On a class of Danielewski surfaces in affine 3-space

Adrien Dubouloz

Institut de Mathématiques de Bourgogne, CNRS-UMR 5584, 9 avenue Alain Savary, BP 47870, 21078 Dijon, France

Pierre-Marie Poloni

Institut de Mathématiques de Bourgogne, CNRS-UMR 5584, 9 avenue Alain Savary, BP 47870, 21078 Dijon, France

Abstract

In [17] and [18], L. Makar-Limanov computed the automorphism groups of surfaces in $\mathbb{C}^3$ defined by the equations $x^n z - P(y) = 0$, where $n \geq 1$ and $P(y)$ is a nonzero polynomial. Similar results have been obtained by A. Crachiola [3] for surfaces with equations $x^n z - y^2 - \sigma(x) y = 0$, where $n \geq 2$ and $\sigma(0) \neq 0$, defined over arbitrary base fields. Here we consider more general surfaces defined by equations $x^n z - Q(x, y) = 0$, where $n \geq 2$ and $Q(x, y)$ is a polynomial with coefficients in an arbitrary base field k. We characterize among them the ones which are Danielewski surfaces in the sense of [10], and we compute their automorphism groups. We study closed embeddings of these surfaces in affine 3-space. We show that in general their automorphisms do not extend to automorphisms of the ambient space. Finally, we give explicit examples of $\mathbb{C}^*$-actions on a surface in $\mathbb{A}^3_{\mathbb{C}}$ which can be extended holomorphically but not algebraically to $\mathbb{C}^*$-actions on $\mathbb{A}^3_{\mathbb{C}}$.

Key words: $\mathbb{A}^1$-fibrations, Danielewski surfaces, automorphism groups, extension of automorphisms.

Introduction

Since they appeared in a counterexample to the Cancellation Problem due to W. Danielewski [7], affine surfaces defined by equations $xz - y(y-1) = 0$ and $x^2 z - y(y-1) = 0$ in the complex affine three space $\mathbb{A}^3$ and their natural generalizations, such as surfaces defined by equations $x^n z - P(y) = 0$, where $P(y)$ is a nonconstant polynomial, have been studied in many different contexts. Of particular interest is the fact that they can be equipped with nontrivial actions of the additive group $\mathbb{C}_+$. General orbits of such actions on affine surfaces essentially coincide with general fibers of

Email addresses: adrien.dubouloz@u-bourgogne.fr (Adrien Dubouloz), pierre-marie.poloni@u-bourgogne.fr (Pierre-Marie Poloni)

$\mathbb{A}^1$-fibrations $\pi : S \to C$ over an affine curve C, that is, surjective morphisms with general fibers isomorphic to an affine line. Normal affine surfaces S equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ over the affine line can be roughly classified into two classes according the following alternative : either $\pi : S \to \mathbb{A}^1$ is a unique $\mathbb{A}^1$-fibration on S up to automorphisms of the base, or there exists a second $\mathbb{A}^1$-fibration $\pi' : S \to \mathbb{A}^1$ with general fibers distinct from the ones of π. It was established by L. Makar-Limanov [18] that on a surface $S_{P,n}$ defined by an equation $x^n z - P(y) = 0$ in $\mathbb{A}^3$, where $n \geq 2$ and where $P(y)$ is a polynomial of degree $r \geq 2$, the projection $\mathrm{pr}_x : S_{P,n} \to \mathbb{A}^1$ is a unique $\mathbb{A}^1$-fibration up to automorphisms of the base. In contrast, a surface defined by an equation $xz - P(y) = 0$ admits at least two distinct $\mathbb{A}^1$-fibrations over the affine line, due to the symmetry between the variables x and z. In his proof, L. Makar-Limanov used the correspondence between algebraic $\mathbb{C}_+$-actions on an affine surface S and locally nilpotent derivations of the algebra of regular functions on S. It turns out that the argument is essentially independent of the base field k, up to replacing locally nilpotent derivations by suitable systems of Hasse-Schmidt derivations when the characteristic of k is positive (see e.g., [3]).

The fact that an affine surface S admits a unique $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ makes its study simpler. For instance, every automorphism of S must preserve this fibration. In this context, a result due to J. Bertin [2] asserts that the identity component of the automorphisms group of such a surface is an algebraic pro-group obtained as an increasing union of solvable algebraic subgroups of rank ≤ 1. For surfaces defined by equations $x^n z - P(y) = 0$ in $\mathbb{A}^3$, the picture has been completed by L. Makar-Limanov [18] who gave explicit generators of their automorphism groups. Similar results have been obtained over arbitrary base fields by A. Crachiola [3] for surfaces defined by equations $x^n z - y^2 - \sigma(x) y = 0$, where $\sigma(x)$ is a polynomial such that $\sigma(0) \neq 0$.

The latter surfaces are particular examples of a class of $\mathbb{A}^1$-fibered surfaces called *Danielewski surfaces* [10], that is, normal integral affine surfaces S equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ over an affine line with a fixed k-rational point o, such that every fiber $\pi^{-1}(x)$, where $x \in \mathbb{A}^1_k \setminus \{o\}$, is geometrically integral, and such that every connected component of $\pi^{-1}(o)$ is geometrically integral. In this article, we consider Danielewski surfaces $S_{Q,n}$ in $\mathbb{A}^3_k$ defined by equations of the form $x^n z - Q(x,y) = 0$, where $n \geq 2$ and where $Q(x,y) \in k[x,y]$ is a polynomial such that $Q(0,y)$ splits with $r \geq 2$ simple roots in k.

The paper is organized as follows. Section one contains definitions about weighted trees and the notion of equivalence of algebraic surfaces in affine 3-space. In section 2, we recall the main facts about Danielewski surfaces and we review the correspondence between these surfaces and certain classes of weighted trees. We also generalize to arbitrary base fields some results which are only stated for fields of characteristic zero in [9] , [10] and [11]. In particular, we give a characterization of Danielewski surfaces admitting two $\mathbb{A}^1$-fibrations with distinct general fibers.

In section 3, we classify Danielewski surfaces $S_{Q,h}$ in $\mathbb{A}^3_k$ defined by equations of the form $x^h z - Q(x,y) = 0$. Such surfaces admit many embeddings as surfaces $S_{Q,h}$ for different polynomials Q, but we establish in Theorem 3.2 that they can always be embedded as surfaces $S_{\sigma,h}$ defined by equations of the form

$$x^h z - \prod_{i=1}^{r} (y - \sigma_i(x)) = 0$$

for suitable collections of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$. We say that these surfaces $S_{\sigma,h}$ are *standard form* of Danielewski surfaces $S_{Q,h}$. Next, we compute the automorphism groups of Danielewski surfaces in standard form (Theorem 3.11). We show in particular that their automorphisms come as the restrictions of algebraic automorphisms of the ambient space $\mathbb{A}^3_k$.

Finally, we consider the problem of extending automorphisms of a general Danielewski surface $S_{Q,h}$ to automorphisms of the ambient space $\mathbb{A}^3_k$. We show that this is always possible in the holomorphic category (when $k = \mathbb{C}$) but not in the algebraic one. We give explicit examples which come from the study of multiplicative group actions on Danielewski surfaces. For instance, we establish that every surface $S \subset \mathbb{A}^3_{\mathbb{C}}$ defined by an equation $x^h z - (1-x) P(y) = 0$, where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial $\mathbb{C}^*$-action which is not algebraically extendable but holomorphically extendable to a $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$.

1. Preliminaries

1.1. Basic facts on weighted trees

Definition 1.1. A *tree* is a nonempty, finite, partially ordered set $\Gamma = (\Gamma, \leq)$ with a unique minimal element e_0 called the *root*, and such that for every $e \in \Gamma$ the subset $(\downarrow e)_\Gamma = \{e' \in \Gamma,\, e' \leq e\}$ is a chain for the induced ordering.

1.2. A minimal sub-chain $\overleftarrow{e'e} = \{e' < e\}$ with two elements of a tree Γ is called *an edge* of Γ. We denote the set of all edges in Γ by $E(\Gamma)$. The *children* of an element $e' \in \Gamma$ are the elements of Γ at relative level 1 with respect to e', i.e., the minimal elements of the subset $\{e \in \Gamma,\, e' < e\}$ of Γ. The maximal elements of Γ are called the *leaves* of Γ. We denote the set of those elements by $L(\Gamma)$. The maximal chains of Γ are the chains

$$(\downarrow \ell)_\Gamma = \{e_{\ell,0} = e_0 < e_{\ell,1} < \cdots < e_{\ell,m_\ell} = \ell\}, \quad \text{where } \ell \in L(\Gamma).$$

We say that Γ has *height* $h = \max\{m_\ell,\, \ell \in L\}$. An element $e \in \Gamma$ such that the chain $(\downarrow e)_\Gamma$ has length $m+1 \geq 1$ is said to be at level m. In particular, the root e_0 of Γ is at level 0.

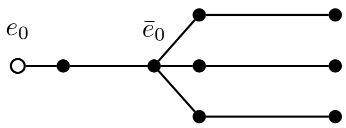
Definition 1.3. A *fine k-weighted tree* $\gamma = (\Gamma, w)$ is a tree Γ equipped with a function $w : E(\Gamma) \to k$ with values in a field k such that $w\left(\overleftarrow{e'e_1}\right) \neq w\left(\overleftarrow{e'e_2}\right)$ whenever e_1 and e_2 are distinct children of a same element e'.

In what follows, we frequently consider the following classes of trees.

