ON THE MAXIMALITY OF THE TRIANGULAR SUBGROUP

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Abstract. We prove that the subgroup of triangular automorphisms of the complex affine n-space is maximal among all solvable subgroups of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ for every n. In particular, it is a Borel subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$, when the latter is viewed as an ind-group. In dimension two, we prove that the triangular subgroup is a maximal closed subgroup and that nevertheless, it is not maximal among all subgroups of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Given an automorphism f of $\mathbb{A}^2_{\mathbb{C}}$, we study the question whether the group generated by f and the triangular subgroup is equal to the whole group $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$.

1. Introduction

The main purpose of this paper is to study the Jonquières subgroup $\mathcal{B}_n$ of the group $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ of polynomial automorphisms of the complex affine n-space, i.e. its subgroup of triangular automorphisms. We will settle the titular question by providing three different answers, depending on to which properties the maximality condition is referring to.

Theorem 1. (1) *For every $n \geq 2$, the subgroup $\mathcal{B}_n$ is maximal among all solvable subgroups of* $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$.
(2) *The subgroup $\mathcal{B}_2$ is maximal among the closed subgroups of* $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$.
(3) *The subgroup $\mathcal{B}_2$ is not maximal among all subgroups of* $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$.

Recall that $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ is naturally an ind-group, i.e. an infinite dimensional algebraic group. It is thus equipped with the usual ind-topology (see Section 2 for the definitions). In particular, since $\mathcal{B}_n$ is a closed connected solvable subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$, the first statement of Theorem 1 can be interpreted as follows:

Corollary 2. *The group $\mathcal{B}_n$ is a Borel subgroup of* $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$.

This generalizes a remark of Berest, Eshmatov and Eshmatov [BEE16] stating that triangular automorphisms of $\mathbb{A}^2_{\mathbb{C}}$, of Jacobian determinant 1, form a Borel subgroup (i.e. a maximal connected solvable subgroup) of the group $\mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$ of polynomial automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ of Jacobian determinant 1. Actually, the proofs in [BEE16] also imply Corollary 2 in the case $n = 2$. Nevertheless, since they are based on results of Lamy [Lam01], which use the Jung-van der Kulk-Nagata structure theorem for $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$, these arguments are specific to the dimension 2 and cannot be generalized to higher dimensions.

The Jonquières subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ is thus a good analogue of the subgroup of invertible upper triangular matrices, which is a Borel subgroup of the classical linear algebraic group $\mathrm{GL}_n(\mathbb{C})$. Moreover, Berest, Eshmatov and Eshmatov strengthen this

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analogy when $n = 2$ by proving that $\mathcal{B}_2$ is, up to conjugacy, the only Borel subgroup of $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}})$. On the other hand, it is well known that there exist, if $n \geq 3$, algebraic additive group actions on $\mathbb{A}^n_{\mathbb{C}}$ that cannot be triangularized [Bas84, Pop87]. Therefore, we ask the following problem.

Problem 3. *Show that Borel subgroups of* $\operatorname{Aut}(\mathbb{A}^n_{\mathbb{C}})$ *are not all conjugate* ($n \geq 3$).

This problem turns out to be closely related to the question of the boundedness of the derived length of solvable subgroups of $\operatorname{Aut}(\mathbb{A}^n_{\mathbb{C}})$. We give such a bound when $n = 2$. More precisely, the maximal derived length of a solvable subgroup of $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is equal to 5 (see Proposition 3.14). As a consequence, we prove that the group $\operatorname{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$ of automorphisms of $\mathbb{A}^3$ fixing the last coordinate admits non-conjugate Borel subgroups (see Corollary 3.22). Note that such a phenomenon has already been pointed out in [BEE16].

The paper is organized as follows. Section 1 is the present introduction. In Section 2, we recall the definitions of ind-varieties and ind-groups given by Shafarevich and explain how the automorphism group of the affine n-space may be endowed with the structure of an ind-group.

In Section 3, we prove the first two statements of Theorem 1 and discuss the question, whether the ind-group $\operatorname{Aut}(\mathbb{A}^n_{\mathbb{C}})$ does admit non-conjugate Borel subgroups. We then study the group of all automorphisms of $\mathbb{A}^3_{\mathbb{C}}$ fixing the last variable, proving that it admits non-conjugate Borel subgroups. In the last part of Section 3, we give examples of maximal closed subgroups of $\operatorname{Aut}(\mathbb{A}^n_{\mathbb{C}})$.

Finally, we consider $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}})$ as an "abstract" group in Section 4. We show that triangular automorphisms do not form a maximal subgroup of $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}})$. More precisely, after defining the affine length of an automorphism in Definition 4.1, we prove the following statement:

Theorem 4. *For any field* $\mathbf{k}$*, the two following assertions hold.*

(1) *If the affine length of an automorphism* $f \in \operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$ *is at least* 1 *(i.e. f is not triangular) and at most* 4*, then the group generated by* $\mathcal{B}_2$ *and* f *satisfies*
$$\langle \mathcal{B}_2, f \rangle = \operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}}).$$
(2) *There exists an automorphism* $f \in \operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$ *of affine length* 5 *such that the group* $\langle \mathcal{B}_2, f \rangle$ *is strictly included into* $\operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$.

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2. Preliminaries: the ind-group of polynomial automorphisms

In [Sha66, Sha81], Shafarevich introduced the notions of ind-varieties and ind-groups, and explained how to endow the group of polynomial automorphisms of the affine n-space with the structure of an ind-group. Since these two papers are well-known to contain several inaccuracies, we now recall the definitions from Shafarevich and describe the ind-group structure of the automorphism group of the affine n-space.

For simplicity, we assume in this section that $\mathbf{k}$ is an algebraically closed field.

2.1. **Ind-varieties and ind-groups.** We first define the category of infinite dimensional algebraic varieties (*ind-varieties* for short).

Definition 2.1 (Shafarevich, 1966)**.**

(1) An ind-variety V (over $\mathbf{k}$) is a set together with an ascending filtration $V_{\leq 0} \subseteq V_{\leq 1} \subseteq V_{\leq 2} \subseteq \cdots \subseteq V$ such that the following holds:
  (a) $V = \bigcup_d V_{\leq d}$.
  (b) Each $V_{\leq d}$ has the structure of an algebraic variety (over $\mathbf{k}$).
  (c) Each $V_{\leq d}$ is Zariski closed in $V_{\leq d+1}$.
(2) A morphism of ind-varieties (or ind-morphism) is a map $\varphi \colon V \to W$ between two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ such that there exists, for every d, an e for which $\varphi(V_{\leq d}) \subseteq W_{\leq e}$ and such that the induced map $V_{\leq d} \to W_{\leq e}$ is a morphism of varieties (over $\mathbf{k}$).

In particular, every ind-variety V is naturally equipped with the so-called ind-topology in which a subset $S \subseteq V$ is closed if and only if every subset $S_{\leq d} := S \cap V_{\leq d}$ is Zariski-closed in $V_{\leq d}$.

We remark that the product $V \times W$ of two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ has the structure of an ind-variety for the filtration $V \times W = \bigcup_d V_{\leq d} \times W_{\leq d}$.

Definition 2.2. An *ind-group* is a group G which is an ind-variety such that the multiplication $G \times G \to G$ and inversion $G \to G$ maps are morphisms of ind-varieties.

If G is an abstract group, we denote by $D(G) = D^1(G)$ its (first) derived subgroup. It is the subgroup generated by all commutators $[g,h] := ghg^{-1}h^{-1}$, $g,h \in G$. The n-th derived subgroup of G is then defined inductively by $D^n(G) = D^1(D^{n-1}(G))$ for $n \geq 1$, where by definition $D^0(G) = G$. A group G is called *solvable* if $D^n(G) = \{1\}$ for some integer $n \geq 0$. Furthermore, the smallest such integer n is called the *derived length* of G.

For later use, we state (and prove) the following results which are well-known for algebraic groups and which extend straightforwardly to ind-groups.

Lemma 2.3. *Let H be a subgroup of an ind-group G. Then, the following assertions hold.*

(1) *The closure $\overline{H}$ of H is again a subgroup of G.*
(2) *We have $D(\overline{H}) \subseteq \overline{D(H)}$.*
(3) *If H is solvable, then $\overline{H}$ is solvable too.*

Proof. (1). The proof for algebraic groups given in [Hum75, Proposition 7.4A, page 54] directly applies to ind-groups. This proof being very short, we give it here. Inversion being a homeomorphism, we get $(\overline{H})^{-1} = \overline{H^{-1}} = \overline{H}$. Similarly, left translation by an element x of H being a homeomorphism, we get $x\overline{H} = \overline{xH} = \overline{H}$, i.e. $H\overline{H} \subseteq \overline{H}$. In turn, right translation by an element x of $\overline{H}$ being a homeomorphism, we get $\overline{H}x = \overline{Hx} \subseteq \overline{H\overline{H}} \subseteq \overline{\overline{H}} = \overline{H}$. This says that $\overline{H}$ is a subgroup.

(2). Fix an element y of H. The map $\varphi \colon G \to G$, $x \mapsto [x,y] = xyx^{-1}y^{-1}$ being an ind-morphism, it is in particular continuous. Since H is obviously contained in $\varphi^{-1}(\overline{D(H)})$, we get $\overline{H} \subseteq \varphi^{-1}(\overline{D(H)})$. Consequently, we have proven that

$$\forall x \in \overline{H},\ \forall y \in H,\ [x,y] \in \overline{D(H)}.$$

In turn (and analogously), for each fixed element x of $\overline{H}$, the map $\psi\colon G \to G$, $y \mapsto [x,y]$ is continuous. Since H is included into $\psi^{-1}(\overline{D(H)})$, we get $\overline{H} \subseteq \psi^{-1}(\overline{D(H)})$ and thus

$$\forall x, y \in \overline{H},\ [x,y] \in \overline{D(H)}.$$

This implies the desired inclusion.

