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REAL FORMS OF SOME GIZATULLIN SURFACES AND KORAS–RUSSELL THREEFOLDS

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Abstract: We describe the real forms of Gizatullin surfaces of the form $xy = p(z)$ and of Koras–Russell threefolds of the first kind. The former admit zero, two, three, four, or six isomorphism classes of real forms, depending on the degree and the symmetries of the polynomial p. The latter, which are threefolds given by an equation of the form $x^d y + z^k + x + t^\ell = 0$, all admit exactly one real form up to isomorphism.

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1. Introduction

Given a complex algebraic variety X, a *real form* of X is a real algebraic variety Y whose complexification is isomorphic to X. It is then natural to ask whether X has one, only one, finitely many or infinitely many isomorphism classes of real forms. Here we study the case where X is affine. The most natural examples to look at in this context are the affine spaces. For any $n \geq 1$, an obvious real form of $\mathbb{A}^n_{\mathbb{C}}$ is $\mathbb{A}^n_{\mathbb{R}}$. For $n \leq 2$, it turns out to be the only one up to isomorphism. This is a nice exercise for $n = 1$, and for $n = 2$ it is a result of Kambayashi in [**16**, Theorem 3] based on the amalgamated free product structure of $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}})$. For $n \geq 3$, it is still unknown whether $\mathbb{A}^n_{\mathbb{C}}$ admits any nontrivial real form.

In this article, we investigate some affine surfaces and threefolds which are close to the affine plane and space.

Recall that a *Gizatullin surface* is a normal complex affine surface completable by a *zigzag*, that is, by a simple normal crossing divisor with rational components and a linear dual graph; for more details see [**12**]. These surfaces are classical generalisations of the affine plane. For instance, a smooth affine surface is quasihomogeneous (that is, its automorphism group admits an open orbit with finite complement) if and

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only if it is a Gizatullin surface or isomorphic to $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2$; see [**14**]. Moreover, by [**7**, Theorem], a normal complex affine surface admits two $(\mathbb{C},+)$-actions with different general fibres if and only if it is a Gizatullin surface not isomorphic to $\mathbb{A}^1_{\mathbb{C}}\times(\mathbb{A}^1_{\mathbb{C}}\setminus\{0\})$. In the latter case, the zigzag can be chosen to have a sequence of self-intersections $(0,-1,-a_1,\dots,-a_r)$, with $a_1,\dots,a_r \geq 2$ (see for instance [**2**]).

The case $r=0$ is the affine plane $\mathbb{A}^2_{\mathbb{C}}$. The case $r=1$ corresponds to the surfaces $D_p = \mathrm{Spec}(\mathbb{C}[x,y,z]/(xy-p(z)))$, where $p\in\mathbb{C}[z]$ is of degree at least 2, called *Danielewski surfaces* by some authors. For $r=2$, there are Gizatullin surfaces with uncountably many nonisomorphic real forms, as the second author recently proved in [**4**]. In this text, we compute the number of isomorphism classes of real forms of all surfaces D_p, and show in particular that this number is finite for all of them.

We first establish in Proposition 3.11 that D_p admits a real form if and only if there exist $a,\lambda\in\mathbb{C}^*$, $b\in\mathbb{C}$, such that $\lambda p(az+b)\in\mathbb{R}[z]$. In this case, we can assume that $p\in\mathbb{R}[z]$, and moreover that p is in *reduced* form as defined in Definition 3.3, i.e., that $p(z)=z^d+s(z)$ for some integer d and some polynomial $s\in\mathbb{R}[z]$ with $\deg(s)\leq d-2$. We then obtain the full list of isomorphism classes of real forms for any such surface in Propositions 3.19, 3.20, and 3.21, summarised as follows:

Theorem A. *Let $p\in\mathbb{R}[z]$ be a polynomial of degree $d\geq 2$ in reduced form. Write $p(z)=z^m q(z^n)$, where $m\geq 0$, $n\geq 1$, $q\in\mathbb{R}[z]$, $q(0)\neq 0$, and where q, n are chosen such that n is maximal if $q\neq 1$. For all $a,b,c\in\{0,1\}$, the surface*

$$S_{abc} = \mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2+(-1)^a y^2+(-1)^b z^m q((-1)^c z^n)))$$

is a real form of the Gizatullin surface $D_p=\mathrm{Spec}(\mathbb{C}[x,y,z]/(xy-p(z)))$. Moreover, the number i of isomorphism classes of real forms of D_p and the representatives are related as follows.

i	*Representatives*	*Conditions on q, n, d*	
2	S_{000}, S_{110}	$q=1$, $d=2$	$q=1$, $d\geq 3$ *odd*
3	S_{000}, S_{010}, S_{110}	$q=1$, $d\geq 4$ *even*	$q\neq 1$, n *odd*
4	S_{abb}, $a,b\in\{0,1\}$	$q\neq 1$, n *even*, d *odd*	$q\neq 1$, $(n,d)=(2,2)$
6	S_{00c}, S_{a1c}, $a,c\in\{0,1\}$	$q\neq 1$, n, d *both even*, $(n,d)\neq(2,2)$	

Just as for the affine plane, the automorphism group of a surface D_p has the structure of a free product of two subgroups amalgamated over their intersection (see Theorem 3.6 below, or [**2**, Theorem 5.4.5]). The situation is, however, more complicated than for $\mathbb{A}^2_{\mathbb{C}}$, since the cohomology pointed sets of the two factors are not trivial.

In the particular case of the affine quadric $\mathrm{Spec}(\mathbb{R}[x,y,z]/(xy-z^2+1))$, Theorem A provides exactly four isomorphism classes of real forms, given by $\mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 + z^2 \pm 1))$. This rectifies a similar claim in the introduction of [**9**], where only three of the four real forms were given.

To complete our study of real forms of affine surfaces, we consider in Section 4 the surfaces $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2$ and $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ mentioned above. We prove that they admit six and four isomorphism classes of real forms, respectively.

Following the examination in dimension two, we move to the study of three-dimensional affine varieties. We investigate the Koras–Russell threefolds of the first kind in Section 5. We recall that they are defined as the hypersurfaces

$$X_{d,k,\ell} = \{x^d y + z^k + x + t^\ell = 0\} \subset \mathbb{A}^4_{\mathbb{C}},$$

where $d \geq 2$ and $2 \leq k < \ell$ are integers with k and ℓ relatively prime, and that they are all smooth affine contractible, and hence diffeomorphic to $\mathbb{R}^6$ when equipped with the Euclidean topology [**5**]. They are furthermore $\mathbb{A}^1_{\mathbb{C}}$-contractible in the $\mathbb{A}^1_{\mathbb{C}}$-homotopy sense [**8**]. Nevertheless, none of them is isomorphic to $\mathbb{A}^3_{\mathbb{C}}$ as an algebraic variety [**19, 15**]. We also recall that two important questions about them are still wide open for all d, k, ℓ: it is not known whether $X_{d,k,\ell}$ is biholomorphic to $\mathbb{A}^3_{\mathbb{C}}$, nor whether its cylinder $X_{d,k,\ell} \times \mathbb{A}^1_{\mathbb{C}}$ is isomorphic to $\mathbb{A}^4_{\mathbb{C}}$ (algebraically or analytically).

We prove in Subsection 5.2 that no Koras–Russell threefold of the first kind admits nontrivial real forms.

Theorem B. *For all integers d, k, ℓ with $d \geq 2$ and $2 \leq k < \ell$ with k and ℓ relatively prime, every real form of the Koras–Russell threefold*

$$X_{d,k,\ell} = \mathrm{Spec}(\mathbb{C}[x,y,z,t]/(x^d y + z^k + x + t^\ell))$$

is isomorphic to the real surface $\mathrm{Spec}(\mathbb{R}[x,y,z,t]/(x^d y + z^k + x + t^\ell))$.

To achieve this result, we use the structure of the automorphism group of the threefold $X_{d,k,\ell}$ as a subnormal series as computed in [**10, 20**] (see Proposition 5.8). The factor groups being isomorphic to $\mathbb{C}^*$, $(\mathbb{C}[x,z],+)$, or $\{f \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x,z,t]) \mid f \equiv \mathrm{id} \mod (x^d)\}$, the key step in the proof of Theorem B is then to show that the first cohomology pointed set of this latter group is trivial for any $d \geq 0$. Note that the triviality of this group for $d = 0$ also implies that every real structure of $\mathbb{A}^3_{\mathbb{C}}$ compatible with the projection along one coordinate is equivalent to the standard real structure of $\mathbb{A}^3_{\mathbb{C}}$; see Proposition 5.4.

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2. Notation, definitions, and reminders

2.1. Polynomial maps and variables.

Notation 2.1. Let $n \geq 1$ be an integer and R be a commutative algebra over a field $\mathbf{k}$. We denote by $\mathrm{End}(\mathbb{A}^n_R) = \mathrm{End}_R(\mathbb{A}^n_R)$ the monoid of algebraic endomorphisms of $\mathbb{A}^n_R = \mathbb{A}^n_{\mathbf{k}} \times_{\mathrm{Spec}(\mathbf{k})} \mathrm{Spec}(R)$. These are the morphisms of the form

$$f\colon (x_1,\ldots,x_n) \longmapsto (f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)),$$

where $f_1,\ldots,f_n \in R[x_1,\ldots,x_n]$. As usual, we shall denote such a morphism simply by $f=(f_1,\ldots,f_n)$ and often replace the variables x_1, x_2, x_3 by x, y, z if $n \leq 3$.

Given $f=(f_1,\ldots,f_n) \in \mathrm{End}(\mathbb{A}^n_R)$, we denote by f^* the corresponding R-algebra endomorphism of $R[x_1,\ldots,x_n]$ defined by $f^*(P)=P(f_1,\ldots,f_n)$ for all $P \in R[x_1,\ldots,x_n]$. In particular, $f^*(x_i) = f_i$ for $i = 1,\ldots,n$.

Notation 2.2. We denote by $\mathrm{Aut}(\mathbb{A}^n_R) = \mathrm{Aut}_R(\mathbb{A}^n_R)$ the group of algebraic automorphisms of $\mathbb{A}^n_R$ over $\mathrm{Spec}(R)$, by

$$\mathrm{Aff}_n(R) = \{f \in \mathrm{Aut}(\mathbb{A}^n_R) \mid \deg(f^*(x_i)) = 1 \text{ for all } 1 \leq i \leq n\}$$

the subgroup of affine automorphisms, and by

$$\mathrm{BA}_n(R) = \{f \in \mathrm{Aut}(\mathbb{A}^n_R) \mid f^*(x_i) \in R[x_1,\ldots,x_i] \text{ for all } 1 \leq i \leq n\}$$

the subgroup of triangular automorphisms.

Another common notation is $GA_n(R) = \mathrm{Aut}_R(\mathbb{A}^n_R)$.

We recall that, in dimension two, affine and triangular automorphisms generate all automorphisms of $\mathbb{A}^2_{\mathbf{k}}$ for any field $\mathbf{k}$. Moreover, $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ then has the structure of an amalgamated product.

Theorem 2.3 (Jung–van der Kulk theorem [**16**, Theorem 2])**.** *Let* $\mathbf{k}$ *be a field. Then, the group* $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ *is the free product*

$$\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) = \mathrm{Aff}_2(\mathbf{k}) \ast_{\cap} \mathrm{BA}_2(\mathbf{k})$$

of its affine and triangular subgroups amalgamated over their intersection.

Notation 2.4. We denote by

$$\mathrm{Jac}(f) = \begin{vmatrix} \frac{\partial f_1}{x_1} & \cdots & \frac{\partial f_1}{x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{x_1} & \cdots & \frac{\partial f_n}{x_n} \end{vmatrix} \in R[x_1, \ldots, x_n]$$

the determinant of the Jacobian matrix of any $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n_R)$. We recall that $\mathrm{Jac}(f) \in R^\times$ if $f \in \mathrm{Aut}(\mathbb{A}^n_R)$.

Definition 2.5. A polynomial $P \in R[x_1, \ldots, x_n]$ is called a *variable* if there exists an automorphism f in $\mathrm{Aut}(\mathbb{A}^n_R)$ such that $f^*(x_1) = P$.

The following result is a consequence of [**13**]. We recall the proof here, as the statement is not explicitly stated in [**13**].

Lemma 2.6. *Let $P \in \mathbb{C}[x, y, z]$ be a polynomial. Suppose that P is a variable, when viewed as an element of $\mathbb{C}(z)[x, y]$, i.e., suppose that there exists an automorphism $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}(z)})$ such that $f^*(x) = P$. Then, there exists for each $q \in \mathbb{A}^1_{\mathbb{C}}$ a variable $v \in \mathbb{C}[x, y]$ such that $P(x, y, q) \in \mathbb{C}[v]$.*

Proof: First we recall briefly how the *ind-topology* of $\mathbb{C}[x, y]$ is defined in [**13**]. For each integer $d \geq 0$, the set $\mathbb{C}[x, y]_{\leq d} = \{f \in \mathbb{C}[x, y] \mid \deg(f) \leq d\}$ is a vector subspace of $\mathbb{C}[x, y]$ of finite dimension and thus it can be equipped with a natural Zariski topology, in which we identify the coefficients of the polynomials with the coordinates of an affine space. We then have a sequence of closed embeddings

$$\mathbb{C}[x, y]_{\leq 0} \hookrightarrow \mathbb{C}[x, y]_{\leq 1} \hookrightarrow \cdots \hookrightarrow \mathbb{C}[x, y]_{\leq d} \hookrightarrow \mathbb{C}[x, y]_{\leq d+1} \hookrightarrow \cdots$$

This allows us to define a natural topology associated to these embeddings by saying that a subset F of $\mathbb{C}[x, y]$ is closed if and only if $F \cap \mathbb{C}[x, y]_{\leq d}$ is closed for each $d \geq 0$.

In [**13**], the set of variables (see Definition 2.5) of $\mathbb{C}[x, y]$ is denoted by $\mathcal{V}$. Moreover, for each integer k, denote by $\mathcal{V}^{\leq k} \subseteq \mathcal{V}$ the set of variables that are components of an automorphism of $\mathbb{A}^2_{\mathbb{C}}$ of length $\leq k$, where the length is here defined using the amalgamated free product structure given by the Jung–van der Kulk theorem (Theorem 2.3).

Setting $\mathcal{W}^0 = \mathbb{C}$ and $\mathcal{W}^k = \bigcup_{v \in \mathcal{V}^{\leq k-1}} \mathbb{C}[v]$ for each $k \geq 1$, we then have the following result (see [**13**, Theorem 4]): for each $k \geq 0$, the closure of $\mathcal{V}^{\leq k}$ in $\mathbb{C}[x, y]$ is equal to $\mathcal{V}^{\leq k} \cup \mathcal{W}^k$.

We now prove the lemma. Let $P \in \mathbb{C}[x, y, z]$ and $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}(z)})$ be such that $f^*(x) = P$. Note that the map

$$\nu \colon \mathbb{A}^1 \longrightarrow \mathbb{C}[x, y], \quad q \longmapsto P(x, y, q)$$

is continuous, since it corresponds to a morphism of algebraic varieties $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{C}[x,y]_{\leq d}$ for some d. By the Jung–van der Kulk theorem, we can write $f = \alpha_1 \circ \cdots \circ \alpha_k$ for some $k \geq 1$ and some $\alpha_1, \ldots, \alpha_k \in \mathrm{Aff}_2(\mathbb{C}(z)) \cup \mathrm{BA}_2(\mathbb{C}(z))$. Noting that $\mathrm{Jac}(\alpha_i) \in \mathbb{C}(z) \setminus \{0\}$ for each $1 \leq i \leq k$, we define

$$U = \bigcap_{i=1}^{k} \{u \in \mathbb{A}^1_{\mathbb{C}} \mid \mathrm{Jac}(\alpha_i)(u) \in \mathbb{C}^*\}.$$

Then, $P(x,y,u) \in \mathcal{V}^{\leq k}$ for each $u \in U$. Moreover, since U is a dense open subset of $\mathbb{A}^1_{\mathbb{C}}$ and since the map ν is continuous, this implies that the polynomial $\nu(q) = P(x,y,q)$ belongs to the closure of $\mathcal{V}^{\leq k}$ for each $q \in \mathbb{A}^1_{\mathbb{C}}$, i.e., to $\mathcal{V}^{\leq k} \cup \mathcal{W}^k \subseteq \mathcal{W}^{k+1}$. This achieves the proof. □

We will apply the next result with $S = \{1\}$ and $\mathbf{k} = \mathbb{C}$ in Lemma 5.2.

Lemma 2.7. *Let $\mathbf{k}$ be a field, and let $S \subseteq \mathbf{k}^*$ be a subgroup. Then, the normal subgroup $N = \{f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}}) \mid \mathrm{Jac}(f) \in S\} \subseteq \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ is the free product*

$$N = (N \cap \mathrm{Aff}_2(\mathbf{k})) \ast_{\cap} (N \cap \mathrm{BA}_2(\mathbf{k}))$$

of its affine and triangular subgroups amalgamated over their intersection.

Proof: We prove that N is generated by $N \cap \mathrm{Aff}_2(\mathbf{k})$ and $N \cap \mathrm{BA}_2(\mathbf{k})$. The structural description then follows from [**26**, Proposition 2]. We can write any $g \in N$ as $g = \alpha_1 \circ \cdots \circ \alpha_n$ for some $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Aff}_2(\mathbf{k}) \cup \mathrm{BA}_2(\mathbf{k})$ by the Jung–van der Kulk theorem (Theorem 2.3). For all $1 \leq i \leq n-1$, we set $\mu_i = \mathrm{Jac}(\alpha_i) \in \mathbf{k}^*$ and $h_i = (\mu_1 \cdots \mu_i x, y) \in \mathrm{Aff}_2(\mathbf{k}) \cap \mathrm{BA}_2(\mathbf{k})$. We may replace $\alpha_1, \ldots, \alpha_n$ with $\alpha_1 \circ h_1^{-1}$, $h_1 \circ \alpha_2 \circ h_2^{-1}, \ldots, h_{n-1} \circ \alpha_{n-1} \circ h_n^{-1}$, $h_n \circ \alpha_n$, and assume that $\mathrm{Jac}(\alpha_i) = 1$ for $1 \leq i \leq n-1$ and $\mathrm{Jac}(\alpha_n) = \mathrm{Jac}(g) \in S$. Hence, $\alpha_1, \ldots, \alpha_n$ belong to $N \cap (\mathrm{Aff}_2(\mathbf{k}) \cup \mathrm{BA}_2(\mathbf{k}))$ as desired. □

2.2. Group cohomology, real structures, and real forms.

Definition 2.8. For each group $(G, \circ)$ on which $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts, we denote by $\alpha \mapsto \overline{\alpha}$ the action of the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and by $Z^1(G) := Z^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G) = \{\nu \in G \mid \nu \circ \overline{\nu} = 1\}$ the set of 1-*cocycles*. We say that two 1-cocycles ν, τ are *equivalent* if there exists $\alpha \in G$ such that $\tau = \alpha^{-1} \circ \nu \circ \overline{\alpha}$. The cohomology set $H^1(G) := H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G)$ is the set of equivalence classes of 1-cocycles. It is a pointed set, with a distinguished trivial element, denoted by 1, which is the class of the identity.