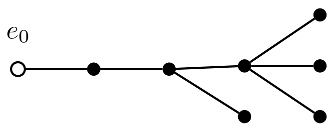
Definition 1.4. Let Γ be a rooted tree.

a) If all the leaves of Γ are at the same level $h \geq 1$ and if there exists a unique element $\bar{e}_0 \in \Gamma$ for which $\Gamma \setminus \{\bar{e}_0\}$ is a nonempty disjoint union of chains then we say that Γ is a *rake*.

b) If $\Gamma \setminus L(\Gamma)$ is a chain then we say that Γ is a *comb*. Equivalently, Γ is a comb if and only if every $e \in \Gamma$ has at most one child which is not a leaf of Γ.



A rake rooted in e_0.



A comb rooted in e_0.

1.2. Algebraic and holomorphic equivalence of closed embeddings

Let S be an irreducible affine surface and let $i_{P_1} : S \hookrightarrow \mathbb{A}^3_k$ and $i_{P_2} : S \hookrightarrow \mathbb{A}^3_k$ be algebraic embeddings of S in a same affine 3-space as closed subvarieties with defining ideals generated by polynomials P_1 and P_2 respectively.

Definition 1.5. The closed embeddings i_{P_1} and i_{P_2} are *algebraically equivalent* if one of the following equivalent conditions is satisfied:

1) There exists an automorphism Φ of $\mathbb{A}^3_k$ such that $i_{P_2} = \Phi \circ i_{P_1}$.

2) There exists an automorphism Φ of $\mathbb{A}^3_k$ and a nonzero constant $\lambda \in k^*$ such that $\Phi^* P_2 = \lambda P_1$.

3) There exists an automorphism Φ of $\mathbb{A}^3_k$ and a *linear* automorphism ϕ of $\mathbb{A}^1_k$ such that $P_2 \circ \Phi = \phi \circ P_1$.

It follows from the definition that if i_{P_1} and i_{P_2} are algebraically equivalent, then the level surfaces of the polynomials P_1 and P_2 considered as regular functions on $\mathbb{A}^3_k$ are pairwise isomorphic. In particular, every automorphism Φ of $\mathbb{A}^3_k$ such that $i_{P_2} = \Phi \circ i_{P_1}$ maps the level surfaces of P_2 isomorphically onto the ones of P_1.

1.6. Over the field $k = \mathbb{C}$ of complex numbers, one can also consider holomorphic automorphisms. With the notation of Definition 1.5, two closed algebraic embeddings i_{P_1} and i_{P_2} of a given affine surface S in $\mathbb{A}^3_{\mathbb{C}}$ are called *holomorphically equivalent* if there exists a biholomorphism $\Phi : \mathbb{A}^3_{\mathbb{C}} \xrightarrow{\sim} \mathbb{A}^3_{\mathbb{C}}$ such that $i_{P_2} = \Phi \circ i_{P_1}$. Equivalently, there exists a biholomorphism Φ of $\mathbb{A}^3_{\mathbb{C}}$ such that $\Phi^* (P_2) = \lambda P_1$ for a certain nowhere vanishing holomorphic function λ on $\mathbb{A}^3_{\mathbb{C}}$. In contrast with the algebraic case, such biholomorphisms need not preserve the families of level surfaces of P_1 and P_2.

2. Danielewski surfaces

For certain authors, a Danielewski surface is an affine surface S isomorphic to a surface in the complex affine space $\mathbb{A}^3$ defined by an equation of the form $x^n z - P(y) = 0$, where $n \geq 1$ and $P(y) \in \mathbb{C}[y]$. The latter come equipped with fibrations $\pi = \mathrm{pr}_x \mid_S : S \to \mathbb{A}^1$ restricting to trivial $\mathbb{A}^1$-bundles over the complement of the origin. If the polynomial P is nonconstant with $r \geq 1$ simple roots, then π factors through a locally trivial fiber bundle over the affine line with an r -fold origin (see e.g., [7] and [13]). In [10], the term Danielewski surface refers to an affine surface S equipped with a morphism $\pi : S \to \mathbb{A}^1$ which factors through a locally trivial fiber bundle in a similar way as above. Here we keep this second point of view. We recall that an $\mathbb{A}^1$-*fibration* over an integral scheme Y is a faithfully flat (i.e., flat and surjective) affine morphism $\pi : X \to Y$ with generic fiber isomorphic to the affine line $\mathbb{A}^1_{K(Y)}$ over the function field $K(Y)$ of Y.

Definition 2.1. (See [10]) A *Danielewski surface* is an integral affine surface S defined over a field k, equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ restricting to a trivial $\mathbb{A}^1$-bundle over the complement of a k-rational point o of $\mathbb{A}^1_k$ and such that the fiber $\pi^{-1}(o)$ is reduced, consisting of a nonempty disjoint union of curves isomorphic to the affine line $\mathbb{A}^1_k$ over k.

Notation 2.2. In what follows, we fix an isomorphism $\mathbb{A}^1_k \simeq \mathrm{Spec}(k[x])$ and we choose the closed k-rational point $o = (x)$ of $\mathrm{Spec}(k[x])$ as the origin of $\mathbb{A}^1_k$.

2.3. In the following subsections, we briefly recall the correspondence between Danielewski surfaces and weighted rooted trees as established in [10]. Although the results given in *loc. cit.* are formulated for surfaces defined over a field of characteristic zero, we just observe below that most of them remain valid without any changes over a field of arbitrary characteristic. As an application, we classify Danielewski surfaces S with a trivial canonical sheaf $\omega_{S/k} = \Lambda^2\Omega^1_{S/k}$ in terms of their associated weighted trees.

2.1. Danielewski surfaces and weighted trees

In what follows we denote by $X(r)$ the affine line with an r-fold origin, that is, the scheme obtained by gluing r copies X_i of $\mathbb{A}^1_k$ by the the identity outside their respective origins o_i, $i = 1, \ldots, r$. We let $\delta : X(r) \to \mathbb{A}^1$ be the natural morphism restricting to an isomorphism on each of the open subsets X_i, $i = 1, \ldots, r$ of $X(r)$.

2.4. Given a fine k-weighted tree $\gamma = (\Gamma, w)$ of height h, with leaves ℓ_i at levels $m_i \leq h$, $i = 1, \ldots, r$, we associate to every maximal sub-chain $(\downarrow \ell_i)$ of γ (see 1.2 for the notation) a polynomial

$$\sigma_i(x) = \sum_{j=0}^{m_i-1} w\left(\overleftarrow{e_{\ell_i,j}e_{\ell_i,j+1}}\right) x^j \in k[x], \quad i = 1, \ldots, r.$$

We let $\rho : S(\gamma) \to X(r)$ be the $\mathbb{A}^1$-bundle over $X(r)$ obtained by gluing r copies $S_i = \mathrm{Spec}(k[x][u_i])$ of the affine plane along the open subsets $S_i^* = \mathrm{Spec}\left(k\left[x, x^{-1}\right][u_i]\right)$ by means of the isomorphisms induced by the $k\left[x, x^{-1}\right]$-algebra isomorphisms

$$\begin{array}{rcl} k\left[x, x^{-1}\right][u_i] & \stackrel{\sim}{\to} & k\left[x, x^{-1}\right][u_j] \\ u_i & \mapsto & f_{ij}u_j + g_{ij} = x^{m_j - m_i}u_j + x^{-m_i}(\sigma_j(x) - \sigma_i(x)). \end{array}$$

This definition makes sense since by construction the transition functions $f_{ij} = x^{m_j - m_i}$ and $g_{ij} = x^{-m_i}(\sigma_j(x) - \sigma_i(x))$ satisfy the relations

$$\left\{ \begin{array}{rcl} f_{ji} & = & f_{ji}^{-1} \\ f_{ik} & = & f_{jk} \cdot f_{ij} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} g_{ji} & = & -f_{ji} \cdot g_{ij} \\ g_{ik} & = & g_{ij} + f_{ij} \cdot g_{jk} \end{array} \right.$$

for every triple of indices i, j and k. Since γ is a fine weighted tree, it follows that for every pair of distinct indices i and j, $g_{ij} \in k\left[x, x^{-1}\right]$ does not extend to a regular function on $\mathbb{A}^1_k$. This condition guarantees that $S(\gamma)$ is a separated scheme, whence an affine surface by virtue of Fieseler's criterion 1.4 in [13]. Therefore, $\pi_\gamma = \delta \circ \rho : S(\gamma) \to \mathbb{A}^1_k = \mathrm{Spec}(k[x])$ is a Danielewski surface, the fiber $\pi_\gamma^{-1}(o)$ being a disjoint union of affine lines

$$C_i = \pi_\gamma^{-1}(o) \cap S_i = \rho^{-1}(o_i) \simeq \mathrm{Spec}(k[u_i]), \quad i = 1, \ldots, r.$$

Remark 2.5. It follows from this description that the divisor class group of a Danielewski surface $S(\gamma)$ is isomorphic to $\mathbb{Z}^{r-1}$, generated by the classes of the irreducible components $C_1, \ldots, C_r$ of $\pi_\gamma^{-1}(o)$ with a unique relation $C_1 + \cdots + C_r = \pi_\gamma^{-1}(o) \sim 0$.