(3). If H is solvable, it admits a sequence of subgroups such that

$$H = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_n = \{1\} \quad \text{and} \quad D(H_i) \subseteq H_{i+1} \text{ for each } i.$$

This yields $\overline{H} = \overline{H}_0 \supseteq \overline{H}_1 \supseteq \cdots \supseteq \overline{H}_n = \{1\}$ and by (2) we get $D(\overline{H}_i) \subseteq \overline{D(H_i)} \subseteq \overline{H}_{i+1}$ for each i. ☐

2.2. **Automorphisms of the affine n-space.** As usual, given an endomorphism $f \in \mathrm{End}(\mathbb{A}^n_{\mathbf{k}})$, we denote by f^* the corresponding endomorphism of the algebra of regular functions $\mathcal{O}(\mathbb{A}^n_{\mathbf{k}}) = \mathbf{k}[x_1,\ldots,x_n]$. Note that every endomorphism $f \in \mathrm{End}(\mathbb{A}^n_{\mathbf{k}})$ is uniquely determined by the polynomials $f_i = f^*(x_i)$, $1 \le i \le n$.

In the sequel, we identify the set $\mathcal{E}_n(\mathbf{k}) := \mathrm{End}(\mathbb{A}^n_{\mathbf{k}})$ with $(\mathbf{k}[x_1,\ldots,x_n])^n$. We thus simply denote by $f = (f_1,\ldots,f_n)$ the element of $\mathcal{E}_n(\mathbf{k})$ whose corresponding endomorphism f^* is given by

$$f^*\colon \mathcal{O}(\mathbb{A}^n_{\mathbf{k}}) \to \mathcal{O}(\mathbb{A}^n_{\mathbf{k}}), \quad P(x_1,\ldots,x_n) \mapsto P \circ f = P(f_1,\ldots,f_n).$$

The composition $g \circ f$ of two endomorphisms $f = (f_1,\ldots,f_n)$ and $g = (g_1,\ldots,g_n)$ is equal to

$$g \circ f = (g_1(f_1,\ldots,f_n),\ldots,g_n(f_1,\ldots,f_n)).$$

Note that for each nonnegative integer d, the following set is naturally an affine space (and therefore an algebraic variety!).

$$\mathbf{k}[x_1,\ldots,x_n]_{\le d} := \{P \in \mathbf{k}[x_1,\ldots,x_n],\ \deg P \le d\}.$$

If $f = (f_1,\ldots,f_n) \in \mathcal{E}_n(\mathbf{k})$, we set $\deg f := \max_i\{\deg f_i\}$ and define

$$\mathcal{E}_n(\mathbf{k})_{\le d} := \{f \in \mathcal{E}_n(\mathbf{k}),\ \deg f \le d\}.$$

The equality $\mathcal{E}_n(\mathbf{k})_{\le d} = (\mathbf{k}[x_1,\ldots,x_n]_{\le d})^n$ shows that $\mathcal{E}_n(\mathbf{k})_{\le d}$ is naturally an affine space. Moreover, the filtration $\mathcal{E}_n(\mathbf{k}) = \bigcup_d \mathcal{E}_n(\mathbf{k})_{\le d}$ defines a structure of ind-variety on $\mathcal{E}_n(\mathbf{k})$.

We denote by $\mathcal{G}_n(\mathbf{k}) = \mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ the automorphism group of $\mathbb{A}^n_{\mathbf{k}}$. The next result allows us to endow $\mathcal{G}_n(\mathbf{k})$ with the structure of an ind-variety.

Lemma 2.4. *Denote by $\mathcal{C}_n(\mathbf{k})$, resp. $\mathcal{J}_n(\mathbf{k})$, the set of elements f in $\mathcal{E}_n(\mathbf{k})$ whose Jacobian determinant* $\mathrm{Jac}(f)$ *is a constant, resp. a nonzero constant. Then, the following assertions hold:*

(1) *The set $\mathcal{C}_n(\mathbf{k})$ is closed in $\mathcal{E}_n(\mathbf{k})$.*
(2) *The set $\mathcal{J}_n(\mathbf{k})$ is open in $\mathcal{C}_n(\mathbf{k})$.*
(3) *The set $\mathcal{G}_n(\mathbf{k})$ is closed in $\mathcal{J}_n(\mathbf{k})$.*

Proof. (1). Since $\deg(\mathrm{Jac}(f)) \le n(\deg(f)-1)$, the map $\mathrm{Jac}\colon \mathcal{E}_n(\mathbf{k}) \to \mathbf{k}[x_1,\ldots,x_n]$ is an ind-morphism. By definition, $\mathcal{C}_n(\mathbf{k})$ is the preimage of the set $\mathbf{k}$ which is closed in $\mathbf{k}[x_1,\ldots,x_n]$.

(2). The Jacobian morphism induces a morphism $\varphi\colon \mathcal{C}_n(\mathbf{k}) \to \mathbf{k}$, $f \mapsto \mathrm{Jac}(f)$. By definition, $\mathcal{J}_n(\mathbf{k})$ is the preimage of the set $\mathbf{k}^*$ which is open in $\mathbf{k}$.

(3). Set $\mathcal{J}_{n,0} := \{f \in \mathcal{J}_n(\mathbf{k}),\ f(0) = 0\}$. Every element $f \in \mathcal{J}_{n,0}$ admits a formal inverse for the composition (see e.g. [vdE00, Theorem 1.1.2]), i.e. a formal power series $g = \sum_{d\geq 1} g_d$, where each $g_d = (g_{d,1}, \dots, g_{d,n})$ is a d-homogeneous element of $\mathcal{E}_n(\mathbf{k})$, meaning that $g_{d,1}, \dots, g_{d,n}$ are d-homogeneous polynomials in $\mathbf{k}[x_1, \dots, x_n]$ such that

$$f \circ g = g \circ f = (x_1, \dots, x_n) \quad \text{(as formal power series)}.$$

Furthermore, for each d, the map $\psi_d \colon \mathcal{J}_{n,0} \to \mathcal{E}_n(\mathbf{k})$ sending f onto g_d is a morphism because each coefficient of every component of g_d can be expressed as a polynomial in the coefficients of the components of f and in the inverse $(\operatorname{Jac} f)^{-1}$ of the polynomial $\operatorname{Jac} f$. Recall furthermore (see [BCW82, Theorem 1.5]) that every automorphism $f \in \mathcal{G}_n(\mathbf{k})$ satisfies

$$\deg(f^{-1}) \leq (\deg f)^{n-1}. \tag{5}$$

Therefore, an element $f \in \mathcal{J}_n(\mathbf{k})_{\leq d}$ is an automorphism if and if $\tilde{f} := f - f(0)$ is an automorphism. This amounts to saying that f is an automorphism if and only if $\psi_e(\tilde{f}) = 0$ for all integers $e > d^{n-1}$. These conditions being closed, we have proven that $\mathcal{G}_n(\mathbf{k})_{\leq d}$ is closed in $\mathcal{J}_n(\mathbf{k})_{\leq d}$ for each d, i.e. that $\mathcal{G}_n(\mathbf{k})$ is closed in $\mathcal{J}_n(\mathbf{k})$. Note that when the field $\mathbf{k}$ has characteristic zero, the Jacobian conjecture (see for example [BCW82, vdE00]) asserts that the equality $\mathcal{G}_n(\mathbf{k}) = \mathcal{J}_n(\mathbf{k})$ actually holds. □

Since the multiplication $\mathcal{G}_n(\mathbf{k}) \times \mathcal{G}_n(\mathbf{k}) \to \mathcal{G}_n(\mathbf{k})$ and inversion $\mathcal{G}_n(\mathbf{k}) \to \mathcal{G}_n(\mathbf{k})$ maps are morphisms (for the inversion, this again relies on the fundamental inequality (5)), we obtain that $\mathcal{G}_n(\mathbf{k})$ is an ind-group.

3. Borel subgroups

Throughout this section, we work over the field $\mathbf{k} = \mathbb{C}$ of complex numbers.

Note that the affine subgroup

$$\mathcal{A}_n = \{f = (f_1, \dots, f_n) \in \mathcal{G}_n(\mathbb{C}) \mid \deg(f_i) = 1 \text{ for all } i = 1 \dots n\}$$

and the Jonquières (or triangular) subgroup

$$\begin{aligned} \mathcal{B}_n &= \{f = (f_1, \dots, f_n) \in \mathcal{G}_n(\mathbb{C}) \mid \forall\, i,\ f_i = a_i x_i + p_i,\ a_i \in \mathbb{C}^*,\ p_i \in \mathbb{C}[x_{i+1}, \dots, x_n]\} \\ &= \{f = (f_1, \dots, f_n) \in \mathcal{G}_n(\mathbb{C}) \mid \forall\, i,\ f_i \in \mathbb{C}[x_i, \dots, x_n]\} \end{aligned}$$

are both closed in $\mathcal{G}_n(\mathbb{C})$.

It is well known that the group $\mathcal{G}_n(\mathbb{C})$ is connected (see e.g. [Sha81, proof of Lemma 4], [Kal92, Proposition 2] or [Pop14, Theorem 6]). The same is true for $\mathcal{B}_n$.

Lemma 3.1. *The groups $\mathcal{G}_n(\mathbb{C}) = \operatorname{Aut}(\mathbb{A}^n_{\mathbb{C}})$ and $\mathcal{B}_n$ are connected.*

Proof. We say that a variety V is curve-connected if for all points $x, y \in V$, there exists a morphism $\varphi \colon C \to V$, where C is a connected curve (not necessarily irreducible) such that x and y both belong to the image of φ. The same definition applies to ind-varieties.

We prove that $\mathcal{G}_n(\mathbb{C})$ and $\mathcal{B}_n$ are curve-connected. Let f be an element in $\mathcal{G}_n(\mathbb{C})$. We first consider the morphism $\alpha \colon \mathbb{A}^1_{\mathbb{C}} \to \mathcal{G}_n(\mathbb{C})$ defined by

$$\alpha(t) = f - t f(0, \dots, 0)$$

which is contained in $\mathcal{B}_n$ if f is triangular. Note that $\alpha(0) = f$ and that the automorphism $\tilde{f} := \alpha(1)$ fixes the origin of $\mathbb{A}^n_{\mathbb{C}}$.