Since we will need them later in the text, we collect here the cohomology sets of some classical groups.

Lemma 2.9. *Let $n \geq 1$ be an integer. We consider the standard action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *on $\mathbb{C}^n$, on polynomials, and on matrices via the complex conjugation of their coefficients.*

(1) *The cohomology pointed sets*

$$H^1(\mathbb{C}^n),\ H^1(\mathbb{C}^*),\ H^1(\mathbb{C}[x_1,\dots,x_n])$$

of the groups $(\mathbb{C}^n,+)$, $(\mathbb{C}^*,\cdot)$, $(\mathbb{C}[x_1,\dots,x_n],+)$ are trivial.

(2) *Let $\mu_n = \{c \in \mathbb{C} \mid c^n = 1\}$ be the group of n-th roots of unity. The cohomology set $H^1(\mu_n)$ is trivial if n is odd and contains two elements if n is even. These two elements are the class of squares, that is, the class of* 1, *and the class of nonsquare elements, namely, the class of any generator of μ_n.*

(3) *The cohomology set* $H^1(\mathrm{PGL}_2(\mathbb{C}))$ *contains exactly two elements. The first one is the set of classes of elements of* $\mathrm{PGL}_2(\mathbb{C})$ *given by matrices* $A \in \mathrm{SL}_2(\mathbb{C})$ *with* $A\cdot\overline{A} = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. *The second one is the set of classes of all* $A \in \mathrm{SL}_2(\mathbb{C})$ *with* $A\cdot\overline{A} = \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$.

Proof: (1) An element of $Z^1((\mathbb{C}^n,+))$ is of the form $\nu \in \mathbb{C}^n$, with $\nu+\overline{\nu} = 0$. Choosing $\alpha = \frac{\nu}{2}$, we obtain $\overline{\alpha} = -\alpha$. Whence, $-\alpha+\nu+\overline{\alpha} = 0$. This shows $\nu \sim 0$. The same argument applies to $(\mathbb{C}[x_1,\dots,x_n],+)$.

An element of $Z^1((\mathbb{C}^*,\cdot))$ is of the form $\nu \in \mathbb{C}^*$ with $\nu\cdot\overline{\nu} = 1$. Hence, $|\nu| = 1$. Choosing α with $\alpha^2 = \nu$, we obtain $|\alpha| = 1$. This implies $\alpha^{-1}\cdot\nu\cdot\overline{\alpha} = 1$ and shows that $\nu \sim 1$.

(2) As every element $\nu \in \mu_n$ satisfies $|\mu_n| = 1$, we have $Z^1(\mu_n) = \mu_n$. Moreover, two elements $\nu,\tau \in \mu_n$ are equivalent if and only if there exists $\alpha \in \mu_n$ such that $\tau = \alpha^{-1}\nu\overline{\alpha} = \nu\alpha^{-2}$, i.e., if and only if $\tau\nu^{-1}$ is a square in μ_n. This implies that $H^1(\mu_n)$ is trivial if n is odd and contains exactly two classes if n is even: the class containing the squares and the one consisting of nonsquare elements.

(3) Every element $\tau \in Z^1(\mathrm{PGL}_2(\mathbb{C}))$ is the class of a matrix $A \in \mathrm{GL}_2(\mathbb{C})$ with $A\cdot\overline{A} = \left(\begin{smallmatrix}\epsilon&0\\0&\epsilon\end{smallmatrix}\right)$ for some $\epsilon \in \mathbb{C}$. Replacing A with μA for some $\mu \in \mathbb{C}^*$, we may assume that $A \in \mathrm{SL}_2(\mathbb{C})$. Moreover, $\epsilon = \pm 1$, as $\epsilon^2 = \det(A\cdot\overline{A}) = 1$. First we prove that τ is equivalent to the class of $\left(\begin{smallmatrix}0&\epsilon\\1&0\end{smallmatrix}\right)$. For this, choose a 2×1 vector v such that $A\overline{v}$, v are linearly independent. To see that such a vector exists, observe that if A is not diagonal, then we can choose $v = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ or $v = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. If A is diagonal, then we can choose $v = \left(\begin{smallmatrix}1\\\mathbf{i}\end{smallmatrix}\right)$ if $\tau \in \mathrm{PGL}_2(\mathbb{C})$ is the identity and $v = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ otherwise. Then, taking the matrix $R = \left(\begin{smallmatrix}v & A\overline{v}\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ whose columns are v and $A\overline{v}$

respectively, one checks that τ is equivalent to the class of $R^{-1} \cdot A \cdot \overline{R} = R^{-1} \cdot \left(A\overline{v} \;\; A\cdot\overline{A}v \right) = \left(\begin{smallmatrix} 0 & \epsilon \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$.

Now, consider two matrices $A_1, A_2 \in \mathrm{SL}_2(\mathbb{C})$ with $A_i \cdot \overline{A_i} = \left(\begin{smallmatrix} \epsilon_i & 0 \\ 0 & \epsilon_i \end{smallmatrix}\right)$, $\epsilon_i \in \{\pm 1\}$, and suppose that their classes are equivalent 1-cocycles $\tau_1, \tau_2 \in Z^1(\mathrm{PGL}_2(\mathbb{C}))$. To conclude the proof, it remains to show that $\epsilon_1 = \epsilon_2$. Since τ_1, τ_2 are equivalent, there exist $B \in \mathrm{GL}_2(\mathbb{C})$ and $\mu \in \mathbb{C}^*$ such that $A_2 = \mu B^{-1} \cdot A_1 \cdot \overline{B}$. This gives $\left(\begin{smallmatrix} \epsilon_2 & 0 \\ 0 & \epsilon_2 \end{smallmatrix}\right) = A_2 \cdot \overline{A_2} = |\mu|^2 B^{-1} \cdot A_1 \cdot \overline{A_1} \cdot B = |\mu|^2 \left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_1 \end{smallmatrix}\right)$, which implies $\epsilon_1 = \epsilon_2$. $\square$

Definition 2.10. If R is a $\mathbb{C}$-algebra, a *real structure* on R is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on R such that the nontrivial element acts by $\rho\colon \mathbb{C} \mapsto \mathbb{C}, \alpha \mapsto \overline{\alpha}$ on $\mathbb{C}$. This corresponds to giving a ring homomorphism $\rho\colon R \to R$ such that $\rho \circ \rho = \mathrm{id}_R$ and $\rho(\alpha \cdot f) = \overline{\alpha} \cdot \rho(f)$ for each $\alpha \in \mathbb{C}$ and each $f \in R$. For each such structure, we obtain an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the group $\mathrm{Aut}_{\mathbb{C}}(R)$ of $\mathbb{C}$-automorphisms by defining $\overline{f} = \rho \circ f \circ \rho$, for each $f \in \mathrm{Aut}_{\mathbb{C}}(R)$.

Definition 2.11. If X is a complex algebraic variety, a *real structure* is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on X such that the action of the nontrivial element is an anti-regular morphism $\rho\colon X \to X$, that is, a morphism of schemes such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "z \mapsto \overline{z}"] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

For each such real structure, the group $\langle \rho \rangle \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathrm{Aut}_{\mathbb{C}}(X)$ by defining $\overline{f} = \rho \circ f \circ \rho$, for each $f \in \mathrm{Aut}_{\mathbb{C}}(X)$.

Fixing a real structure ρ – if at least one exists – we have a bijection between the set of equivalence classes of real structures on X and $H^1(\mathrm{Aut}_{\mathbb{C}}(X))$: each real structure is of the form $\nu \circ \rho$ with $\nu \in Z^1(\mathrm{Aut}_{\mathbb{C}}(X))$ and two real structures $\nu \circ \rho$, $\tau \circ \rho$ are equivalent if and only if the classes of ν and τ in $H^1(\mathrm{Aut}_{\mathbb{C}}(X))$ are equal, which means that $\nu \circ \rho$, $\tau \circ \rho$ are conjugate with respect to some automorphism $\alpha \in \mathrm{Aut}_{\mathbb{C}}(X)$, i.e., $\tau \circ \rho = \alpha^{-1} \circ (\nu \circ \rho) \circ \alpha$.

Remark 2.12. Giving a real structure on an affine complex variety X is the same as giving a real structure on the $\mathbb{C}$-algebra $\mathbb{C}[X]$ of regular functions. Fixing such a real structure, the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{C}[X]$ via ring-automorphisms, and the natural $\mathbb{C}$-anti-isomorphism between $\mathrm{Aut}_{\mathbb{C}}(X)$ and $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[X])$ induces an isomorphism of pointed sets

$$H^1(\mathrm{Aut}_{\mathbb{C}}(X)) \xrightarrow{\simeq} H^1(\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[X])),$$

i.e., a bijection sending the identity to the identity.

Definition 2.13. A *real form* of a complex algebraic variety X is a real algebraic variety X_0 together with a $\mathbb{C}$-isomorphism

$$\varphi\colon X_0 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \xrightarrow{\sim} X.$$

Real forms and real structures of a quasiprojective complex algebraic variety X correspond to one another: for any real structure ρ on X, the variety $X/\langle\rho\rangle$ is a real form of X, and, given a real form (X_0, φ) of X, the map $\varphi \circ (\mathrm{id} \times \mathrm{Spec}(z \mapsto \overline{z})) \circ \varphi^{-1}$ defines a real structure on X. We refer to [**1**] for a description of the equivalence of categories between quasiprojective complex varieties with a real structure and quasiprojective real varieties.

Example 2.14. It is an easy exercise to check that $H^1(\mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}}))$ is trivial, and hence that $\mathbb{A}^1_{\mathbb{R}}$ is the only real form of $\mathbb{A}^1_{\mathbb{C}}$ up to isomorphism. However, the affine curve $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ has three different isomorphism classes of real forms; see Proposition 4.2.

Notation 2.15 (Usual complex conjugation)**.** For the rest of the text, we shall always denote the standard action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the affine space $\mathbb{A}^n_{\mathbb{C}} \simeq \mathbb{C}^n$ by $\rho\colon z = (z_1, \dots, z_n) \mapsto \overline{z} = (\overline{z_1}, \dots, \overline{z_n})$, where, in a slight abuse of notation, we write ρ for any $n \geq 1$. This provides the standard real structures on $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{C}[\mathbb{A}^n_{\mathbb{C}}] = \mathbb{C}[x_1, \dots, x_n]$.

Accordingly, we denote by $\overline{p} = \rho \circ p \circ \rho$ and $\overline{f} = \rho \circ f \circ \rho = (\overline{f_1}, \dots, \overline{f_n})$ the conjugate of a polynomial $p \in \mathbb{C}[\mathbb{A}^n_{\mathbb{C}}]$ and of an endomorphism $f = (f_1, \dots, f_n) \in \mathrm{End}(\mathbb{A}^n_{\mathbb{C}})$. If $p = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}$, then we simply have $\overline{p} = \sum_{i_1,\dots,i_n \geq 0} \overline{a_{i_1,\dots,i_n}} x_1^{i_1} \cdots x_n^{i_n}$.

Notation 2.16. If X is a quasiprojective real variety, its *real locus* is the set $X(\mathbb{R})$, which is a topological space for the Euclidean topology. If X is smooth, then $X(\mathbb{R})$ is a manifold.

3. The surfaces D_p

3.1. Reduced form.

Notation 3.1. Given a nonconstant polynomial $p \in \mathbf{k}[z]$, we denote by D_p the hypersurface in $\mathbb{A}^3_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[x, y, z])$ defined by the equation $xy = p(z)$.

Theorem 3.2 ([**6**, Lemma 2.10] and [**2**, Theorem 5.4.5(1)])**.** *Let* $\mathbf{k}$ *be a field and let* $p, q \in \mathbf{k}[z]$*. The surfaces* D_p *and* D_q *are isomorphic over* $\mathbf{k}$ *if and only if there exist* $a, \lambda \in \mathbf{k}^*$ *and* $b \in \mathbf{k}$ *such that* $p(az + b) = \lambda q(z)$*.*

Definition 3.3. A nonconstant polynomial $p \in \mathbf{k}[z]$ is called *in reduced form* if $p(z) = z^d + s(z)$ for some integer $d \geq 1$ and some polynomial $s \in \mathbf{k}[z]$ with $\deg(s) \leq d-2$.

Lemma 3.4. *If $\mathbf{k}$ is a field of characteristic zero, then every surface D_p defined over $\mathbf{k}$ is isomorphic to a surface D_q with q in reduced form.*

Proof: For every $p \in \mathbf{k}[z]$, there exist $\lambda \in \mathbf{k}^*$ and $\mu \in \mathbf{k}$ such that the polynomial $q(z) = \lambda p(z+\mu)$ is in reduced form. Then, the affine map $\varphi = \left(x, \frac{1}{\lambda}y, z+\mu\right) \in \mathrm{Aff}_3(\mathbf{k})$ induces an isomorphism between the hypersurfaces D_q and D_p, as $\varphi^*(xy - p(z)) = \frac{1}{\lambda}(xy - q(z))$. $\square$

3.2. Automorphisms. A list of generators for the automorphism groups of the surfaces D_p was first given in [**18**]. Note that in his article Makar-Limanov assumes that the ground field is algebraically closed (of any characteristic). At the end of the paper he gives: "*Remark.* (1) Though we assumed that [the ground field] is algebraically closed it is not really essential. It is not difficult to show that all roots necessary in Lemma 9 belong to the field itself."

Theorem 3.5 ([**18**])**.** *Let $\mathbf{k}$ be a field and let $p \in \mathbf{k}[z]$ be a polynomial of degree at least 2. Then, every automorphism of the surface $D_p \subset \mathbb{A}^3_{\mathbf{k}}$ extends to an automorphism of $\mathbb{A}^3_{\mathbf{k}}$. Moreover, the group $\mathrm{Aut}_{\mathbf{k}}(D_p)$ is generated by the following subgroups:*

- $\left\{\left(x, y + \frac{p(z+xr(x))-p(z)}{x}, z + xr(x)\right) \mid r(x) \in \mathbf{k}[x]\right\} \simeq (\mathbf{k}[x], +)$;
- $\{(x,y,z), (y,x,z)\} \simeq \mathbb{Z}/2\mathbb{Z}$;
- $\{(ax, by, cz+d) \mid a,b,c \in \mathbf{k}^*,\ d \in \mathbf{k},\ p(cz+d) = abp(z)\}$.

Furthermore, it is a folklore result that $\mathrm{Aut}_{\mathbf{k}}(D_p)$ is equal to the free product of two subgroups, amalgamated over their intersection; see for instance [**12**]. As we did not find the precise statement we need in the literature, we re-prove it here. Theorem 3.6 below essentially follows from [**2**, Theorem 5.4.5] (see also [**17**], for a slightly weaker statement).

Theorem 3.6. *Let $\mathbf{k}$ be a field and $p \in \mathbf{k}[z]$ be a polynomial of degree at least 2. Let $D_p = \mathrm{Spec}(\mathbf{k}[x,y,z]/(xy - p(z)))$ and define the subgroups*

$$A_{\mathbf{k}}(p) = \{f \in \mathrm{Aut}_{\mathbf{k}}(D_p) \mid \exists g \in \mathrm{Aff}_3(\mathbf{k})\colon f = g|_{D_p}\}$$

and

$$B_{\mathbf{k}}(p) = \{\psi_{a,b,c,d,r} \mid a,b,c \in \mathbf{k}^*,\ d \in \mathbf{k},\ r \in \mathbf{k}[x],\ abp(z) = p(cz+d)\},$$

where

$$\psi_{a,b,c,d,r} = \left(ax, by + \frac{p(cz+d+xr(x)) - abp(z)}{ax}, cz+d+xr(x)\right).$$

Then, the automorphism group of D_p is the free product

$$\mathrm{Aut}_{\mathbf{k}}(D_p) = A_{\mathbf{k}}(p) *_{\cap} B_{\mathbf{k}}(p)$$

of $A_{\mathbf{k}}(p)$ and $B_{\mathbf{k}}(p)$ amalgamated over their intersection $A_{\mathbf{k}}(p) \cap B_{\mathbf{k}}(p)$, which is the set of all elements $\psi_{a,b,c,d,r}$, with $a, b, c \in \mathbf{k}^*$, $d \in \mathbf{k}$, such that $abp(z) = p(cz + d)$, and $r \in \mathbf{k}$ such that $r = 0$ if $\deg(p) \geq 3$.

Remark 3.7. One can check, using the birational morphism $D_p \to \mathbb{A}^2_{\mathbf{k}}$, $(x, y, z) \mapsto (x, z)$, that $B_{\mathbf{k}}(p)$ consists of all automorphisms of D_p that preserve the fibration $(x, y, z) \mapsto x$.

Proof: First, we check that the set $B_{\mathbf{k}}(p)$ is indeed a subgroup of $\mathrm{Aut}_{\mathbf{k}}(D_p)$. For this, it suffices to remark that $\psi_{a,b,c,d,r}$ defines an endomorphism of $\mathbb{A}^3_{\mathbf{k}}$ satisfying $\psi^*_{a,b,c,d,r}(xy - p(z)) = ab(xy - p(z))$, and to compute $\psi_{1,1,1,0,0} = \mathrm{id}_{\mathbb{A}^3_{\mathbf{k}}}$ and

$$\psi_{a,b,c,d,r} \circ \psi_{a',b',c',d',r'} = \psi_{aa',bb',cc',cd'+d,cr'(x)+a'r(a'x)}$$

for all $\psi_{a,b,c,d,r}, \psi_{a',b',c',d',r'} \in B_{\mathbf{k}}(p)$.

Let us consider the open embedding $\mathbb{A}^3_{\mathbf{k}} \hookrightarrow \mathbb{P}^3_{\mathbf{k}}$, $(x, y, z) \mapsto [1 : x : y : z]$ and denote by X_p the closure of D_p in $\mathbb{P}^3_{\mathbf{k}}$. Writing $s = \deg(p)$ and $p(z) = \sum_{i=0}^{s} p_i z^i$ with $p_0, \ldots, p_s \in \mathbf{k}$ and $p_s \neq 0$, we obtain that X_p is the hypersurface in $\mathbb{P}^3_{\mathbf{k}}$ given by the equation $w^{s-2}xy = \sum_{i=0}^{s} p_i w^{s-i} z^i$. So, $C_p = X_p \setminus D_p$ is either the conic defined by $\{w = 0,\ xy = p_2 z^2\}$ in the case where $s = 2$, or the line given by $\{w = z = 0\}$ in the case where $s \geq 3$. In both cases, C_p is a curve isomorphic to $\mathbb{P}^1_{\mathbf{k}}$ that contains the point $q = [0 : 0 : 1 : 0]$.

We will prove the two following statements.

(1) The birational map $\hat{\beta}$ of X_p induced by any $\beta \in B_{\mathbf{k}}(p) \setminus A_{\mathbf{k}}(p)$ contracts $C_p \setminus \{q\}$ onto q.