2.6. A Danielewski surface $\pi_\gamma : S(\gamma) \to \mathbb{A}^1_k$ above comes canonically equipped with a regular function ψ_γ whose restrictions to each of the open subsets S_i of S are given by polynomials

$$\psi_{\gamma,i} = x^{n_i} u_i + \sigma_i(x) \in k[x, u_i], \quad i = 1, \ldots, r.$$

Clearly, ψ_γ restricts to a coordinate function on every fiber of π_γ except $\pi_\gamma^{-1}(o)$. If γ is not the trivial tree with one element, then ψ_γ is locally constant on $\pi_\gamma^{-1}(o)$. It takes the same value on two distinct irreducible components of $\pi_\gamma^{-1}(o)$ if and only if the corresponding leaves of γ belong to a same subtree of γ rooted in an element at level 1. The canonical morphism $(\pi_\gamma, \psi_\gamma) : S \to \mathbb{A}^2_k$ induces an isomorphism between $S(\gamma) \setminus \pi_\gamma^{-1}(o)$ and the trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_k \setminus \{o\}$. Proposition 3.4 in [10], which remains valid over arbitrary base fields k, implies the following result.

Theorem 2.7. *For every pair consisting of a Danielewski surface $\pi : S \to \mathbb{A}^1_k$ and a birational morphism $(\pi, \psi) : S \to \mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k$ inducing an isomorphism between $S \setminus \pi^{-1}(o)$ and the trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_k \setminus \{o\}$, there exists a unique fine k-weighted tree γ and an isomorphism $\phi : S \xrightarrow{\sim} S(\gamma)$ such that $\pi = \pi_\gamma \circ \phi$ and $\psi = \psi_\gamma \circ \phi$.*

Remark 2.8. A Danielewski surface nonisomorphic to $\mathbb{A}^2_k$ admits a birational morphism (π, ψ) as above for which ψ is locally constant but not constant on the fiber $\pi^{-1}(o)$. So such pairs (π, ψ) correspond to fine k-weighted trees with at least two elements at level 1.

2.2. $\mathbb{A}^1$-fibrations on Danielewski surfaces

If the structural $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ on a Danielewski surface S is unique up to automorphisms of the base, then every isomorphism Φ between S and another Danielewski surface $\pi' : S' \to \mathbb{A}^1_k$ is necessarily an isomorphism of fibered surfaces, in the sense that there exists an automorphism ϕ of $\mathbb{A}^1_k$ preserving the origin o such that $\pi' \circ \Phi = \phi \circ \pi$. This greatly simplifies the study of isomorphism classes of such surfaces. So it is useful to a have a characterization of Danielewski surfaces admitting two $\mathbb{A}^1$-fibrations with distinct general fibers. The first result toward such a classification has been obtained by T. Bandman and L. Makar-Limanov [1] who established that a complex Danielewski surface S with a trivial canonical sheaf ω_S admits two $\mathbb{A}^1$-fibrations with distinct general fibers if and only if it is isomorphic to a surface $S_{P,1}$ in $\mathbb{A}^3_{\mathbb{C}}$ defined by the equation $xz - P(y) = 0$, where P is a polynomial with simple roots. More generally, we have the following result, which is an extension to arbitrary base fields of a characterization given in [10] and [11].

Theorem 2.9. *A Danielewski surface $\pi : S \to \mathbb{A}^1_k$ admits two $\mathbb{A}^1$-fibrations with distinct general fibers if and only if it is isomorphic to a one $S(\gamma)$ defined by a fine k-weighted comb $\gamma = (\Gamma, w)$.*

Furthermore, if S admits two such fibrations, then there exists an integer $h \geq 1$ and a collection of monic polynomials $P_i \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \ldots, h -$

1, $j=1,\ldots,\deg_t(P_i)$, *such that* S *is isomorphic as a fibered surface to the surface* $S_{P_0,\ldots,P_{h-1}} \subset$ *Spec* $(k[x][y_{-1},\ldots,y_{h-2}][z])$ *defined by the equations*

$$\begin{cases} xz - y_{h-2}\prod_{l=0}^{h-1} P_l(y_{l-1}) = 0 & \\ zy_{i-1} - y_i y_{h-2}\prod_{l=i+1}^{h-1} P_l(y_{l-1}) = 0 \quad xy_i - y_{i-1}\prod_{l=0}^{i} P_l(y_{l-1}) = 0 & 0 \le i \le h-2 \\ y_{i-1}y_j - y_i y_{j-1}\prod_{l=i+1}^{j} P_l(y_{l-1}) = 0 & 0 \le i < j \le h-2 \end{cases}$$

equipped with the $\mathbb{A}^1$*-fibration* $\mathrm{pr}_x : S_{P_0,\ldots,P_{h-1}} \to \mathbb{A}^1_k$.

Proof. The construction given in 4.6 in [10] implies that every Danielewski surface defined by a fine k-weighted tree $\gamma = (\Gamma, w)$ can be canonically completed into a smooth projective surface $\bar{S}(\gamma)$ in such a way that the boundary divisor $B(\gamma) = \bar{S}(\gamma) \setminus S(\gamma)$ is a tree of nonsingular proper k-rational curves, whose dual graph is obtained from Γ by deleting its leaves and replacing its root by a chain of length 2. Furthermore $B(\gamma)$ contains no (-1)-curve as an irreducible component. It follows from a result M.H. Gizatullin [15] (see also [9]) that $S(\gamma)$ admits two $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_k$ if and only if $B(\gamma)$ is a *zigzag*, that is, a chain of nonsingular proper k-rational curves. Actually, the results in *loc. cit* are only stated for surfaces defined over an algebraically closed field but one checks that it holds in our more general context due to the fact that the arguments of the proofs only involve blow-ups of k-rational points and blow-downs of k-rational curves. Clearly, $B(\gamma)$ is a zigzag if and only if Γ is a comb, and so, the first assertion follows.

The existence of an embedding of a Danielewski surface associated with a fine k-weighted comb in an affine space as a surface $S_{P_0,\ldots,P_{h-1}}$ is a particular case of a more general construction described in 4.5-4.7 in [11], which does not depend on the characteristic of the base field. □

2.3. Special Danielewski surfaces

2.10. Here we consider Danielewski surface S with a trivial canonical sheaf $\omega_{S/k} = \Lambda^2\Omega^1_{S/k}$. We call them *special Danielewski surfaces*. For instance, every Danielewski surface S in $\mathbb{A}^3_k$ is special as a consequence of adjunction formula. Special Danielewski surfaces correspond to a distinguished class of weighted trees γ. Indeed, it follows from the gluing construction described in 2.4 above that a Danielewski surface $S(\gamma)$ admits a nowhere vanishing differential 2-form if and only if all the leaves of γ are at the same level. In turn, this means that these surfaces S are the total spaces of $\mathbb{A}^1$-bundles $\rho : S \to X(r)$ over $X(r)$ defined by means of transition isomorphisms

$$\tau_{ij} : k\left[x, x^{-1}\right][u_i] \to k\left[x, x^{-1}\right][u_j], \quad u_i \mapsto u_j + g_{ij}(x), \quad i,j = 1,\ldots,r,$$

where $g = \{g_{ij}\}_{i,j} \in C^1\left(\{X_i\}_{i=1,\ldots,r}, \mathcal{O}_{X(r)}\right) \simeq \mathbb{C}\left[x, x^{-1}\right]^{2r}$ is a Čech 1-cocycle with values in the sheaf $\mathcal{O}_{X(r)}$ for the canonical open covering by the open subsets X_i, $i = 1,\ldots,r$. So such surfaces can be equivalently characterized among Danielewski surfaces by the fact that the underlying $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ is actually the structural morphism of a principal homogeneous $\mathbb{G}_{a,k}$-bundle.

Example 2.11. A fine k-weighted comb γ with all its leaves at the same level is either a chain or a rake with all its leaves at relative level 1 from the vertex $\bar{e}_0$ (see definition 1.4 above). The associated surfaces $S(\gamma)$ are isomorphic to surfaces $S_{P,1}$ in $\mathbb{A}^3_k$ with equations $xz - P(y) = 0$ for nonconstant polynomials P with simple roots in k (see 3.1 below). According to Theorem 2.9, the latter are the only special Danielewski surfaces admitting two $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_k$ with distinct general fibers. Such surfaces $S_{P,1}$ admit many $\mathbb{A}^1$-fibrations $q : S_{P,1} \to \mathbb{A}^1_k$ with distinct general fibers but it follows from Theorem 2.3 in [4] that for every such fibration q, there exists an automorphism Ψ of $S_{P,1}$ such that $q = \mathrm{pr}_x|_{S_{P,1}} \circ \Psi : S_{P,1} \to \mathbb{A}^1_k$.

2.12. The group $\mathrm{Aut}(X(r))$ acts on the set $\mathbb{P}H^1(X(r), \mathcal{O}_{X(r)})$ of isomorphism classes of $\mathbb{A}^1$-bundle over $X(r)$ of the above type. Namely, the image by an element $\phi \in \mathrm{Aut}(X(r))$ of a class represented by an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ is the isomorphism class of the fiber product bundle $\mathrm{p}_2 : \phi^* S = S \times_{X(r)} X(r) \to X(r)$. The following criterion generalizes a result of J. Wilkens [20].