Therefore the morphism $\beta\colon \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathcal{G}_n(\mathbb{C})$, $t \mapsto (t^{-1} \cdot \mathrm{id}_{\mathbb{A}^n_{\mathbb{C}}}) \circ \tilde{f} \circ (t \cdot \mathrm{id}_{\mathbb{A}^n_{\mathbb{C}}})$ extends to a morphism $\beta\colon \mathbb{A}^1_{\mathbb{C}} \to \mathcal{G}_n(\mathbb{C})$ (with values in $\mathcal{B}_n$ if f, thus $\tilde{f}$, is triangular) such that $\beta(1) = \tilde{f}$ and such that $\beta(0)$ is a linear map, namely the linear part of $\tilde{f}$. This concludes the proof since $\mathrm{GL}_n(\mathbb{C})$ (resp. the set of all invertible upper triangular matrices) is curve-connected. □

Recall that the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$ is solvable and has derived length $\lceil \log_2(n) \rceil + 1$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number x (see e.g. [Weh73, page 16]). In contrast, we have the following result.

Lemma 3.2. *The group $\mathcal{B}_n$ is solvable of derived length $n + 1$.*

Proof. For each integer $k \in \{0, \dots, n\}$, denote by U_k the subgroup of $\mathcal{B}_n$ whose elements are of the form $f = (f_1, \dots, f_n)$ where $f_i = x_i$ for all $i > k$ and $f_i = x_i + p_i$ with $p_i \in \mathbb{C}[x_{i+1}, \dots, x_n]$ for all $i \leq k$. We will prove $D(\mathcal{B}_n) = U_n$ and $D^j(U_n) = U_{n-j}$ for all $j \in \{0, \dots, n\}$.

For this, we consider the dilatation $d(j, \lambda_j)$ and the elementary automorphism $e(j, q_j)$ which are defined for every integer $j \in \{1, \dots, n\}$, every nonzero constant $\lambda_j \in \mathbb{C}^*$ and every polynomial $q_j \in \mathbb{C}[x_{j+1}, \dots, x_n]$ by

$$d(j, \lambda_j) = (g_1, \dots, g_n) \quad \text{and} \quad e(j, q_j) = (h_1, \dots, h_n),$$

where $g_j = \lambda_j x_j$, $h_j = x_j + q_j$ and $g_i = h_i = x_i$ for $i \neq j$. Note that an element $f \in U_k$ as above is equal to

$$f = e(k, p_k) \circ \cdots \circ e(2, p_2) \circ e(1, p_1).$$

In particular, this tells us that U_k is generated by the elements $e(j, q_j)$, $j \leq k$, $q_j \in \mathbb{C}[x_{j+1}, \dots, x_n]$.

The inclusion $D(\mathcal{B}_n) \subseteq U_n$ is straightforward and left to the reader. The converse inclusion $U_n \subseteq D(\mathcal{B}_n)$ follows from the equality

$$[e(j, q_j), d(j, \lambda_j)] = e(j, (1 - \lambda_j) q_j).$$

Finally, we prove $D^j(U_n) = U_{n-j}$ by proving that the equality $D(U_{k+1}) = U_k$ holds for all $k \in \{0, \dots, n-1\}$. The inclusion $D(U_{k+1}) \subseteq U_k$ is straightforward and left to the reader. To prove the converse inclusion, let us introduce the map $\Delta_i\colon \mathbb{C}[x_i, \dots, x_n] \to \mathbb{C}[x_i, \dots, x_n]$, $q \mapsto q(x_i, \dots, x_n) - q(x_i - 1, x_{i+1}, \dots, x_n)$. Note that Δ_i is surjective and that

$$[e(j, q_j), e(j+1, 1)] = e(j, \Delta_{j+1}(q_j))$$

for all $j \in \{1, \dots, n-1\}$ and all $q_j \in \mathbb{C}[x_{j+1}, \dots, x_n]$. This implies $U_k \subseteq D(U_{k+1})$ and concludes the proof. □

3.1. **Triangular automorphisms form a Borel subgroup.** In this section, we prove the first two statements of the theorem 1 from the introduction. For this, we need the following result.

Proposition 3.3. *Let $n \geq 2$ be an integer. If a closed subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ strictly contains $\mathcal{B}_n$, then it also contains at least one linear automorphism that is not triangular.*

Proof. Let H be a closed subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ strictly containing $\mathcal{B}_n$. We first prove that H contains an automorphism whose linear part is not triangular. Let $f = (f_1, \dots, f_n)$ be an element in $H \setminus \mathcal{B}_n$. Then, there exists at least one component f_i of f that depends on an indeterminate x_j with $j < i$, i.e. such that $\frac{\partial f_i}{\partial x_j} \neq 0$. Now, choose $c = (c_1, \dots, c_n) \in \mathbb{A}^n_{\mathbb{C}}$ such that $\frac{\partial f_i}{\partial x_j}(c) \neq 0$ and consider the translation $t_c := (x_1 + c_1, \dots, x_n + c_n) \in \mathcal{B}_n$. Since

$$f_i(x + c) = f_i(c) + \sum_k \frac{\partial f_i}{\partial x_k}(c) x_k + \text{(terms of higher order)},$$

the linear part l of $f \circ t_c$ is not triangular because it corresponds to the (non-triangular) invertible matrix $\left(\frac{\partial f_i}{\partial x_k}(c)\right)_{ik}$. Composing on the left hand side by another translation t', we obtain an element $g := t' \circ f \circ t \in H$ which fixes the origin of $\mathbb{A}^n_{\mathbb{C}}$ and whose linear part is again l.

For every $\varepsilon \in \mathbb{C}^*$, set $h_\varepsilon := (\varepsilon x_1, \dots, \varepsilon x_n) \in \mathcal{B}_n$. We can finally conclude by noting that

$$\lim_{\varepsilon \to 0} h_\varepsilon^{-1} \circ g \circ h_\varepsilon = l \in H,$$

where the limit means that the ind-morphism $\varphi\colon \mathbb{C}^* \to \mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$, $\varepsilon \mapsto h_\varepsilon^{-1} \circ g \circ h_\varepsilon$ extends to a morphism $\psi\colon \mathbb{C} \to \mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ such that $\psi(0) = l$. Since we have $\psi(\varepsilon) \in H$ for each $\varepsilon \in \mathbb{C}^*$, it is clear that $\psi(0)$ must also belong to H. Indeed, note that the set $\{\varepsilon \in \mathbb{C},\ \psi(\varepsilon) \in H\}$ is Zariski-closed in $\mathbb{C}$. □

Proposition 3.4. *Let $n \geq 2$ be an integer. Then, the Jonquières group $\mathcal{B}_n$ is maximal among all solvable subgroups of* $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$.

Proof. Suppose by contradiction that there exists a solvable subgroup H of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ that strictly contains $\mathcal{B}_n$. Up to replacing H by its closure $\overline{H}$ (see Lemma 2.3), we may assume that H is closed. By Proposition 3.3, the group $H \cap \mathcal{A}_n$ strictly contains $\mathcal{B}_n \cap \mathcal{A}_n$. But since $\mathcal{B}_n \cap \mathcal{A}_n$ is a Borel subgroup of $\mathcal{A}_n$, this prove that $H \cap \mathcal{A}_n$ is not solvable, thus that H itself is not solvable. Notice that we have used the fact that every Borel subgroup of a connected linear algebraic group is a maximal solvable subgroup. Indeed, every parabolic subgroup (i.e. a subgroup containing a Borel subgroup) of a connected linear algebraic group is necessarily closed and connected. See e.g. [Hum75, Corollary B of Theorem (23.1), page 143]. □

In dimension two, we establish another maximality property of the triangular subgroup which is actually stronger than the above one (see Remark 3.7 below).

Proposition 3.5. *The Jonquières group $\mathcal{B}_2$ is maximal among the closed subgroups of* $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$.

Proof. Let H be a closed subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ strictly containing $\mathcal{B}_2$. By Proposition 3.3 above, H contains a linear automorphism which is not triangular. This implies that H contains all linear automorphisms, hence $\mathcal{A}_2$, and it is therefore equal to $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Recall indeed that the subgroup $B_2 = \mathcal{B}_2 \cap \mathrm{GL}_2(\mathbb{C})$ of invertible upper triangular matrices is a maximal subgroup of $\mathrm{GL}_2(\mathbb{C})$, since the Bruhat decomposition expresses $\mathrm{GL}_2(\mathbb{C})$ as the disjoint union of two double cosets of B_2, which are namely B_2 and $B_2 \circ f \circ B_2$, where f is any element of $\mathrm{GL}_2(\mathbb{C}) \setminus B_2$. □

Remark 3.6. Proposition 3.5 can not be generalized to higher dimension, since $\mathcal{B}_n$ is strictly contained into the (closed) subgroup of automorphisms of the form $f = (f_1, \dots, f_n)$ such that $f_n = a_n x_n + b_n$ for some $a_n, b_n \in \mathbb{C}$ with $a_n \neq 0$.

Remark 3.7. Proposition 3.5 implies Proposition 3.4 for $n = 2$. Indeed, suppose that $\mathcal{B}_2$ is strictly included into some solvable subgroup H of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Up to replacing H by $\overline{H}$ (see Lemma 2.3), we may further assume that H is closed. By Proposition 3.5, we would thus get that $H = \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. But this is a contradiction because the group $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is obviously not solvable, since it contains the linear group $\mathrm{GL}(2, \mathbb{C})$ which is not solvable.

By Proposition 3.4, we can say that the triangular group $\mathcal{B}_n$ is a Borel subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$. This was already observed, in the case $n = 2$ only, by Berest, Eshmatov and Eshmatov in the nice paper [BEE16] in which they obtained the following strong results. (In [BEE16], these results are stated for the group $\mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$ of polynomial automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ of Jacobian determinant 1, but all the proofs remain valid for $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$.)