(2) The birational map $\hat{\alpha}$ of X_p induced by any $\alpha \in A_{\mathbf{k}}(p)$ preserves the curve C_p, and if it fixes the point q, then $\alpha \in A_{\mathbf{k}}(p) \cap B_{\mathbf{k}}(p)$.

Before proving them, let us show that $\mathrm{Aut}_{\mathbf{k}}(D_p) = A_{\mathbf{k}}(p) *_{\cap} B_{\mathbf{k}}(p)$ follows from these two claims. Recall that, by Theorem 3.5, $\mathrm{Aut}_{\mathbf{k}}(D_p)$ is generated by $A_{\mathbf{k}}(p)$ and $B_{\mathbf{k}}(p)$. Letting $m \geq 1$, $\alpha_1, \ldots, \alpha_{m-1} \in A_{\mathbf{k}}(p) \setminus B_{\mathbf{k}}(p)$ and $\beta_1, \ldots, \beta_m \in B_{\mathbf{k}}(p) \setminus A_{\mathbf{k}}(p)$, it then suffices to prove that

$$\varphi = \beta_m \circ \alpha_{m-1} \circ \cdots \circ \alpha_1 \circ \beta_1 \notin A_{\mathbf{k}}(p).$$

For this, we prove by induction on m that the extension of $\varphi \in \mathrm{Aut}_{\mathbf{k}}(D_p)$ to a birational map $\hat{\varphi} \in \mathrm{Bir}_{\mathbf{k}}(X_p)$ contracts $C_p \setminus \{q\}$ onto q. For $m = 1$, this is given by (1). For $m \geq 2$, write $\varphi = \beta_m \circ \alpha_{m-1} \circ \varphi'$. The result follows, since the extensions $\hat{\beta}_m$, $\hat{\alpha}_{m-1}$, $\hat{\varphi}'$ are elements of $\mathrm{Bir}_{\mathbf{k}}(X_p)$ such that $\hat{\varphi}'(C_p \setminus \{q\}) = \{q\}$ (by the induction hypothesis), $\hat{\alpha}_{m-1}(q) \in C_p \setminus \{q\}$ (by (2)), and $\hat{\beta}_m(C_p \setminus \{q\}) = \{q\}$ (by (1)).

We now prove (1). First, note that an automorphism $\psi_{a,b,c,d,r}$ is in $A_{\mathbf{k}}(p)$ if $r=0$, or if $\deg(r)=0$ and $\deg(p)=2$. Therefore, we consider an automorphism $\psi=\psi_{a,b,c,d,r}$ with either $\deg(r)\geq 1$, or with $\deg(r)=0$ and $\deg(p)\geq 3$.

In both cases, the second component of the birational map $\hat{\psi}\in \mathrm{Bir}_{\mathbf{k}}(X_p)$ induced by ψ is of degree $D\coloneqq \deg(p)\cdot(\deg(r)+1)-1>\deg(r)+1$, strictly greater than the degree of any other component of the map, and its leading term is ξx^D for some $\xi\in\mathbf{k}^*$.

Extending ψ to a rational map $\tilde{\psi}\colon \mathbb{P}^3_{\mathbf{k}} \dashrightarrow \mathbb{P}^3_{\mathbf{k}}$ by homogenising its components, we obtain $\tilde{\psi}([0:x:y:z])=[0:0:\xi x^D:0]$ for any $[0:x:y:z]\in C_p$. As every point of $C_p\setminus\{q\}$ satisfies $x\neq 0$, the equality $\hat{\psi}(C_p\setminus\{q\})=\{q\}$ follows. This proves (1).

We remark that we have proved above that no map $\hat{\psi}_{a,b,c,d,r}\in\mathrm{Bir}_{\mathbf{k}}(X_p)$ is an automorphism if $\deg(r)\geq 1$ or if $\deg(r)=0$ and $\deg(p)\geq 3$. In particular, it is not an element of $A_{\mathbf{k}}(p)$ in these cases. Hence, we get the desired description of $A_{\mathbf{k}}(p)\cap B_{\mathbf{k}}(p)$.

Finally, it remains to prove (2). Let $\alpha\in A_{\mathbf{k}}(p)$. As it is the restriction of an element of $\mathrm{Aff}_3(\mathbf{k})$, which itself is the restriction of an element $\tilde{\alpha}\in\mathrm{Aut}(\mathbb{P}^3_{\mathbf{k}})$ that preserves the curve C_p, the automorphism α induces a map $\hat{\alpha}\in\mathrm{Bir}_{\mathbf{k}}(X_p)$ that preserves C_p. Suppose that $\hat{\alpha}(q)=q$. Then, the birational morphism $\kappa\colon D_p\to\mathbb{A}^2_{\mathbf{k}},(x,y,z)\mapsto(x,z)$ conjugates α to an affine automorphism $\alpha'\in\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$, because this morphism is the restriction of the projection $\mathbb{P}^3_{\mathbf{k}}\dashrightarrow\mathbb{P}^2_{\mathbf{k}}$, $[w:x:y:z]\mapsto[w:x:z]$ from the point q. For each $(x,z)\in\mathbb{A}^2_{\mathbf{k}}$, the fibre $\kappa^{-1}(x,z)$ consists of one single point if and only if $x\neq 0$. Hence, α' is of the form $(x,z)\mapsto(ax,cz+d+r_0x)$ for some $a,c\in\mathbf{k}^*$ and some $d,r_0\in\mathbf{k}$. This gives $\alpha=(ax,by+h(x,z),cz+d+r_0x)$ for some $b\in\mathbf{k}^*$ and some $h\in\mathbf{k}[x,z]$ of degree 1. As $\alpha^*(xy-p(z))=abxy+axh(x,z)-p(cz+d+r_0x)$ lies in the ideal generated by $xy-p(z)$, it must be equal to $ab(xy-p(z))$. This implies, by setting $x=0$, that $abp(z)=p(cz+d)$, and then that $h(x,z)=\frac{p(cz+d+r_0x)-abp(z)}{ax}$; hence $\alpha\in B_{\mathbf{k}}(p)$. □

The aim of the next three results is to give a precise description of the subgroup $A_{\mathbf{k}}(p)$ of "affine" automorphisms of a surface D_p. We start with the case where $\deg(p)\geq 3$.

Lemma 3.8. *Let $\mathbf{k}$ be a field and $p\in\mathbf{k}[z]$ with $\deg(p)\geq 3$. Then,*

$$A_{\mathbf{k}}(p)=(A_{\mathbf{k}}(p)\cap B_{\mathbf{k}}(p))\rtimes\langle(y,x,z)\rangle,$$

where

$$A_{\mathbf{k}}(p)\cap B_{\mathbf{k}}(p)=\{(ax,by,cz+d)\mid a,b,c\in\mathbf{k}^*,\ d\in\mathbf{k},\ abp(z)=p(cz+d)\}.$$

Proof: Theorem 3.6 gives the explicit description of the intersection of $A_{\mathbf{k}}(p)$ and $B_{\mathbf{k}}(p)$.

As the involution (y,x,z) is an element of $A_{\mathbf{k}}(p)\setminus B_{\mathbf{k}}(p)$ that normalises $A_{\mathbf{k}}(p)\cap B_{\mathbf{k}}(p)$, the subgroup of $A_{\mathbf{k}}(p)$ generated by $A_{\mathbf{k}}(p)\cap B_{\mathbf{k}}(p)$ and (y,x,z) is isomorphic to $(A_{\mathbf{k}}(p)\cap B_{\mathbf{k}}(p))\rtimes\langle(y,x,z)\rangle$. It remains to see that every element $\alpha\in A_{\mathbf{k}}(p)$ is in that subgroup, i.e., is of the form $\alpha=(ax,by,cz+d)$ or $\alpha=(ay,bx,cz+d)$ for some $a,b,c\in\mathbf{k}^*$ and $d\in\mathbf{k}$.

Write $\alpha=(\ell_1,\ell_2,\ell_3)$, where $\ell_1,\ell_2,\ell_3\in\mathbf{k}[x,y,z]$ are of degree 1. Then,

$$\ell_1\ell_2-p(\ell_3)=\alpha^*(xy-p(z))=\mu(xy-p(z))$$

for some $\mu\in\mathbf{k}^*$. Since $\deg(p)\geq 3$ and $\deg(\ell_1\ell_2)=2$, we obtain that $\ell_3=cz+d$ for some $c\in\mathbf{k}^*, d\in\mathbf{k}$, and we have that

$$\ell_1\ell_2=\mu xy+p(cz+d)-\mu p(z).$$

Observe that the right-hand side of the above equality is an irreducible polynomial, unless $p(cz+d)-\mu p(z)=0$. Thus, $p(cz+d)=\mu p(z)$ and $\ell_1\ell_2=\mu xy$. In turn, the latter equality implies that either $\ell_1=ax$ and $\ell_2=by$ or $\ell_1=ay$ and $\ell_2=bx$ for some $a,b\in\mathbf{k}^*$ with $ab=\mu$. $\square$

We now investigate the case where $\deg(p)=2$.

Lemma 3.9. *Let $\mathbf{k}$ be a field of characteristic not equal to 2, and let $p=z^2-1=(z-1)(z+1)\in\mathbf{k}[z]$. The surface $D_p=\operatorname{Spec}(\mathbf{k}[x,y,z]/(xy-p(z)))$ is isomorphic to $(\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}})\setminus\Delta$, where Δ denotes the diagonal, via*

$$\begin{aligned}(\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}})\setminus\Delta &\xrightarrow{\simeq} D_p\\ ([a:b],[c:d]) &\longmapsto \left(\tfrac{2ac}{ad-bc},\tfrac{2bd}{ad-bc},\tfrac{ad+bc}{ad-bc}\right).\end{aligned}$$

Moreover, $A_{\mathbf{k}}(p)$ is isomorphic to $\mathrm{PGL}_2(\mathbf{k})\times\langle\sigma\rangle$, where $\sigma=(-x,-y,-z)\in\operatorname{Aut}_{\mathbf{k}}(D_p)$ acts on $\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$ via the exchange of the two factors and where $\mathrm{PGL}_2(\mathbf{k})$ acts diagonally on $(\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}})\setminus\Delta$ and via

$$\begin{aligned}\mathrm{PGL}_2(\mathbf{k})\times D_p &\longrightarrow D_p\\ \left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right),\left(\begin{smallmatrix}x\\ y\\ z\end{smallmatrix}\right) &\longmapsto \tfrac{1}{\alpha\delta-\beta\gamma}\begin{pmatrix}\alpha^2&\beta^2&2\alpha\beta\\ \gamma^2&\delta^2&2\gamma\delta\\ \alpha\gamma&\beta\delta&\alpha\delta+\beta\gamma\end{pmatrix}\cdot\left(\begin{smallmatrix}x\\ y\\ z\end{smallmatrix}\right)\end{aligned}$$

on D_p.

Proof: As $\operatorname{char}(\mathbf{k})\neq 2$, we may consider $\mathbb{A}^3_{\mathbf{k}}$ embedded into $\mathbb{P}^3_{\mathbf{k}}$, via the open embedding $(x,y,z)\mapsto[x:y:z+1:z-1]$, and obtain that $D_p=Q\setminus H$, where $Q,H\subseteq\mathbb{P}^3_{\mathbf{k}}$ are given respectively by $x_0x_1=x_2x_3$ and $x_2=x_3$.

We then use the classical isomorphism $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \xrightarrow{\simeq} Q$, $([a:b],[c:d]) \mapsto [ac:bd:ad:bc]$, which restricts to the isomorphism $(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) \setminus \Delta \xrightarrow{\simeq} Q \setminus H = D_p$ described in the statement.

By definition, $A_{\mathbf{k}}(p) = \{f \in \mathrm{Aut}_{\mathbf{k}}(D_p) \mid \exists g \in \mathrm{Aff}_3(\mathbf{k}) : f = g|_{D_p}\}$ corresponds to the group of automorphisms of $\mathbb{P}^3$ which preserve H and Q, and thus to the group of automorphisms of Q that preserve $Q \cap H$; it is conjugate via the above isomorphism to the group of automorphisms of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ that preserve the diagonal.

As $\mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}) = (\mathrm{PGL}_2(\mathbf{k}) \times \mathrm{PGL}_2(\mathbf{k})) \rtimes \langle \sigma \rangle$, where σ is the exchange of the two factors, which corresponds to $(-x,-y,-z) \in \mathrm{Aut}_{\mathbf{k}}(D_p)$, we obtain that $A_{\mathbf{k}}(p)$ corresponds, via the isomorphism, to the group $\mathrm{PGL}_2(\mathbf{k}) \times \langle \sigma \rangle$, where $\mathrm{PGL}_2(\mathbf{k})$ acts diagonally on $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$. Conjugating the action gives the explicit description of the action of $\mathrm{PGL}_2(\mathbf{k})$ on D_p. □

Lemma 3.10. *Let $\mathbf{k}$ be a field of characteristic not equal to 2, and let $p = z^2 \in \mathbf{k}[z]$. The group $A_{\mathbf{k}}(p)$ is isomorphic to $\mathrm{PGL}_2(\mathbf{k}) \times \mathbf{k}^*$, and the action of this latter group on the surface $D_p = \mathrm{Spec}(\mathbf{k}[x,y,z]/(xy - z^2))$ is*

$$\begin{aligned} (\mathrm{PGL}_2(\mathbf{k}) \times \mathbf{k}^*) \times D_p &\longrightarrow D_p \\ \left(\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \mu \right), \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) &\longmapsto \frac{\mu}{\alpha\delta - \beta\gamma} \begin{pmatrix} \alpha^2 & \beta^2 & 2\alpha\beta \\ \gamma^2 & \delta^2 & 2\gamma\delta \\ \alpha\gamma & \beta\delta & \alpha\delta + \beta\gamma \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

Proof: As observed in Lemma 3.9, the above formula gives an embedding $\mathrm{PGL}_2(\mathbf{k}) \hookrightarrow \mathrm{GL}_3(\mathbf{k})$ whose action on $\mathbb{A}^3_{\mathbf{k}}$ preserves $xy - z^2 - 1$, and thus also $xy - z^2$. Its image moreover lies in $\mathrm{SL}_3(\mathbf{k})$. The action of $\mathbf{k}^*$ on $\mathbb{A}^3_{\mathbf{k}}$ by homotheties gives another embedding $\mathbf{k}^* \hookrightarrow \mathrm{GL}_3(\mathbf{k})$. Since both groups commute and have a trivial intersection, we get an embedding $\varphi\colon \mathrm{PGL}_2(\mathbf{k}) \times \mathbf{k}^* \hookrightarrow A_{\mathbf{k}}(p)$.

It remains to see that every element $f \in A_{\mathbf{k}}(p)$ lies in the image of φ. As it is the only singular point of D_p, the point $(0,0,0) \in D_p \subseteq \mathbb{A}^3_{\mathbf{k}}$ is fixed by any $f \in A_{\mathbf{k}}(p)$. Hence, $f = g|_{D_p}$ for some $g \in \mathrm{GL}_3(\mathbf{k})$ whose action on $\mathbb{P}^2_{\mathbf{k}}$ preserves the conic Γ given by $xy = z^2$, isomorphic to $\mathbb{P}^1_{\mathbf{k}}$ via $[u:v] \mapsto [u^2 : v^2 : uv]$. The induced action of g on $\mathbb{P}^1_{\mathbf{k}}$ is of the form $[u:v] \mapsto [\alpha u + \beta v : \gamma u + \delta v]$ for some $R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbf{k})$. Hence, the action of g on $\mathbb{P}^1_{\mathbf{k}}$ coincides with that of the image of R in $\mathrm{PGL}_2(\mathbf{k}) \subseteq \mathrm{SL}_3(\mathbf{k})$, i.e., with that of $\varphi((R,1))$. Hence, the map $f \circ \varphi((R,1))^{-1} \in A_{\mathbf{k}}(p)$ acts trivially on Γ and thus on $\mathbb{P}^2_{\mathbf{k}}$ (a nontrivial automorphism of $\mathbb{P}^2_{\mathbf{k}}$ only fixes points and lines), and is then a homothety. □

3.3. Existence of real forms.

Proposition 3.11. *Let $p \in \mathbb{C}[z] \setminus \mathbb{C}$ be a nonconstant polynomial. The following conditions are equivalent:*

(1) *The complex affine surface $D_p = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - p(z)))$ admits a real structure.*
(2) *There exist $a, \lambda \in \mathbb{C}^*$, $b \in \mathbb{C}$, such that $\lambda p(az+b) \in \mathbb{R}[z]$.*
(3) *There exists $q \in \mathbb{R}[z]$ such that the complex affine surfaces $D_p = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - p(z)))$ and $D_q = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - q(z)))$ are isomorphic.*

Proof: The equivalence between (2) and (3) follows from Theorem 3.2.

The implication (3) $\Rightarrow$ (1) follows from the fact that $(x,y,z) \mapsto (\overline{x}, \overline{y}, \overline{z})$ is a real structure on $D_q = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - q(z)))$, since $q \in \mathbb{R}[z]$.

It remains to prove (1) $\Rightarrow$ (2). Applying a suitable affine automorphism of the form $(\lambda x, y, az+b)$ we can assume that p is in reduced form. Let $d = \deg(p) \geq 1$. Since (2) is satisfied when $p = z^d$, we may further assume that p is not a monomial.

We take a real structure on D_p which is of the form

$$(x,y,z) \longmapsto (\overline{f_1(x,y,z)}, \overline{f_2(x,y,z)}, \overline{f_3(x,y,z)}),$$

for some polynomials $f_1, f_2, f_3 \in \mathbb{C}[x,y,z]$. This provides an isomorphism of complex affine surfaces

$$\begin{aligned} D_p &\longmapsto D_{\overline{p}} \\ (x,y,z) &\longmapsto (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)). \end{aligned}$$

Hence, by Theorem 3.2, there exist $a, \lambda \in \mathbb{C}^*$ and $b \in \mathbb{C}$ such that $\overline{p}(z) = \lambda p(az+b)$. Since p is in reduced form and is not a monomial, we have $b = 0$, $\lambda = a^{-d}$, and $|a| = 1$. Let $\alpha \in \mathbb{C}^*$ be such that $\alpha^2 = a$. We now conclude the proof by showing that the polynomial $q(z) = \alpha^{-d} p(\alpha z)$ lies in $\mathbb{R}[z]$.

Indeed, since $|a|=|\alpha|=1$, we get

$$\overline{q}(z) = \overline{\alpha^{-d}} \cdot \overline{p}(\overline{\alpha} z) = \alpha^d \lambda p(a\overline{\alpha} z) = \alpha^d a^{-d} p(a \alpha^{-1} z) = \alpha^{-d} p(\alpha z) = q(z),$$

as desired. $\square$

By Proposition 3.11, we may assume a surface D_p to have no real forms or its defining polynomial p to lie in $\mathbb{R}[z]$. We shall classify the number of real forms for the latter case. First we prove that if $p \in \mathbb{R}[z]$, then both subgroups $A_{\mathbb{C}}(p)$ and $B_{\mathbb{C}}(p)$ of $\operatorname{Aut}_{\mathbb{C}}(D_p)$ defined in Theorem 3.6 are invariant under the action of $\rho\colon (x,y,z) \mapsto (\overline{x}, \overline{y}, \overline{z})$.

