AN INTRODUCTION TO AUTOMORPHISMS WITH POSITIVE ENTROPY ON COMPACT COMPLEX SURFACES

by

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Introduction

Let $\mathcal{M}$ be a complex projective manifold. Let f be a rational or holomorphic map on $\mathcal{M}$. When we iterate this map we obtain a "dynamical system": a point p of $\mathcal{M}$ moves to $p_1 = f(p)$, then to $p_2 = f(p_1)$, to $p_3 = f(p_2)$ $\ldots$ So f "induces a movement on $\mathcal{M}$". The set $\{p, p_1, p_2, p_3, \ldots\}$ is the *orbit* of p.

Let A be a smooth cubic in the complex projective plane defined by an homogeneous equation with rational coefficients; assume that A contains at least a point with rational coordinates. Up to a linear change of coordinates (with rational coefficients) such a A can be described by

$$y^2z = 4x^3 + axz^2 + bz^3, \qquad a, b \in \mathbb{Q}.$$

The set of complex solutions of this equation is a RIEMANN surface which is holomorphically diffeomorphic to the quotient $\mathbb{C}/\Lambda$ where Λ is a cocompact lattice ([**34, 42**]). Let us denote $A(\mathbb{C})$ the quotient $\mathbb{C}/\Lambda$. We can choose the isomorphism such that the class of 0 in $\mathbb{C}$ modulo Λ corresponds to the point $(0:1:0)$ of A. Thus $A(\mathbb{C})$ has a structure of additive group induced by $(\mathbb{C},+)$ for which $(0:1:0)$ is the identity element. Under these assumptions if $p_1,\ p_2,\ p_3$ are three points of $A(\mathbb{C})$ then $p_1+p_2+p_3 = 0$ if and only if $p_1,\ p_2,\ p_3$ are aligned in $\mathbb{P}^2(\mathbb{C})$. Let A be a projective manifold; A is an *Abelian variety* of dimension k if $A(\mathbb{C})$ is isomorphic to a compact quotient of $\mathbb{C}^k$.

Multiplication by an integer $m > 1$ on an Abelian variety, endomorphisms of degree $d > 1$ on projective spaces are studied since XIXth century in particular by JULIA and FATOU ([**2**]). These two families of maps "have an interesting dynamic". Consider the first case; let f_m denote the multiplication by m. Periodic points of f_m are repulsive and dense in $A(\mathbb{C})$: a point is periodic if and only if it is a torsion point of A; the differential of f^n at a periodic point of period n is an homothety of ratio $m^n > 1$. Around 1964 ADLER, KONHEIM and MCANDREW introduce a new way to measure the complexity of a dynamical system: the topological entropy ([**1**]). Let X be a compact metric space and let f be a continuous map from X into itself. Let n be a positive integer and let ε be a real number strictly positif. Two points x and y of X have the same orbit of period n with precision ε if

$$\mathrm{dist}(f^j(x), f^j(y)) \leq \varepsilon, \qquad \forall\ 0 \leq j \leq n.$$

Assume that ε is fixed; as $n \to +\infty$ the number of orbits which can be distinguished grows at most exponentially. The *topological entropy* $\mathrm{h_{top}}(f)$ measures this exponential growth as $\varepsilon \to 0$. For an isometry of X the topological entropy is zero. For the multiplication by m on a complex Abelian variety of dimension k we

have: $\mathrm{h_{top}}(f) = 2k\log m$. For an endomorphism of $\mathbb{P}^k(\mathbb{C})$ defined by homogeneous polynomials of degree d we have: $\mathrm{h_{top}}(f) = k\log d$ (*see* [**24**]).
Let $\mathcal{M}$ be a complex projective manifold. On which conditions do rational maps with chaotic behavior exist ? The existence of such rational maps implies a lot of constraints on $\mathcal{M}$:

Theorem 0.1 **([4])**. — *A smooth complex projective hypersurface of dimension greater than* 1 *and degree greater than* 2 *admits no endomorphism of degree greater than* 1.

Let us consider the case of compact homogeneous manifolds $\mathcal{M}$: the group of holomorphic diffeomorphisms acts faithfully on $\mathcal{M}$ and there are a lot of holomorphic maps on it. Meanwhile in this context all endomorphisms with topological degree strictly greater than 1 come from endomorphisms on projective manifolds and nilvarieties.
So the "idea” is that complex projective manifolds with rich polynomial dynamic are rare; moreover it is not easy to describe the set of rational or holomorphic maps on such manifolds.

Acknowledgment. — I would like to thank D. CERVEAU for fruitful and helpful discussions. I also thank J.-L. LIN for comments and references. Thanks to J. GRIVAUX for the explanation about the result of NAGATA.

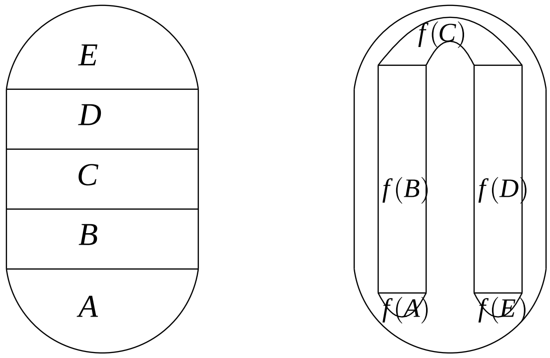
Contents

1. Some dynamic

1.1. SMALE horseshoe. — The SMALE horsehoe is the hallmark of chaos. Let us describe what it is (*see for example* [**40**]). Consider the embedding f of the disc Δ into itself. Assume that

- f contracts the semi-discs $f(A)$ and $f(E)$ in A;
- f sends the rectangles B and D linearly to the rectangles $f(B)$ and $f(D)$ stretching them vertically and shrinking them horizontally, in the case of D it also rotates by 180 degrees.

We don't care what the image $f(C)$ of C is as long as $f(C)\cap(B\cup C\cup D)=\emptyset$. In other words we have the following situation