Theorem 3.8 ([BEE16])**.** (1) *All Borel subgroups of* $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ *are conjugate to* $\mathcal{B}_2$.
(2) *Every connected solvable subgroup of* $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ *is conjugate to a subgroup of* $\mathcal{B}_2$.

Recall that there exist, for every $n \geq 3$, connected solvable subgroups of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ that are not conjugate to subgroups of $\mathcal{B}_n$ [Bas84, Pop87]. Hence, the second statement of the above theorem does not hold for $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$, $n \geq 3$. Similarly, we believe that not all Borel subgroups of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ are conjugate to $\mathcal{B}_n$ if $n \geq 3$. This would be clearly the case, if we knew that the following question has a positive answer.

Question 3.9. *Is every connected solvable subgroup of* $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$, $n \geq 3$, *contained into a maximal connected solvable subgroup?*

The natural strategy to attack the above question would be to apply Zorn's lemma, as we do in the proof of the following general proposition.

Proposition 3.10. *Let G be a group endowed with a topology. Suppose that there exists an integer $c > 0$ such that every solvable subgroup of G is of derived length at most c. Then, every solvable (resp. connected solvable) subgroup of G is contained into a maximal solvable (resp. maximal connected solvable) subgroup.*

Proof. Let H be a solvable (resp. connected solvable) subgroup of G. Denote by $\mathcal{F}$ the set of solvable (resp. connected solvable) subgroups of G that contain H. Our hypothesis, on the existence of the bound c, implies that the poset $(\mathcal{F}, \subseteq)$ is inductive. Indeed, if $(H_i)_{i \in I}$ is a chain in $\mathcal{F}$, i.e. a totally ordered family of $\mathcal{F}$, then the group $\bigcup_i H_i$ is solvable, because we have that

$$D^j\Big(\bigcup_i H_i\Big) = \bigcup_i D^j(H_i)$$

for each integer $j \geq 0$. Moreover, if all H_i are connected, then so is their union. Thus, $\mathcal{F}$ is inductive and we can conclude by Zorn's lemma. □

Remark 3.11. Proposition 3.10 does not require any compatibility conditions between the group structure and the topology on G. Let us moreover recall that an algebraic group (and all the more an ind-group) is in general not a topological group.

We are now left with another concrete question.

Definition 3.12. Let G be a group. We set

$$\psi(G) := \sup\{l(H) \mid H \text{ is a solvable subgroup of } G\} \in \mathbb{N} \cup \{+\infty\},$$

where $l(H)$ denotes the derived length of H.

Question 3.13. *Is $\psi(\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}}))$ finite?*

Recall that $\psi(\mathrm{GL}(n,\mathbb{C}))$ is finite. This classical result has been first established in 1937 by Zassenhaus [Zas37, Satz 7] (see also [Mal56]). More recently, Martelo and Ribón have proved in [MR14] that $\psi\big((\mathcal{O}_{\mathrm{ana}}(\mathbb{C}^n),0)\big) < +\infty$, where $(\mathcal{O}_{\mathrm{ana}}(\mathbb{C}^n),0)$ denotes the group of germs of analytic diffeomorphisms defined in a neighbourhood of the origin of $\mathbb{C}^n$.

Our next result answers Question 3.13 in the case $n = 2$.

Proposition 3.14. *We have $\psi(\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})) = 5$.*

Proof. The proof relies on a precise description of all subgroups of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$, due to Lamy, that we will recall below. Using this description, the equality $\psi(\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})) = 5$ directly follows from the equality $\psi(\mathcal{A}_2) = 5$ that we will establish in the next section (see Proposition 3.16). The description of all subgroups of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ given by Lamy uses the amalgamated structure of this group, generally known as the theorem of Jung, van der Kulk and Nagata: The group $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is the amalgamated product of its subgroups $\mathcal{A}_2$ and $\mathcal{B}_2$ over their intersection

$$\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}}) = \mathcal{A}_2 *_{\mathcal{A}_2\cap\mathcal{B}_2} \mathcal{B}_2.$$

In the discussion below, we will use the Bass-Serre tree associated to this amalgamated structure. We refer the reader to [Ser03] for details on Bass-Serre trees in full generality and to [Lam01] for details on the particular tree associated to the above amalgamated structure. That latter tree is the tree whose vertices are the left cosets $g\circ\mathcal{A}_2$ and $h\circ\mathcal{B}_2$, $g,h \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Two vertices $g\circ\mathcal{A}_2$ and $h\circ\mathcal{B}_2$ are related by an edge if and only if there exists an element $k \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ such that $g\circ\mathcal{A}_2 = k\circ\mathcal{A}_2$ and $h\circ\mathcal{B}_2 = k\circ\mathcal{B}_2$, i.e. if and only if $g^{-1}\circ h \in \mathcal{A}_2\circ\mathcal{B}_2$. The group $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ acts on the Bass-Serre tree by left translation: For all $g,h \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$, we set $g.(h\circ\mathcal{A}_2) = (g\circ h)\circ\mathcal{A}_2$ and $g.(h\circ\mathcal{B}_2) = (g\circ h)\circ\mathcal{B}_2$. Each element of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ satisfies one property of the following alternative:

(1) It is triangularizable, i.e. conjugate to an element of $\mathcal{B}_2$. This is the case where the automorphism fixes at least one point on the Bass-Serre tree.
(2) It is a Hénon automorphism, i.e. it is conjugate to an element of the form

$$g = a_1\circ b_1\circ\cdots\circ a_k\circ b_k,$$

where $k \geq 1$, each a_i belongs to $\mathcal{A}_2 \smallsetminus \mathcal{B}_2$ and each b_i belongs to $\mathcal{B}_2 \smallsetminus \mathcal{A}_2$. This is the case where the automorphism acts without fixed points, but preserves a (unique) geodesic of the Bass-Serre tree on which it acts as a translation of length $2k$.

Furthermore, according to [Lam01, Theorem 2.4], every subgroup H of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ satisfies one and only one of the following assertions:

(1) It is conjugate to a subgroup of $\mathcal{A}_2$ or of $\mathcal{B}_2$.

(2) Every element of H is triangularizable and H is not conjugate to a subgroup of $\mathcal{A}_2$ or of $\mathcal{B}_2$. In that case, H is Abelian.
(3) The group H contains some Hénon automorphisms (i.e. non triangularizable automorphisms) and all those have the same geodesic on the Bass-Serre tree. The group H is then solvable.
(4) The group H contains two Hénon automorphisms having different geodesics. Then, H contains a free group with two generators.

Let H be now a solvable subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. If we are in case (1), then we may assume that H is a subgroup of $\mathcal{A}_2$ or of $\mathcal{B}_2$. Since $\psi(\mathcal{A}_2) = 5$ and $\psi(\mathcal{B}_2) = 3$ (the group $\mathcal{B}_2$ being solvable of derived length 3), this settles this case. In case (2), H is Abelian hence of derived length at most 1. In case (3), there exists a geodesic Γ which is globally fixed by every element of H. Therefore, we may assume without restriction that

$$H = \{f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}}),\ f(\Gamma) = \Gamma\}.$$

Note that $D^2(H)$ is included into the group K that fixes pointwise the geodesic Γ. Up to conjugation, we may assume that Γ contains the vertex $\mathcal{B}_2$, i.e. that K is included into $\mathcal{B}_2$. By [Lam01, Proposition 3.3], each element of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ fixing an unbounded set of the Bass-Serre tree has finite order. If $f, g \in K$, their commutator is of the form $(x + p(y), y + c)$. This latter automorphism being of finite order, it must be equal to the identity, showing that K is Abelian. Therefore, we get $D^3(H) = \{1\}$.

Finally, we cannot be in case (4), because a free group with two generators is not solvable. □

From Propositions 3.10 and 3.14, we get at once the following result, which also follows from Theorem 3.8 above.

Corollary 3.15. *Every solvable connected subgroup of* $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ *is contained into a Borel subgroup.*

3.2. **Proof of the equality $\psi(\mathcal{A}_2) = 5$.** Recall that Newman [New72] has computed the exact value $\psi(\mathrm{GL}(n, \mathbb{C}))$ for all n. It turns out that $\psi(\mathrm{GL}(n, \mathbb{C}))$ is equivalent to $5 \log_9(n)$ as n goes to infinity (see [Weh73, Theorem 3.10]). Let us give a few particular values for $\psi(\mathrm{GL}(n, \mathbb{C}))$ taken from [New72].

n	1	2	3	4	5	6	7	8	9	10	18	26	34	66	74
$\psi(\mathrm{GL}(n, \mathbb{C}))$	1	4	5	6	7	7	7	8	9	10	11	12	13	14	15

We now consider the affine group $\mathcal{A}_n$. On the one hand, observe that $\mathcal{A}_n$ is isomorphic to a subgroup of $\mathrm{GL}(n+1, \mathbb{C})$. Hence, $\psi(\mathcal{A}_n) \le \psi(\mathrm{GL}(n+1, \mathbb{C}))$. On the other hand, we have the short exact sequence

$$1 \to \mathbb{C}^n \to \mathcal{A}_n \xrightarrow{L} \mathrm{GL}_n(\mathbb{C}) \to 1,$$

where $L \colon \mathcal{A}_n \to \mathrm{GL}(n, \mathbb{C})$ is the natural morphism sending an affine transformation to its linear part. Thus, if H is a solvable subgroup of $\mathcal{A}_n$, we have a short exact sequence

$$1 \to H \cap (\mathbb{C}^n) \to H \xrightarrow{L} L(H) \to 1.$$

Since $L(H)$ is solvable of derived length at most $\psi(\mathrm{GL}_n(\mathbb{C}))$ and since $H\cap(\mathbb{C}^n)$ is Abelian, this implies that $l(H) \leq \psi(\mathrm{GL}_n(\mathbb{C}))+1$. Therefore, we have proved the general formula

$$\psi(\mathrm{GL}_n(\mathbb{C})) \leq \psi(\mathcal{A}_n) \leq \min\{\psi(\mathrm{GL}(n,\mathbb{C}))+1, \psi(\mathrm{GL}(n+1,\mathbb{C}))\}.$$

For $n=2$, this yields $\psi(\mathcal{A}_2)=4$ or 5. We shall now prove that $\mathcal{A}_2$ contains solvable subgroups of derived length 5 (see Lemma 3.19 below), hence the following desired result.