Lemma 3.12. *Assume that $p \in \mathbb{R}[z]$. Then, the subgroups $A_{\mathbb{C}}(p)$ and $B_{\mathbb{C}}(p)$ of $\mathrm{Aut}_{\mathbb{C}}(D_p)$ given in Theorem 3.6 are invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.*

Proof: Since p is real, we have $\rho(D_p) = D_p$, and thus $\overline{f}(D_p) = \rho \circ f \circ \rho(D_p) = D_p$ for all $f \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$ satisfying $f(D_p) = D_p$. Since any element of $A_{\mathbb{C}}(p)$ comes from the restriction of an element of $\mathrm{Aff}_3(\mathbb{C})$, this implies that $A_{\mathbb{C}}(p)$ is invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Similarly, as $\overline{\psi_{a,b,c,d,r}} = \psi_{\bar{a},\bar{b},\bar{c},\bar{d},\bar{r}}$ for all $\psi_{a,b,c,d,r} \in B_{\mathbb{C}}(p)$, the group $B_{\mathbb{C}}(p)$ is also invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. $\square$

The next result shows that it will actually be sufficient to compute the cohomology set $H^1(A_{\mathbb{C}}(p))$ to determine all real forms of D_p.

Lemma 3.13. *Let $p \in \mathbb{C}[z]$ be a polynomial with $\deg(p) \geq 2$. The homomorphisms of pointed sets*

$$H^1(A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)) \longrightarrow H^1(B_{\mathbb{C}}(p))$$
$$H^1(A_{\mathbb{C}}(p)) \longrightarrow H^1(\mathrm{Aut}_{\mathbb{C}}(D_p))$$

given by the inclusions $A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p) \hookrightarrow B_{\mathbb{C}}(p)$ and $A_{\mathbb{C}}(p) \hookrightarrow \mathrm{Aut}_{\mathbb{C}}(D_p)$ are isomorphisms of pointed sets.

Proof: Recall that by definition every element ψ of $B_{\mathbb{C}}(p)$ is of the form $\psi = \psi_{a,b,c,d,r}$ for some $a, b, c \in \mathbb{C}^*$, $d \in \mathbb{C}$, and $r \in \mathbb{C}[x]$ such that $p(cz + d) = abp(z)$. Moreover, for all $\psi_{a,b,c,d,r}, \psi_{a',b',c',d',r'}$ in $B_{\mathbb{C}}(p)$, we have

$$\psi_{a,b,c,d,r} \circ \psi_{a',b',c',d',r'} = \psi_{aa',bb',cc',cd'+d,cr'(x)+a'r(a'x)}$$

and

$\psi_{a,b,c,d,r} = \psi_{a',b',c',d',r'}$ if and only if $a = a'$, $b = b'$, $c = c'$, $d = d'$, $r = r'$.

The latter claim can be proved using the birational morphism $D_p \to \mathbb{A}^2_{\mathbb{C}}$, $(x, y, z) \mapsto (x, z)$, or by saying that if the two maps are equal, then $\psi_{a,b,c,d,r}$ and $\psi_{a',b',c',d',r'}$ have the same components modulo $xy - p(z)$. Note also that any element of $B_{\mathbb{C}}(p)$ of the form $\psi_{a,b,c,d,0}$ belongs to $A_{\mathbb{C}}(p)$.

By [**16**, Theorem 1], the fact that $\mathrm{Aut}_{\mathbb{C}}(D_p)$ is the free product of $A_{\mathbb{C}}(p)$ and $B_{\mathbb{C}}(p)$ amalgamated over their intersection as in Theorem 3.6 implies that we have the following co-Cartesian diagram of morphisms of pointed sets.

$$\begin{array}{ccc} H^1(A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)) & \longrightarrow & H^1(A_{\mathbb{C}}(p)) \\ \downarrow & & \downarrow \\ H^1(B_{\mathbb{C}}(p)) & \longrightarrow & H^1(\mathrm{Aut}_{\mathbb{C}}(D_p)) \end{array}$$

Therefore, it suffices to prove that $H^1(A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)) \to H^1(B_{\mathbb{C}}(p))$ is a bijection to obtain that $H^1(A_{\mathbb{C}}(p)) \to H^1(\mathrm{Aut}_{\mathbb{C}}(D_p))$ is a bijection.

For this, we will show that:

(1) Each element of $Z^1(B_{\mathbb{C}}(p))$ is equivalent to an element ψ of the form $\psi = \psi_{1,b,c,d,0}$ in $Z^1(A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p))$.

(2) Two such elements $\psi_{1,b,c,d,0}$, $\psi_{1,b',c',d',0}$ of $Z^1(A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p))$ are equivalent in $B_{\mathbb{C}}(p)$ if and only if they are equivalent in $A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)$.

Let $\tau = \psi_{a,b,c,d,r}$ be a 1-cocycle in $Z^1(B_{\mathbb{C}}(p))$. This implies $a\overline{a} = 1$, as $\tau \circ \overline{\tau} = \mathrm{id}_{D_p}$. Therefore, we can find $\varepsilon \in \mathbb{C}^*$ with $\varepsilon^2 = a$ and define $\theta = \psi_{\varepsilon,\varepsilon^{-1},1,0,0} = (\varepsilon x, \varepsilon^{-1} y, z) \in A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)$. Then,

$$\begin{aligned}\tilde{\tau} &= \theta^{-1} \circ \tau \circ \overline{\theta}\\ &= \psi_{\varepsilon^{-1},\varepsilon,1,0,0} \circ \psi_{a,b,c,d,r} \circ \psi_{\varepsilon^{-1},\varepsilon,1,0,0}\\ &= \psi_{\varepsilon^{-2}a,\varepsilon^2 b,c,d,\varepsilon^{-1}r(\varepsilon^{-1}x)}\\ &= \psi_{1,ab,c,d,\varepsilon^{-1}r(\varepsilon^{-1}x)}\end{aligned}$$

is a 1-cocycle in $Z^1(B_{\mathbb{C}}(p))$ equivalent to τ.

Denote $s(x) = r(\varepsilon^{-1}x) \in \mathbb{C}[x]$. Computing the third component of $\tilde{\tau} \circ \overline{\tilde{\tau}} = \mathrm{id}_{D_p}$, we see that $c\overline{s}(x) + s(x) = 0$. Define $\psi = \psi_{1,1,1,0,\frac{1}{2}s}$. Then, $\tau' = \psi^{-1} \circ \tilde{\tau} \circ \overline{\psi}$ is a 1-cocycle in $Z^1(B_{\mathbb{C}}(p))$ equivalent to τ. Moreover, one checks that

$$\begin{aligned}\tau' &= \psi^{-1} \circ \tilde{\tau} \circ \overline{\psi}\\ &= \psi_{1,1,1,0,-\frac{1}{2}s(x)} \circ \psi_{1,ab,c,d,s(x)} \circ \psi_{1,1,1,0,\frac{1}{2}\overline{s}(x)}\\ &= \psi_{1,ab,c,d,\frac{1}{2}s(x)} \circ \psi_{1,1,1,0,\frac{1}{2}\overline{s}(x)}\\ &= \psi_{1,ab,c,d,c\frac{1}{2}\overline{s}(x)+\frac{1}{2}s(x)}\\ &= \psi_{1,ab,c,d,0}.\end{aligned}$$

This proves (1).

Now, let $\tau = \psi_{1,b,c,d,0}$ and $\sigma = \psi_{1,b',c',d',0}$ be two elements in $Z^1(B_{\mathbb{C}}(p))$ and suppose that $\varphi^{-1} \circ \tau \circ \overline{\varphi} = \sigma$ for some $\varphi = \psi_{\alpha,\beta,\gamma,\delta,\pi}$ in $B_{\mathbb{C}}(p)$. It is then straightforward to check that $\psi^{-1} \circ \tau \circ \overline{\psi} = \sigma$, where ψ is the element of $A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)$ defined by $\psi = \psi_{\alpha,\beta,\gamma,\delta,0}$. This proves (2). ☐

3.4. Cohomology set of the group $A_{\mathbb{C}}(p)$. First we deal with the case where $\deg(p) = 2$. In view of Lemma 3.9 and Lemma 3.10, we proceed in two distinct cases.

Lemma 3.14. *If $p = z^2 - 1$, then $H^1(A_{\mathbb{C}}(p))$ contains exactly four elements, namely the classes of (x, y, z), $(-x, -y, -z)$, $(y, x, -z)$, $(-y, -x, z)$.*

Proof: Lemma 3.9 provides an explicit isomorphism $\mathrm{PGL}_2(\mathbb{C}) \times \langle\sigma\rangle \xrightarrow{\simeq} A_{\mathbb{C}}(p)$, where σ is an involution, the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\langle\sigma\rangle \simeq \mathbb{Z}/2$ is trivial and the one on $\mathrm{PGL}_2(\mathbb{C})$ is the standard one. As by Lemma 2.9(3), $H^1(\mathrm{PGL}_2(\mathbb{C}))$ consists of two elements, which are the class of the identity and that of $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we find that $H^1(A_{\mathbb{C}}(p))$ consists of exactly four elements, which are the classes of the images of $(\mathrm{id}, \mathrm{id})$, (id, σ), (M, id), and (M, σ) under the above isomorphism. It moreover follows from the explicit action of $\mathrm{PGL}_2(\mathbb{C}) \times \langle\sigma\rangle$ on D_p given in Lemma 3.9 that these four images are equal to (x, y, z), $(-x, -y, -z)$, $(y, x, -z)$, and $(-y, -x, z)$, respectively. $\square$

Lemma 3.15. *If $p = z^2$, then $H^1(A_{\mathbb{C}}(p))$ contains exactly two elements, which are the classes of (x, y, z) and of $(y, x, -z)$.*

Proof: Lemma 3.10 provides an explicit isomorphism $\mathrm{PGL}_2(\mathbb{C}) \times \mathbb{C}^* \xrightarrow{\simeq} A_{\mathbb{C}}(p)$. As $H^1(\mathrm{PGL}_2(\mathbb{C}))$ consists of two elements, which are the class of the identity and that of $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ – see Lemma 2.9(3)) – and as $H^1(\mathbb{C}^*) = \{1\}$, the pointed set $H^1(A_{\mathbb{C}}(p))$ contains exactly two elements, which are the classes of the identity and that of the image of $(M, 1)$ under the above isomorphism. This latter is equal to the class of $(y, x, -z)$; compare with Lemma 3.10. $\square$

To describe $H^1(A_{\mathbb{C}}(p))$ when $\deg(p) \geq 3$, we will need the group $H_p \subseteq \mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}})$ associated to p. It corresponds to the group of symmetries of the polynomial.

Definition 3.16. Let $p \in \mathbb{C}[z]$ be a polynomial. We denote by $H_p \subseteq \mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}}) = \mathrm{Aut}_{\mathbb{C}}(\mathrm{Spec}(\mathbb{C}[z]))$ the subgroup

$$H_p = \{(cz + d) \mid c \in \mathbb{C}^*,\, d \in \mathbb{C},\, \exists \lambda \in \mathbb{C}^* : p(cz + d) = \lambda p(z)\}.$$

As the following lemma shows, the shape of H_p is particularly simple for polynomials in reduced form. A similar statement can be obtained for all polynomials of $\mathbb{C}[z]$ and even for other Galois field extensions.

Lemma 3.17. *Let $p \in \mathbb{C}[z]$ be in reduced form.*

(1) *If p has a unique root, then $p = z^d$ is a monomial and $H_p = \{(cz) \mid c \in \mathbb{C}^*\}$. In particular, H_p is then isomorphic to $\mathbb{C}^*$ and $H^1(H_p)$ contains only one element, namely the class of (z).*

(2) *If p has at least two roots, then $H_p = \{(cz) \mid c \in \mathbb{C}^*,\, c^n = 1\}$ is cyclic of finite order $n \geq 1$. In particular, $H^1(H_p)$ contains either a single element when n is odd or two elements when n is even, namely the classes of (z) and (cz) where c denotes any primitive n-th root of unity. Moreover, p is of the form $p(z) = z^m q(z^n)$ for some integer $m \geq 0$ and some polynomial $q \in \mathbb{C}[t]$ with $q(0) \neq 0$.*

Proof: (1) Recall that $H^1(\mathbb{C}^*)$ is trivial by Lemma 2.9.

(2) Let $p(z) = \sum_{i=0}^{\ell} p_i z^i \in \mathbb{C}[z]$ with $p_\ell = 1$ and $p_{\ell-1} = 0$ and suppose that p is not a monomial. Suppose that $c, \lambda \in \mathbb{C}^*$ and $d \in \mathbb{C}$ are such that $p(cz + d) = \lambda p(z)$. Then, $d = 0$ because $p_{\ell-1} = 0$. Moreover, for any i, j with $p_i, p_j \neq 0$, we find $c^i = \lambda = c^j$. This implies that c is of finite order, say $n \geq 1$, and that $i \equiv j \pmod{n}$. Hence, p is of the form $p(z) = z^m q(z^n)$, as claimed in the statement. In turn, $H_p = \{(cz) \mid c \in \mathbb{C}^*, c^n = 1\}$ is cyclic of order n. Finally, the claims about $H^1(H_p)$ follow from Lemma 2.9. $\square$

Lemma 3.18. *Let $p \in \mathbb{R}[z]$ be a polynomial of degree at least 3 in reduced form. Then, the following holds:*

(1) *If H_p is infinite and $\deg(p)$ is odd, then $H^1(A_{\mathbb{C}}(p))$ contains exactly two elements, namely the classes of*
$$(x, y, z),\ (y, x, z).$$
(2) *If H_p is infinite and $\deg(p)$ is even, or H_p is finite of odd order, then $H^1(A_{\mathbb{C}}(p))$ contains exactly three elements, namely the classes of*
$$(x, y, z),\ (y, x, z),\ (-y, -x, z).$$
(3) *If H_p is of even order $n \geq 2$ and $\deg(p)$ is odd, then $H^1(A_{\mathbb{C}}(p))$ contains exactly four elements, namely the classes of*
$$(x, y, z),\ (ax, ay, cz),\ (y, x, z),\ (ay, ax, cz),$$
for any $c \in \mathbb{C}^*$ of order n, and any $a \in \mathbb{C}^*$ such that $a^2 = c^{\deg(p)}$.

(4) *If H_p is of even order $n \geq 2$ and $\deg(p)$ is even, then $H^1(A_{\mathbb{C}}(p))$ contains exactly six elements, namely the classes of*
$$(x, y, z),\ (ax, ay, cz),\ (y, x, z),\ (-y, -x, z),\ (ay, ax, cz),\ (-ay, -ax, cz),$$
for any $c \in \mathbb{C}^*$ of order n, and any $a \in \mathbb{C}^*$ such that $a^2 = c^{\deg(p)}$.

Proof: As p is in reduced form, every element of H_p is of the form (cz) for some $c \in \mathbb{C}^*$ thanks to Lemma 3.17. Since $\deg(p) \geq 3$, we have by Lemma 3.8 $A_{\mathbb{C}}(p) = (A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)) \rtimes \langle (y, x, z) \rangle$, with $A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p) = \{(ax, by, cz) \mid a, b, c \in \mathbb{C}^*, abp(z) = p(cz)\}$. Thus, we can define a surjective group homomorphism $\varphi\colon A_{\mathbb{C}}(p) \twoheadrightarrow H_p \times \langle (y, x, z) \rangle$ by sending (ax, by, cz) onto (cz, id) and (y, x, z) onto $(\mathrm{id}, (y, x, z))$.

There are two cases to distinguish, both following from Lemma 3.17:

(i) If H_p is infinite or finite of odd order, then $H^1(H_p) = \{1\}$.

(ii) If H_p is finite of even order $n \geq 2$, then $H^1(H_p)$ contains exactly two classes, namely the class of the identity and a second class that contains (cz) for each $c \in \mathbb{C}^*$ of order n.

In case (ii), we fix $c \in \mathbb{C}^*$ of order n.

For each 1-cocycle $\tau \in Z^1(A_{\mathbb{C}}(p))$, we may assume that $\sigma = \varphi(\tau)$ belongs to $\{\mathrm{id}\} \times \langle (y,x,z) \rangle$ in case (i) and to $\{\mathrm{id},(cz)\} \times \langle (y,x,z) \rangle$ in case (ii). This gives two or four possibilities for σ, respectively. Moreover, two 1-cocycles that get mapped to different elements in $H_p \times \langle (y,x,z) \rangle$ cannot be equivalent. So, we may study the different possibilities for σ separately.

First we consider the case where $\tau \in Z^1(A_{\mathbb{C}}(p))$ with $\sigma = \varphi(\tau) = (\mathrm{id},\mathrm{id})$. Then, $\tau = \left(ax, \frac{1}{\overline{a}}y, z\right)$ for some $a \in \mathbb{C}^*$ with $a\overline{a} = 1$. Choosing $\lambda \in \mathbb{C}$ with $\lambda^2 = a$ and defining $\theta = \left(\lambda x, \frac{1}{\lambda}y, z\right) \in A_{\mathbb{C}}(p)$, we obtain $\theta^{-1} \circ \tau \circ \overline{\theta} = \left(\frac{a}{\lambda^2}x, \frac{\lambda^2}{a}y, z\right) = (x,y,z)$, since $\lambda\overline{\lambda} = 1$.

Now, consider the case where $\tau \in Z^1(A_{\mathbb{C}}(p))$ with $\sigma = ((cz),\mathrm{id})$. Then, $\tau = (ax, by, cz)$ for some $a, b \in \mathbb{C}^*$ with $a\overline{a} = b\overline{b} = 1$ and $abp(z) = p(cz) = c^{\deg(p)}p(z)$. Let $\lambda \in \mathbb{C}$ with $\lambda\overline{\lambda} = 1$ and define $\theta = \left(\lambda x, \frac{1}{\lambda}y, z\right)$. Then, $\theta^{-1} \circ \tau \circ \overline{\theta} = \left(\frac{a}{\lambda^2}x, \lambda^2 by, cz\right)$. Choosing λ with $\lambda^4 = \frac{a}{b}$, we may thus assume that $b = a$, i.e., that $\tau = (ax, ay, cz)$ with $a^2 = c^{\deg(p)}$. Repeating the same argument with $\lambda = \mathbf{i}$, we see that the two 1-cocycles (ax, ay, cz) and $(-ax, -ay, cz)$ are equivalent. Hence, there is only one class of 1-cocycles associated to $\sigma = ((cz),\mathrm{id})$.

Finally, we consider the case where $\tau \in Z^1(A_{\mathbb{C}}(p))$ with $\sigma = (\mathrm{id},(y,x,z))$ or $\sigma = ((cz),(y,x,z))$. Then, $\tau = \left(ay, \frac{1}{\overline{a}}x, \mu z\right)$ for some $a \in \mathbb{C}^*$ satisfying $a \cdot \frac{1}{\overline{a}} = \mu^{\deg(p)}$, where $\mu = 1$ or $\mu = c$. Choosing $\lambda \in \mathbb{R}_{>0}$ with $\lambda^2 = |a|$ and defining $\theta = \left(\lambda x, \frac{1}{\lambda}y, z\right)$, we obtain $\theta^{-1} \circ \tau \circ \overline{\theta} = \left(\frac{a}{\lambda^2}y, \frac{\lambda^2}{\overline{a}}x, \mu z\right)$. So, we may assume that $|a| = 1$, and hence that $\tau = (ay, ax, \mu z)$.