There are three fixed points: $p\in f(B)$, $q\in A$, $s\in f(D)$. The points q is a *sink* in the sense that forall $z\in A\cup C\cup E$ we have $\lim_{n\to+\infty} f^n(z)=q$. The points p and s are *saddle points*: if m lies on the horizontal through p then f^n squeezes it to p as $n\to+\infty$, while if m lies on the vertical through p then f^{-n} squeezes it to p as $n\to+\infty$. In some coordinates centered in p we have

$$\forall (x,y)\in B, \qquad\qquad f(x,y)=(kx,my)$$

for some $0<k<1<m$; similarly $f(x,y)=(-kx,-my)$ on D for some coordinates centered at s. The sets

$$W^s=\{z \mid f^n(z)\to p \text{ as } n\to+\infty\}, \qquad\qquad W^u=\{z \mid f^n(z)\to p \text{ as } n\to-\infty\}$$

are called *stable* and *unstable* manifolds of p. They intersect at r, which is what POINCARÉ called a *homoclinic point*. Homoclinic points are dense in $\{m\in\Delta \mid f^n(m)\in\Delta,\, n\in\mathbb{Z}\}$.
The keypart of the dynamic of f happens on the horseshoe

$$\Lambda=\{z \mid f^n(z)\in B\cup D\ \forall n\in\mathbb{Z}\}.$$

Let us introduce shift map on the space of two symbols. Take two symbols 0 and 1, and look at the set $\Sigma=\{0,1\}^{\mathbb{Z}}$ of all bi-infinite sequences $a=(a_n)_{n\in\mathbb{Z}}$ where, for each n, a_n is 0 or 1. The map $\sigma\colon\Sigma\to\Sigma$ that sends $a=(a_n)$ to $\sigma(a)=(a_{n+1})$ is a homeomorphism called the *shift map*. Let us consider the itinerary map $i\colon\Lambda\to\Sigma$ defined as follows: $i(p)=(s_n)_{n\in\mathbb{Z}}$ where $s_n=1$ if $f^n(p)$ is in B and $s_n=0$ if $f^n(p)$ belongs to D. The diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "i"'] & \Sigma \arrow[d, "i"] \\ \Lambda \arrow[r, "f"] & \Lambda \end{tikzcd}$$

commutes so every dynamical property of the shift map is possessed equally by $f_{|\Lambda}$. Due to conjugacy the chaos of σ is reproduced exactly in the horseshoe: the map σ has positive entropy: $\log 2$; it has 2^n periodic orbits of period n, and so must be the set of periodic orbits of $f_{|\Lambda}$.
To summarize: every dynamical system having a transverse homoclinic point also has a horseshoe and thus has a shift chaos, even in higher dimensions. The mere existence of a transverse intersection between the stable and unstable manifolds of a periodic orbit implies a horseshoe; since transversality persists under perturbation, it follows that so does the horseshoe and so does the chaos.
The concepts of horseshoe and hyperbolicity are related. In the description of the horseshoe the derivative of f stretches tangent vectors that are parallel to the vertical and contracts vectors parallel to the horizontal,

not only at the saddle points, but uniformly throughout Λ. In general, *hyperbolicity* of a compact invariant set such as Λ is expressed in terms of expansion and contraction of the derivative on subbundles of the tangent bundle.

1.2. Two examples. — Let us consider $P_c(z) = z^2 + c$. A periodic point p of P_c with period n is *repelling* if $|(P_c^n(p))'| > 1$ and the JULIA *set* of f is the closure of the set of repelling periodic points. P_c is a complex horseshoe if it is hyperbolic (*i.e.* uniformly expanding on the JULIA set) and conjugated to the shift on two symbols. The MANDELBROT *set* M is defined as the set of all points c such that the sequence $(P_c^n(0))_n$ does not escape to infinity

$$M = \{c \in \mathbb{C} \,|\, \exists s \in \mathbb{R}, \forall n \in \mathbb{N}, |P_c^n(0)| \leq s\}.$$

The complex horseshoe locus is the complement of the MANDELBROT set.

Let us consider the HÉNON family of quadratic maps

$$f_{a,b} \colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad\qquad f_{a,b}(x,y) = (x^2 + a - by, x).$$

For fixed parameters a and b, $f_{a,b}$ define a dynamical system, and we are interested in the way that the dynamic varies with the parameters. The parameter b is equal to $\det \mathrm{jac}\, f_{a,b}$; when $b = 0$, the map has a one-dimensional image and is equivalent to P_c. As soon as b is non zero, these maps are diffeomorphisms, and maps similar to SMALE's horseshoe example occur when $a << 0$ (*see* [**13**]).

In the 60's it was hoped that uniformly hyperbolic dynamical systems might be in some sense typical. While they form a large open sets on all manifolds, they are not dense. The search for typical dynamical systems continues to be a great problem, in order to find new phenomena we try the framework of complex compact surfaces.

2. Some algebraic geometry

2.1. Compact complex surfaces. — A *proper modification* is a proper surjective holomorphic map whose generic fiber is a point. Let $\mathcal{Z}$ be a compact complex surface and let p be a point of $\mathcal{Z}$; the *blow-up* of $\mathcal{Z}$ at p is the proper modification $\pi \colon \mathrm{Bl}_p\mathcal{Z} \to \mathcal{Z}$ which replaces p with the set $\pi^{-1}(p) \simeq \mathbb{P}^1(\mathbb{C})$ of holomorphic tangent directions at p and is a biholomorphism elsewhere. The rational curve $\pi^{-1}(p)$ is called *exceptional divisor*. An irreducible curve is said *exceptional* if it is the exceptional set for some blow-up. If $\mathcal{C}$ is a curve through p, then $\widetilde{C} = \overline{\pi^{-1}(\mathcal{C} \setminus \{p\})}$ is the *strict transform* of $\mathcal{C}$. The blow-ups play an important role as we can not in the following statement.

Theorem 2.1. — *Any proper modification between compact complex surfaces is a composition of finitely many blow-ups.*

Let $\mathcal{X}$, $\mathcal{Z}$ be two compact complex surfaces. A *meromorphic map* is defined by its graph $\Gamma(f) \subset \mathcal{X} \times \mathcal{Z}$, an irreducible subvariety for which the projection $\pi_1 \colon \Gamma(f) \to \mathcal{X}$ is a proper modification ([**20**]). The map f is *dominating* when the second projection $\pi_2 \colon \Gamma(f) \to \mathcal{Z}$ is surjective. The *indeterminacy set* is the finite set of points where π_1 does not admit a local inverse; we will denote it $\mathrm{Ind}\, f$. Let $\mathrm{Exc}\,\pi_2$ be the set of points where π_2 is not a finite map; the *exceptional set* of f is given by $\mathrm{Exc}\, f = \pi_1(\mathrm{Exc}\,\pi_2)$.
When $f \colon \mathcal{X} \dashrightarrow \mathcal{Z}$ admits a meromorphic inverse, we say that f is *bimeromorphic*. If f is a bimeromorphic map and Γ a desingularization of its graph, then the two induced projections $\pi_1 \colon \Gamma \to \mathcal{X}$, $\pi_2 \colon \Gamma \to \mathcal{Z}$ are proper modifications and Theorem 2.1 implies the following statement.