Theorem 2.13. *Two special Danielewski surfaces $\pi_i : S_i \to \mathbb{A}^1_k$ with underlying $\mathbb{A}^1$-bundle structures $\rho_i : S_i \to X(r_i)$, $i = 1, 2$, are isomorphic as abstract surfaces if and only if $r_1 = r_2 = r$ and the corresponding isomorphism classes in $\mathbb{P}H^1(X(r), \mathcal{O}_{X(r)})$ belong to a same Aut$(X(r))$-orbit.*

Proof. We let $g_s = \{g_{s,ij}(x)\}_{i,j=1,\ldots,r} \in C^1\left(\{X_i\}_{i=1,\ldots,r}, \mathcal{O}_{X(r)}\right)$, $s = 1, 2$ be Čech cocycles representing the isomorphism classes of the $\mathbb{A}^1$-bundle structures on S_1 and S_2 respectively, and we let $S_{s,i} \simeq \mathrm{Spec}(k[x, u_{s,i}])$, $s = 1, 2$, $i = 1, \ldots, r$, be the corresponding trivializing open subsets of S_1 and S_2 respectively. The condition that the isomorphism classes of S_1 and S_2 belong to a same $\mathrm{Aut}(X(r))$-orbit means that there exists $\lambda \in k^*$ and an element $\phi = (a, \alpha) \in \mathrm{Aut}(X(r)) \simeq \mathbb{G}_{m,k} \times \mathfrak{s}_r$, where $\mathfrak{s}_r$ denotes the symmetric group of r elements, such that the Čech cocycles $\{g_{2,ij}(x)\}_{i,j=1,\ldots,r}$ and $\lambda \phi^* g_1 = \left\{\lambda g_{1,\alpha(i)\alpha(j)}(ax)\right\}_{i,j=1,\ldots,r}$ are cohomologous. Equivalently, there exists polynomials $\tau_i \in k[x]$, $i = 1, \ldots, r$ such that $g_{2,ij}(ax) = \lambda g_{1,\alpha(i)\alpha(j)}(x) + \tau_i(x) - \tau_j(x)$ for every, $i, j = 1, \ldots, r$. It follows that the local isomorphisms $\Phi_i : S_{1,i} \xrightarrow{\sim} S_{2,\alpha(i)}$ defined by the k-algebra isomorphisms $k\left[x, u_{2,\alpha(i)}\right] \xrightarrow{\sim} k[x, u_{1,i}]$, $x \mapsto ax$, $u_{2,\alpha(i)} \mapsto \lambda u_{1,i} + \tau_i(x)$, $i = 1, \ldots, r$ glue to a global one $\Phi : S_1 \xrightarrow{\sim} S_2$.

Conversely, suppose that there exists an isomorphism $\Phi : S_1 \xrightarrow{\sim} S_2$. Then it follows from remark 2.5 that $r_1 = r_2 = r$ for a certain $r \geq 1$. Example 2.11 implies that if one of the S_i's, say S_1, is isomorphic to a surface $S_{P_1,1} \subset \mathbb{A}^3_k$ defined by the equation $xz - P_1(y) = 0$, then so is S_2. For such surfaces, the assertion follows from lemma 2.10 in [5]. Otherwise, Theorem 2.9 and example 2.11 guarantee that the $\mathbb{A}^1$-fibrations $\pi_i : S_i \to \mathbb{A}^1_k$, $i = 1, 2$, are unique up to automorphisms of the base. It follows that Φ induces an isomorphism $\phi : X(r) \xrightarrow{\sim} X(r)$ such that $\phi \circ \rho_1 = \rho_2 \circ \Phi$. Therefore, $\Phi : S_1 \xrightarrow{\sim} S_2$ factors through an isomorphism $\tilde{\phi} : S_1 \xrightarrow{\sim} \phi^* S_2$ of $\mathbb{A}^1$-bundles over $X(r)$. Clearly, $\phi^* S_2$ is isomorphic to S_2, and so, the assertion follows. □

2.14. Let $\pi_l : S_l \to \mathbb{A}^1_k$, $l = 1, 2$ be special Danielewski surfaces with $\mathbb{A}^1$-bundle structures $\rho_l : S_l \to X(r)$ associated respectively with Čech cocycles

$$\left\{g_{l,ij} = x^{-h_l}(\sigma_{l,j} - \sigma_{l,i})\right\}_{i,j=1,\ldots,r} \in C^1\left(\{X_i\}_{i=1,\ldots,r}, \mathcal{O}_{X(r)}\right)$$

as in 2.4 (see also 2.10). It follows from the above result that S_1 and S_2 are isomorphic if and only if $h_1 = h_2 = h$ and there exists a datum $(\alpha, \mu, a) \in \mathfrak{s}_r \times k^* \times k^*$ such that the polynomial $c(x) = \sigma_{1,\alpha(i)}(ax) - \mu\sigma_{2,i}(x)$ does not depend on the index $i = 1, \ldots, r$. Futhermore, if neither S_1 nor S_2 is isomorphic to a surface $S_{P,1}$, then every isomorphism $\Phi : S_2 \stackrel{\sim}{\to} S_1$ is an isomorphism of fibered surfaces which is uniquely determined on the coverings of the S_l's by the open subsets $S_{l,i} = \mathrm{Spec}(k[x, u_{l,i}])$, $l = 1, 2$, $i = 1, \ldots, r$ by a collection of local isomorphisms $\Phi_i : S_{2,i} \stackrel{\sim}{\to} S_{1,\alpha(i)}$ induced by k-algebra isomorphisms

$$k\left[x, u_{1,\alpha(i)}\right] \stackrel{\sim}{\to} k[x, u_{2,i}], \quad x \mapsto ax, \ u_{\alpha(i)} \mapsto \mu a^{-h} u_i + b(x)$$

for suitable data $(\alpha, \mu, a, b(x)) \in \mathfrak{s}_r \times k^* \times k^* \times k[x]$ such that the polynomial $\sigma_{1,\alpha(i)}(ax) - \mu\sigma_{2,i}(x)$ does not depend on the index $i = 1, \ldots, r$.

3. Danielewski surfaces in $\mathbb{A}^3_k$ defined by an equation of the form $x^h z - Q(x, y) = 0$ and their automorphisms

In this section, we study Danielewski surfaces $\pi : S \to \mathbb{A}^1_k$ non isomorphic to $\mathbb{A}^2_k$ admitting a closed embedding $i : S \hookrightarrow \mathbb{A}^3_k$ as a surface $S_{Q,h}$ defined by an equation of the form $x^h z - Q(x, y) = 0$. A same abstract Danielewski surface S admits many such closed embeddings but we establish that it always admit a one as a surface $S_{\sigma,h}$ defined by an equation of the form $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$ for a suitable collection of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$. We study the automorphism groups of the above surfaces S. We show in particular that in an embedding of S as surface $S_{\sigma,h}$, every automorphism arises explicitly as the restriction of an automorphism of the ambient space. The results of the next section will show on the contrary that it is not true for a general embedding as a surface $S_{Q,h}$.

3.1. Danielewski surfaces $S_{Q,h}$

A surface $S = S_{Q,h}$ in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x, y) = 0$ is a Danielewski surface $\pi = \mathrm{pr}_x|_S : S \to \mathbb{A}^1_k$ if and only if the polynomial $Q(0, y)$ splits with simple roots $y_1, \ldots, y_r \in k$, where $r = \deg_y(Q(0, y))$. Such a surface is isomorphic to the affine plane $\mathbb{A}^2_k$ if and only if $r = 1$.

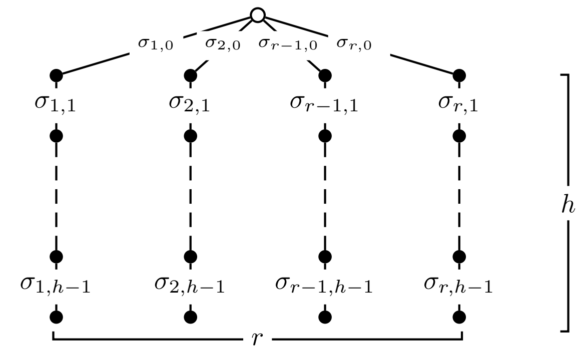
Example 3.1. Let $h \geq 1$ be an integer and let $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ be a collection of $r \geq 2$ polynomials $\sigma_i(x) = \sum_{j=0}^{h-1} \sigma_{i,j} x^j \in k[x]$ such that $\sigma_i(0) \neq \sigma_j(0)$ for every $i \neq j$. The surface $S = S_{\sigma,h}$ in $\mathbb{A}^3_k = \mathrm{Spec}(k[x, y, z])$ defined by the equation

$$x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$$

is a Danielewski surface $\pi = \mathrm{pr}_x|_S : S \to \mathbb{A}^1_k$. The fiber $\pi^{-1}(o)$ consists of r copies C_i of the affine line with defining equations $\{x = 0, y = \sigma_i(0)\}_{i=1,\ldots,r}$ respectively. For every index $i = 1, \ldots, r$, the open subset $S_i = S \setminus \bigcup_{j \neq i} C_j$ of S is isomorphic to the affine plane $\mathbb{A}^2_k = \mathrm{Spec}(k[x, u_i])$, where u_i denotes the regular function on S_i induced by the rational function

$$u_i = x^{-h}(y - \sigma_i(x)) = z \prod_{j \neq i} (y - \sigma_j(x))^{-1} \in k(S).$$

Thus $\pi : S \to \mathbb{A}^1_k$ factors through an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ isomorphic to the one with transition functions $(f_{ij}, g_{ij}) = \left(1, x^{-h}(\sigma_j(x) - \sigma_i(x))\right)$, $i, j = 1, \ldots, r$ (see 2.4). The collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ is exactly the one associated with the following fine k-weighted tree $\gamma = (\Gamma, w)$.



So S is isomorphic to the corresponding Danielewski surface $\pi_\gamma : S(\gamma) \to \mathbb{A}^1_k$. By construction, the canonical function ψ_γ on $S(\gamma)$ (see 2.6 above) coincides with the restriction of y on S under the above isomorphism. In the setting of Theorem 2.7, this means that γ corresponds to the Danielewski surface S equipped with the birational morphism $\mathrm{pr}_{x,y} : S \to \mathbb{A}^2_k$.

The following results shows that up to isomorphisms, the above class of Danielewski surfaces $S_{\sigma,h}$ contains all possible Danielewski surfaces $S_{Q,h}$.