As $p(\mu z) = a^2 p(z)$, we get $\mu^{\deg(p)} = a^2$. In particular, $a = \pm 1$ if $\mu = 1$. To conclude the proof, it only remains to prove that the 1-cocycles $(ay, ax, \mu z)$ and $(-ay, -ax, \mu z)$ are equivalent if and only if the following holds:

($\clubsuit$) $\deg(p)$ is odd and H_p is either infinite or finite of even order.

First suppose that ($\clubsuit$) holds. In this case, $(-z) \in H_p$, and $p(-z) = (-1)^{\deg(p)}p(z) = -p(z)$. Hence, $\theta = (x, -y, -z) \in A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)$, and $\theta^{-1} \circ (ay, ax, \mu z) \circ \overline{\theta} = (-ay, -ax, \mu z)$.

Suppose now that ($\clubsuit$) does not hold, and suppose, by contradiction, that $\theta^{-1} \circ (ay, ax, \mu z) \circ \overline{\theta} = (-ay, -ax, \mu z)$ for some $\theta \in A_{\mathbb{C}}(p)$.

If $\theta \in A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)$, then $\theta = (\alpha x, \beta y, \gamma z)$ for some $\alpha, \beta, \gamma \in \mathbb{C}^*$ such that $\alpha\beta p(z) = p(\gamma z)$. This implies that $\theta^{-1} \circ (ay, ax, \mu z) \circ \overline{\theta} = \left(\frac{\overline{\beta}}{\alpha}ay, \frac{\overline{\alpha}}{\beta}ax, \frac{\overline{\gamma}}{\gamma}\mu z\right)$. Hence, $\beta = -\overline{\alpha}$ and $\gamma \in \mathbb{R}$. In particular, we have that $\alpha\beta = -\alpha\overline{\alpha} \in \mathbb{R}_{<0}$. Since $\alpha\beta p(z) = p(\gamma z) = \gamma^{\deg(p)}p(z)$, we also have $\alpha\beta = \gamma^{\deg(p)}$. This implies that $\deg(p)$ is odd and $\gamma < 0$. As we assumed that (♣) does not hold, H_p is finite of odd order. But then, $(\gamma z) \notin H_p$, a contradiction.

If $\theta \in A_{\mathbb{C}}(p) \setminus B_{\mathbb{C}}(p)$, then write $\theta = (y, x, z) \circ \theta'$ with $\theta' \in A_{\mathbb{C}}(p) \cap B_{\mathbb{C}}(p)$. Since (y, x, z) commutes with $\tau = (ay, ax, \mu z)$, the equality $\theta^{-1} \circ \tau \circ \overline{\theta} = \theta'^{-1} \circ \tau \circ \overline{\theta'}$ holds and we get a contradiction as above. $\square$

3.5. Real forms.

Proposition 3.19. *Let $p = z^2 - 1$. The complex surface*

$$D_p = \operatorname{Spec}(\mathbb{C}[x, y, z]/(xy - z^2 + 1))$$

has exactly four nonisomorphic classes of real forms, which are those of the four real surfaces

$$\begin{aligned} S_1 &= \operatorname{Spec}(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 + 1)), \\ S_2 &= \operatorname{Spec}(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)), \\ S_3 &= \operatorname{Spec}(\mathbb{R}[x, y, z]/(x^2 - y^2 + z^2 - 1)), \\ S_4 &= \operatorname{Spec}(\mathbb{R}[x, y, z]/(x^2 - y^2 + z^2 + 1)). \end{aligned}$$

All four are pairwise nonhomeomorphic: their real loci are diffeomorphic to

$$S_1(\mathbb{R}) = \varnothing,\ S_2(\mathbb{R}) \simeq \mathbb{S}^2,\ S_3(\mathbb{R}) \simeq \mathbb{R}^2 \setminus \{(0, 0)\},\ S_4(\mathbb{R}) \simeq \mathbb{R}^2 \amalg \mathbb{R}^2.$$

Proof: By Lemma 3.13 and Lemma 3.14, $H^1(\operatorname{Aut}_{\mathbb{C}}(D_p))$ contains exactly four elements, namely the classes of $\tau_3 = (x, y, z)$, $\tau_4 = (-x, -y, -z)$, $\tau_1 = (y, x, -z)$, $\tau_2 = (-y, -x, z)$. Therefore, there are exactly four nonisomorphic real forms of D_p.

To see that they correspond to the real surfaces $S_1, \ldots, S_4$, we produce, for every $i = 1, 2, 3, 4$, an element $\theta_i \in \mathrm{GL}_3(\mathbb{C}) \subseteq \operatorname{Aut}(\mathbb{A}^3_{\mathbb{C}})$ such that $\tau_i \circ \rho = \theta_i \circ \rho \circ \theta_i^{-1}$, where ρ is the standard real form $(x, y, z) \mapsto (\overline{x}, \overline{y}, \overline{z})$ on $\mathbb{A}^3_{\mathbb{C}}$, and such that $\theta_i^{-1}(D_p)$ is the complexification of S_i, i.e., is $S_i \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C})$.

i	τ_i	θ_i with $\tau_i \circ \rho \circ \theta_i = \theta_i \circ \rho$	$\theta_i^*(xy - z^2 + 1)$
1	$(y, x, -z)$	$(x + \mathbf{i}y, x - \mathbf{i}y, \mathbf{i}z)$	$x^2 + y^2 + z^2 + 1$
2	$(-y, -x, z)$	$(x + \mathbf{i}y, -x + \mathbf{i}y, z)$	$-(x^2 + y^2 + z^2 - 1)$
3	(x, y, z)	$(x + y, y - x, z)$	$-(x^2 - y^2 + z^2 - 1)$
4	$(-x, -y, -z)$	$(\mathbf{i}(-x + y), \mathbf{i}(x + y), \mathbf{i}z)$	$x^2 - y^2 + z^2 + 1$

From the equations of S_1 and S_2, we see that $S_1(\mathbb{R}) = \varnothing$ and $S_2(\mathbb{R}) = \mathbb{S}^2$. The map $(x, y, z) \mapsto \left(y, \frac{x}{\sqrt{x^2+z^2}}, \frac{z}{\sqrt{x^2+z^2}}\right)$ provides an explicit diffeomorphism from $S_3(\mathbb{R})$ to the cylinder $\mathbb{R} \times \mathbb{S}^1$, which is diffeomorphic to the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$. For $S_4(\mathbb{R})$, note that $x^2 + z^2 = y^2 - 1$

implies that $y \neq 0$. Then, $S_4(\mathbb{R}) = \{(x,y,z) \mid y > 0\} \amalg \{(x,y,z) \mid y < 0\}$ is diffeomorphic to the disjoint union of two copies of $\mathbb{R}^2$. $\square$

Proposition 3.20. *Let $p = z^2$. The complex surface*

$$D_p = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - z^2))$$

has exactly two nonisomorphic classes of real forms, which are those of the two real surfaces

$$T_1 = \operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + z^2)),$$
$$T_2 = \operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 - y^2 - z^2)).$$

Both are pairwise nonhomeomorphic: $T_1(\mathbb{R})$ consists of only one point, while $T_2(\mathbb{R})$ is infinite; it is a cone over $\mathbb{S}^1$.

Proof: By Lemma 3.13 and Lemma 3.15, $H^1(\operatorname{Aut}_{\mathbb{C}}(D_p))$ contains exactly two elements, namely the classes of $\tau_2 = (x,y,z)$, $\tau_1 = (y,x,-z)$. Therefore, there are exactly two nonisomorphic real forms of D_p.

To see that they correspond to the real surfaces T_1, T_2, we give, for every $i = 1,2$, an element $\theta_i \in \mathrm{GL}_3(\mathbb{C}) \subseteq \operatorname{Aut}(\mathbb{A}^3_{\mathbb{C}})$ such that $\tau_i \circ \rho = \theta_i \circ \rho \circ \theta_i^{-1}$, where ρ is the standard real form $(x,y,z) \mapsto (\overline{x}, \overline{y}, \overline{z})$ on $\mathbb{A}^3_{\mathbb{C}}$, and such that $\theta_i^{-1}(D_p)$ is the complexification of T_i, i.e., is $T_i \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C})$.

i	τ_i	θ_i with $\tau_i \circ \rho \circ \theta_i = \theta_i \circ \rho$	$\theta_i^*(xy - z^2)$
1	$(y,x,-z)$	$(x + \mathbf{i}y, x - \mathbf{i}y, \mathbf{i}z)$	$x^2 + y^2 + z^2$
2	(x,y,z)	$(x - y, x + y, z)$	$x^2 - y^2 - z^2$

The equation of T_1 directly gives $T_1(\mathbb{R}) = \{(0,0,0)\}$, whereas $T_2(\mathbb{R})$ is a cone over the conic $x^2 - y^2 = z^2$ in $\mathbb{P}^2_{\mathbb{R}}$ whose set of real points is diffeomorphic to $\mathbb{S}^1$. $\square$

Proposition 3.21. *Let $p \in \mathbb{R}[z]$ be a polynomial of degree $d \geq 3$ in reduced form and define $D_p = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - p(z)))$.*

(1) *If H_p is infinite, then $p = z^d$ and there are two cases:*

(i) *If d is odd, then D_p has exactly two isomorphism classes of real forms, namely those of*

$$\operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 - z^d)).$$

(ii) *If d is even, then D_p has exactly three isomorphism classes of real forms, namely those of*

$$\operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + z^d)),$$
$$\textit{and} \quad \operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 - z^d)).$$

(2) *If H_p is cyclic of order n, then $p = z^m q(z^n)$ for some integer $m \geq 0$ and some monic polynomial $q \in \mathbb{R}[z] \setminus \mathbb{R}$ with $q(0) \neq 0$, and there are three cases:*

(i) *If n is odd, then D_p has exactly three isomorphism classes of real forms, namely those of*

$$\begin{aligned} &\operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + z^m q(z^n))), \\ \textit{and} \quad &\operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 - z^m q(z^n))). \end{aligned}$$

(ii) *If n is even and* $\deg(p)$ *– and thus m – is odd, then D_p has exactly four isomorphism classes of real forms, namely those of*

$$\operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 - z^m q(\pm z^n))).$$

(iii) *If n is even and* $\deg(p)$ *– and thus m – is even, then D_p has exactly six isomorphism classes of real forms, namely those of*

$$\begin{aligned} &\operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + z^m q(\pm z^n))), \\ \textit{and} \quad &\operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 - z^m q(\pm z^n))). \end{aligned}$$

Proof: Define

$$\begin{aligned} &\tau_1 = (x,y,z), && \tau_2 = (ax, ay, cz), && \tau_3 = (y,x,z), \\ &\tau_4 = (-y,-x,z), && \tau_5 = (ay, ax, cz), && \tau_6 = (-ay, -ax, cz), \end{aligned}$$

which are the 1-cocycles appearing in Lemma 3.18.

(1) Suppose that H_p is infinite. Then, D_p is the surface of equation $xy = z^d$, and by Lemma 3.18(1)-(2) we only need to consider τ_1, τ_3, and τ_4. In the table below, we produce, for every $i \in \{1,3,4\}$, an element $\theta_i \in \mathrm{GL}_3(\mathbb{C}) \subseteq \operatorname{Aut}(\mathbb{A}^3_{\mathbb{C}})$ such that $\tau_i \circ \rho = \theta_i \circ \rho \circ \theta_i^{-1}$, where ρ is the standard real form $(x,y,z) \mapsto (\overline{x}, \overline{y}, \overline{z})$ on $\mathbb{A}^3_{\mathbb{C}}$, and compute the equation of the hypersurface $\theta_i^{-1}(D_p) \subset \mathbb{A}^3_{\mathbb{C}}$. Combining Lemma 3.13 with Lemma 3.18, this proves (1).

i	τ_i	θ_i with $\tau_i \circ \rho \circ \theta_i = \theta_i \circ \rho$	$\theta_i^*(xy - z^d)$
1	(x,y,z)	$(x+y, x-y, z)$	$x^2 - y^2 - z^d$
3	(y,x,z)	$(x+\mathbf{i}y, x-\mathbf{i}y, z)$	$x^2 + y^2 - z^d$
4	$(-y,-x,z)$	$(x+\mathbf{i}y, -x+\mathbf{i}y, z)$	$-(x^2 + y^2 + z^d)$

(2) Suppose that H_p is cyclic of finite order $n \geq 1$. Then, D_p is given by an equation of the form $xy = z^m q(z^n)$ with $m \geq 0$ and $\deg(q) \geq 1$ such that $q(0) \neq 0$. Let $c = \mathrm{e}^{2\pi\mathbf{i}/n}$ be a primitive n-th root of unity and set $a = \mathrm{e}^{2\pi\mathbf{i}m/2n}$, which satisfies $a^2 = c^m = c^{\deg(p)}$.

Fix $\alpha = \mathrm{e}^{2\pi\mathbf{i}/2n}$ and $\beta = \mathrm{e}^{2\pi\mathbf{i}m/4n}$, for which $\alpha^2 = c$ and $\alpha^n = -1$, and $\beta^2 = a = \alpha^m$, respectively. In the table below, we produce, for every $i \in \{1, \dots, 6\}$, an element $\theta_i \in \mathrm{GL}_3(\mathbb{C}) \subseteq \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$ such that $\tau_i \circ \rho = \theta_i \circ \rho \circ \theta_i^{-1}$, where ρ is the standard real form $(x, y, z) \mapsto (\overline{x}, \overline{y}, \overline{z})$ on $\mathbb{A}^3_{\mathbb{C}}$ and compute the equation of the hypersurface $\theta_i^{-1}(D_p) \subset \mathbb{A}^3_{\mathbb{C}}$. Combining Lemma 3.13 with Lemma 3.18, this proves (2).

i	τ_i	θ_i with $\tau_i \circ \rho \circ \theta_i = \theta_i \circ \rho$	$\theta_i^*(xy - z^m q(z^n))$
1	(x, y, z)	$(x + y, x - y, z)$	$x^2 - y^2 - z^m q(z^n)$
2	(ax, ay, cz)	$(\beta(x + y), \beta(x - y), \alpha z)$	$\beta^2(x^2 - y^2 - z^m q(-z^n))$
3	(y, x, z)	$(x + \mathbf{i}y, x - \mathbf{i}y, z)$	$x^2 + y^2 - z^m q(z^n)$
4	$(-y, -x, z)$	$(x + \mathbf{i}y, -x + \mathbf{i}y, z)$	$-(x^2 + y^2 + z^m q(z^n))$
5	(ay, ax, cz)	$(\beta(x + \mathbf{i}y), \beta(x - \mathbf{i}y), \alpha z)$	$\beta^2(x^2 + y^2 - z^m q(-z^n))$
6	$(-ay, -ax, cz)$	$(\beta(x + \mathbf{i}y), \beta(-x + \mathbf{i}y), \alpha z)$	$-\beta^2(x^2 + y^2 + z^m q(-z^n))$

□

We finalise this section by proving Theorem A, which summarises Propositions 3.19, 3.20, and 3.21.

Proof of Theorem A: We recall that $p \in \mathbb{R}[z]$ is a polynomial in reduced form of degree $d \geq 2$, $p(z) = z^m q(z^n)$, where $m \geq 0$, $n \geq 1$, $q \in \mathbb{R}[z]$, $q(0) \neq 0$, and where q and n are chosen such that n is maximal if $q \neq 1$. In particular, q, n, and m are uniquely determined by p.

First we remark that S_{abc} is a real form of D_p for all $a, b, c \in \{0, 1\}$. Indeed, the linear map $(x + \mathbf{i}^{a-1}y, x - \mathbf{i}^{a-1}y, z) \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$ sends the hypersurface $x^2 + (-1)^a y^2 + (-1)^b z^m q((-1)^c z^n) = 0$ onto that of equation $xy + (-1)^b z^m q((-1)^c z^n) = 0$, which is isomorphic to D_p by Theorem 3.2. Propositions 3.19, 3.20, and 3.21 then give the number $2 \leq i \leq 6$ of isomorphism classes together with a list of representatives.

First suppose that $q = 1$. Then $p(z) = z^d = z^m$ and H_p is thus infinite. If $d = 2$, then Proposition 3.20 gives $i = 2$ together with the representatives S_{000} and S_{110}. If $d \geq 3$, Proposition 3.21(1) gives $i = 2$ when d is odd and $i = 3$ when d is even. In the case where d is odd, Proposition 3.21(1)(i) gives the two representatives $\mathrm{Spec}(\mathbb{R}[x, y, z]/(x^2 + y^2 - z^d)) = S_{010}$ and $\mathrm{Spec}(\mathbb{R}[x, y, z]/(x^2 - y^2 - z^d)) = S_{110}$. Using the isomorphism $(x, y, -z)\colon S_{000} \xrightarrow{\simeq} S_{010}$, we obtain the two representatives given in the statement of Theorem A. In the case where d is even, the three representatives of Proposition 3.21(1)(ii) are precisely S_{000}, S_{010}, and S_{110}.

Suppose now that $q \neq 1$. Hence, $\deg(q) \geq 1$, and as n was chosen maximal, the group

$$H_p = \{\lambda \in \mathbb{C}^* \mid p(\lambda z) = \lambda^d p(z)\} = \{(\lambda z) \mid \lambda \in \mathbb{C}^*, \lambda^n = 1\}$$

is cyclic of order n by Lemma 3.17. If $d = 2$, then $p = z^2+\mu$, for some $\mu \in \mathbb{R}^*$. So $(m,n) = (0,2)$ and the surface D_p is isomorphic to $D_{p'}$ with $p' = z^2 - 1$ by Theorem 3.2. Hence, Proposition 3.19 gives $i = 4$ and provides the four representatives $\mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 + z^2 \pm 1))$. We now need to check that these surfaces are isomorphic to the four surfaces S_{abb}, $a, b \in \{0,1\}$ that are given in the statement of Theorem A. Since these latter are defined by $\mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2 \pm y^2 + z^2 \pm \mu))$, it actually suffices to apply the linear automorphism $(\xi x, \xi y, \xi z) \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{R}})$, where $\xi = \sqrt{|\mu|}$.

If $d \geq 3$, Proposition 3.21(2) specifies three different cases.

If n is odd, then $i = 3$ and the representatives in Proposition 3.21(2)(i) are precisely the surfaces S_{000}, S_{010}, and S_{110}.

If n is even and d is odd, then $i = 4$ and the representatives given by Proposition 3.21(2)(ii) are the surfaces S_{a1c} with $a, c \in \{0,1\}$. As the map $(x, y, -z) \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{R}})$ sends S_{a1c} to S_{a0c}, we obtain $S_{a1c} \simeq S_{a0c}$, and in particular $S_{a1c} \simeq S_{acc}$. This gives the result.

