = x_i$ for all $1 \leq i \leq n$, $\Psi(z) = (1 + \underline{x}^{[1]}r(P_q))z - \underline{x}^{[2]}w$ and $\Psi(w) = (1 - \underline{x}^{[1]}r(P_q))w + (r(P_q))^2z$.

Now, one checks that $\Phi \circ \Psi(z) = z$ and that $\Phi \circ \Psi(w) = w$. Moreover, since $\Phi \circ \Psi(P_{q(0)}) = P_{q(0)}$, we have $\underline{x}^{[2]}\Phi \circ \Psi(y) + z^2 + \underline{x}^{[1]}q(0) = \Phi \circ \Psi(P_{q(0)}) = P_{q(0)} = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}q(0)$. This implies that $\Phi \circ \Psi(y) = y$. Therefore, Ψ is the inverse morphism of Φ. This proves the lemma. □

Together with Propositions 2.5 and 3.2, Lemma 3.4 leads to many counterexamples to the "stable equivalence problem" of every dimension $d \geq 2$. Finally, let us emphasize two particular examples.

Example 3.5.

(1) The polynomials $P = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}(z^2 - 1) - 1$ and $Q = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}(z^2 - 2) - 1$ of $\mathbb{C}[\underline{x}, y, z]$ are stably equivalent, but the hypersurfaces $V(P)$ and $V(Q)$ in $\mathbb{C}^{n+2}$ are not equivalent. Indeed, they are even non-isomorphic varieties.

(2) The polynomials $Q_k = \underline{x}^{[2]}y + z^2 + \underline{x}^{[1]}(z^2 - 1)^k \in \mathbb{C}[\underline{x}, y, z]$ are stably equivalent for all $k \geq 1$, whereas the hypersurfaces $V(Q_k) \subset \mathbb{C}^{n+2}$ are pairwise non-equivalent. However, the varieties $V(Q_k - c)$ and $V(Q_{k'} - c)$ are isomorphic for all $k, k' \geq 1$ and every $c \in \mathbb{C}$.

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