Theorem 3.2. *Let $S_{Q,h}$ be a Danielewski surface in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$, where $Q(x,y) \in k[x,y]$ is a polynomial such that $Q(0,y)$ splits with $r \geq 2$ simple roots in k. Then there exists a collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ of polynomials of degrees $\deg(\sigma_i(x)) < h$ such that $S_{Q,h}$ is isomorphic to the surface $S_{\sigma,h}$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$.*

Proof. Using the fact that $Q(0,y)$ splits with simple roots $y_1, \ldots, y_r$ in k, one checks by succesive liftings modulo x^n, $n = 0, \ldots, h$, that the polynomial $Q(x,y)$ can be written in a unique way as

$$Q(x,y) = R_1(x,y) \prod_{i=1}^{r} (y - \sigma_i(x)) + x^h R_2(x,y),$$

where $R_1(x,y) \in k[x,y] \setminus \left(x^h k[x,y]\right)$ is a polynomial such that $R_1(0,y)$ is a nonzero constant and where $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ is a collection of polynomials of degree strictly lower than h such that $\sigma_i(0) = y_i$ for every index $i = 1, \ldots, r$. Since $y_i \neq y_j$ for every $i \neq j$ and $R_1(0,y)$ is a nonzero constant, it follows that for every index $i = 1, \ldots, r$, the rational function

$$u_i = x^{-h}(y - \sigma_i(x)) = \prod_{j \neq i} (y - \sigma_j(x))^{-1} R_1(x,y)^{-1} (z - R_2(x,y)) \in k(S_{Q,h})$$

restricts to a regular function on the complement S_i in $S_{Q,h}$ of the irreducible components of the fiber $\mathrm{pr}_x^{-1}(o)$ with defining equations $\{x = 0, y = y_j\}_{j \neq i}$ and induces an isomorphism $S_i \simeq \mathrm{Spec}(k[x, u_i])$. Therefore, the collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ coincides with the one derived from the fine k-weighted rake $\gamma = (\Gamma, w)$ with all its leaves at a same level h corresponding to the Danielewski surface $\mathrm{pr}_x : S_{Q,h} \to \mathbb{A}^1_k$ equipped with the birational morphism $\psi = \mathrm{pr}_{x,y} : S_{Q,h} \to \mathbb{A}^2_k$ (see 2.7 and 2.10 above). In turn, we deduce from Example 3.1 that the corresponding Danielewski surface $S(\gamma)$ embeds in $\mathbb{A}^3_k$ as the surface $S_{\sigma,h}$ in $\mathbb{A}^3_k$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$. □

Definition 3.3. Given a Danielewski surface S isomorphic to a certain surface $S_{Q,h}$ in $\mathbb{A}^3_k$, a closed embedding $i_s : S \hookrightarrow \mathbb{A}^3_k$ of S in $\mathbb{A}^3_k$ as a surface $S_{\sigma,h}$ defined by the equation

$$x^h z - \prod_{i=1}^{r} (y - \sigma_i(x)) = 0$$

is called a *standard embedding of* S. We say that $S_{\sigma,h}$ is a *standard form of* S *in* $\mathbb{A}^3_k$.

3.4. It follows from Theorem 3.2 above that every Danielewski surface $S_{Q,h}$ admits a standard embedding in $\mathbb{A}^3_k$. Isomorphisms between a Danielewski surface $S_{Q,h}$ and one of its standard forms $S_{\sigma,h}$ can be constructed explicitly as follows. Letting $Q(x,y) = R_1(x,y)\prod_{i=1}^r (y - \sigma_i(x)) + x^h R_2(x,y)$ be as in the proof of Theorem 3.2, the endomorphism Φ^s of $\mathbb{A}^3_k$ defined by

$$\Phi^s(x,y,z) \quad = \quad (x, y, R_1(x,y) z + R_2(x,y))$$

induces an isomorphism ϕ^s between $S_{\sigma,h}$ and $S_{Q,h}$. Indeed, one checks conversely that for every pair (f,g) of polynomials such that $R_1(x,y) f(x,y) + x^h g(x,y) = 1$, the endomorphism Φ_s of $\mathbb{A}^3_k$ defined by

$$\Phi_s(x,y,z) = \left(x, y, f(x,y) z + g(x,y)\prod_{i=1}^r (y - \sigma_i(x)) - f(x,y) R_2(x,y)\right)$$

induces an isomorphism ϕ_s between $S_{Q,h}$ and $S_{\sigma,h}$ such that $\phi^s \circ \phi_s = \mathrm{id}_{S_{Q,h}}$ and $\phi_s \circ \phi^s = \mathrm{id}_{S_{\sigma,h}}$.

3.5. The use of standard forms makes the study of isomorphism classes of Danielewski surfaces $S_{Q,h}$ simpler. For instance, we have the following characterization which generalises a result due to D. Daigle [5] and L. Makar-Limanov [18] for surfaces $S_{P,h}$ defined by equations $x^h z - P(y) = 0$.

Proposition 3.6. *Two Danielewski surfaces S_{σ_1,h_1} and S_{σ_2,h_2} in $\mathbb{A}^3_k$ defined by equations*

$$x^{h_1} z = P_1(x,y) = \prod_{i=1}^{r_1} (y - \sigma_{1,i}(x)) \quad \text{and} \quad x^{h_2} z = P_2(x,y) = \prod_{i=1}^{r_2} (y - \sigma_{2,i}(x))$$

are isomorphic if and only if $h_1 = h_2 = h$, $r_1 = r_2 = r$ and there exists a triple $(a, \mu, \tau(x)) \in k^* \times k^* \times k[x]$ such that $P_1(ax, y) = \mu^r P_2\left(x, \mu^{-1} y + \tau(x)\right)$.

If so, then the automorphism $(x,y,z) \mapsto \left(ax, \mu(y - \tau(x)), \mu^r a^{-h} z\right)$ of $\mathbb{A}^3_k$ induces an isomorphism between S_{σ_2,h_2} and S_{σ_1,h_1}.

Proof. If $S_1 = S_{\sigma_1,h_1}$ and $S_2 = S_{\sigma_2,h_2}$ are isomorphic, then $r_1 = r_2 = r$ by virtue of Remark 2.5 above. If $r = 1$, then the assertion holds trivially. So we may assume from now on that $r \geq 2$. It follows from example 3.1 and Theorem 2.9 that S_i admits two $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_k$ with distinct general fibers if and only if $h_i = 1$. Thus either $h_1 = h_2 = 1$ or $h_1, h_2 \geq 2$. In the first case, the result follows from Lemma 2.10 in [5]. Otherwise, if $h_1, h_2 \geq 2$, then it follows from Theorem 2.13 and 2.14 that $h_1 = h_2 = h$ and that there exists a datum $(\alpha, \mu, a) \in \mathfrak{s}_r \times k^* \times k^*$ such that the polynomial $c(x) = \sigma_{1,\alpha(i)}(ax) - \mu\sigma_{2,i}(x)$ does not depend on the index i. Letting $\tau(x) = \mu^{-1}c(x)$, this means exactly that $P_1(ax, y) = \mu^r P_2\left(x, \mu^{-1}y + \tau(x)\right)$. □

Corollary 3.7. *The standard embeddings of a same Danielewski surface $S_{Q,h}$ are all algebraically equivalent.*

3.8. It is known that there exists embeddings of Danielewski surfaces S as surfaces of the form $S_{Q,h}$ which are not equivalent to a standard one. For instance, this holds for the surface S_1 in $\mathbb{A}^3_{\mathbb{C}}$ defined by the equation $f_1 = x^2z - (1-x)\left(y^2-1\right) = 0$ as shown by G. Freudenburg and L. Moser-Jauslin [14]. Indeed, one checks that a standard form for S_1 is the Danielewski surface S_0 defined by the equation $f_0 = x^2z - \left(y^2-1\right) = 0$. So S_0 and S_1 can be considered as the images of two distinct closed embeddings of a same abstract Danielewski surface S in $\mathbb{A}^3_{\mathbb{C}}$. But the latter are not equivalent, due to the fact that the level surfaces of f_1 are all smooth whereas the level surface $f_0^{-1}(1)$ of f_0 is singular (so condition 3) in Definition 1.5 cannot be satisfied). The classification of embeddings with images $S_{Q,h}$ up to algebraic equivalence seems to be a difficult problem in general (see e.g. [19] for the case $h = r = 2$). This contrast with the following result.

Theorem 3.9. *The embeddings $i_{Q,h} : S \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$ of a Danielewski surface S as a surface defined by the equation $x^hz - Q(x,y) = 0$ are all* holomorphically *equivalent.*

Proof. Since all standard embeddings of a Danielewski surface are algebraically equivalent, it is enough to show that every embedding $i_{Q,h}$ is holomorphically equivalent to a standard one. Similarly as in the proof of Theorem 3.2, we may write $Q(x,y) = R_1(x,y)\prod_{i=1}^r(y-\sigma_i(x)) + x^hR_2(x,y)$, where $R_1(0,y) = \lambda \in \mathbb{C}^*$. Letting $f(x,y) \in \mathbb{C}[x,y]$ be a polynomial such that $\lambda\exp(xf(x,y)) \equiv R_1(x,y) \bmod x^h$, the result follows from the fact that the holomorphic automorphism Ψ of $\mathbb{A}^3_{\mathbb{C}}$ defined by

$$\Psi(x,y,z) = \left(x, y, \lambda\exp(xf(x,y))\,z - x^{-h}[\lambda\exp(xf(x,y)) - R_1(x,y)]\prod_{i=1}^r(y-\sigma_i(x)) + R_2(x,y)\right)$$

satisfies $\Psi^*\left(x^hz - Q(x,y)\right) = \lambda\exp(xf(x,y))\left(x^hz - \prod_{i=1}^r(y-\sigma_i(x))\right)$. □

Example 3.10. With the notation of 3.8 above, the biholomorphism

$$(x,y,z) \mapsto \left(x, y, e^{-x}z - x^{-2}\left(e^{-x} - 1 + x\right)\left(y^2-1\right)\right)$$

of $\mathbb{A}^3_{\mathbb{C}}$ maps the surface S_0 defined by the equation $x^2z - \left(y^2-1\right) = 0$ isomorphically onto the one S_1 defined by the equation $x^2z - (1-x)\left(y^2-1\right) = 0$.