The remaining case is when n and d are both even. Here, $i = 6$ and the real forms are S_{00c}, S_{a1c}, $a, c \in \{0,1\}$ by Proposition 3.21(2)(iii). □

In Proposition 3.19, of the given complex surface, there is only one real form whose real locus is compact and nondegenerate in the sense that the dimension of the real locus as a manifold is equal to 2. The following examples illustrate that we can also construct complex surfaces with two nonisomorphic real forms having compact and nondegenerate loci. In the first example, the corresponding manifolds are diffeomorphic. In the second example, they are not.

Example 3.22. Choose $p(z) = (z^2-1)(z^2+4) = q(z^2)$ with $q(z) = (z-1)(z+4)$. By Theorem A, the surface $D_p = \mathrm{Spec}(\mathbb{C}[x,y,z]/(xy - p(z)))$ admits six isomorphism classes of real forms. In particular, the surfaces

$$\begin{aligned}
S_{000} &= \mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + q(z^2))) \\
&= \mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + (z^2-1)(z^2+4))), \\
S_{001} &= \mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + q(-z^2))) \\
&= \mathrm{Spec}(\mathbb{R}[x,y,z]/(x^2 + y^2 + (z^2+1)(z^2-4)))
\end{aligned}$$

are two nonisomorphic real forms of D_p. As D_p is smooth, their real loci are the manifolds $S_{000}(\mathbb{R})$ and $S_{001}(\mathbb{R})$. Both are diffeomorphic to the sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, via the diffeomorphisms

$$\begin{aligned}
&\mathbb{S}^2 \xrightarrow{\simeq} S_{000}(\mathbb{R}), \quad (x,y,z) \longmapsto (x\sqrt{z^2+4}, y\sqrt{z^2+4}, z), \\
&\mathbb{S}^2 \xrightarrow{\simeq} S_{001}(\mathbb{R}), \quad (x,y,z) \longmapsto (2x\sqrt{4z^2+1}, 2y\sqrt{4z^2+1}, 2z).
\end{aligned}$$

Example 3.23. Choose $p(z) = (z^2-1)(z^2+1)(z^2+4) = q(z^2)$ with $q(z) = (z-1)(z+1)(z+2)$. By Theorem A, $D_p = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy-p(z)))$ admits six isomorphism classes of real forms. In particular, the surfaces

$$\begin{aligned} S_{000} &= \operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2+y^2+q(z^2))) \\ &= \operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2+y^2+(z^2-1)(z^2+1)(z^2+4))), \\ S_{011} &= \operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2+y^2-q(-z^2))) \\ &= \operatorname{Spec}(\mathbb{R}[x,y,z]/(x^2+y^2+(z^2+1)(z^2-1)(z^2-4))) \end{aligned}$$

are two nonisomorphic real forms of D_p. Similarly as in Example 3.22, $S_{000}(\mathbb{R})$ is diffeomorphic to the sphere $\mathbb{S}^2$. However, $S_{011}(\mathbb{R})$ has two connected components $U_+ = \{(x,y,z) \in S_{011}(\mathbb{R}) \mid z > 0\}$ and $U_- = \{(x,y,z) \in S_{011}(\mathbb{R}) \mid z < 0\}$. Hence, the two compact manifolds $S_{000}(\mathbb{R})$ and $S_{011}(\mathbb{R})$ are not diffeomorphic. One can check that U_+ and U_- are both diffeomorphic to $\mathbb{S}^2$, and thus that $S_{011}(\mathbb{R})$ is diffeomorphic to the union of two spheres.

4. The surfaces $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2$ and $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$

In this section, we compute the real forms of the two affine surfaces $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2$ and $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$. In Propositions 4.2 and 4.3, we prove that these surfaces have respectively six and four isomorphism classes of real forms. In the case of $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2$, a partial result, together with a sketch of the proof, is given in [**21**, Lemma 1.5 and Remark 1.6]. Our proof follows essentially the same lines.

The following well-known result is an easy exercise. We give the proof for the sake of completeness.

Lemma 4.1. *There are exactly three conjugacy classes of elements of order* 2 *in* $\mathrm{GL}_2(\mathbb{Z})$*, namely those of* $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$*,* $\sigma_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$*, and* $\sigma_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$*.*

Proof: First we prove that the involutions σ_1, σ_2, σ_3 are pairwise non-conjugate. As $\det(\sigma_2) = 1$ and $\det(\sigma_1) = \det(\sigma_3) = -1$, we only need to prove that σ_1 and σ_3 are not conjugate. If they were, we would have a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ such that $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ are eigenvectors of σ_3 of eigenvalue 1 and -1 respectively. This would imply $c = a$ and $d = -b$, which is impossible, as $\det(M) = ad - bc = -2ab \notin \{\pm 1\}$.

It remains to prove that every element $M \in \mathrm{GL}_2(\mathbb{Z})$ of order 2 is conjugate to σ_1, σ_2, or σ_3. If $M \neq \sigma_2$, then the eigenvalues of M are 1 and -1. Consider an eigenvector of M with integer entries prime to each other and complete it to a matrix of $\mathrm{GL}_2(\mathbb{Z})$ that conjugates M to $M' = \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right)$ for some $b, d \in \mathbb{Z}$. Note that $d = -1$, since M has eigenvalues 1 and -1. Conjugating M' by $\left(\begin{smallmatrix}1 & \mu\\ 0 & 1\end{smallmatrix}\right)$ with $\mu \in \mathbb{Z}$, we get the matrix $\left(\begin{smallmatrix}1 & b-2\mu\\ 0 & -1\end{smallmatrix}\right)$. If b is even, then M is conjugate to σ_1. If b is odd, then M is conjugate to $\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)$, which is conjugate to σ_3 by $\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$. $\square$

Proposition 4.2.

(1) *The affine complex curve $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ has exactly three equivalence classes of real structures, namely those of*

$$\rho_1 \colon x \longmapsto \overline{x},\ \rho_2 \colon x \longmapsto \overline{x}^{-1},\ \rho_3 \colon x \longmapsto -\overline{x}^{-1}.$$

The corresponding real forms of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ are the three affine conics $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathbb{A}^2_{\mathbb{R}}$ given by

$$xy - 1 = 0,\ x^2 + y^2 - 1 = 0,\ \textit{and } x^2 + y^2 + 1 = 0,$$

whose real loci are diffeomorphic to $\Gamma_1(\mathbb{R}) \simeq \mathbb{R}^*$, $\Gamma_2(\mathbb{R}) \simeq \mathbb{S}^1$, and $\Gamma_3(\mathbb{R}) = \varnothing$, respectively.

(2) *The affine complex surface $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2$ has exactly six isomorphism classes of real forms, namely those of*

$$\Gamma_1 \times \Gamma_1,\ \Gamma_1 \times \Gamma_2,\ \Gamma_1 \times \Gamma_3,\ \Gamma_2 \times \Gamma_2,\ \Gamma_3 \times \Gamma_3,\ \textit{and } \mathbb{A}^2_{\mathbb{R}} \setminus \{x^2 + y^2 = 0\}.$$

Moreover, the real form $\Gamma_2 \times \Gamma_3$ is isomorphic to $\Gamma_3 \times \Gamma_3$.

Proof: We recall that for $n \geq 1$, the invertible regular functions on $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^n$ are the Laurent monomials $\mu x_1^{a_1} \cdots x_n^{a_n}$, with $\mu \in \mathbb{C}^*$, $a_1, \dots, a_n \in \mathbb{Z}$. This implies that $\operatorname{Aut}((\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^n) \simeq (\mathbb{C}^*)^n \rtimes \mathrm{GL}_n(\mathbb{Z})$, and gives in particular

$$\begin{aligned}\operatorname{Aut}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) &= \{\lambda x^a \mid \lambda \in \mathbb{C}^*,\ a = \pm 1\},\\ \operatorname{Aut}((\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2) &= \{(a x^{m_{11}} y^{m_{12}}, b x^{m_{21}} y^{m_{22}})\\ &\qquad \mid a, b \in \mathbb{C}^*,\ \left(\begin{smallmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})\}.\end{aligned}$$

We prove (1). As the complexification of Γ_i is a smooth affine conic with two points at infinity, it is isomorphic to $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, and thus Γ_i is a real form of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Since Γ_1, Γ_2, and Γ_3 have nonhomeomorphic real loci, we get three pairwise nonisomorphic real forms. We now prove that these are the only ones. We fix the standard real structure ρ_1 that corresponds to the real form $\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}$, isomorphic to Γ_1. The description of $\operatorname{Aut}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ implies that every element of $Z^1(\operatorname{Aut}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}))$ is either of the form $\nu = (\mu x)$ with $\mu \in \mathbb{C}^*$, $\mu\overline{\mu} = 1$, or of the form $\nu = (\mu x^{-1})$ with $\mu \in \mathbb{R}^*$. In the first case, we reduce to $\mu = 1$, as $H^1(\mathbb{C}^*) = \{1\}$ (Lemma 2.9), and obtain the trivial real form Γ_1. In the second case, we choose $\alpha = (\lambda x)$ with $\lambda \in \mathbb{R}$, $\lambda^2 = |\mu|$, and obtain $\alpha^{-1} \circ \nu \circ \overline{\alpha} = (\pm x^{-1})$. This gives the two real structures ρ_2 and ρ_3, which then necessarily correspond to Γ_2 and Γ_3. As $\Gamma_3(\mathbb{R}) = \varnothing$ and as no $x \in \mathbb{C}^*$ satisfies $x = \rho_3(x) = -\overline{x}^{-1}$, we find that ρ_i corresponds to Γ_i for $i = 1, 2, 3$.

It remains to prove (2). We fix the standard real form $\rho_1 \times \rho_1$ and compute $H^1(\mathrm{Aut}((\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2))$. Let $\nu \in Z^1(\mathrm{Aut}((\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2))$ be a 1-cocycle. As $\mathrm{Aut}((\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2) \simeq (\mathbb{C}^* \times \mathbb{C}^*) \rtimes \mathrm{GL}_2(\mathbb{Z})$, the 1-cocycle ν gives rise to an involution $\sigma \in \mathrm{GL}_2(\mathbb{Z})$. Up to conjugation, σ is equal to precisely one of $\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, or $\sigma_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; see Lemma 4.1. These four being pairwise nonconjugate in $\mathrm{GL}_2(\mathbb{Z})$, two 1-cocycles arising from two different σ_i, σ_j are not equivalent, so we can study each σ_i separately. For σ_0, σ_1, and σ_2, we can, on each component of the map ν, apply the same reduction as we did above for $\mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$.

If $\sigma = \sigma_0$, then $\nu = (\lambda x, \mu y)$, where $\lambda, \mu \in \mathbb{C}^*$ have modulus 1. As $H^1(\mathbb{C}^*) = \{1\}$, we can reduce to the case $\lambda = \mu = 1$, and get the real structure $\rho_1 \times \rho_1$, and thus the real form $\Gamma_1 \times \Gamma_1 \simeq (\mathbb{A}^1_{\mathbb{R}} \setminus \{0\})^2$.

If $\sigma = \sigma_1$, then $\nu = (\lambda x, \mu y^{-1})$, where $\lambda \in \mathbb{C}^*$ has modulus 1 and $\mu \in \mathbb{R}^*$. We reduce to $\lambda = 1$ and $\mu = \pm 1$, get two real structures $\rho_1 \times \rho_2$ and $\rho_1 \times \rho_3$, and thus the real forms $\Gamma_1 \times \Gamma_2$ and $\Gamma_1 \times \Gamma_3$. These real forms are not isomorphic, as the second one has no real points, whereas the first has.

If $\sigma = \sigma_2$, then $\nu = (\lambda x^{-1}, \mu y^{-1})$, where $\lambda, \mu \in \mathbb{R}^*$. We reduce to $\lambda, \mu \in \{\pm 1\}$, get the four real structures $\rho_i \times \rho_j$, where $i, j = 2, 3$, and hence four real forms $\Gamma_i \times \Gamma_j$. With $\alpha = (x, xy)$, we obtain $\alpha^{-1} \circ (-x^{-1}, y^{-1}) \circ \overline{\alpha} = (-x^{-1}, -y^{-1})$, and hence an isomorphism $\Gamma_2 \times \Gamma_3 \xrightarrow{\simeq} \Gamma_3 \times \Gamma_3$. Similarly, $\alpha = (y, x)$ provides an isomorphism $\Gamma_2 \times \Gamma_3 \xrightarrow{\simeq} \Gamma_3 \times \Gamma_2$. As $\Gamma_2 \times \Gamma_2$ has real points and $\Gamma_2 \times \Gamma_3$ does not, we obtain exactly two isomorphism classes of real forms in this case.

If $\sigma = \sigma_3$, then $\nu = \left(\frac{1}{\lambda} y, \overline{\lambda} x\right)$, for some $\lambda \in \mathbb{C}^*$. With $\alpha = \left(\frac{1}{\lambda} x, y\right)$, we obtain $\alpha^{-1} \circ \nu \circ \overline{\alpha} = (y, x)$, resulting in the real structure $\rho' \colon (x, y) \mapsto (\overline{y}, \overline{x})$. We use the isomorphism $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})^2 = \mathbb{A}^2_{\mathbb{C}} \setminus \{xy = 0\} \xrightarrow{\simeq} \mathbb{A}^2_{\mathbb{C}} \setminus \{x^2 + y^2 = 0\}$, $(x, y) \mapsto (x + y, \mathbf{i}(x - y))$. It conjugates the real structure ρ' to the standard real structure $(x, y) \mapsto (\overline{x}, \overline{y})$. The real form induced is then isomorphic to $\mathbb{A}^2_{\mathbb{R}} \setminus \{x^2 + y^2 = 0\}$. □

Proposition 4.3. *The affine complex surface $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ has exactly four isomorphism classes of real forms, namely those of*

$$\mathbb{A}^1_{\mathbb{R}} \times \Gamma_1,\ \mathbb{A}^1_{\mathbb{R}} \times \Gamma_2,\ \mathbb{A}^1_{\mathbb{R}} \times \Gamma_3,\ \textit{and}\ \mathbb{P}^2_{\mathbb{R}} \setminus \{x^2 + y^2 = 0\},$$

where Γ_1, Γ_2, and Γ_3 are the real forms of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, given in Proposition 4.2(1).

Proof: First, recall that $\mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}))$ is equal to

$$\{(\lambda x y^m + c(y), \mu y^{\pm 1}) \mid \lambda, \mu \in \mathbb{C}^*,\ m \in \mathbb{Z},\ c \in \mathbb{C}[y, y^{-1}] \subseteq \mathbb{C}(y)\}.$$

To obtain this, we can use the fact that every morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ is constant, so any automorphism φ of $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ sends a fibre of the first projection to another fibre. Thus, φ is of the form $(a(x,y), b(y))$, where $x \mapsto a(x,y)$ is an automorphism of $\mathbb{A}^1_{\mathbb{C}}$ for every y, and where $y \mapsto b(y)$ is an automorphism of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, since the inverse of φ is of the same form.

We fix the standard real structure $(x,y) \mapsto (\overline{x}, \overline{y})$ on $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$, corresponding to the real form $\mathbb{A}^1_{\mathbb{R}} \times (\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}) \simeq \mathbb{A}^1_{\mathbb{R}} \times \Gamma_1$, see Proposition 4.2(1), and compute $H^1(\mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})))$. We consider the group homomorphism $\theta\colon \mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})) \to \mathrm{GL}_2(\mathbb{Z})$ that sends $(\lambda x y^m + c(y), \mu y^{\pm 1})$ onto $\left(\begin{smallmatrix} 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right)$.

Let $\nu \in Z^1(\mathrm{Aut}(\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})))$ be a 1-cocycle. Then, the matrix $\theta(\nu)$ is an involution in the group $H = \left\{\left(\begin{smallmatrix} 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right) \mid m \in \mathbb{Z}\right\} \subseteq \mathrm{GL}_2(\mathbb{Z})$. This involution is either $\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$, or more generally $\left(\begin{smallmatrix} 1 & m \\ 0 & -1 \end{smallmatrix}\right)$ for any $m \in \mathbb{Z}$. Conjugating the latter by $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ gives the matrix $\left(\begin{smallmatrix} 1 & m-2a \\ 0 & -1 \end{smallmatrix}\right)$, so we may reduce to the cases of σ_0, σ_1, or σ_2. Since $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is conjugate to σ_2, using $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, Lemma 4.1 implies that the involutions σ_1, σ_2 are not conjugate in $\mathrm{GL}_2(\mathbb{Z})$, and thus also not conjugate in H. We then obtain three disjoint families of real forms, up to isomorphism, and may consider the three cases separately.

First consider the case where $\theta(\nu) = \sigma_0$. Thus, $\nu = (\lambda x + c(y), \mu y)$ for some $\lambda, \mu \in \mathbb{C}^*$ of modulus 1 and $c \in \mathbb{C}[y, y^{-1}]$. Considering $\alpha^{-1} \circ \nu \circ \overline{\alpha}$ with $\alpha = (\xi_1 x, \xi_2 y)$ where $\xi_1^2 = \lambda$, $\xi_2^2 = \mu$, we may reduce to the case where $\lambda = \mu = 1$. Then, the 1-cocycle condition $\nu \circ \overline{\nu} = 1$ gives $c(y) + \overline{c}(y) = 0$. Considering $\alpha^{-1} \circ \nu \circ \overline{\alpha}$ with $\alpha = (x + c(y)/2, y)$, we further reduce to the trivial real structure, corresponding to the real form $\mathbb{A}^1_{\mathbb{R}} \times (\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}) \simeq \mathbb{A}^1_{\mathbb{R}} \times \Gamma_1$.

We now consider the case where $\theta(\nu) = \sigma_1$. Thus, $\nu = (\lambda x + c(y), \mu y^{-1})$ for some $\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$, $\mu \in \mathbb{R}^*$, and $c \in \mathbb{C}[y, y^{-1}]$. Considering $\alpha^{-1} \circ \nu \circ \overline{\alpha}$ with $\alpha = (\xi_1 x, \xi_2 y)$, $\xi_1 \in \mathbb{C}^*$, $\xi_2 \in \mathbb{R}^*$, $\xi_1^2 = \lambda$, $\xi_2^2 = |\mu|$, we reduce to the case where $\lambda = 1$, $\mu \in \{\pm 1\}$. Then, the 1-cocycle condition $\nu \circ \overline{\nu} = 1$ gives $\overline{c}(y) + c(\mu y^{-1}) = 0$. Considering $\alpha^{-1} \circ \nu \circ \overline{\alpha}$ with $\alpha = (x - \overline{c}(y)/2, y)$, we reduce to $c = 0$. This gives the two real structures $(x,y) \mapsto (\overline{x}, \overline{y}^{-1})$ and $(x,y) \mapsto (\overline{x}, -\overline{y}^{-1})$ and the real forms $\mathbb{A}^1_{\mathbb{R}} \times \Gamma_2$ and $\mathbb{A}^1_{\mathbb{R}} \times \Gamma_3$. The first one has real points and the second does not, so these are not isomorphic.