Theorem 2.2. — *Any bimeromorphic map $f\colon \mathcal{X} \dashrightarrow \mathcal{Z}$ between smooth compact complex surfaces can be written as composition $f = f_1 \dots f_k$ where f_i is either a blow-up, or the inverse of a blow-up.*

Let $\mathcal{Z}$ be a compact complex surface. A *divisor* E on $\mathcal{Z}$ is a linear combination $\sum c_i\mathcal{C}_i$ where $\mathcal{C}_i$ is an irreducible curve (singular or not) and c_i an element of $\mathbb{Z}$. The two divisors D and D' are *linearly equivalent*, if $\mathrm{D}-\mathrm{D}'$ is the divisor of a rational function f, in other words if $\mathrm{D}-\mathrm{D}'$ is the set of zeroes of a rational function f minus the poles. If D and D' are linearly equivalent, we will denote it $\mathrm{D}\sim\mathrm{D}'$. The PICARD *group* of $\mathcal{Z}$, denoted $\mathrm{Pic}\,\mathcal{Z}$, is the quotient group of divisors modulo linear equivalence. If $\mathcal{C}$ and $\mathcal{C}'$ are two distinct curves on $\mathcal{Z}$, then $\mathcal{C}\cdot\mathcal{C}'$ is the number of intersections counted with multiplicity; note that in this case $\mathcal{C}\cdot\mathcal{C}'\geq 0$. We can naturally extend this definition to give a sense to the intersection of two divisors; one particular case is the self intersection, we denote $\mathrm{D}\cdot\mathrm{D}$ by D^2. This intersection number satisfies the following properties:
- if $\mathrm{D}\sim\mathrm{D}'$, then $\mathrm{D}\cdot\mathrm{D}''=\mathrm{D}'\cdot\mathrm{D}''$;
- if $\pi\colon \mathrm{Bl}_p\mathcal{Z}\to\mathcal{Z}$ is the blow-up of p in $\mathcal{Z}$ and E the exceptional divisor, then
$$\pi^*\mathrm{D}\cdot\pi^*\mathrm{D}'=\mathrm{D}\cdot\mathrm{D}', \qquad \mathrm{E}\cdot\pi^*\mathrm{D}=0, \qquad \mathrm{E}^2=-1, \qquad \widetilde{\mathcal{C}}^2=\mathcal{C}^2-1$$
where $\widetilde{\mathcal{C}}$ is the strict transform of a curve $\mathcal{C}$ through p.

We have a criterion to detect exceptional curves.

***Theorem 2.3* (CASTELNUOVO's criterion)**. — *An irreducible curve $\mathcal{C}\subset\mathcal{Z}$ is exceptional if and only if it is a smooth rational curve of self-intersection -1.*

To any surface $\mathcal{Z}$ one associates its DOLBEAULT cohomology groups $\mathrm{H}^{p,q}(\mathcal{Z})$ and the cohomological groups $\mathrm{H}^k(\mathcal{Z},\mathbb{Z})$, $\mathrm{H}^k(\mathcal{Z},\mathbb{R})$ and $\mathrm{H}^k(\mathcal{Z},\mathbb{C})$. Set $\mathrm{H}^{1,1}_{\mathbb{R}}(\mathcal{Z})=\mathrm{H}^{1,1}(\mathcal{Z})\cap\mathrm{H}^2(\mathcal{Z},\mathbb{R})$. Let $f\colon\mathcal{X}\dashrightarrow\mathcal{Z}$ be a dominating meromorphic map between compact complex surfaces, Γ a desingularization of its graph and π_1, π_2 the natural projections. A smooth form α in $\mathcal{C}^\infty_{p,q}(\mathcal{Z})$ can be *pulled back* as a smooth form $\pi_2^*\alpha\in\mathcal{C}^\infty_{p,q}(\Gamma)$ and then pushed forward as a current. We define f^* by
$$f^*\alpha=\pi_{1*}\pi_2^*\alpha$$
which gives a $\mathrm{L}^1_{\mathrm{loc}}$ form on $\mathcal{X}$ that is smooth outside $\mathrm{Ind}\,f$. The action of f^* satisfies: $f^*(\mathrm{d}\alpha)=\mathrm{d}(f^*\alpha)$ so descends to a linear action on DOLBEAULT cohomology.
Let $\{\alpha\}\in\mathrm{H}^{p,q}(\mathcal{Z})$ be the DOLBEAULT class of some smooth form α. We set
$$f^*\{\alpha\}=\{\pi_{1*}\pi_2^*\alpha\}\in\mathrm{H}^{p,q}(\mathcal{X}).$$
This defines a linear map f^* from $\mathrm{H}^{p,q}(\mathcal{Z})$ into $\mathrm{H}^{p,q}(\mathcal{X})$. Similarly one can define the *push-forward* $f_*=\pi_{2*}\pi_1^*$ from $\mathrm{H}^{p,q}(\mathcal{X})$ into $\mathrm{H}^{p,q}(\mathcal{Z})$. When f is bimeromorphic, one has $f_*=(f^{-1})^*$. The operation $(\alpha,\beta)\mapsto\int\alpha\wedge\overline{\beta}$ on smooth 2-forms induced a quadratic intersection form, called *product intersection*, denoted $\langle .,.\rangle$ on $\mathrm{H}^2(\mathcal{Z},\mathbb{C})$. Its structure is given by the following fundamental statement.

***Theorem 2.4* ([3])**. — *Let $\mathcal{Z}$ be a compact* KÄHLER *surface and let* $\mathrm{h}^{1,1}$ *denote the dimension of* $\mathrm{H}^{1,1}(\mathcal{Z},\mathbb{R})\subset\mathrm{H}^2(\mathcal{Z},\mathbb{R})$. *Then the signature of the restriction of the intersection product to* $\mathrm{H}^{1,1}(\mathcal{Z},\mathbb{R})$ *is* $(1,\mathrm{h}^{1,1}-1)$. *In particular, there is no* 2*-dimensional linear subspace* L *in* $\mathrm{H}^{1,1}(\mathcal{Z},\mathbb{R})$ *with the property that* $\langle v,v\rangle=0$ *forall v in* L.

***Theorem-Definition 2.5* ([15])**. — *Let $f\colon\mathcal{Z}\to\mathcal{Z}$ be a dominating meromorphic map on a* KÄHLER *surface and let ω be a* KÄHLER *form. Then f is algebraically stable if and only if any of the following hold:*
- *for any $\alpha\in\mathrm{H}^{1,1}(\mathcal{Z})$ and any k in $\mathbb{N}$, one has $(f^*)^k\alpha=(f^k)^*\alpha$;*
- *there is no curve $\mathcal{C}$ in $\mathcal{Z}$ such that $f^k(\mathcal{C})\subset\mathrm{Ind}\,f$ for some integer $k\geq 0$;*
- *for all $k\geq 0$ one has $(f^k)^*\omega=(f^*)^k\omega$.*

2.2. Projective plane and blow-ups of the projective plane. — A *rational map* from $\mathbb{P}^2(\mathbb{C})$ into itself is a map of the form

$$(x:y:z) \dashrightarrow (f_0(x,y,z):f_1(x,y,z):f_2(x,y,z))$$

where the f_i are homogeneous polynomials of the same degree without common factor.
A *birational map* is a rational map which admits an inverse of the same type. Let $\mathrm{Bir}(\mathbb{P}^2)$ denote the group of such maps; it is also called CREMONA *group*.