3.2. Automorphisms of Danielewski surfaces $S_{Q,h}$ in $\mathbb{A}^3_k$

In [17] and [18], L. Makar-Limanov computed the automorphism groups of surfaces in

$\mathbb{A}^3$ defined by equations $x^h z - P(y) = 0$, where $h \geq 1$ and where $P(y)$ is an arbitrary polynomial. In particular, he established that every automorphism of such a surface is induced by the restriction of an automorphism of the ambient space. Recently, A. Crachiola [3] established that this also holds for surfaces defined by equations $x^h z - y^2 - \sigma(x) y = 0$, where $h \geq 1$ and where $\sigma(x)$ is a polynomial such that $\sigma(0) \neq 0$. This subsection is devoted to the study of automorphism groups of Danielewski surfaces $S_{Q,h}$. We begin with the case of standard forms.

Theorem 3.11. *Every automorphism of a Danielewski surface $S_{\sigma,h}$ defined by the equation*

$$x^h z - P(x,y) = 0, \qquad \text{where} \quad P(x,y) = \prod_{i=1}^{r} (y - \sigma_i(x))$$

is the restriction of an automorphism of $\mathbb{A}^3_k$ belonging to the subgroup $G_{\sigma,h}$ of $\mathrm{Aut}\left(\mathbb{A}^3_k\right)$ generated by the following automorphisms:

(a) $\Delta_b(x,y,z) = \left(x, y + x^h b(x), z + x^{-h}\left(P\left(x, y + x^h b(x)\right) - P(x,y)\right)\right)$, *where* $b(x) \in k[x]$.

(b) *If there exists a polynomial $\tau(x)$ such that $P(x, y + \tau(x)) = \tilde{P}(y)$ then the automorphisms $H_a(x,y,z) = \left(ax, y + \tau(ax) - \tau(x), a^{-h} z\right)$, where $a \in k^*$ should be added.*

(c) *If there exists a polynomial $\tau(x)$ such that $P(x, y + \tau(x)) = \tilde{P}(x^{q_0}, y)$, then the automorphisms $\tilde{H}_a(x,y,z) = \left(ax, y + \tau(ax) - \tau(x), a^{-h} z\right)$, where $a \in k^*$ and $a^{q_0} = 1$ should be added.*

(d) *If there exists a polynomial $\tau(x)$ such that $P(x, y + \tau(x)) = y^i \tilde{P}(x, y^s)$, where $i = 0, 1$ and $s \geq 2$, then the automorphisms $S_\mu(x,y,z) = \left(x, \mu y + (1-\mu)\tau(x), \mu^i z\right)$, where $\mu \in k^*$ and $\mu^s = 1$ should be added.*

(e) *If $char(k) = s > 0$ and $P(x,y) = \tilde{P}\left(y^s - c(x)^{s-1} y\right)$ for a certain polynomial $c(x) \in k[x]$ such that $c(0) \neq 0$, then the automorphism $T_c(x,y,z) = (x, y + c(x), z)$ should be added.*

(f) If $h = 1$, then the involution $I(x,y,z) = (z,y,x)$ should be added.

Remark 3.12. Automorphisms of type a) in Theorem 3.11 correspond to algebraic actions of the additive group $\mathbb{G}_{a,k}$ on the surface $S_{\sigma,h}$. More precisely, for every polynomial $b \in k[x]$, the subgroup $\left\{\Delta_{tb(x)}, t \in k\right\}$ of $\mathrm{Aut}(S_{\sigma,h})$ is isomorphic to $\mathbb{G}_{a,k}$, the corresponding $\mathbb{G}_{a,k}$-action on $S_{\sigma,h}$ being defined by $t \star (x,y,z) = \Delta_{tb(x)}(x,y,z)$. Similarly, automorphisms of type b) correspond to algebraic actions of the multiplicative group $\mathbb{G}_{m,k}$.

Proof. Clearly, every automorphism of $\mathbb{A}^3_k$ of type (a)-(f) above leaves $S_{\sigma,h}$ invariant, whence induces an automorphism of $S_{\sigma,h}$. If $h = 1$, then the converse follows from [17]. Otherwise, if $h \geq 2$, then by virtue of 2.14, every automorphism Φ of $S_{\sigma,h}$ is uniquely determined on the covering of $S_{\sigma,h}$ by open subsets $S_i = \mathrm{Spec}(k[x, u_i])$, $i = 1, \ldots, r$ by a collection of local isomorphisms induced by k-algebra isomorphisms $k\left[x, u_{\alpha(i)}\right] \xrightarrow{\sim} k[x, u_i]$, $x \mapsto ax$, $u_{\alpha(i)} \mapsto \mu a^{-h} u_i + b(x)$ for a datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x)) \in \mathfrak{s}_r \times k^* \times k^* \times k[x]$ such that the polynomial $c(x) = \sigma_{\alpha(i)}(ax) - \mu\sigma_i(x)$ does not depend on the index $i = 1, \ldots, r$. By construction of the closed embedding of $S_{\sigma,h}$ in $\mathbb{A}^3_k$ given in Example

3.1, Φ is induced by the restriction to $S_{\sigma,h}$ of the automorphism

$$\Psi(x,y,z)=\left(ax,\mu y+\tilde{c}(x),a^{-h}\mu^{r}z+(ax)^{-h}\left(\prod_{i=1}^{r}(\mu y+\tilde{c}(x)-\sigma_i(ax))-\mu^{r}\prod_{i=1}^{r}(y-\sigma_i(x))\right)\right)$$

of $\mathbb{A}_k^3$, where $\tilde{c}(x)=c(x)+x^h b(x)$. Using these descriptions, one checks that the composition of two automorphisms Φ_1 and Φ_2 of $S_{\sigma,h}$ with data $\mathcal{A}_{\Phi_1}=(\alpha_1,\mu_1,a_1,b_1)$ and $\mathcal{A}_{\Phi_2}=(\alpha_2,\mu_2,a_2,b_2)$ respectively is the automorphism of $S_{\sigma,h}$ with datum $\mathcal{A}=\left(\alpha_2\circ\alpha_1,\mu_2\mu_1,a_2a_1,a_2^{-h}\mu_2 b_1(x)+b_2(a_1x)\right)$. Automorphisms of type (a) coincide with the ones determined by data $\mathcal{A}=(\mathrm{Id},1,1,b(x))$, where $b(x)\in k[x]$. In view of the composition rule above, it suffices to consider from now on automorphisms Φ corresponding to data $\mathcal{A}_\Phi=(\alpha,\mu,a,0)$.

1) If α is trivial, then $\mu=1$ by virtue of Lemma 3.14 below, and so $\mathcal{A}_\Phi=(\mathrm{Id},1,a,0)$. Then, the relation $c(x)=\sigma_i(ax)-\sigma_i(x)$ holds for every $i=1,\ldots,r$.

1a) If $a^q\neq 1$ for every $q=1,\ldots,h-1$, then there exists a polynomial $\tau(x)\in k[x]$ such that $\sigma_i(x)=\sigma_i(0)+\tau(x)$ for every $i=1,\ldots,r$. Thus $c(x)=\tau(ax)-\tau(x)$, $P(x,y+\tau(x))=\tilde{P}(y)=\prod_{i=1}^{r}(y-\sigma_i(0))$, and Φ is of type (b).

1b) If $a\neq 1$ but $a^{q_0}=1$ for a minimal $q_0\in\{2,\ldots,h-1\}$, then there exists polynomials $\tau(x)$ and $\tilde{\sigma}_i(x)$, $i=1,\ldots,r$, such that $\sigma_i(x)=\tilde{\sigma}_i(x^{q_0})+\tau(x)$ for every $i=1,\ldots,r$. So there exists a polynomial $\tilde{P}$ such that $P(x,y+\tau(x))=\tilde{P}(x^{q_0},y)$. Moreover, $c(x)=\tau(ax)-\tau(x)$, and so, Φ is of type (c).

2) If α is not trivial then, by virtue of Lemma 3.14 below, all the nontrivial cycles occuring in a decomposition of α have the same length $s\geq 2$ and $\mu^s=1$. Since $\Phi=\Phi_2\circ\Phi_1$, where Φ_1 and Φ_2 denote the automorphisms with data $\mathcal{A}_{\Phi_1}=(\mathrm{Id},1,a,0)$ and $\mathcal{A}_{\Phi_2}=(\alpha,\mu,1,0)$ respectively, it suffices to consider the situation that Φ is determined by a datum $\mathcal{A}_\Phi=(\alpha,\mu,1,0)$, where $\mu\in k^*$ and $\mu^s=1$. So the relation $\sigma_{\alpha(i)}(x)=\mu\sigma_i(x)+c(x)$ holds for every $i=1,\ldots,r$.