Proposition 3.16. *The maximal derived length of a solvable subgroup of the affine group $\mathcal{A}_2$ is* 5*, i.e. we have $\psi(\mathcal{A}_2)=5$.*

As explained above, it still remains to provide an example of a solvable subgroup of $\mathcal{A}_2$ of derived length 5. In that purpose, recall that the group $\mathrm{PSL}(2,\mathbb{C})$ contains a subgroup isomorphic to the symmetric group S_4 and that all such subgroups are conjugate (see for example [Bea10]).

Definition 3.17. The *binary octahedral group* 2O is the pre-image of the symmetric group S_4 by the $(2:1)$-cover $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$.

The following result is also well-known.

Lemma 3.18. *The derived length of the binary octahedral group $G=2\mathrm{O}$ is* 4*.*

Proof. Using the short exact sequence

$$0 \to \{\pm I\} \to G \xrightarrow{\pi} S_4 \to 0,$$

we get $\pi(D^2G)=D^2(\pi(G))=D^2(S_4)=\mathrm{V}_4$, where $V_4 \simeq \mathbb{Z}_2\times\mathbb{Z}_2$ is the Klein group. One could also easily check that $\pi^{-1}(\mathrm{V}_4)$ is isomorphic to the quaternion group Q_8. The equality $\pi(D^2G)=\mathrm{V}_4$ is then sufficient for showing that $D^2G=\pi^{-1}(\mathrm{V}_4)$. Indeed, if D^2G was a strict subgroup of $\pi^{-1}(\mathrm{V}_4)\simeq \mathrm{Q}_8$, it would be cyclic, hence $\pi(D^2G)=\mathrm{V}_4$ would be cyclic too. A contradiction. Since $D^2G\simeq \mathrm{Q}_8$ has derived length 2, this shows us that the derived length of G is $2+2=4$. □

Lemma 3.19. *Consider the pre-image $L^{-1}(G)\simeq G\ltimes\mathbb{C}^2$ of the binary octahedral group $G:=2\mathrm{O}\subseteq \mathrm{SL}(2,C)$ by the natural morphism $L\colon \mathcal{A}_2\to \mathrm{GL}(2,\mathbb{C})$ sending an affine transformation onto its linear part. Then, the derived length of $L^{-1}(G)$ is equal to* 5*.*

Proof. By Lemma 3.18, the derived length of G is 4. The short exact sequence

$$1\to\mathbb{C}^2\to G\ltimes\mathbb{C}^2\to G\to 1$$

implies that the derived length of $G\ltimes\mathbb{C}^2$ is at most $4+1=5$. Moreover, the strictly decreasing sequence $G=D^0(G)>D^1(G)>D^2(G)>D^3(G)>D^4(G)=1$ shows that the group $D^2(G)$ is non-Abelian and in particular non-cyclic. By Lemma 3.20 below, we thus have $D^i(G\ltimes\mathbb{C}^2)=D^i(G)\ltimes\mathbb{C}^2$ for every $i\leq 3$. But since $D^3(G)$ is non-trivial, the group $D^3(G\ltimes\mathbb{C}^2)=D^3(G)\ltimes\mathbb{C}^2$ strictly contains the subgroup $(\mathbb{C}^2,+)$ of translations and cannot be Abelian, because the group $\mathbb{C}^2$ is its own centralizer in $\mathcal{A}_2$. Finally, we get $D^4(G\ltimes\mathbb{C}^2)\neq 1$, proving that the derived length of $G\ltimes\mathbb{C}^2$ is indeed 5. □

Lemma 3.20. *Let H be a finite non-cyclic subgroup of $\mathrm{GL}(2,\mathbb{C})$. Then the derived subgroup of $L^{-1}(H)=H\ltimes\mathbb{C}^2\subseteq\mathcal{A}_2$ is the group $D(H)\ltimes\mathbb{C}^2$.*

Proof. Set $K := D(H \ltimes \mathbb{C}^2) \cap \mathbb{C}^2$. Note that K contains the commutator $[\mathrm{id} + v, h]$ for all $v \in \mathbb{C}^2$, $h \in H$, i.e. it contains all elements $h \cdot v - v$. It is enough to show that these vectors generate $\mathbb{C}^2$. Indeed, it would then imply that there exist h_1, v_1, h_2, v_2 such that the vectors $h_1 \cdot v_1 - v_1$ and $h_2 \cdot v_2 - v_2$ are linearly independent. But then, K would also contain the vectors $h_1 \cdot (\lambda_1 v_1) - (\lambda_1 v_1) + h_2 \cdot (\lambda_2 v_2) - \lambda_2 v_2$ for any $\lambda_1, \lambda_2 \in \mathbb{C}$, proving that $K = \mathbb{C}^2$. Therefore, let us assume by contradiction that there exists a non-zero vector $w \in \mathbb{C}^2$ such that $h \cdot v - v$ is a multiple of w for all $h \in H$, $v \in \mathbb{C}^2$. Take $w' \in \mathbb{C}^2$ such that (w, w') is a basis of $\mathbb{C}^2$. In this basis, any element of H admits a matrix of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Therefore, by the theory of representations of finite group, we may assume, up to conjugation, that each element of H admits a matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

This would imply that H is isomorphic to a finite subgroup of $\mathbb{C}^*$, hence that it is cyclic. A contradiction. □

3.3. **An ind-group with nonconjugate Borel subgroups.** In this section, we consider the subgroup $\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$ of $\mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$ of all automorphisms $f = (f_1, f_2, z)$ fixing the last coordinate of $\mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x, y, z])$. Since it is clearly a closed subgroup, it is also an ind-group. Note that $\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$ is naturally isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}(z)})$. In its turn, the field $\mathbb{C}(z)$ can be embedded into the field $\mathbb{C}$, so that the group $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}(z)})$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Therefore, by Proposition 3.14, we get

$$\psi\big(\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})\big) \leq \psi\big(\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}(z)})\big) \leq \psi\big(\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})\big) = 5.$$

Recall moreover that $\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$ contains nontriangularizable additive group actions [Bas84]. Let us briefly describe the example given by Bass. Consider the following locally nilpotent derivation of $\mathbb{C}[x, y, z]$:

$$\Delta = -2y\partial_x + z\partial_y.$$

Then, the derivation $(xz + y^2)\Delta$ is again locally nilpotent. We associate it with the morphism

$$(\mathbb{C}, +) \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[x, y, z]), \quad t \mapsto \exp(t(xz + y^2)\Delta).$$

The automorphism of $\mathbb{A}^3_{\mathbb{C}}$ corresponding to $\exp(t(xz + y^2)\Delta)$ is given by

$$f_t := (x - 2ty(xz + y^2) - t^2 z(xz + y^2)^2, y + tz(xz + y^2), z) \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}}).$$

For $t = 1$, we get the famous Nagata automorphism. Note that the fixed point set of the corresponding $(\mathbb{C}, +)$-action on $\mathbb{A}^3_{\mathbb{C}}$ is the hypersurface $\{xz + y^2 = 0\}$ which has an isolated singularity at the origin. On the other hand, the fixed point set of a triangular $(\mathbb{C}, +)$-action on $\mathbb{A}^3_{\mathbb{C}}$

$$t \mapsto g_t = \exp(t(a(y, z)\partial_x + b(z)\partial_y)) \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$$

is the set $\{a(y, z) = b(z) = 0\}$, which is isomorphic to a cylinder $\mathbb{A}^1_{\mathbb{C}} \times Z$ for some variety Z. This implies that the $(\mathbb{C}, +)$-action $t \mapsto f_t$ is not triangularizable.

By Proposition 3.10, it follows that $\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$ contains Borel subgroups that are not conjugate to a subgroup of the group

$$\mathcal{B}_z = \{(f_1, f_2, z) \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}}) \mid f_1 \in \mathbb{C}[x,y,z], f_2 \in \mathbb{C}[y,z]\}$$

of triangular automorphisms of $\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$.

Proposition 3.21. *The group $\mathcal{B}_z$ is a Borel subgroup of $\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$.*

Proof. With the same proof as for Lemma 3.1, we obtain easily that $\mathcal{B}_z$ is connected. It is also solvable, since it can be seen as a subgroup of the Jonquières subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}(z)})$, which is solvable.

Now, we simply follow the proof of Proposition 3.3. Let $H \subset \mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$ be a closed subgroup containing strictly $\mathcal{B}_z$ and take an element f in $H \setminus \mathcal{B}_z$, i.e. an element $f = (f_1, f_2, z)$ with $f_2 \in \mathbb{C}[x,y,z] \setminus \mathbb{C}[y,z]$. Arguing as before, we can find suitable translations $t_c = (x+c_1, y+c_2, z)$ and $t_{c'} = (x+c'_1, y+c'_2, z)$ such that the automorphism $g = t_c \circ f \circ t_{c'}$ fixes the point $(0,0,0)$ and is of the form $g = (g_1, g_2, z)$ with $g_2 = xc(z)+yd(z)+h(x,y,z)$ for some $c(z), d(z) \in \mathbb{C}[z]$, $c(z) \not\equiv 0$, and some polynomial $h(x,y,z)$ belonging to the ideal (x^2, xy, y^2) of $\mathbb{C}[x,y,z]$.

Conjugating this g by the automorphism $(tx, ty, z) \in H$, $t \neq 0$, and taking the limit when t goes to 0, we obtain an element of the form $(a(z)x + b(z)y, c(z)x + d(z)y, z)$ with $c(z) \not\equiv 0$ in H. By Lemma 3.23 below, this implies that the group H is not solvable. □

Corollary 3.22. *The ind-group $\mathrm{Aut}_z(\mathbb{A}^3_{\mathbb{C}})$ contains non-conjugate Borel subgroups.*

In the course of the proof of Proposition 3.21, we have used the following lemma that we prove now.