We now study the case where $\theta(\nu) = \sigma_2$. Thus, $\nu = (\lambda x y^{-1} + c(y), \mu y^{-1})$ for some $\lambda, \mu \in \mathbb{C}^*$. As $\nu \circ \overline{\nu} = 1$, we obtain $\lambda\overline{\lambda}/\overline{\mu} = 1$ and $\mu = \overline{\mu}$, whence $\mu \in \mathbb{R}_{>0}$. Considering $\alpha^{-1} \circ \nu \circ \overline{\alpha}$ with $\alpha = (x, \xi y)$, where $\xi \in \mathbb{R}^*$, $\xi^2 = \mu$, we may reduce to the case where $\mu = 1$, and consequently $|\lambda| = 1$. Considering $\alpha^{-1} \circ \nu \circ \overline{\alpha}$ with $\alpha = (\varepsilon x, y)$, where $\varepsilon \in \mathbb{C}^*$ and $\varepsilon^2 = \lambda$, we may further assume that $\lambda = 1$. Then, the 1-cocycle condition implies $\overline{c}(y)y + c(y^{-1}) = 0$. With $\alpha = (x - \overline{c}(y)y/2, y)$, we get $\alpha^{-1} \circ \nu \circ \overline{\alpha} = (xy^{-1}, y^{-1})$. Taking the morphism $\mathbb{A}^1_{\mathbb{C}} \times (\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$, $(x, y) \mapsto [x : y : 1]$, we obtain the real structure $\rho' \colon [x : y : z] \mapsto [\overline{x} : \overline{z} : \overline{y}]$ on $\mathbb{P}^2_{\mathbb{C}} \setminus \{yz = 0\}$. It remains to apply the automorphism $[x : y : z] \mapsto [y + z : \mathbf{i}(y - z) : x]$ of $\mathbb{P}^2_{\mathbb{C}}$, which gives an isomorphism $\mathbb{P}^2_{\mathbb{C}} \setminus \{yz = 0\} \xrightarrow{\simeq} \mathbb{P}^2_{\mathbb{C}} \setminus \{x^2 + y^2 = 0\}$, and conjugate the real structure ρ' to the standard one. The corresponding real form is then isomorphic to $\mathbb{P}^2_{\mathbb{R}} \setminus \{x^2 + y^2 = 0\}$. □

5. Koras–Russell threefolds of the first kind

5.1. Automorphisms of the three-space fixing the last coordinate. Throughout this section, $\mathbf{k}$ is a field and we denote by x, y, z the coordinates of the affine three-space $\mathbb{A}^3_{\mathbf{k}} = \operatorname{Spec}(\mathbf{k}[x, y, z])$.

Notation 5.1. Let $\pi \colon \mathbb{A}^3_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ be the projection $(x, y, z) \mapsto z$. Then, denote by $G_{\mathbf{k},z}$ the subgroup

$$\begin{aligned} G_{\mathbf{k},z} &= \{f \in \operatorname{Aut}(\mathbb{A}^3_{\mathbf{k}}) \mid \pi \circ f = \pi\} \\ &= \{f \in \operatorname{Aut}(\mathbb{A}^3_{\mathbf{k}}) \mid f^*(z) = z\} \\ &= \{f \in \operatorname{Aut}(\mathbb{A}^3_{\mathbf{k}}) \mid f = (P_1(x, y, z), P_2(x, y, z), z) \\ &\qquad\qquad\qquad\qquad \text{with } P_1, P_2 \in \mathbf{k}[x, y, z]\} \end{aligned}$$

of all automorphisms of $\mathbb{A}^3_{\mathbf{k}}$ that fix the last coordinate. Note that we have a natural isomorphism $GA_2(\mathbf{k}[z]) = \operatorname{Aut}_{\mathbf{k}[z]}(\mathbb{A}^2_{\mathbf{k}[z]}) \simeq G_{\mathbf{k},z}$.

We recall that the Jung–van der Kulk theorem applies to $\operatorname{Aut}_{\mathbf{k}(z)}(\mathbb{A}^2_{\mathbf{k}(z)})$ but not to $\operatorname{Aut}_{\mathbf{k}[z]}(\mathbb{A}^2_{\mathbf{k}[z]})$, as shown by Nagata [**22**, Theorem 1.4].

Let $\mathbf{k} \subseteq \mathbf{K}$ be a field extension and let $f \in G_{\mathbf{k},z}$. Then, for each $q \in \mathbf{K}$, we can define an automorphism $f|q$ of $\mathbb{A}^2_{\mathbf{K}} = \operatorname{Spec}(\mathbf{K}[x, y])$ by setting

$$f|q \colon (x, y) \longmapsto (P_1(x, y, q), P_2(x, y, q)).$$

We remark that $(f|q)^{-1} = f^{-1}|q$ and $\operatorname{Jac}(f|q) = \operatorname{Jac}(f) \in \mathbf{k}^*$.

Lemma 5.2. *Let $q \in \mathbb{C} \setminus \mathbb{R}$. Then, the map*

$$\begin{aligned} \Psi_q \colon G_{\mathbb{R},z} &\longrightarrow \operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}}) \\ f &\longmapsto f|q \end{aligned}$$

is a group homomorphism whose image consists of all elements of $\operatorname{Aut}(\mathbb{A}^2_{\mathbb{C}})$ that have a real Jacobian determinant.

Proof: By construction, Ψ_q is a group homomorphism and $\mathrm{Jac}(\Psi_q(f)) = \mathrm{Jac}(f) \in \mathbb{R}^*$ for all $f \in G_{\mathbb{R},z}$. So, we only need to prove that every element f in $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ with a real Jacobian determinant is indeed in the image of Ψ_q.

(a) We prove that any element $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ of Jacobian determinant 1 is in $\Psi_q(G_{\mathbb{R},q})$. By Lemma 2.7, it suffices to consider the case where $f \in \mathrm{Aff}_2(\mathbb{C}) \cup \mathrm{BA}_2(\mathbb{C})$.

First suppose that f is an elementary triangular map of the form $f = (x, y + \xi x^n)$ for some integer $n \geq 0$ and some constant $\xi \in \mathbb{C}$. Since q is not real, there exist $s, t \in \mathbb{R}$ such that $sq + t = \xi$ and we then have $f = \Psi_q(g)$ where $g \in G_{\mathbb{R},z}$ is defined by $g = (x, y + (sz + t)x^n, z)$. Since Ψ_q is a group homomorphism, this implies that all triangular maps of the form $(x, y + p(x))$ with $p \in \mathbb{C}[x]$ also belong to $\Psi_q(G_{\mathbb{R},q})$.

We now consider affine maps. We have already proved that $(x, y + \lambda x) \in \Psi_q(G_{\mathbb{R},q})$ for each $\lambda \in \mathbb{C}$. Let us write $\sigma = (-y, x) = \Psi_q((-y, x, z))$. As $\mathrm{SL}_2(\mathbb{C})$ is generated by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and by $\{\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \mid \lambda \in \mathbb{C}\}$, we can infer that every element $(ax + by, cx + dy)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ belongs to $\Psi_q(G_{\mathbb{R},q})$. As the translations are generated by σ and by $(x, y + \nu)$ with $\nu \in \mathbb{C}$, every element of $\mathrm{Aff}_2(\mathbb{C})$ of Jacobian determinant 1 lies in $\Psi_q(G_{\mathbb{R},q})$. With the above, we can deduce that any element of $\mathrm{BA}_2(\mathbb{C})$ of Jacobian determinant 1 is also in $\Psi_q(G_{\mathbb{R},q})$, as it is of the form $\big(ax + b, \frac{1}{a}y + p(x)\big)$ with $a \in \mathbb{C}^*$, $b \in \mathbb{C}$, and $p \in \mathbb{C}[x]$. This shows the claim.

(b) Let $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ be such that $\mathrm{Jac}(f) \in \mathbb{R}^*$. We consider the map $\gamma = (\nu x, y) = \Psi_q((\nu x, y, z))$, where $\nu = \mathrm{Jac}(f)$. As $\gamma^{-1} \circ f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ has Jacobian 1 and $f = \gamma \circ (\gamma^{-1} \circ f)$, we conclude with (a). □

Lemma 5.3. *The following propositions hold true.*

(1) *Let $v \in \mathbb{C}[x, y]$ be a variable and let $q \in \mathbb{C} \setminus \mathbb{R}$. Then, there exists $f = (P_1(x, y, z), P_2(x, y, z), z) \in G_{\mathbb{R},z}$ such that $v = P_1(x, y, q)$.*

(2) *Let $v \in \mathbb{R}[x, y] \subset \mathbb{C}[x, y]$ be a variable of $\mathbb{C}[x, y]$ and let $q \in \mathbb{R}$. Then, there exists $f = (P_1(x, y, z), P_2(x, y, z), z) \in G_{\mathbb{R},z}$ such that $v = P_1(x, y, q)$.*

Proof: (1) Suppose that $v \in \mathbb{C}[x, y]$ is a variable. Let $w \in \mathbb{C}[x, y]$ be such that $\varphi = (v, w)$ is an automorphism of $\mathbb{A}^2_{\mathbb{C}}$. Replacing w with ξw for some $\xi \in \mathbb{C}^*$, we may assume that $\mathrm{Jac}(\varphi) = 1$. Then, for any $q \in \mathbb{C} \setminus \mathbb{R}$, there exist by Lemma 5.2 polynomials $P_1, P_2 \in \mathbb{R}[x, y, z]$ such that the automorphism

$$f = (P_1(x, y, z), P_2(x, y, z), z) \in G_{\mathbb{R},z}$$

satisfies $f|q = \varphi$. In particular, $v = P_1(x, y, q)$ as desired.

(2) It is a well-known fact that a polynomial $v \in \mathbb{R}[x,y]$ is a variable of $\mathbb{R}[x,y]$ if and only if it is a variable of $\mathbb{C}[x,y]$. For instance, this is an immediate consequence of [**25**, Theorem 3.2]. Hence, there exists $w \in \mathbb{R}[x,y]$ such that $(v,w) \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{R}})$ and the result follows. Indeed, the map $f = (P_1(x,y,z), P_2(x,y,z), z) \in G_{\mathbb{R},z}$ with $P_1(x,y,z) = v(x,y)$ and $P_2(x,y,z) = w(x,y)$ satisfies $v = P_1(x,y,q)$ for any $q \in \mathbb{R}$. $\square$

Proposition 5.4. *Consider the standard real structure $\rho\colon (x,y,z) \mapsto (\overline{x},\overline{y},\overline{z})$ on $\mathbb{A}^3_{\mathbb{C}}$. Then, the first Galois cohomology set of $G_{\mathbb{C},z} \simeq \mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{A}^2_{\mathbb{C}[z]}) = \mathrm{GA}_2(\mathbb{C}[z])$ is trivial:*

$$H^1(\mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{A}^2_{\mathbb{C}[z]})) = \{1\}.$$

Consequently, every real structure $\hat{\rho}$ on $\mathbb{A}^3_{\mathbb{C}}$ that makes commutative the diagram

$$\begin{array}{ccc} \mathbb{A}^3_{\mathbb{C}} & \xrightarrow{\hat{\rho}} & \mathbb{A}^3_{\mathbb{C}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{A}^1_{\mathbb{C}} & \xrightarrow{z \mapsto \overline{z}} & \mathbb{A}^1_{\mathbb{C}} \end{array}$$

is equivalent to the standard real structure ρ.

Proof: Let $\nu \in Z^1(G_{\mathbb{C},z})$ be a 1-cocycle, that is, an element $\nu \in G_{\mathbb{C},z}$ such that $\nu \circ \overline{\nu} = \mathrm{id}_{\mathbb{A}^3_{\mathbb{C}}}$. We need to show that there exists $f \in G_{\mathbb{C},z}$ such that $\nu = f^{-1} \circ \overline{f}$.

Consider $G_{\mathbb{C},z}$ as a subgroup of $\mathrm{Aut}(\mathbb{A}^2_K)$, where $K = \mathbb{C}(z)$ and $\mathbb{A}^2_K = \mathrm{Spec}(K[x,y])$. Since $H^1(\mathrm{Aut}_K(K[x,y])) = 1$ by [**16**, Theorem 3], there is an element $f \in \mathrm{Aut}_{\mathbb{C}(z)}(\mathbb{A}^2_{\mathbb{C}(z)})$ such that $\nu = f^{-1} \circ \overline{f}$. In other words, there exist $f_1, f_2, g_1, g_2 \in \mathbb{C}(z)[x,y]$ such that

$$\begin{aligned} f\colon \mathbb{A}^3_{\mathbb{C}} &\dashrightarrow \mathbb{A}^3_{\mathbb{C}} \\ (x,y,z) &\longmapsto (f_1(x,y,z), f_2(x,y,z), z) \end{aligned}$$

and

$$\begin{aligned} g = f^{-1}\colon \mathbb{A}^3_{\mathbb{C}} &\dashrightarrow \mathbb{A}^3_{\mathbb{C}} \\ (x,y,z) &\longmapsto (g_1(x,y,z), g_2(x,y,z), z) \end{aligned}$$

are inverse birational maps and $\nu = f^{-1} \circ \overline{f}$.

We may actually assume that f_1 and f_2 are both elements of $\mathbb{C}[x,y,z]$. Indeed, there exists $c \in \mathbb{R}[z] \setminus \{0\}$ such that $c(z)f_1(x,y,z)$ and $c(z)f_2(x,y,z)$ belong to $\mathbb{C}[x,y,z]$, and the equality $\nu = f^{-1} \circ \overline{f}$ remains true when we replace f with $\gamma \circ f$, where $\gamma \in \mathrm{Bir}(\mathbb{A}^3_{\mathbb{C}})$ is defined by $(x,y,z) \mapsto (c(z)x, c(z)y, z)$, because $\overline{\gamma} = \gamma$.

Let us write $g_i = \frac{h_i}{a_i}$ for each $i = 1,2$, where $h_i \in \mathbb{C}[x,y,z]$ and $a_i \in \mathbb{C}[z] \setminus \{0\}$ are without common factors.

If $\deg(a_1 \cdot a_2) = 0$, i.e., if a_1 and a_2 are nonzero constants, then g is a morphism too. In this case, f is in $G_{\mathbb{C},z}$ and we are done. If $\deg(a_1 \cdot a_2) \geq 1$, we proceed by decreasing induction on $\deg(a_1 \cdot a_2)$. To prove the proposition, it suffices to find a suitable birational map $\varphi \in \mathrm{Bir}(\mathbb{A}^3_{\mathbb{C}})$ with the following four properties:

(1) $\varphi^*(z) = z$;
(2) all components of $\varphi \circ f$ are in $\mathbb{C}[x, y, z]$;
(3) $\varphi = \overline{\varphi}$, which implies $\nu = (\varphi \circ f)^{-1} \circ \overline{(\varphi \circ f)}$;
(4) the degree of the product of the denominators appearing in the components of $(\varphi \circ f)^{-1}$ is strictly smaller than that of $a_1 \cdot a_2$.

So, suppose from now on that $\deg(a_1 \cdot a_2) \geq 1$ and let $q \in \mathbb{C}$ be such that $a_1(q)a_2(q) = 0$. Without loss of generality, we may assume that $a_1(q) = 0$. Since $g \circ f = \mathrm{id}_{\mathbb{A}^3_{\mathbb{C}}}$, we then obtain that

$$a_1(z)x = h_1(f_1(x, y, z), f_2(x, y, z), z)$$

and thus that the equality

$$(\spadesuit) \qquad h_1(f_1(x, y, q), f_2(x, y, q), q) = 0$$

holds in $\mathbb{C}[x, y]$.

For each $p \in \mathbb{A}^1_{\mathbb{C}}$, we consider the set $\Delta_{f,p} \subseteq \mathbb{A}^3_{\mathbb{C}}$ defined by

$$\Delta_{f,p} = \{(f_1(x, y, p), f_2(x, y, p), p) \mid (x, y) \in \mathbb{A}^2_{\mathbb{C}}\} = f(\mathbb{A}^2_{\mathbb{C}} \times \{p\}).$$

We remark that applying the complex conjugation to the set $\Delta_{f,p}$ gives

$$(\heartsuit) \qquad \overline{\Delta_{f,p}} = \Delta_{f,\overline{p}}$$

for all $p \in \mathbb{C}$. Indeed, as $\nu = f^{-1} \circ \overline{f}$, we have $f \circ \nu = \overline{f} = \rho \circ f \circ \rho$ and therefore

$$\overline{\Delta_{f,p}} = \rho(\Delta_{f,p}) = \rho \circ f(\mathbb{A}^2_{\mathbb{C}} \times \{p\}) = f \circ \nu \circ \rho(\mathbb{A}^2_{\mathbb{C}} \times \{p\}) = f(\mathbb{A}^2_{\mathbb{C}} \times \{\overline{p}\}) = \Delta_{f,\overline{p}}.$$

We shall prove later that if $\Delta_{f,q}$, with $a_1(q) = 0$ as above, is not a point, then it is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$.

(a) Let us first consider the case where the set $\Delta_{f,q}$ is a point. Then, there exist $r_1, r_2 \in \mathbb{C}$ such that $\Delta_{f,q} = \{(r_1, r_2, q)\}$ and $R_1, R_2 \in \mathbb{C}[x, y, z]$ such that

$$f_i(x, y, z) = r_i + (z - q)R_i(x, y, z)$$

for both $i = 1, 2$.

(a1) Suppose now that $q \in \mathbb{R}$. By the equality ($\heartsuit$), we then have that $r_1, r_2 \in \mathbb{R}$. Therefore, the birational map

$$\varphi = \left(\frac{x - r_1}{z - q}, \frac{y - r_2}{z - q}, z\right) \in \mathrm{Bir}(\mathbb{A}^3_{\mathbb{C}})$$

satisfies $\varphi = \overline{\varphi}$ and we compute

$$\varphi \circ f = (R_1(x, y, z), R_2(x, y, z), z).$$

The inverse map of $\varphi \circ f$ is given by

$$\begin{aligned}(\varphi \circ f)^{-1} &= \left(\frac{h_1(x(z-q)+r_1, y(z-q)+r_2, z)}{a_1(z)},\right.\\ &\qquad \left.\frac{h_2(x(z-q)+r_1, y(z-q)+r_2, z)}{a_2(z)}, z\right)\\ &= \left(\frac{\widetilde{h}_1(x,y,z)}{a_1(z)}, \frac{\widetilde{h}_2(x,y,z)}{a_2(z)}, z\right),\end{aligned}$$

where $\widetilde{h}_1, \widetilde{h}_2 \in \mathbb{C}[x, y, z]$. We obtain that

$$\widetilde{h}_1(x, y, q) = h_1(r_1, r_2, q) = h_1(f_1(x, y, q), f_2(x, y, q), q) \overset{(\spadesuit)}{=} 0.$$

Therefore, $\widetilde{h}_1(x, y, z)$ is divisible by $(z - q)$ and the map φ fulfils the four desired properties (1)–(4).