2a) If $\mu^s=1$ but $\mu^{s'}\neq 1$ for every $s'\in\{1,\ldots,s-1\}$, then, letting $\tau(x)=(1-\mu)^{-1}c(x)$ and $\tilde{\sigma}_i(x)=\sigma_i(x)-\tau(x)$ for every $i=1,\ldots,r$, we arrive at the relation $\tilde{\sigma}_{\alpha(i)}(x)=\mu\tilde{\sigma}_i(x)$ for every $i=1,\ldots,r$. Furthermore, if i_0 is a unique fixed point of α (see Lemma 3.14 below), then $\tilde{\sigma}_{i_0}(x)=0$. It follows that $P(x,y+\tau(x))=y^i\tilde{P}(x,y^s)$ where i is either 0 or 1. Clearly, Φ is of type (d).

2b) If $\mu=1$ then $\mathrm{char}(k)=s$ and α is fixed point free by virtue of Lemma 3.14. Moreover, $s'\cdot c(0)\neq 0$ for every $s'\in\{1,\ldots,s-1\}$ and $\sigma_{i_m}(x)=\sigma_{i_1}(x)+(m-1)\cdot c(x)$ for every index i_m occurring in a cycle $(i_1,\ldots,i_s)$ of length s in α. Letting $r=ds$, we may suppose up to a reordering that α decomposes as the product of the standard cycles $(is+1,is+2\ldots,(i+1)s)$, where $i=0,\ldots,d-1$. Letting $G(x,y)=\prod_{m=1}^{s}(y-m\cdot c(x))=y^s-c(x)^{s-1}y$, we conclude that

$$P(x,y)=\prod_{i=0}^{d-1}G(x,y-\sigma_{is}(x))=\tilde{P}\left(x,y^s-c(x)^{s-1}y\right)$$

for a suitable polynomial $\tilde{P}(x,y)\in k[x,y]$, and that Φ is of type (e).

□

3.13. In the proof of Theorem 3.11 above, we used the fact that every automorphism Φ of a Danielewski surface $S_{\sigma,h}$, where $h\geq 2$, is determined by a certain datum $\mathcal{A}_\Phi=(\alpha,\mu,a,b(x))\in\mathfrak{s}_r\times k^*\times k^*\times k[x]$ satisfying the following properties.

Lemma 3.14. *For a datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x))$ corresponding to an automorphism Φ of $S_{\sigma,h}$, the following assertions hold :*

1) The permutation α is either trivial or has at most one fixed point. If it is nontrivial then all nontrivial cycles with disjoint support occurring in a decomposition of α have the same length $s \geq 2$ and $\mu^s = 1$. Otherwise, α is trivial and $\mu = 1$.

2) If α is nontrivial then either $\mu^{s'} \neq 1$ for every $1 \leq s' < s$, or $\mu = 1$, char$(k) = s$, and α is fixed point free.

Proof. To simplify the notation, we let $y_i = \sigma_i(0)$ for every $i = 1, \ldots, r$. Note that by hypothesis, $y_i \neq y_j$ for every $i \neq j$. If $\alpha \in \mathfrak{s}_r$ has at least two fixed points, say i_0 and i_1, then $y_{i_0}(1-\mu) = y_{i_1}(1-\mu) = c(0)$, and so, $\mu = 1$ and $c(0) = 0$ (recall that $c(x) = \sigma_{\alpha(i)}(ax) - \mu\sigma_i(x)$). In turn, this implies that α is trivial. Indeed, otherwise there would exist an index i such that $\alpha(i) \neq i$ but $y_{\alpha(i)} = y_i$, in contradiction with our hypothesis. Suppose from now that α is nontrivial and let $s \geq 2$ be the infimum of the length's of the nontrivial cycles occurring in decomposition of α into a product of cycles with disjoint supports. It follows that $y_i(1-\mu^s) = y_j(1-\mu^s)$ for every pair of distinct indices i and j in the support of a same cycle of length s. Thus $\mu^s = 1$ as $y_i \neq y_j$ for every $i \neq j$.

If $\mu = 1$ then $s' \cdot c(0) \neq 0$ for every $s' = 1, \ldots, s-1$. Indeed, otherwise we would have $y_{\alpha^{s'}(i)} = y_i + s' \cdot c(0) = y_i$ for every index $i = 1, \ldots, r$ which is impossible since α is nontrivial. In particular, α is fixed-point free. On the other hand $s \cdot c(0) = 0$ as $y_i = y_{\alpha^s(i)} = y_i + s \cdot c(0)$ for every index i in the support of a cycle of length s in α. This is possible if and only if the characteristic of the base field k is exactly s. We also conclude that every cycle in α have length s for otherwise there would exist an index i such that $\alpha^s(i) \neq i$ but $y_{\alpha^s(i)} = y_i + s \cdot c(0) = y_i$ in contradiction with our hypothesis.

If $\mu \neq 1$ then $\mu^{s'} \neq 1$ for every $s' < s$. Indeed, otherwise there would exist an index i such that $\alpha^{s'}(i) \neq i$ but $y_{\alpha^{s'}(i)} = \mu^{s'} y_i + c(0) \sum_{p=0}^{s'-1} \mu^p = y_i$, which is impossible. The same argument also implies that all the nontrivial cycles in α have length s. □

3.15. By combining Theorems 3.2 and 3.11, we obtain the following description of automorphism groups of Danielewski surfaces $S_{Q,h}$.

Corollary 3.16. *Let $S_{Q,h}$ be a Danielewski surface in $\mathbb{A}^3_k$ with equation $x^h z - Q(x,y) = 0$ and let $S_{\sigma,h}$ be one of its standard forms. Then, every automorphism of $S_{Q,h}$ is induced by an endomorphism $\Phi^s \circ \psi \circ \Phi_s$ of $\mathbb{A}^3_k$, where ψ belongs to the subgroup $G_{\sigma,h}$ of the automorphism group of $\mathbb{A}^3_k$ defined in Theorem 3.11 and where Φ^s and Φ_s are endomorphisms of $\mathbb{A}^3_k$ inducing isomorphisms between $S_{Q,h}$ and $S_{\sigma,h}$ as in 3.4.*

3.17. It follows from 3.8 that there exists embeddings $S_{Q,h}$ which are not equivalent to a standard one. This may lead to suspect that in a general embedding $S_{Q,h}$, certain automorphisms of $S_{Q,h}$ do not extend to algebraic automorphisms of $\mathbb{A}^3_k$. We give examples for which this phenomenon occurs in the next section. In contrast, if $k = \mathbb{C}$, then Theorem 3.9 implies the following result.

Corollary 3.18. *Every algebraic automorphism of a Danielewski surface $S_{Q,h}$ in $\mathbb{A}^3_{\mathbb{C}}$ is extendable to a* holomorphic *automorphism of $\mathbb{A}^3_{\mathbb{C}}$.*

4. Special Danielewski surfaces and multiplicative group actions

In this section, we fix a base field k of characteristic zero and we consider special Danielewski surfaces S admitting a nontrivial action of the multiplicative group $\mathbb{G}_m = \mathbb{G}_{m,k}$. We establish that every such surface is isomorphic to a one $S_{Q,h}$ admitting a standard embedding in $\mathbb{A}^3_k$ as a surface defined by an equation of the form $x^h z - P(y) = 0$ for a suitable polynomial $P(y) \in k[y]$. In this embedding, every multiplicative group action on S arises as the restriction of a linear $\mathbb{G}_m$-action on $\mathbb{A}^3_k$. We show on the contrary that there exists embeddings $S_{Q,h}$ for which the $\mathbb{G}_m$-action on $S_{Q,h}$ cannot be extended to an action on $\mathbb{A}^3_k$.

4.1. Multiplicative group actions on special Danielewski surfaces

4.1. By combining Theorem 2.13, Proposition 3.6 and Theorem 3.11, we deduce that a Danielewski surface $S_{Q,h}$ admits a nontrivial $\mathbb{G}_m$-action if and only if it is isomorphic to a surface $S_{P,h}$ in $\mathbb{A}^3_k$ defined by an equation of the form $x^h z - P(y) = 0$ for a certain polynomial $P(y)$. For such surfaces, the $\mathbb{G}_m$-action even arises as the restriction of a linear $\mathbb{G}_m$-action Ψ on $\mathbb{A}^3_k$ defined by $\Psi(a; x, y, z) = H_a(x, y, z) = \left(ax, y, a^{-h} z\right)$. More generally, we have the following result.

Theorem 4.2. *A special Danielewski surface S admits a nontrivial action of the multiplicative group $\mathbb{G}_m$ if and only if it is isomorphic to a surface $S_{P,h}$ in $\mathbb{A}^3_k$ with equation $x^h z - P(y) = 0$.*