Lemma 3.23. *The subgroup $\mathrm{B}_2(\mathbb{C}[z])$ of upper triangular matrices of $\mathrm{GL}_2(\mathbb{C}[z])$ is a maximal solvable subgroup.*

Proof. For every $\alpha \in \mathbb{C}$, denote by $\mathrm{ev}_\alpha \colon \mathrm{GL}_2(\mathbb{C}[z]) \to \mathrm{GL}_2(\mathbb{C})$ the evaluation map that associates to an element $M(z) \in \mathrm{GL}_2(\mathbb{C}[z])$ the constant matrix $M(\alpha)$ obtained by replacing z by α. Let H be a subgroup of $\mathrm{GL}_2(\mathbb{C}[z])$ strictly containing the group $\mathrm{B}_2(\mathbb{C}[z])$. By definition, H contains a non-triangular matrix, i.e. a matrix of the form

$$M = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \text{ with } c \not\equiv 0.$$

Choose a complex number α such that $c(\alpha) \neq 0$. Then, the group $\mathrm{ev}_\alpha(H)$ contains the upper triangular constant matrices $\mathrm{B}_2(\mathbb{C})$ and a non-triangular matrix. Therefore, $\mathrm{ev}_\alpha(H) = \mathrm{GL}_2(\mathbb{C})$ and H is not solvable. □

Remark 3.24. By Nagao's theorem (see [Nag59] or e.g. [Ser03, Chapter II, no 1.6]), we have an amalgamated product structure

$$\mathrm{GL}_2(\mathbb{C}[z]) = \mathrm{GL}_2(\mathbb{C}) *_{\mathrm{B}_2(\mathbb{C})} \mathrm{B}_2(\mathbb{C}[z]).$$

However, contrarily to the case of $\mathrm{Aut}(\mathbb{A}^2)$, the group $\mathrm{B}_2(\mathbb{C}[z])$ is not a maximal closed subgroup. Indeed, for every complex number α, this group is strictly included into the group $\mathrm{ev}_\alpha^{-1}(\mathrm{B}_2(\mathbb{C}))$.

3.4. **Maximal closed subgroups.** In this section, we mainly focus on the following question.

Question 3.25. *What are the maximal closed subgroups of* $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$*?*

First of all, it is easy to observe that, since the action of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ on $\mathbb{A}^n_{\mathbb{C}}$ is infinite transitive, i.e. m-transitive for all integers $m \geq 1$, the stabilizers of a finite number of points are examples of maximal closed subgroups.

Proposition 3.26. *For every finite subset* Δ *of* $\mathbb{A}^n_{\mathbb{C}}$*,* $n \geq 2$*, the group*

$$\mathrm{Stab}(\Delta) = \{f \in \mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}}),\ f(\Delta) = \Delta\}$$

is a maximal subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$*. Furthermore, it is closed.*

Proof. Let $\Delta = \{a_1, \ldots, a_k\}$ be a finite subset of $\mathbb{A}^n_{\mathbb{C}}$. Let $f \in \mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}}) \setminus \mathrm{Stab}(\Delta)$. We will prove that $\langle \mathrm{Stab}(\Delta), f\rangle = \mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$, where $\langle \mathrm{Stab}(\Delta), f\rangle$ denotes the subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ that is generated by $\mathrm{Stab}(\Delta)$ and f. We will use repetitively the well-known fact that $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ acts $2k$-transitively on $\mathbb{A}^n_{\mathbb{C}}$.

We first observe that $\langle \mathrm{Stab}(\Delta), f\rangle$ contains an element g such that $g(\Delta) \cap \Delta = \emptyset$. To see this, denote by $m := |\Delta \cap f(\Delta)|$ the cadinality of the set $\Delta \cap f(\Delta)$. Up to composing it by an element of $\mathrm{Stab}(\Delta)$, we can suppose that f fixes the points $a_1, \ldots, a_m$ and maps $a_{m+1}, \ldots, a_k$ outside Δ. If $m \geq 1$, then we consider an element $\alpha \in \mathrm{Stab}(\Delta)$ that maps the point a_m onto a_{m+1} and sends all points $f(a_{m+1}), \ldots, f(a_k)$ outside the set $f^{-1}(\Delta)$. Remark that $g = f \circ \alpha \circ f$ is an element of $\langle \mathrm{Stab}(\Delta), f\rangle$ with $|\Delta \cap g(\Delta)| < m$. By descending induction on m, we can further suppose that $|\Delta \cap g(\Delta)| = 0$ as desired.

Now, consider any $\varphi \in \mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$. Let us prove that φ belongs to the subgroup $\langle \mathrm{Stab}(\Delta), g\rangle$. Take an element $\beta \in \mathrm{Stab}(\Delta)$ such that $\beta(\varphi(\Delta)) \cap g^{-1}(\Delta) = \emptyset$. Then, $g(\beta(\varphi(\Delta)) \cap \Delta = \emptyset$ and we can find an element $\gamma \in \mathrm{Stab}(\Delta)$ such that $(\gamma \circ g \circ \beta \circ \varphi)(a_i) = g(a_i)$ for all i. We have $\varphi = \beta^{-1} \circ g^{-1} \circ \gamma^{-1} \circ g \circ \delta \in \langle \mathrm{Stab}(\Delta), g\rangle$, where $\delta := g^{-1} \circ (\gamma \circ g \circ \beta \circ \varphi)$ is an element of $\mathrm{Stab}(\Delta)$, proving that $\langle \mathrm{Stab}(\Delta), g\rangle$ is equal to the whole group $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$. Therefore, the group $\mathrm{Stab}(\Delta)$ is actually maximal in $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$. Finally, note that for each point $a \in \mathbb{A}^n_{\mathbb{C}}$ the evaluation map $\mathrm{ev}_a \colon \mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}}) \to \mathbb{A}^n_{\mathbb{C}}$, $f \mapsto f(a)$ is and ind-morphism. Since Δ is a closed subset of $\mathbb{A}^n_{\mathbb{C}}$ the equality

$$\mathrm{Stab}(\Delta) = \bigcap_i (\mathrm{ev}_{a_i})^{-1}(\Delta)$$

implies that $\mathrm{Stab}(\Delta)$ is closed in $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$. □

Besides the above examples and the triangular subgroup $\mathcal{B}_2$, the only other maximal closed subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ that we are aware of is the affine subgroup $\mathcal{A}_2$. The fact that $\mathcal{A}_2$ is maximal among all closed subgroups of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is a particular case of the following recent result of Edo [Edo16]. (We recall that the so-called *tame subgroup* of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is its subgroup generated by $\mathcal{A}_n$ and $\mathcal{B}_n$.)

Theorem 3.27 ([Edo16])**.** *If a closed subgroup of* $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$*,* $n \geq 2$*, contains strictly the affine subgroup* $\mathcal{A}_n$*, then it also contains the whole tame subgroup, hence its closure. In particular, for* $n = 2$*, the affine group* $\mathcal{A}_2$ *is maximal among the closed subgroups of* $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$*.*

Remark 3.28. Note that Theorem 3.27 does not allow us to settle the question of the (non) maximality of $\mathcal{A}_n$ among the closed subgroups of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$ when $n \geq 3$. Indeed, on the one hand, it was recently shown that, in dimension 3, the tame subgroup is not closed (see [EP15]). But, on the other hand, it is still unknown whether it is dense in $\mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$ or not. For $n \geq 4$, the three questions, whether the tame subgroup is closed, whether it is dense, or even whether it is a strict subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbb{C}})$, are all open.

Let us finally remark that the affine group $\mathcal{A}_2$ is not a maximal among all abstract subgroups of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Indeed, using the amalgamated structure

$$\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}}) = \mathcal{A}_2 *_{\mathcal{A}_2 \cap \mathcal{B}_2} \mathcal{B}_2$$

and following [FM89], we can define the multidegree (or polydegree) of any automorphism $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ in the following way. If f admits an expression

$$f = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \circ a_{k+1},$$

where each a_i belongs to $\mathcal{A}_2$, each b_i belongs to $\mathcal{B}_2$ and $a_i \notin \mathcal{B}_2$ for $2 \leq i \leq k$, $b_i \notin \mathcal{A}_2$ for $1 \leq i \leq k$, the multidegree of f is defined as the finite sequence (possibly empty) of integers at least equal to 2:

$$\mathrm{mdeg}(f) = (\deg b_1, \deg b_2, \ldots, \deg b_k).$$

Then, the subgroup $M_r := \langle \mathcal{A}_2, (\mathcal{B}_2)_{\leq r} \rangle \subseteq \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ coincides with the set of automorphisms whose multidegree is of the form $(d_1, \ldots, d_k)$ for some k with $d_1, \ldots, d_k \leq r$. We thus have a strictly increasing sequence of subgroups

$$\mathcal{A}_2 = M_1 < M_2 < \cdots < M_d < \cdots,$$

showing in particular that $\mathcal{A}_2$ is not a maximal abstract subgroup.

4. Non-maximality of the Jonquières subgroup in dimension 2

Throughout this section, we work over an arbitrary ground field $\mathbf{k}$.

Recall that by the famous Jung-van der Kulk-Nagata theorem [Jun42, vdK53, Nag72], the group $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$, of algebraic automorphisms of the affine plane, is the amalgamated free product of its affine subgroup

$$A = \{(ax + by + c, a'x + b'y + c') \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \mid a, b, c, a', b', c' \in \mathbf{k}\}$$

and its Jonquières subgroup

$$B := \{(ax + p(y), b'y + c') \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \mid a, b', c' \in \mathbf{k}, p(y) \in \mathbf{k}[y]\}$$

above their intersection. Therefore, every element $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ admits a *reduced expression* as a product of the form

$$f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1}, \qquad (*)$$

where $a_1, \ldots, a_n$ belong to $A \setminus A \cap B$, and $t_1, \ldots, t_{n+1}$ belong to B with $t_2, \ldots, t_n \notin A \cap B$.