Examples 2.6. —
– Each element of $\mathrm{Aut}(\mathbb{P}^2)=\mathrm{PGL}_3(\mathbb{C})$ is a birational map.
– The map $\sigma\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C})$, $(x:y:z)\mapsto(yz:xz:xy)$ is a rational map; in the affine chart $z=1$ one has $\sigma=\left(\frac{1}{x},\frac{1}{y}\right)$. One notes that σ is an involution; in particular σ is birational.

Example 2.7. — A *polynomial automorphism* of $\mathbb{C}^2$ is a bijective application of the following type

$$f\colon \mathbb{C}^2\to\mathbb{C}^2, \qquad (x,y)\mapsto(f_1(x,y),f_2(x,y)), \qquad f_i\in\mathbb{C}[x,y].$$

The *degree* of $f=(f_1,f_2)$ is defined by $\deg f=\max(\deg f_1,\deg f_2)$. The set of the polynomial automorphisms is a group denoted $\mathrm{Aut}[\mathbb{C}^2]$.
The map

$$\mathbb{C}^2\to\mathbb{C}^2, \qquad (x,y)\mapsto(a_1x+b_1y+c_1,a_2x+b_2y+c_2), \qquad a_i,\,b_i,\,c_i\in\mathbb{C},\,a_1b_2-a_2b_1\neq 0$$

is an automorphism of $\mathbb{C}^2$. The set of all these maps is the *affine group* A.
The map

$$\mathbb{C}^2\to\mathbb{C}^2, \qquad (x,y)\mapsto(\alpha x+P(y),\beta y+\gamma), \qquad \alpha,\,\beta,\,\gamma,\,\alpha\beta\neq 0,\,P\in\mathbb{C}[y]$$

is an automorphism of $\mathbb{C}^2$. The set of all these maps is a group, the *elementary group* E.
Of course

$$\mathrm{S}=\mathrm{A}\cap\mathrm{E}=\{(a_1x+b_1y+c_1,b_2y+c_2)\,|\,a_i,\,b_i,\,c_i\in\mathbb{C},\,a_1b_2\neq 0\}$$

is a subgroup of $\mathrm{Aut}[\mathbb{C}^2]$.
The group $\mathrm{Aut}[\mathbb{C}^2]$ has a very special structure.

***Theorem 2.8* ([29])**. — *The group* $\mathrm{Aut}[\mathbb{C}^2]$ *is the amalgated product of* A *and* E *along* S :

$$\mathrm{Aut}[\mathbb{C}^2]=\mathrm{A}*_{\mathrm{S}}\mathrm{E}.$$

In other words A *and* E *generate* $\mathrm{Aut}[\mathbb{C}^2]$ *and each element* f *in* $\mathrm{Aut}[\mathbb{C}^2]\setminus\mathrm{S}$ *can be written as follows*

$$f=(a_1)e_1\dots a_n(e_n), \qquad e_i\in\mathrm{E}\setminus\mathrm{A},\,a_i\in\mathrm{A}\setminus\mathrm{E}.$$

Moreover this decomposition is unique modulo the following relations

$$a_ie_i=(a_is)(s^{-1}e_i), \qquad e_{i-1}a_i=(e_{i-1}s')(s'^{-1}a_i), \qquad s,\,s'\in\mathrm{S}.$$

From a dynamical point of view affine automorphisms and elementary automorphisms are simple. Nevertheless there exist some elements in $\mathrm{Aut}[\mathbb{C}^2]$ with a rich dynamic; this is the case of HÉNON *automorphisms,* automorphisms of the type

$$\varphi g_1\dots g_p\varphi^{-1}, \qquad \varphi\in\mathrm{Aut}[\mathbb{C}^2],\,g_i=(y,P_i(y)-\delta_ix),\,P_i\in\mathbb{C}[y],\,\deg P_i\geq 2,\,\delta_i\in\mathbb{C}^*.$$

Note that $g_i=\overbrace{(y,x)}^{\in\mathrm{A}\setminus\mathrm{E}}\ \overbrace{(-\delta_ix+P_i(y),y)}^{\in\mathrm{E}\setminus\mathrm{A}}$.
Using JUNG's theorem, FRIEDLAND and MILNOR proved the following statement.

***Proposition 2.9* ([21]).** — *Let f be an element of* $\mathrm{Aut}[\mathbb{C}^2]$.
Either f is conjugate to an element of E, *or f is a* HÉNON *automorphism.*

The polynomial automorphisms of $\mathbb{C}^2$ can be viewed as birational maps of $\mathbb{P}^2(\mathbb{C})$, in other words $\mathrm{Aut}[\mathbb{C}^2]$ is a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.

***Exercise 2.1*.** — Let $\mathcal{H}$ be the subset of $\mathrm{Aut}[\mathbb{C}^2]$ defined by

$$\mathcal{H} = \{f = (y, P(y) - \delta x) \,|\, P(y) = y^\nu + p_{\nu-2}y^{\nu-2} + \ldots + p_0,\ \nu \geq 2,\ \delta \in \mathbb{C}^*\}.$$

- Show that the jacobian of an element f of $\mathcal{H}$ is δ, and in particular it is constant.
- Show that the degree of the first (resp. second) component of f^n is d^{n-1} (resp. d^n).
- Consider the dilation $d(x,y) = (\alpha x, \alpha y)$ by α and the translation $t(x,y) = (x+\beta, y+\beta)$ by β. Show that if f is in $\mathcal{H}$, then the conjugate $tdfd^{-1}t^{-1}$ has the form of an element of $\mathcal{H}$ except that the coefficient of y^d is an arbitrary nonzero number, and the coefficient of y^{d-1} is arbitrary.
- If f is an element of $\mathcal{H}$, its inverse is not. Let τ be the involution defined by $\tau(x,y) = (y,x)$. The conjugate $\tau f^{-1}\tau$ has the form of an element of $\mathcal{H}$, except that P is not monic.

The *degree* of a birational map $f \in \mathrm{Bir}(\mathbb{P}^2)$ is equal to the degree of the f_i's. This is not a birational invariant, but the degree growth is: forall f, g in $\mathrm{Bir}(\mathbb{P}^2)$ there exist two strictly positive constants α and β such that

$$\alpha \deg f^n \leq \deg(gf^ng^{-1}) \leq \beta \deg f^n.$$

So we introduce the birational invariant defined by

$$\lambda(f) = \liminf_{n \to +\infty} (\deg f^n)^{1/n}$$

which is called *first dynamical degree* of f.

***Exercise 2.2*.** —
- Find the first dynamical degree of an element of $\mathrm{Aut}(\mathbb{P}^2)$.
- Find the first dynamical degree of $\sigma\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C})$, $(x:y:z) \dashrightarrow (yz:xz:xy)$.
- Find the first dynamical degree of an elementary automorphism of $\mathbb{C}^2$.
- Find the first dyamical degree of the HÉNON automorphism $(y, P(y) - \delta x)$ with δ in $\mathbb{C}^*$, P in $\mathbb{C}[y]$, $\nu = \deg P \geq 2$.