(a2) We now consider the case where $q \notin \mathbb{R}$. For each $i = 1, 2$, we define two real numbers $s_i = \frac{\overline{r_i} - r_i}{\overline{q} - q}$ and $t_i = \frac{\overline{q} r_i - q \overline{r_i}}{\overline{q} - q}$. Then, the polynomials $p_i(z) = s_i z + t_i \in \mathbb{R}[z]$ satisfy that $p_i(q) = r_i$ and $p_i(\overline{q}) = \overline{r_i}$. We recall that the equality $f_i(x, y, q) = r_i$ holds true in $\mathbb{C}[x, y]$. Similarly, it follows from ($\heartsuit$) that $f_i(x, y, \overline{q}) = \overline{r_i}$. Therefore the polynomials $f_i(x, y, z) - s_i z - t_i$ are divisible by $(z - q)(z - \overline{q}) \in \mathbb{R}[z]$. This implies that the birational map

$$\varphi = \left(\frac{x - s_1 z - t_1}{(z - q)(z - \overline{q})}, \frac{y - s_2 z - t_2}{(z - q)(z - \overline{q})}, z\right)$$

satisfies $\varphi = \overline{\varphi}$ and that all components of $\varphi \circ f$ are elements of $\mathbb{C}[x, y, z]$. Moreover, the inverse map of $\varphi \circ f$ is then given by

$$\begin{aligned}(\varphi \circ f)^{-1} = &\bigg(\frac{h_1(x(z-q)(z-\overline{q})+s_1 z + t_1, y(z-q)(z-\overline{q}) + s_2 z + t_2, z)}{a_1(z)},\\ &\frac{h_2(x(z-q)(z-\overline{q})+s_1 z+t_1, y(z-q)(z-\overline{q}) + s_2 z + t_2, z)}{a_2(z)}, z\bigg)\\ = &\left(\frac{\widetilde{h}_1(x, y, z)}{a_1(z)}, \frac{\widetilde{h}_2(x, y, z)}{a_2(z)}, z\right),\end{aligned}$$

where $\widetilde{h}_1, \widetilde{h}_2 \in \mathbb{C}[x, y, z]$. We obtain the two equalities

$$\widetilde{h}_1(x, y, q) = h_1(r_1, r_2, q) = h_1(f_1(x, y, q), f_2(x, y, q), q) \overset{(\spadesuit)}{=} 0,$$
$$\widetilde{h}_1(x, y, \overline{q}) = h_1(\overline{r_1}, \overline{r_2}, \overline{q}) = h_1(f_1(x, y, q), f_2(x, y, q), q) \overset{(\spadesuit)}{=} 0.$$

Therefore, $\widetilde{h}_1(x, y, z)$ is divisible by $(z-q)(z-\overline{q})$ and the map φ fulfils the four desired properties (1)–(4).

(b) We now proceed with the case where $\Delta_{f,q}$ is not a point. For every $p \in \mathbb{A}^1_{\mathbb{C}}$ and every variable $u \in \mathbb{C}[x, y]$, we define the curve

$$\Gamma_{p,u} = \{(x, y, p) \in \mathbb{A}^3_{\mathbb{C}} \mid u(x, y) = 0\} \simeq \mathbb{A}^1_{\mathbb{C}}.$$

By Lemma 2.6, there exists a variable $v \in \mathbb{C}[x, y]$ such that $h_1(x, y, q) \in \mathbb{C}[v]$. Note that $h_1(x, y, q)$ is not the zero-polynomial because h_1 and a_1 were chosen without common factors. Setting $\mu \in \mathbb{C}^*$, $m \geq 1$, and $\xi_1, \ldots, \xi_m \in \mathbb{C}$ such that

$$h_1(x, y, q) = \mu \prod_{i=1}^{m} (v(x, y) - \xi_i) \in \mathbb{C}[x, y],$$

it then follows from $(\spadesuit)$ that there exists $1 \leq i \leq m$ such that the equality

$$v(f_1(x, y, q), f_2(x, y, q)) - \xi_i = 0$$

holds true in $\mathbb{C}[x, y]$. Therefore, the set $\Delta_{f,q}$ is contained in the curve $\Gamma_{q,w}$, where $w = v - \xi_i$. As a nonconstant morphism $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ is surjective, and since $\Delta_{f,q}$ is not a point, this implies that

$$\Delta_{f,q} = \Gamma_{q,w} \simeq \mathbb{A}^1_{\mathbb{C}}.$$

We now prove that we can assume that $w \in \mathbb{R}[x, y]$ if $q \in \mathbb{R}$, so that we may apply Lemma 5.3. Indeed, suppose $q \in \mathbb{R}$. In this case, we have $\overline{\Delta_{f,q}} \overset{(\heartsuit)}{=} \Delta_{f,\overline{q}} = \Delta_{f,q}$. Thus, $\Gamma_{q,w} = \Gamma_{q,\overline{w}}$. As both polynomials w and $\overline{w}$ are variables, they are irreducible. Since their zero-sets are equal, there exists a constant $\mu \in \mathbb{C}^*$ such that $\overline{w} = \mu w$. It then follows that $w = \overline{\overline{\mu}\,\overline{w}} = \overline{\mu}\,\overline{w} = \mu\overline{\mu}w$, whence $\mu\overline{\mu} = 1$. As $H^1(\mathbb{C}^*) = \{1\}$ by Lemma 2.9, we may choose $\eta \in \mathbb{C}^*$ with $\eta/\overline{\eta} = \mu$. The variable $w' = \eta w$ then satisfies $\overline{w'} = \overline{\eta}\,\overline{w} = \overline{\eta}\mu w = w'$ and $\Delta_{f,q} = \Gamma_{q,w'}$. Thus, we may replace w by $w' \in \mathbb{R}[x,y]$ if necessary, as desired.

By Lemma 5.3, there exists an element $\psi = (P_1(x,y,z), P_2(x,y,z), z)$ in $G_{\mathbb{R},z}$ such that $P_1(x,y,q) = w$. Observe that $\psi(\Gamma_{q,w}) \subseteq \Gamma_{q,x}$. As these two curves are isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, they are actually equal, i.e., $\psi(\Gamma_{q,w}) = \Gamma_{q,x}$. We may thus replace f with $\psi \circ f$ and suppose that $\Delta_{f,q} = \Gamma_{q,x}$. Note that, as $\psi \in G_{\mathbb{R},z}$ is defined over $\mathbb{R}$ and is an automorphism, the equality $\nu = f^{-1} \circ \overline{f}$ is preserved when replacing f with $\psi \circ f$, and we do not change the denominators a_1, a_2 appearing in the expression of the inverse of f. Moreover, the fact that $\Delta_{f,q} = \Gamma_{q,x}$ implies that $f_1(x,y,q) = 0$, or, equivalently, that $z - q$ divides f_1 in $\mathbb{C}[x,y,z]$. We note that in the case where $q \notin \mathbb{R}$, we also have $\Delta_{f,\overline{q}} = \overline{\Delta_{f,q}} = \Gamma_{\overline{q},x}$, and so $(z - \overline{q})$ also divides f_1. Defining $u(z) = z - q$ if $q \in \mathbb{R}$ and $u(z) = (z-q)(z-\overline{q})$ if $q \notin \mathbb{R}$, we thus get a polynomial $u \in \mathbb{R}[z]$ with $u(q) = 0$ that divides f_1.

Finally, since the birational map $\varphi \in \mathrm{Bir}(\mathbb{A}^3_{\mathbb{C}})$ defined by $\varphi\colon (x,y,z) \mapsto \left(\frac{x}{u(z)}, y, z\right)$ satisfies the four properties (1)–(4), we can conclude the proof. $\square$

Corollary 5.5. *Taking the standard action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *on* $\mathbb{C}[x,y,z]$, *we obtain*

$$H^1(\mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{C}[x,y,z])) = \{1\}.$$

Proof: The map $f \mapsto (f^{-1})^*$ defines an isomorphism between the groups $G_{\mathbb{C},z}$ and $\mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{C}[x,y,z])$. As the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on both groups is compatible with this isomorphism, $H^1(\mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{C}[x,y,z])) = \{1\}$ then follows from $H^1(G_{\mathbb{C},z}) = \{1\}$. $\square$

Question 5.6. *Do we have* $H^1(\mathrm{Aut}_{\mathbb{C}[z,w]}(\mathbb{C}[x,y,z,w])) = \{1\}$*?*

Lemma 5.7. *Let* $r \geq 1$ *and let*

$$G_r = \{f \in \mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{C}[x,y,z]) \mid f \equiv \mathrm{id} \pmod{z^r}\}.$$

Taking the standard action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *on* $\mathbb{C}[x,y,z]$, *we obtain*

$$H^1(G_r) = \{1\}.$$

Proof: (a) First we prove the result in the case where $r = 1$. Let $\nu \in Z^1(G_1)$. By Corollary 5.5, there exists $\alpha \in \mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{C}[x,y,z])$ such that $\nu = \alpha^{-1} \circ \overline{\alpha}$. Since $\alpha \circ \nu = \overline{\alpha}$ and since $\nu \equiv \mathrm{id} \pmod{z}$, we have that $\alpha \equiv \overline{\alpha} \pmod{z}$.

Denoting $\alpha(x) = a(x,y,z)$ and $\alpha(y) = b(x,y,z)$, we can define an automorphism $\varphi \in \mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{C}[x,y,z])$ by letting $\varphi(x) = a(x,y,0)$ and $\varphi(y) = b(x,y,0)$. Note that $\varphi \equiv \alpha \pmod{z}$ and that $\overline{\varphi} = \varphi$.

Thus, $\beta = \varphi^{-1} \circ \alpha$ defines an element in G_1 and we check that

$$\beta^{-1} \circ \overline{\beta} = \alpha^{-1} \circ \varphi \circ \overline{\varphi^{-1}} \circ \overline{\alpha} = \alpha^{-1} \circ \varphi \circ \varphi^{-1} \circ \overline{\alpha} = \alpha^{-1} \circ \overline{\alpha} = \nu.$$

Hence, $H^1(G_1) = \{1\}$ is proved.

(b) We prove the lemma for every $r \geq 2$ by induction on r. We fix $r \geq 2$ and suppose that $H^1(G_{r-1}) = \{1\}$ holds. Let $\nu \in Z^1(G_r)$. We want to find an element $\beta \in G_r$ such that $\nu = \beta^{-1} \circ \overline{\beta}$. By our induction hypothesis, there exists $\alpha \in G_{r-1}$ such that $\nu = \alpha^{-1} \circ \overline{\alpha}$.

Now, it suffices to construct an element $\varphi \in \mathrm{Aut}_{\mathbb{C}[z]}(\mathbb{C}[x,y,z])$ with $\varphi = \overline{\varphi}$ such that $\varphi \equiv \alpha \pmod{z^r}$. The lemma will indeed follow since the automorphism $\beta = \varphi^{-1} \circ \alpha$ is then in G_r and satisfies

$$\beta^{-1} \circ \overline{\beta} = \alpha^{-1} \circ \varphi \circ \overline{\varphi^{-1}} \circ \overline{\alpha} = \alpha^{-1} \circ \varphi \circ \varphi^{-1} \circ \overline{\alpha} = \alpha^{-1} \circ \overline{\alpha} = \nu,$$

as desired.

Let $a, b \in \mathbb{C}[x,y]$ be such that $\alpha(x) \equiv x + z^{r-1}a(x,y)$ and $\alpha(y) \equiv y + z^{r-1}b(x,y) \pmod{z^r}$. Since $\alpha \circ \nu = \overline{\alpha}$ and $\nu \equiv \mathrm{id} \pmod{z^r}$, we have that $a(x,y)$ and $b(x,y)$ both belong to $\mathbb{R}[x,y]$. Therefore, α induces an endomorphism $\tilde{\alpha} \in \mathrm{End}_{\mathbb{R}[z]/(z^r)}(\mathbb{R}[z]/(z^r)[x,y])$ defined by $\tilde{\alpha}(x) = x + z^{r-1}a(x,y)$ and $\tilde{\alpha}(y) = y + z^{r-1}b(x,y)$. In fact, $\tilde{\alpha}$ is an isomorphism. Indeed, one can check that its inverse map is simply defined by $\tilde{\alpha}^{-1}(x) = x - z^{r-1}a(x,y)$ and $\tilde{\alpha}^{-1}(y) = y - z^{r-1}b(x,y)$. Moreover, the Jacobian determinant of $\tilde{\alpha}$ is equal to $1 \in \mathbb{R}[z]/(z^r)$ because $\tilde{\alpha} \equiv \alpha \pmod{z^r}$ and $\mathrm{Jac}(\alpha) = 1 \in \mathbb{R}[z]$.

By the main result of [**24**], there thus exists $\varphi \in \mathrm{Aut}_{\mathbb{R}[z]}(\mathbb{R}[z][x,y])$ with $\varphi(x) \equiv x + z^{r-1}a(x,y) \equiv \alpha(x)$ and $\varphi(y) \equiv y + z^{r-1}b(x,y) \equiv \alpha(y) \pmod{z^r}$. This concludes the proof. □

5.2. Real forms of Koras–Russell threefolds of the first kind. The Koras–Russell threefolds of the first kind are the hypersurfaces $X_{d,k,\ell}$ in $\mathbb{A}^4_{\mathbb{C}}$ defined by an equation of the form $x^d y + z^k + x + t^\ell = 0$, where $d \geq 2$ and $2 \leq k < \ell$ are integers with k and ℓ relatively prime. Their automorphism groups are computed in [**10, 20**]; see also [**11**], where the following notations are introduced. We fix the integers d, k, ℓ as above, and denote by $\mathcal{A} \subseteq \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[x,z,t])$ the subgroup of all automorphisms of $\mathbb{C}[x,z,t]$ that preserves the ideals (x) and $(x^d, z^k + x + t^\ell)$. For every $1 \leq n \leq d$ we further denote by $\mathcal{A}_n$ the normal subgroup of $\mathcal{A}$ defined by

$$\mathcal{A}_n = \{f \in \mathcal{A} \mid f \equiv \mathrm{id} \mod (x^n)\} \subset \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x,z,t]).$$

Proposition 5.8 ([**20**]).

(1) $\mathrm{Aut}_{\mathbb{C}}(X_{d,k,\ell}) \simeq \mathcal{A}$.

(2) $\mathcal{A} \simeq \mathcal{A}_1 \rtimes \mathbb{C}^*$, *where* $\mathbb{C}^*$ *acts on* $\mathbb{C}[x,z,t]$ *by*

$$a \cdot P(x,z,t) = P(a^{k\ell}x, a^{\ell}z, a^{k}t)$$

for all $a \in \mathbb{C}^*$, $P \in \mathbb{C}[x,z,t]$.

(3) $\mathcal{A}_{n+1}$ *is a normal subgroup of* $\mathcal{A}_n$ *and* $\mathcal{A}_n/\mathcal{A}_{n+1} \simeq (\mathbb{C}[z,t],+)$ *for all* $1 \leq n \leq d-1$.

Moreover, the above isomorphisms are compatible with the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^*$ and on the polynomial rings $\mathbb{C}[z,t] \subset \mathbb{C}[x,z,t] \subset \mathbb{C}[x,x^{-1},y,z,t]$, where we see the ring $\mathbb{C}[X_{d,k,\ell}]$ of regular functions on $X_{d,k,\ell}$ as the subalgebra of $\mathbb{C}[x,x^{-1},z,t]$ that is generated by x, z, t, and $y = -(z^k + x + t^{\ell})/x^d$. We may now prove Theorem B.

Proof of Theorem B: By Proposition 5.8(3), we have a subnormal series

$$\{1\} \triangleleft \mathcal{A}_d \triangleleft \mathcal{A}_{d-1} \triangleleft \cdots \triangleleft \mathcal{A}_1 \triangleleft \mathcal{A},$$

where $\mathcal{A}_n/\mathcal{A}_{n+1} \simeq (\mathbb{C}[z,t],+)$ for each $1 \leq n \leq d-1$. We may write the latter isomorphism in the form of a short exact sequence of group homomorphisms

$$\{1\} \longrightarrow \mathcal{A}_{n+1} \longrightarrow \mathcal{A}_n \longrightarrow (\mathbb{C}[z,t],+) \longrightarrow \{1\},$$

that gives rise to a short exact sequence of homomorphisms of pointed sets

$$\{1\} \longrightarrow H^1(\mathcal{A}_{n+1}) \longrightarrow H^1(\mathcal{A}_n) \longrightarrow H^1(\mathbb{C}[z,t]) \longrightarrow \{1\}$$

(see for example [**3**, Proposition 1.17.]). Observe that

$$\mathcal{A}_d = \{f \in \mathrm{Aut}_{\mathbb{C}[x]}(\mathbb{C}[x,z,t]) \mid f \equiv \mathrm{id} \mod (x^d)\},$$

for which Lemma 5.7 implies that the first cohomology pointed set $H^1(\mathcal{A}_d)$ is trivial. Since, by Lemma 2.9, $H^1(\mathbb{C}[z,t])$ is trivial, $H^1(\mathcal{A}_{d-1})$ is too. By repeating the same argument, we see that the triviality of $H^1(\mathcal{A}_{n+1})$ implies that of $H^1(\mathcal{A}_n)$. Hence, we successively find that all cohomology pointed sets $H^1(\mathcal{A}_d), \ldots, H^1(\mathcal{A}_1)$ are trivial.

Now, by Proposition 5.8(2), we again obtain a short exact sequence of group homomorphisms

$$\{1\} \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A} \longrightarrow \mathbb{C}^* \longrightarrow \{1\},$$

and thus a short exact sequence of homomorphisms of pointed sets

$$\{1\} \longrightarrow H^1(\mathcal{A}_1) \longrightarrow H^1(\mathcal{A}) \longrightarrow H^1(\mathbb{C}^*) \longrightarrow \{1\}.$$

As $H^1(\mathcal{A}_1) = \{1\}$ by the preceding argument, and since $H^1(\mathbb{C}^*)$ is trivial by Lemma 2.9, we can deduce that $H^1(\mathcal{A})$ is trivial.

Therefore, $H^1(\mathrm{Aut}_{\mathbb{C}}(X_{d,k,\ell}))$ is also trivial and we obtain that all real forms of $X_{d,k,\ell}$ are isomorphic to the standard one. $\square$

Remark 5.9. The so-called Koras–Russell threefolds of the second kind are the hypersurfaces in $\mathbb{A}^4_{\mathbb{C}}$ defined by an equation of the form $x+y(x^d+z^{\alpha_2})^\ell+t^{\alpha_3}=0$, where $d\geq 2$, $\ell\geq 1$, and $\alpha_3\geq\alpha_2\geq 2$ are integers with $\gcd(\alpha_2,d)=\gcd(\alpha_2,\alpha_3)=1$. Their automorphism groups are computed in the main theorem of [**23**]. If X is such a threefold, then $\operatorname{Aut}(X)$ is isomorphic to a semi-direct product of two of its subgroups. One subgroup is isomorphic to $\mathbb{C}^*$, the other one to $(\mathbb{C}[x,z],+)$. Moreover, as in the case of Koras–Russell threefolds of the first kind, all isomorphisms are compatible with the natural action of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$. Therefore, arguing as in the proof of Theorem B, it follows that every Koras–Russell threefold of the second kind admits a unique (up to isomorphism) real form.

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