Definition 4.1. The number n of affine non-triangular automorphisms appearing in such an expression for f is unique. We call it the *affine length* of f and denote it by $\ell_A(f)$.

Remark 4.2. Instead of counting affine elements to define the length of an automorphism of $\mathbb{A}^2$, one can of course also consider the Jonquières elements and define the triangular length $\ell_B(f)$ of every $f \in \operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$. Actually, this is the triangular length, that one usually uses in the literature. Let us in particular recall that this length map ℓ_B : $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}}) \to \mathbb{N}$ is lower semicontinuous [Fur02], when considering $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}})$ as an ind-group. Since

$$\ell_A(f) = \max_{b_1,b_2 \in B} \ell_B(b_1 \circ f \circ b_2) - 1$$

for every $f \in \operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$ and since the supremum of arbitrarily many lower semicontinuous maps is lower semicontinuous, we infer that ℓ_A has also this property.

Proposition 4.3. *The affine length map* $\ell_A \colon \operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}}) \to \mathbb{N}$ *is lower semicontinuous.*

The next result shows that the Jonquières subgroup is not a maximal subgroup of $\operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$.

Proposition 4.4. *Let* $p \in \mathbf{k}[y]$ *be a polynomial that fulfils the following property:*

(WG) $\quad \forall\, \alpha, \beta, \gamma \in \mathbf{k},\ \deg[p(y) - \alpha p(\beta y + \gamma)] \leq 1 \Longrightarrow \alpha = \beta = 1$ *and* $\gamma = 0,$

and consider the following elements of $\operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$*:*

$$\sigma = (y, x), \quad t = (-x + p(y), y), \quad f = (\sigma \circ t)^2 \circ \sigma \circ (t \circ \sigma)^2.$$

Then, the subgroup generated by B *and* f *is a strict subgroup of* $\operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$*, i.e.* $\langle B, f\rangle \neq \operatorname{Aut}(\mathbb{A}^2_{\mathbf{k}})$*.*

Remark 4.5. Polynomials satisfying the above property (WG) are called *weakly general* in [FL10], where a stronger notion of a general polynomial is also given (see [FL10, Definition 15, page 585]). In particular, by [FL10, Example 65, page 608], the polynomial $q = y^5 + y^4$ is weakly general if $\mathbf{k}$ is a field of characteristic zero.

Moreover, the polynomial $q = y^{2p} - y^{2p-1}$ is weakly general if $\operatorname{char}(\mathbf{k}) = p > 0$. This follows directly from the fact that the coefficients of y^{2p}, y^{2p-1} and y^{2p-2} in the polynomial $q(y) - \alpha q(\beta y + \gamma)$ are equal to $1 - \alpha\beta^{2p}$, $1 - \alpha\beta^{2p-1}$ and $-\alpha\beta^{2p-2}\gamma$, respectively.

Proof of Proposition 4.4. Remark that σ and t, hence f, are involutions. Therefore, every element $g \in \langle B, f\rangle$ can be written as

$$g = b_1 \circ f \circ b_2 \circ f \circ \cdots \circ b_k \circ f \circ b_{k+1},$$

where the elements b_i belong to B and where we can assume without restriction that $b_2, \ldots, b_k$ are different from the identity (otherwise, the expression for g could be shortened using that $f^2 = \mathrm{id}$).

In order to prove the proposition, it is enough to show that no element g as above is of affine-length equal to 1. Note that $\ell_A(g) = 0$ if $k = 0$ and that $\ell_A(g) = \ell_A(f) = 5$ if $k = 1$. It remains to consider the case where $k \geq 2$.

For this, let us define four subgroups $B_0, \ldots, B_3$ of B by

$$\begin{aligned}
&B_0 = B,\\
&B_1 = A \cap B = \{(ax + by + c, b'y + c') \mid a, b, c, b', c' \in \mathbf{k}, a, b' \neq 0\},\\
&B_2 = (A \cap B) \cap [\sigma \circ (A \cap B) \circ \sigma] = \{(ax + c, b'y + c') \mid a, c, b', c' \in \mathbf{k}, a, b' \neq 0\},\\
&B_3 = \{(x, y + c') \mid c' \in \mathbf{k}\}.
\end{aligned}$$

Note that $B = B_0 \supseteq B_1 \supseteq B_2 \supseteq B_3$. We will now give a reduced expression of $u_i := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2$ for each $i \in \{2, \dots, k\}$. We do it by considering successively the four following cases:

1. $b_i \in B_0 \smallsetminus B_1$; 2. $b_i \in B_1 \smallsetminus B_2$; 3. $b_i \in B_2 \smallsetminus B_3$; 4. $b_i \in B_3 \smallsetminus \{\mathrm{id}\}$.

Case 1: $b_i \in B_0 \smallsetminus B_1$.
Since $b_i \in B \smallsetminus A$, the element u_i admits the following reduced expression

$$u_i = (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2.$$

Case 2: $b_i \in B_1 \smallsetminus B_2$.
Since $\widehat{b_i} := \sigma \circ b_i \circ \sigma \in A \smallsetminus B$, the element u_i has the following reduced expression

$$u_i = t \circ \sigma \circ t \circ \widehat{b_i} \circ t \circ \sigma \circ t.$$

Case 3: $b_i \in B_2 \smallsetminus B_3$.
Let us check that $\overline{b_i} := t \circ \sigma \circ b_i \circ \sigma \circ t \in B \smallsetminus A$. We are in the case where $b_i = (ax + c, b'y + c')$ with $(a, c, b') \neq (1, 0, 1)$. A direct calculation gives that

$$\overline{b_i} = (b'x + p(ay + c) - b'p(y) - c', ay + c).$$

By the assumption made on p, we have that $\deg[p(ay + c) - b'p(y)] \geq 2$, hence that $\overline{b_i} \in B \smallsetminus A$. Therefore u_i admits the following reduced expression

$$u_i = t \circ \sigma \circ \overline{b_i} \circ \sigma \circ t.$$

Case 4. $b_i \in B_3 \smallsetminus \{\mathrm{id}\}$.
Let us check that $\widetilde{b_i} := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2 \in B \smallsetminus A$. We are in the case where $b_i = (x, y + c')$ with $c' \in \mathbb{C}^*$. Using the computation in case 3 with $(a, c, b') = (1, 0, 1)$, we then obtain that

$$\widetilde{b_i} = t \circ \sigma \circ (x - c', y) \circ \sigma \circ t = t \circ (x, y - c') \circ t = (x + p(y - c') - p(y), y - c') \in B \smallsetminus A.$$

Therefore, the element u_i has the following reduced expression

$$u_i = \widetilde{b_i}.$$

Finally we obtain a reduced expression for an element $g \in \langle B, f \rangle$ from the above study of cases, since we can express

$$\begin{aligned} g &= b_1 \circ f \circ b_2 \circ f \circ \cdots \circ b_k \circ f \circ b_{k+1} \\ &= b_1 \circ (\sigma \circ t)^2 \circ \sigma \circ u_2 \circ \sigma \circ \cdots \circ \sigma \circ u_k \circ \sigma \circ (t \circ \sigma)^2 \circ b_{k+1}. \end{aligned}$$

In particular, observe that $\ell_A(g) \geq 6$ if $k \geq 2$. This concludes the proof. □

Note that the element f such that $\langle B, f \rangle \neq \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$, that we constructed in Proposition 4.4, is of affine-length $\ell_A(f) = 5$. Our next result shows that 5 is precisely the minimal length for elements $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \setminus B$ with that property.

Proposition 4.6. *Suppose that $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ is an automorphism of affine length ℓ with $1 \leq \ell \leq 4$. Then, the subgroup generated by B and f is equal to the whole group $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$, i.e. $\langle B, f \rangle = \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$.*

In order to prove the above proposition, it is useful to remark that we can impose extra conditions on the elements $t_1, \dots, t_{n+1}, a_1, \dots, a_n$ appearing in a reduced expression $(*)$ of an automorphism $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$. We do it in Proposition 4.10 below. First, we need to introduce some notations.

Notation 4.7. In the sequel, we will denote, as in the proof of Proposition 4.4, by σ the involution

$$\sigma = (y, x) \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$$

and by B_2 the subgroup

$$B_2 = \{(ax + c, b'y + c') \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \mid a, c, b', c' \in \mathbf{k}\} \subset A \cap B.$$

Moreover, we denote by I the subset

$$I = \{(-x + p(y), y) \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \mid p(y) \in \mathbf{k}[y], \deg p(y) \geq 2\} \subset B \setminus A \cap B.$$

Note that the elements of I are all involutions.

Lemma 4.8. *The followings hold:*

(1) $B_2 \circ \sigma = \sigma \circ B_2$.
(2) $B \setminus A \cap B = I \circ B_2 = B_2 \circ I = B_2 \circ I \circ B_2$.
(3) $A \setminus A \cap B \subset (A \cap B) \circ \sigma \circ (A \cap B)$.

Remark 4.9. In particular, Assertion (3) implies that the group generated by σ and all triangular automorphisms is equal to the whole $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$, i.e. $\langle B, \sigma \rangle = \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$.