Let $f = (f_0 : f_1 : f_2)$ a birational map of $\mathbb{P}^2(\mathbb{C})$. The *indeterminacy set* of f is the set

$$\{m \in \mathbb{P}^2(\mathbb{C}) \,|\, f_0(m) = f_1(m) = f_2(m) = 0\}.$$

The *exceptional set* of f is given by the zeroes of $\det \mathrm{jac}\, f$.

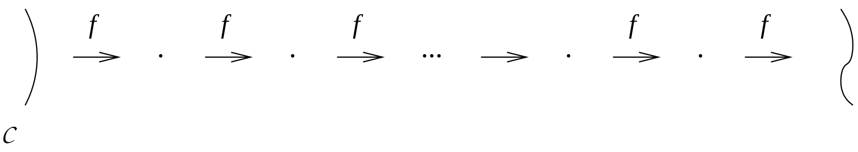
***Exercise 2.3*.** — Let f be an element of $\mathrm{Aut}(\mathbb{P}^2)$; check that $\mathrm{Ind}\, f = \mathrm{Exc}\, f = \emptyset$.

***Exercise 2.4*.** —
- Consider the birational map given by $\sigma\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C})$, $(x:y:z) \dashrightarrow (yz:xz:xy)$; describe $\mathrm{Ind}\,\sigma$ and $\mathrm{Exc}\,\sigma$.
- Consider the birational map given by $\rho\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C})$, $(x:y:z) \dashrightarrow (xy:z^2:yz)$; describe $\mathrm{Ind}\,\rho$ and $\mathrm{Exc}\,\rho$.
- Consider the birational map given by $\tau\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C})$, $(x:y:z) \dashrightarrow (x^2:xy:y^2-xz)$; describe $\mathrm{Ind}\,\tau$ and $\mathrm{Exc}\,\tau$.
- Are there some relationships between the exceptional set of a birational map of f and the degree of f?
- Are there some relationships between the indetermicay set of a birational map of f and the degree of f?

***Exercise 2.5*.** —
- Let f be the HÉNON automorphism $(y, P(y) - \delta x)$ with δ in $\mathbb{C}^*$, P in $\mathbb{C}[y]$, $\nu = \deg P \geq 2$; the map f can be viewed as a birational map, describe $\mathrm{Ind}\, f$ and $\mathrm{Exc}\, f$.

– Let f be an elementary map $(\alpha x+P(y),\beta y+\delta)$ with α, β, γ in $\mathbb{C}$, $\alpha\beta\neq 0$, P in $\mathbb{C}[y]$; the map f can be viewed as a birational map, describe $\mathrm{Ind}\, f$ and $\mathrm{Exc}\, f$.
– What can you say about the indeterminacy and exceptional sets of a polynomial automorphism ?

Let $\mathcal{Z}$ be a surface and $f\colon \mathcal{Z} \dashrightarrow \mathcal{Z}$ be a birational map. We say that f is *analytically stable* if for any curve $\mathcal{C}$ in $\mathcal{Z}$ and for any positive integer n, $f^n(\mathcal{C})$ is not in the indeterminacy set. In other words for an analytically stable map the following does not happen



Exercise 2.6. — Let f be a CREMONA transformation. The map f is not analytically stable if and only if there exists an integer k such that $\deg f^k < (\deg f)^k$. So if f is analytically stable, then $\lambda(f)=\deg f$.

Exercise 2.7. — Let A be an automorphism of the complex projective plane and let σ be the birational map given by $\sigma\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C})$, $(x:y:z) \dashrightarrow (yz:xz:xy)$. Assume that the coefficients of A are positive real numbers. Show that $A\sigma$ is analytically stable.
Let A be an automorphism of the complex projective plane and let σ be the birational map given by

$$\rho\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C}), \qquad (x:y:z) \dashrightarrow (xy:z^2:yz).$$

Assume that the coefficients of A are positive real numbers. Show that $A\sigma$ is analytically stable.
Let A be an automorphism of the complex projective plane and let σ be the birational map given by

$$\tau\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C}), \qquad (x:y:z) \dashrightarrow (x^2:xy:y^2-xz).$$

Assume that the coefficients of A are positive real numbers. Show that $A\tau$ is analytically stable.
Let us say that the coefficients of an automorphism A of $\mathbb{P}^2(\mathbb{C})$ are algebraically independent if it has a representative A in $\mathrm{GL}_3(\mathbb{C})$ whose coefficients are algebraically independent over $\mathbb{Q}$. One can deduce the following: let A be an automorphism of the projective plane whose coefficients are algebraically independent over $\mathbb{Q}$, then $A\sigma$ and $(A\sigma)^{-1}$ are analytically stable.

The PICARD group $\mathrm{Pic}\,\mathbb{P}^2(\mathbb{C})$ is isomorphic to $\mathbb{Z}$; similarly $\mathrm{H}^2(\mathbb{P}^2(\mathbb{C}),\mathbb{Z})$ is isomorphic to $\mathbb{Z}$. We may identify $\mathrm{Pic}\,\mathbb{P}^2(\mathbb{C})$ and $\mathrm{H}^2(\mathbb{P}^2(\mathbb{C}),\mathbb{Z})$.

Let $\pi\colon \mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^2(\mathbb{C})$ be the blow-up of p in $\mathbb{P}^2(\mathbb{C})$ and $\mathrm{E}=\pi^{-1}(p)$ be the exceptional divisor. One can describe the set of rational functions from $\mathrm{Bl}_p\mathbb{P}^2$ into itself:

$$\mathrm{Rat}\,(\mathrm{Bl}_p\mathbb{P}^2)=\pi^*(\mathrm{Rat}\,\mathbb{P}^2(\mathbb{C})).$$

The group $\mathrm{Pic}\,(\mathrm{Bl}_p\mathbb{P}^2)$ is generated by E and $\widetilde{\mathrm{L}}$ where $\widetilde{\mathrm{L}}=\pi^*\mathrm{L}=\{\ell\circ\pi=0\}$ is the pull back of a line $\mathrm{L}=\{\ell=0\}$ which does not contain p.
Let us give a concrete presentation of the blow-up of $p=(0:0:1)$ in $\mathbb{P}^2(\mathbb{C})$. We will work in the affine chart $z=1$. Consider

$$\Gamma=\{((x,y),[r:v])\in\mathbb{C}^2\times\mathbb{P}^1(\mathbb{C})\,|\,xv=yr\}$$

and π the projection on the first factor. The pair (Γ,π) is the blow-up of p in the affine chart $z=1$. Note that $\pi^{-1}\colon \mathbb{C}^2\setminus\{0\}\to\Gamma$ is given by $\pi^{-1}(x,y)=((x,y),[x,y])$; one can also write

$$\pi^{-1}(x,y)=((x,y),[1:y/x])=((x,y),[x/y:1])$$

These two representations allow to define two charts $\mathcal{U}=\mathbb{C}^2_{u,v}$ and $\mathcal{U}'=\mathbb{C}^2_{r,s}$ where coordinates are given by

$$x=u,\ y=uv, \qquad\qquad x=rs,\ y=s.$$

One has $\Gamma=\mathcal{U}\cup\mathcal{U}'$ and if we want to work near $\{x=0\}$ (resp. $\{y=0\}$) we use the coordinates (r,s) (resp. (u,v)).