Proof. We may suppose that $S = S(\gamma)$ is the Danielewski surface associated to a fine k-weighted tree $\gamma = (\Gamma, w)$ with $r \geq 2$ elements at level 1 and with all its leaves at a same level $h \geq 1$. We denote by $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ the collection of polynomial associated with γ. It follows from the gluing construction 2.4 that the collection $\tilde{\sigma}$ defined by

$$\tilde{\sigma}_i(x) = \sigma_i(x) - \frac{1}{r}\sum_{i=1}^{r} \sigma_i(x) \quad i = 1, \ldots, r$$

leads to a Danielewski surface isomorphic to S. So we may suppose from the beginning that $\sigma_1(x) + \cdots + \sigma_r(x) = 0$. If $h = 1$, then S is isomorphic to a surface in $\mathbb{A}^3_k$ defined by an equation of the form $xz - P(y) = 0$. Otherwise, if $h \geq 2$ then it follows from Theorem 2.9 that the structural $\mathbb{A}^1$-fibration $\pi_\gamma : S = S(\gamma) \to \mathbb{A}^1_k$ is unique up to automorphisms of the base. We consider S as an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ defined by the cocycle $g = \left\{g_{ij} = x^{-h}(\sigma_j(x) - \sigma_i(x))\right\}_{i,j=1,\ldots,r}$. Again, see 2.14, every automorphism Φ of S is determined by a datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x)) \in \mathfrak{s}_r \times k^* \times k^* \times k[x]$ for which the polynomial $\sigma_{\alpha(i)}(ax) - \mu\sigma_i(x) \in k[x]$ does not depend on the index i. In view of the composition rule given in the proof of Theorem 2.14, an automorphism Φ of S can belong to a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathbb{G}_m$ only if it corresponds to a datum of the form $\mathcal{A}_\Phi = (\alpha, \mu, a, 0)$. Let Φ be a nontrivial automorphism determined by such a datum $\mathcal{A}_\Phi$. Then, since $\alpha \in \mathfrak{s}_r$, there exists an integer $N \geq 1$ such that the polynomial $d(x) = \sigma_i\left(a^N x\right) - \mu^N \sigma_i(x)$ does not depend on the index $i = 1, \ldots, r$. Since $\sigma_1(x) + \cdots + \sigma_r(x) = 0$, it follows that $d(x) = 0$, whence that $\sigma_i\left(a^N x\right) = \mu^N \sigma_i(x)$ for every $i = 1, \ldots, r$. In turn, this implies that $\mu^N = 1$. Indeed, there exists at least one index i such that $\sigma_i(0) \neq 0$ as γ is a fine k-weighted tree with at least two elements a level 1. Suppose that one of the polynomials σ_i is not constant. Then the above identity implies that $a^{Np} = 1$ for a certain integer p. Therefore, every automorphism Φ of S with associated datum $(\alpha, \mu, a, 0)$ has finite order, and so, $\mathrm{Aut}(S)$ cannot contain a subgroup isomorphic to $\mathbb{G}_m$. So, S admits a nontrivial $\mathbb{G}_m$-action only if the polynomials σ_i, $i = 1, \ldots, r$ are constant. This completes the proof since these collections $\{\sigma_i\}_{i=1,\ldots,r}$ correspond to Danielewski surfaces $S_{P,h}$ (see example 3.1). □

4.2. Extensions of multiplicative group actions on a Danielewski surface

It follows from Theorems 3.11 and 4.2 that every special Danielewski surface S equipped with a nontrivial $\mathbb{G}_m$-action admits an equivariant embedding in $\mathbb{A}^3_k$ as a surface $S_{P,h}$. In contrast, the following result shows that there exists closed embeddings of S as surfaces $S_{Q,h}$ for which no nontrivial $\mathbb{G}_m$-action on S extends to an action on the ambient space $\mathbb{A}^3_k$.

Theorem 4.3. *Every Danielewski surface $S \subset \mathbb{A}^3_k$ defined by an equation of the form*

$$x^h z - (1-x) P(y) = 0,$$

where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial $\mathbb{G}_m$-action $\tilde{\theta} : \mathbb{G}_m \times S \to S$ which is not algebraically extendable to $\mathbb{A}^3_k$. More precisely, for every $a \in k^* \setminus \{1\}$ the automorphism $\tilde{\theta}_a = \tilde{\theta}(a, \cdot)$ of S does not extend to an algebraic automorphism of $\mathbb{A}^3_k$.

Proof. The endomorphisms Φ^s and Φ_s of $\mathbb{A}^3_k$ defined by $\Phi^s(x,y,z) = (x, y, (1-x)z)$ and $\Phi_s(x,y,z) = \left(x, y, (\sum_{i=0}^{h-1} x^i) z + P(y)\right)$ respectively induce isomorphisms $\phi^s : S_{P,h} \xrightarrow{\sim} S$ and $\phi_s : S \xrightarrow{\sim} S_{P,h}$ between S and the surface $S_{P,h}$ defined by the equation $x^h z - P(y) = 0$ (see 3.4). The latter admits an action $\theta : \mathbb{G}_m \times S_{P,h} \to S_{P,h}$ of the multiplicative group $\mathbb{G}_m$ defined by $\theta(a,x,y,z) = H_a(x,y,z) = \left(ax, y, a^{-h} z\right)$ for every $a \in k^*$. The corresponding action $\tilde{\theta}$ on S is therefore defined by $\tilde{\theta}(a,x,y,z) = \tilde{\theta}_a(x,y,z) = \phi^s \circ H_a(x,y,z)\mid_{S_{P,h}} \circ \phi_s$. Since by construction, $(\tilde{\theta}_a)^*(x) = ax$ for every $a \in k^*$, the assertion is a consequence of the next Lemma. □

Lemma 4.4. *An algebraic automorphism Φ of $\mathbb{A}^3_k$ extending a one of S satisfies $\Phi^*(x) = x$.*

Proof. The proof is similar to the one of Theorem 2.1 in [19]. We let Φ be an automorphism of $\mathbb{A}^3_k$ extending an arbitrary automorphism of S. Since $f_1 = x^h z - (1-x) P(y)$ is an irreducible polynomial, there exists $\mu \in k^*$ such that $\Phi^*(f_1) = \mu f_1$. Therefore, for every $t \in k$, the automorphism Φ induces an isomorphism between the level surfaces $f_1^{-1}(t)$ and $f_1^{-1}\left(\mu^{-1} t\right)$ of f_1. There exists an open subset $U \subset \mathbb{A}^1_k$ such that for every $t \in U$, $f_1^{-1}(t)$ is a special Danielewski surface isomorphic to a one defined by a fine k-weighted rake γ whose underlying tree Γ is isomorphic to the one associated with S. Since Γ is not a comb, it follows from Theorem 2.9 that for every $t \in U$, the projection $\mathrm{pr}_x : f_1^{-1}(t) \to \mathbb{A}^1_k$ is a unique $\mathbb{A}^1$-fibration on $f_1^{-1}(t)$ up to automorphisms of the base. Furthermore, $\mathrm{pr}_x : f_1^{-1}(t) \to \mathbb{A}^1_k$ has $\mathrm{pr}_x^{-1}(o)$ as a unique degenerate fiber. Therefore, for every $t \in U$, the image of the ideal $(x, f_1 - t)$ of $k[x,y,z]$ by Φ^* is contained in the ideal $(x, \mu f_1 - t) = \left(x, P(y) + \mu^{-1}t\right)$, and so $\Phi^*(x) \in \bigcap_{t \in U} \left(x, P(y) + \mu^{-1}t\right) = (x)$. Since Φ is an automorphism of $\mathbb{A}^3_k$, we conclude that there exists $c \in k^*$ such that $\Phi^*(x) = cx$. In turn, this implies that for every $t, u \in k$, Φ induces an isomorphism between the surfaces $S_{t,u}$ and $\tilde{S}_{t,u}$ defined by the equations $f_1 + tx + u = x^h z - (1-x) P(y) + tx + u = 0$ and $f_1 + \mu^{-1}ctx + \mu^{-1}u = x^h z - (1-x) P(y) + \mu^{-1}ctx + \mu^{-1}u = 0$ respectively. Since $\deg(P) \geq 2$ there exists $y_0 \in k$ such that $P'(y_0) = 0$. Note that y_0 is not a root of P as these ones are simple. We let $t = -u = -P(y_0)$. Since $h \geq 2$, it follows from the Jacobian Criterion that $S_{t,u}$ is singular, and even non normal, along the nonreduced component of the fiber $\mathrm{pr}_x^{-1}(o)$ defined by the equation $\{x = 0;\ y = y_0\}$. Therefore $\tilde{S}_{t,u}$ must be singular along a multiple component of the fiber $\mathrm{pr}_x^{-1}(o)$. This is the case if and only if the polynomial $P(y) - \mu^{-1}P(y_0)$ has a multiple root, say y_1, such that $P(y_1) + \mu^{-1}cP(y_0) = 0$. Since $P(y_0) \neq 0$ this condition is satisfied if and only if $c = 1$. This completes the proof. □

Example 4.5. Let S be the surface defined by the equation $x^2 z - (1-x) P(y) = 0$. It follows from the above results that even the involution of S induced by the endomorphism

$$J(x,y,z) = (-x, y, (1+x)((1+x)z + P(y)))$$

of $\mathbb{A}^3_k$ does not extend to an algebraic automorphism of $\mathbb{A}^3_k$.

If $k = \mathbb{C}$, then it follows from Theorem 3.9 that every embedding of a Danielewski surface S as a surface $S_{Q,h}$ is holomorphically equivalent to a standard one. Combined with the above discussion, this leads to the following result.

Proposition 4.6. *Every surface $S \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $x^h z - (1-x) P(y) = 0$, where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial algebraic $\mathbb{G}_m$-action which is algebraically non-extendable but holomorphically extendable to $\mathbb{A}^3_{\mathbb{C}}$.*

Remark 4.7. The above phenomenon does not occur with additive group actions. Indeed, one checks that every $\mathbb{G}_{a,k}$-action on a Danielewski $S_{Q,h}$ in $\mathbb{A}^3_k$ with equation $x^h z - Q(x,y) = 0$ arises as the restriction of a $\mathbb{G}_{a,k}$-action on $\mathbb{A}^3_k$ defined by

$$\tilde{\Delta}(t,x,y,z) = \left(x, y + x^h b(x)t, z + x^{-h}(Q(x, y + x^h b(x)t) - Q(x,y))\right),$$

for a certain polynomial $b(x) \in k[x]$. This contrasts with an example, given by H. Derksen, F. Kutzschebauch and J. Winkelmann in [8], of a non-extendable $\mathbb{G}_{a,\mathbb{C}}$-action on an hypersurface in $\mathbb{A}^5_{\mathbb{C}}$ which is even not holomorphically extendable.

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