Proof. The first assertion is an easy consequence of the following equalities:

$$(ax + c, b'y + c') \circ \sigma = (ay + c, b'x + c') = \sigma \circ (b'x + c', ay + c).$$

Let us now prove the second assertion. It is easy to check that $I \circ B_2 = B_2 \circ I = B_2 \circ I \circ B_2 \subset B \setminus A \cap B$. On the other hand, let $f = (ax + p(y), b'y + c')$ be an element of $B \setminus A \cap B$. Then f belongs to $I \circ B_2$, since we can write

$$f = (-x + p(\frac{y - c'}{b'}), y) \circ (-ax, b'y + c').$$

It remains to prove the last assertion. For this, it suffices to write, given an element $f = (ax + by + c, a'x + b'y + c')$ of $A \setminus A \cap B$ with $a' \neq 0$, that

$$f = (ax + by + c, a'x + b'y + c') = (x + \frac{a}{a'}y + c, y + c') \circ \sigma \circ (a'x + b'y, \frac{ba' - ab'}{a'}y).$$

$\square$

Proposition 4.10. *Let $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ be an automorphism of affine length $\ell = n + 1$ with $n \geq 0$. Then there exist triangular automorphisms $\tau_1, \tau_2 \in B$ and triangular involutions $i_1, \dots, i_n \in I$ such that*

$$(**) \qquad f = \tau_1 \circ \sigma \circ i_1 \circ \sigma \circ \cdots \circ \sigma \circ i_n \circ \sigma \circ \tau_2.$$

In particular, the inverse of f is given by

$$f^{-1} = \tau_2^{-1} \circ \sigma \circ i_n \circ \sigma \circ \cdots \circ \sigma \circ i_1 \circ \sigma \circ \tau_1^{-1}.$$

Proof. Let f be an automorphism of affine length $\ell = n+1$. By definition,

$$f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1},$$

for some $a_1, \ldots, a_n \in A \setminus A \cap B$, $t_1, t_{n+1} \in B$ and $t_2, \ldots, t_n \in B \setminus A \cap B$. Using Assertion (3) of Lemma 4.8, we may replace every a_i by σ. The proposition then follows from Assertions (1) and (2) of Lemma 4.8. □

We can now proceed to the proof of Proposition 4.6.

Proof of Proposition 4.6. Case $\ell = 1$. Let $f \in B$ with $\ell_A(f) = 1$. By Proposition 4.10, we can write $f = \tau_1 \circ \sigma \circ \tau_2$ for some $\tau_1, \tau_2 \in B$. Thus, $\langle B, f \rangle = \langle B, \sigma \rangle = \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ follows from Remark 4.9.

The proofs for affine length $\ell = 2, 3, 4$ will be based on explicit computations. In particular, it will be useful to observe that all $i = (-x + p(y), y) \in I$ satisfy that

$$i \circ (x+1, y) \circ i = (x-1, y), \tag{6}$$

$$\sigma \circ i \circ (x+1, y) \circ i \circ \sigma = (x, y-1) \tag{7}$$

and

$$i \circ (x, y-1) \circ i \circ (-x, y+1) = (-x + (p(y) - p(y+1)), y). \tag{8}$$

Case $\ell = 2$. Let $f \in B$ with $\ell_A(f) = 2$. By Proposition 4.10, we can suppose that $f = \sigma \circ i \circ \sigma$ for some involution $i = (-x + p(y), y) \in I$. Consider the elements $b_1 = \sigma \circ (x, y-1) \circ \sigma$ and $b_2 = \sigma \circ (-x, y+1) \circ \sigma$ of B_2. Since

$$f \circ b_1 \circ f \circ b_2 = \sigma \circ i \circ (x, y-1) \circ i \circ (-x, y+1) \circ \sigma,$$

it follows from Equality (8) above that the automorphism $\sigma \circ (-x + (p(y) - p(y+1), y) \circ \sigma$ belongs to $\langle B, f \rangle$. By induction, we thus obtain an element in $\langle B, f \rangle$ of the form $\sigma \circ (-x + q(y), y) \circ \sigma$ with $\deg(q) = 1$. This element is in fact an element of $A \setminus A \cap B$ and has therefore affine length 1. This implies that $\langle B, f \rangle = \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$.

Case $\ell = 3$. Let $f \in B$ with $\ell_A(f) = 3$. By Proposition 4.10, we can suppose that $f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma$ for some $i_1 = (-x + p_1(y), y), i_2 = (-x + p_2(y), y) \in I$. We first use Equality (7), which implies that

$$\sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma = (x, y-1), \tag{9}$$

where b denotes the element $b = \sigma \circ (x+1, y) \circ \sigma \in B_2$. Hence, denoting by b' the element $b' = \sigma \circ (-x, y+1) \circ \sigma$ in B_2 and using Equalities (8) and (9), we obtain that

$$\begin{aligned} f \circ b \circ f^{-1} \circ b' &= \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \circ b' \\ &= \sigma \circ i_1 \circ (x, y-1) \circ i_1 \circ \sigma \circ b' \\ &= \sigma \circ i_1 \circ (x, y-1) \circ i_1 \circ (-x, y+1) \circ \sigma \\ &= \sigma \circ (-x + (p_1(y) - p_1(y+1)), y) \circ \sigma \end{aligned}$$

is an element of affine length 2 (or 1 in the case where $\deg(p_1) = 2$), which belongs to $\langle B, f \rangle$. Consequently, $\langle B, f \rangle = \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$.

Case $\ell = 4$. Let $f \in B$ with $\ell_A(f) = 4$. By Proposition 4.10, we can suppose that $f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma$ for some $i_j = (-x + p_j(y), y) \in I$, $j = 1, 2, 3$. Letting $b = \sigma \circ (x + 1, y) \circ \sigma$ as above, one get that

$$\begin{aligned} f \circ b \circ f^{-1} &= \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma \circ b \circ \sigma \circ i_3 \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\ &= \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\ &= \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ (-x, y + 1) \circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\ &= \sigma \circ i_1 \circ \sigma \circ i_2' \circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\ &= \sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ (x - 1, -y) \circ i_1 \circ \sigma \\ &= \sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ (x + 1, -y) \circ \sigma \\ &= \sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ \sigma \circ (-x, y + 1), \end{aligned}$$

where $i_2' = (-x + p_2'(y), y)$ and $i_1' = (-x + p_1'(y), y)$ for the polynomials $p_2'(y) = p_2(y) - p_2(y + 1)$ and $p_1'(y) = p_1(-y)$, respectively. In particular, $\langle B, f \rangle$ contains the element $\sigma \circ i_1 \circ \sigma \circ i_2' \circ \sigma \circ i_1' \circ \sigma$. Since $\deg(p_2') = \deg(p_2) - 1$, we obtain by induction an element in $\langle B, f \rangle$ of the form $\sigma \circ i_1 \circ \sigma \circ \widetilde{i_2} \circ \sigma \circ i_1' \circ \sigma$ with $\widetilde{i_2} = (-x + \widetilde{p_2}(y), y)$ and $\deg(\widetilde{p_2}) = 1$. Since $\sigma \circ \widetilde{i_2} \circ \sigma$ is an element of $A \setminus A \cap B$, the above $\sigma \circ i_1 \circ \sigma \circ \widetilde{i_2} \circ \sigma \circ i_1' \circ \sigma$ is an automorphism of affine length 3, and the proposition follows. □

To conclude, let us emphasize that, as pointed to us by S. Lamy, our results concerning the non-maximality of B are related to those of [FL10] about the existence of normal subgroups for the group $\mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$ of automorphisms of the complex affine plane whose Jacobian determinant is equal to 1. Indeed, the subgroup $\langle B, f \rangle$, generated by B and a given automorphism f, is contained into the subgroup $B \circ \langle f \rangle_N = \{h \circ g \mid h \in B, g \in \langle f \rangle_N\}$, where $\langle f \rangle_N$ denotes the normal subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ that is generated by f.

Combined with Proposition 4.6, the above observation gives us a short proof of the following result.

Theorem 4.11 ([FL10, Theorem 1])**.** *If $f \in \mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$ is of affine length at most 4 and $f \neq \mathrm{id}$, then the normal subgroup $\langle f \rangle_N$ generated by f in $\mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$ is equal to the whole group $\mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$.*

Proof. The case where f is a triangular automorphism being easy to treat (see [FL10, Lemma 30, p. 590]), suppose that $f \in \mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$ is of affine length at most 4 and at least 1. By Proposition 4.6, we have $\langle B, f \rangle = \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Since the group $B \circ \langle f \rangle_N$ contains B and f, we get $B \circ \langle f \rangle_N = \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. In particular, the element $(-y, x)$ can be written as $(-y, x) = b \circ g$ for some $b \in B$ and $g \in \langle f \rangle_N$. Consequently, $\langle f \rangle_N$ contains the element $g = b^{-1} \circ (-y, x)$ which is of affine length 1.

Remark that the Jacobian determinant of b is equal to 1. Therefore, we can write $b^{-1} = (ax + P(y), a^{-1}y + c)$ for some $a \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $P(y) \in \mathbb{C}[y]$. Thus, g is given by

$$g = (-ay + P(x), a^{-1}x + c).$$

Next, we consider the translation $\tau = (x + 1, y)$ and compute the commutator $[\tau, g] = \tau \circ g \circ \tau^{-1} \circ g^{-1}$, which is an element of $\langle f \rangle_N$. Since

$$\begin{aligned} [\tau, g] &= (x + 1, y) \circ (-ay + P(x), a^{-1}x + c) \circ (x + 1, y) \circ (ay - ac, -a^{-1}x + a^{-1}P(ay - ac)) \\ &= (x - P(ay - ac) + P(ay - ac - 1) + 1, y - a^{-1}) \end{aligned}$$

is a triangular automorphism different from the identity, the theorem follows directly from [FL10, Lemma 30, p. 590]. □

On the other hand, we can retrieve the fact that the Jonquières subgroup is not a maximal subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ as a corollary of [FL10, Theorem 2]. Indeed, the latter produces elements $f \in \mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$ of affine length $\ell_A(f) = 7$ such that $\langle f \rangle_N \neq \mathrm{SAut}(\mathbb{A}^2_{\mathbb{C}})$. In particular, by [FL10, Theorem 1] above, the identity is the only automorphism of affine length smaller than or equal to 4 contained in $\langle f \rangle_N$. Therefore, since $\langle B, f \rangle \subset B \circ \langle f \rangle_N$, the subgroup $\langle B, f \rangle$ does not contain any non-triangular automorphism of affine length ≤ 4. Consequently, $\langle B, f \rangle$ is a strict subgroup of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$.

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