More generally let us consider n distinct points $p_1,\ldots,p_n$ in $\mathbb{P}^2(\mathbb{C})$. Let $\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2$ be the surface obtained from $\mathbb{P}^2(\mathbb{C})$ by blowing up $p_1,\ldots,p_n$ and denote by $\pi\colon \mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2\to\mathbb{P}^2(\mathbb{C})$ the composition of these blow-ups. Let H be a line in $\mathbb{P}^2(\mathbb{C})$ and set $\mathrm{E}_j=\pi^{-1}(p_j)$. If H contains no p_i then $\pi^*\mathrm{H}$ is represented by the strict transform $\widetilde{\mathrm{H}}$ of H, otherwise $\pi^*\mathrm{H}=\widetilde{\mathrm{H}}+\displaystyle\sum_{j\,|\,p_j\in\mathrm{H}}\mathrm{E}_j$. In this case one can describe $\mathrm{Pic}(\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2)$.

Theorem 2.10. — *Let $p_1,\ldots,p_n$ be n distinct points in $\mathbb{P}^2(\mathbb{C})$. Denote by $\pi\colon \mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2\to\mathbb{P}^2(\mathbb{C})$ the sequence of blow-ups of the p_i's. If $\mathrm{E}_j=\pi^{-1}(p_j)$ are the exceptional divisors and H a generic line of $\mathbb{P}^2(\mathbb{C})$, then*

$$\mathrm{Pic}(\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2)=\mathbb{Z}\mathrm{E}_1\oplus\ldots\oplus\mathbb{Z}\mathrm{E}_n\oplus\mathbb{Z}\pi^*\mathrm{H}.$$

Theorem 2.11. — *Let $p_1,\ldots,p_n$ be n distinct points in $\mathbb{P}^2(\mathbb{C})$. Denote by $\pi\colon \mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2\to\mathbb{P}^2(\mathbb{C})$ the sequence of blow-ups of the p_i's and by $\mathrm{E}_j=\pi^{-1}(p_j)$ the exceptional divisors. Consider two elements $\mathrm{T}=\mathrm{D}+\sum_j a_j\mathrm{E}_j$ and $\mathrm{T}'=\mathrm{D}'+\sum_j b_j\mathrm{E}_j$ in $\mathrm{Pic}(\mathrm{Bl}_{p_1,\ldots,p_n}\mathbb{P}^2)$, where D and D' denote strict transforms of divisors in $\mathbb{P}^2(\mathbb{C})$.*
We have: $\mathrm{T}\sim\mathrm{T}'$ if and only if D and D' have the same degrees, and $a_j=b_j$ for all j.

2.3. Exceptional configurations and characteristic matrices. — Let f be a birational map from $\mathbb{P}^2(\mathbb{C})$ into itself of degree ν. By Theorem 2.2 there exist π and η two sequences of blow-ups such that

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2(\mathbb{C}) \arrow[rr, dashed, "f"'] & & \mathbb{P}^2(\mathbb{C}) \end{tikzcd}$$

We can rewrite π as follows

$$\pi\colon \mathcal{Z}=\mathcal{Z}_k\stackrel{\pi_k}{\to}\mathcal{Z}_{k-1}\stackrel{\pi_{k-1}}{\to}\ldots\stackrel{\pi_2}{\to}\mathcal{Z}_1\stackrel{\pi_1}{\to}\mathcal{Z}_0=\mathbb{P}^2(\mathbb{C})$$

where π_i is the blow-up of the point p_{i-1} in $\mathcal{Z}_{i-1}$. Let

$$\mathrm{E}_i=\pi_i^{-1}(p_i), \qquad\qquad \mathcal{E}_i=(\pi_{i+1}\circ\ldots\circ\pi_k)^*\mathrm{E}_i$$

The divisors $\mathcal{E}_i$ are called the *exceptional configurations* of π and the p_i base points of f.
For any effective divisor $\mathrm{D}\neq 0$ on $\mathbb{P}^2(\mathbb{C})$ let $\mathrm{mult}_{p_i}\mathrm{D}$ be defined inductively in the following way. We set $\mathrm{mult}_{p_1}\mathrm{D}$ to be the usual multiplicity of D at p_1 : it is defined as the largest integer m such that the local equation of D at p_1 belongs to the m-th power of the maximal ideal $\mathfrak{m}_{\mathbb{P}^2,p_1}$. Suppose $\mathrm{mult}_{p_1}\mathrm{D}$ is defined. We

take the proper inverse transform $\pi_i^{-1}\mathrm{D}$ of D in $\mathcal{Z}_i$ and define $\mathrm{mult}_{p_{i+1}}\mathrm{D}=\mathrm{mult}_{p_{i+1}}\pi_i^{-1}\mathrm{D}$. It follows from the definition that

$$\pi^{-1}\mathrm{D}=\pi^*(\mathrm{D})-\sum_{i=1}^k m_i\mathcal{E}_i$$

where $m_i=\mathrm{mult}_{p_i}\mathrm{D}$.
There are two relationships between ν and the m_i's:

$$1=\nu^2-\sum_{i=1}^k m_i^2, \qquad\qquad 3=3\nu-\sum_{i=1}^k m_i.$$

Considered a resolution of a birational map f from $\mathbb{P}^2(\mathbb{C})$ into itself of degree ν :

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2(\mathbb{C}) \arrow[rr, dashed, "f"'] & & \mathbb{P}^2(\mathbb{C}) \end{tikzcd}$$

We can rewrite π as follows

$$\pi\colon \mathcal{Z}=\mathcal{Z}_k\stackrel{\pi_k}{\to}\mathcal{Z}_{k-1}\stackrel{\pi_{k-1}}{\to}\ldots\stackrel{\pi_2}{\to}\mathcal{Z}_1\stackrel{\pi_1}{\to}\mathcal{Z}_0=\mathbb{P}^2(\mathbb{C})$$

where π_i is the blow-up of the point p_{i-1} in $\mathcal{Z}_{i-1}$; similarly, for $j=1,\ldots,k$, there exists $\eta_j\colon\mathcal{X}_j\to\mathcal{X}_{j-1}$ blow-up of the point p'_{j-1} in $\mathcal{X}_{j-1}$ such that

$$\eta\colon \mathcal{Z}=\mathcal{X}_k\stackrel{\eta_k}{\to}\mathcal{X}_{k-1}\stackrel{\eta_{k-1}}{\to}\ldots\stackrel{\eta_2}{\to}\mathcal{X}_1\stackrel{\eta_1}{\to}\mathcal{X}_0=\mathbb{P}^2(\mathbb{C}).$$

Note that $\mathrm{E}_1,\ldots,\mathrm{E}_k$ (resp. $\mathrm{E}'_1,\ldots,\mathrm{E}'_k$) are the exceptional divisors obtained by blowing up $p_1,\ldots,p_k$ (resp. $p'_1,\ldots,p'_k$). An *ordered resolution* of f is a decomposition $f=\eta\pi^{-1}$ where η and π are ordered sequences of blow-ups. An ordered resolution of f induces two basis of $\mathrm{Pic}(\mathcal{Z})$

- $\mathcal{B}=\{e_0=\pi^*\mathrm{H}, e_1=[\mathcal{E}_1],\ldots,e_k=[\mathcal{E}_k]\}$,
- $\mathcal{B}'=\{e'_0=\eta^*\mathrm{H}, e'_1=[\mathcal{E}'_1],\ldots,e'_k=[\mathcal{E}'_k]\}$,

where H is a generic line. We can write e'_i as follows

$$e'_0=\nu e_0-\sum_{i=1}^k m_ie_i, \qquad\qquad e'_j=\nu_je_0-\sum_{i=1}^k m_{ij}e_i,\ j\geq 1.$$

The matrix of change of basis

$$M=\begin{bmatrix}\nu & \nu_1 & \ldots & \nu_k\\ -m_1 & -m_{11} & \ldots & -m_{1k}\\ \vdots & \vdots & & \vdots\\ -m_k & -m_{k1} & \ldots & -m_{kk}\end{bmatrix}$$

is called *characteristic matrix* of f. The first column of M, which is the *characteristic vector* of f, is the vector $(\nu,-m_1,\ldots,-m_k)$. The other columns $(\nu_i,-m_{1i},\ldots,-m_{ki})$ describe the "behavior of $\mathcal{E}'_i$": if $\nu_j>0$, then $\pi(\mathcal{E}'_j)$ is a curve of degree ν_j in $\mathbb{P}^2(\mathbb{C})$ through the points p_ℓ of f with multiplicity $m_{\ell j}$.

Example 2.12. — Consider the birational map

$$\sigma\colon\mathbb{P}^2(\mathbb{C})\dashrightarrow\mathbb{P}^2(\mathbb{C}), \qquad\qquad (x:y:z)\dashrightarrow(yz:xz:xy).$$

The points of indeterminacy of σ are $P=(1:0:0)$, $Q=(0:1:0)$ and $R=(0:0:1)$; the exceptional set is the union of the three lines $\Delta=\{x=0\}$, $\Delta'=\{y=0\}$ and $\Delta''=\{z=0\}$.

First we blow up P; let us denote E the exceptional divisor and $\mathcal{D}_1$ the strict transform of $\mathcal{D}$. Set

$$\begin{cases} y=u_1 & \mathrm{E}=\{u_1=0\} \\ z=u_1v_1 & \Delta_1''=\{v_1=0\} \end{cases} \qquad \begin{cases} y=r_1s_1 & \mathrm{E}=\{s_1=0\} \\ z=s_1 & \Delta_1'=\{r_1=0\} \end{cases}$$

On the one hand

$$(u_1,v_1)\to(u_1,u_1v_1)_{(y,z)}\to(u_1v_1:v_1:1)=\left(\frac{1}{u_1},\frac{1}{u_1v_1}\right)_{(y,z)}\to\left(\frac{1}{u_1},\frac{1}{v_1}\right)_{(u_1,v_1)};$$

on the other hand

$$(r_1,s_1)\to(r_1s_1,s_1)_{(y,z)}\to(r_1s_1:1:r_1)=\left(\frac{1}{r_1s_1},\frac{1}{s_1}\right)_{(y,z)}\to\left(\frac{1}{r_1},\frac{1}{s_1}\right)_{(r_1,s_1)}.$$

Hence E is sent on Δ_1; as σ is an involution Δ_1 is sent on E.

Now blow up Q_1; this time let us denote F the exceptional divisor and $\mathcal{D}_2$ the strict transform of $\mathcal{D}_1$:

$$\begin{cases} x=u_2 & \mathrm{F}=\{u_2=0\} \\ z=u_2v_2 & \Delta_2''=\{v_2=0\} \end{cases} \qquad \begin{cases} x=r_2s_2 & \mathrm{E}=\{s_2=0\} \\ z=s_2 & \Delta_2=\{r_2=0\} \end{cases}$$

One has

$$(u_2,v_2)\to(u_2,u_2v_2)_{(x,z)}\to(v_2:u_2v_2:1)=\left(\frac{1}{u_2},\frac{1}{u_2v_2}\right)_{(x,z)}\to\left(\frac{1}{u_2},\frac{1}{v_2}\right)_{(u_2,v_2)}$$

and

$$(r_2,s_2)\to(r_2s_2,s_2)_{(x,z)}\to(1:r_2s_2:r_2)=\left(\frac{1}{r_2s_2},\frac{1}{s_2}\right)_{(x,z)}\to\left(\frac{1}{r_2},\frac{1}{s_2}\right)_{(r_2,s_2)}.$$

Therefore $\mathrm{F}\to\Delta_2'$ and $\Delta_2'\to\mathrm{F}$.
Finally we blow up R_2; let us denote G the exceptional divisor and set

$$\begin{cases} x=u_3 & \mathrm{G}=\{u_3=0\} \\ y=u_3v_3 & \Delta_3''=\{v_3=0\} \end{cases} \qquad \begin{cases} x=r_3s_3 & \mathrm{E}=\{s_3=0\} \\ z=s_3 & \Delta_2=\{r_3=0\} \end{cases}$$

Note that

$$(u_3,v_3)\to(u_3,u_3v_3)_{(x,y)}\to(v_3:1:u_3v_3)=\left(\frac{1}{u_3},\frac{1}{u_3v_3}\right)_{(x,y)}\to\left(\frac{1}{u_3},\frac{1}{v_3}\right)_{(u_3,v_3)}$$

and

$$(r_3,s_3)\to(r_3s_3,s_3)_{(x,y)}\to(1:r_3:r_3s_3)=\left(\frac{1}{r_3s_3},\frac{1}{s_3}\right)_{(x,y)}\to\left(\frac{1}{r_3},\frac{1}{s_3}\right)_{(r_3,s_3)}.$$

Thus $\mathrm{G}\to\Delta_3'$ and $\Delta_3'\to\mathrm{G}$. There is no more point of indeterminacy, no more exceptional curve; in other words σ is conjugate to an automorphism of $\mathrm{Bl}_{P,Q_1,R_2}\mathbb{P}^2$.

Let H be a generic line. Note that $\mathcal{E}_1=\mathrm{E}$, $\mathcal{E}_2=\mathrm{F}$, $\mathcal{E}_3=\mathrm{H}$. Consider the basis $\{\mathrm{H},\mathrm{E},\mathrm{F},\mathrm{G}\}$. After the first blow-up Δ and E are swapped; the point blown up is the intersection of Δ' and Δ'' so $\Delta\to\Delta+\mathrm{F}+\mathrm{G}$. Then $\sigma^*\mathrm{E}=\mathrm{H}-\mathrm{F}-\mathrm{G}$. Similarly we have:

$$\sigma^*\mathrm{F}=\mathrm{H}-\mathrm{E}-\mathrm{G} \qquad \text{and} \qquad \sigma^*\mathrm{G}=\mathrm{H}-\mathrm{E}-\mathrm{F}.$$

It remains to determine $\sigma^*\mathrm{H}$. The image of a generic line by σ is a conic hence $\sigma^*\mathrm{H}=2\mathrm{H}-m_1\mathrm{E}-m_2\mathrm{F}-m_3\mathrm{G}$. Let L be a generic line described by $a_0x+a_1y+a_2z$. A computation shows that

$$(u_1,v_1)\to(u_1,u_1v_1)_{(y,z)}\to(u_1^2v_1:u_1v_1:u_1)\to u_1(a_0v_2+a_1u_2v_2+a_2)$$

vanishes to order 1 on $\mathrm{E}=\{u_1=0\}$ thus $m_1=1$. Note also that

$$(u_2,v_2)\to(u_2,u_2v_2)_{(x,z)}\to(u_2v_2:u_2^2v_2:u_2)\to u_2(a_0v_2+a_1u_2v_2+a_2),$$

resp.

$$(u_3,v_3)\to(u_3,u_3v_3)_{(x,y)}\to(u_3v_3:u_3:u_3^2v_3)\to u_3(a_0v_3+a_1+a_2u_3v_3)$$

vanishes to order 1 on $\mathrm{F}=\{u_2=0\}$, resp. $\mathrm{G}=\{u_3=0\}$ so $m_2=1$, resp. $m_3=1$. Therefore $\sigma^*\mathrm{H}=2\mathrm{H}-\mathrm{E}-\mathrm{F}-\mathrm{G}$ and the characteristic matrix of σ in the basis $\{\mathrm{H},\mathrm{E},\mathrm{F},\mathrm{G}\}$ is

$$M_\sigma=\begin{bmatrix}2&1&1&1\\-1&0&-1&-1\\-1&-1&0&-1\\-1&-1&-1&0\end{bmatrix}.$$

Exercise 2.8. — Let us consider the involution given by

$$\rho\colon\mathbb{P}^2(\mathbb{C})\dashrightarrow\mathbb{P}^2(\mathbb{C}),\qquad\qquad(x:y:z)\dashrightarrow(xy:z^2:yz).$$

One can show that $M_\rho=M_\sigma$.

Exercise 2.9. — Consider the birational map

$$\tau\colon\mathbb{P}^2(\mathbb{C})\dashrightarrow\mathbb{P}^2(\mathbb{C}),\qquad\qquad(x:y:z)\dashrightarrow(x^2:xy:y^2-xz).$$

One can verify that $M_\tau=M_\sigma$.

3. Where can we find automorphisms with positive entropy ?

3.1. Notion of entropy. — Let X be a compact metric space. Let f be a continuous map from X into itself. Let ε be a strictly positif real number. For all integer n let $N(n,\varepsilon)$ be the minimal cardinal of a part X_n of X such that for all y in X there exists x in X satisfying

$$\mathrm{dist}(f^j(x),f^j(y))\leq\varepsilon,\qquad\qquad\forall\ 0\leq j\leq n.$$

We introduce $\mathrm{h}_{\mathrm{top}}(f,\varepsilon)$ defined by

$$\mathrm{h}_{\mathrm{top}}(f,\varepsilon)=\limsup_{n\to+\infty}\frac{1}{n}\log N(n,\varepsilon).$$

The topological entropy of f is given by

$$\mathrm{h}_{\mathrm{top}}(f)=\lim_{\varepsilon\to0}\mathrm{h}_{\mathrm{top}}(f,\varepsilon).$$

Let f be a map of class $\mathcal{C}^\infty$ on a compact manifold $\mathcal{M}$; we have this inequality

$$\mathrm{h}_{\mathrm{top}}(f)\geq\log r(f^*)$$

in other words the topological entropy is smaller than the logarithm of the spectral radius of the linear map induced by f on $\mathrm{H}^*(\mathcal{M},\mathbb{R})$, direct sum of the cohomological groups of $\mathcal{M}$. Remark that the inequality $\mathrm{h}_{\mathrm{top}}(f)\geq\log r(f^*)$ is still true in the meromorphic case ([**16**]). Before stating a more precise result when $\mathcal{M}$ is KÄHLER we introduce some notation: for all integer p such that $0\leq p\leq\dim_{\mathbb{C}}\mathcal{M}$ we denote by $\lambda_p(f)$ the spectral radius of the map f^* acting on the DOLBEAULT cohomological group $\mathrm{H}^{p,p}(\mathcal{M},\mathbb{R})$.

***Theorem 3.1* ([24, 23, 45]).** — *Let f be a holomorphic map on a complex compact* KÄHLER *manifold $\mathcal{M}$; we have*

$$\mathrm{h}_{top}(f) = \max_{0\leq p\leq \dim_{\mathbb{C}}\mathcal{M}} \log\lambda_p(f).$$

***Remark 3.2*.** — The spectral radius of f^* is strictly greater than 1 if and only if one of the $\lambda_p(f)$'s is and, in fact, if and only if $\lambda(f) = \lambda_1(f)$ is. In other words in order to know if the entropy of f is positive we just have to study the growth of $(f^n)^*\{\alpha\}$ where $\{\alpha\}$ is a KÄHLER form.

***Examples 3.3*.** —
- Let $\mathcal{M}$ be a compact KÄHLER manifold and $\mathrm{Aut}^0(\mathcal{M})$ be the connected component of $\mathrm{Aut}\,\mathcal{M}$ which contains the identity element. The topological entropy of each element of $\mathrm{Aut}^0(\mathcal{M})$ is zero.
- The topological entropy of an holomorphic endomorphism f of the projective sapce is equal to the logarithm of the topological degree of f.
- Whereas the topological entropy of an elementary automorphism is zero, the topological entropy of an HÉNON automorphism is positive.

3.2. A theorem of CANTAT. — Before describing the pairs $(\mathcal{Z}, f)$ of complex compact surfaces $\mathcal{Z}$ carrying an automorphism f with positive entropy, let us recall one definition: a surface $\mathcal{Z}$ is *rational* if one can find a surface $\mathcal{X}$ and proper modifications $\pi_1\colon \mathcal{X}\to\mathbb{P}^2(\mathbb{C})$ and $\pi_2\colon \mathcal{X}\to\mathcal{Z}$. A rational surface is always projective ([**3**]). A *K*3 *surface* is a complex, compact, simply connected surface $\mathcal{Z}$ with a trivial canonical bundle. In particular there exists a holomorphic 2-form ω on $\mathcal{Z}$ which is never zero; ω is unique modulo multiplication by a scalar. Let $\mathcal{Z}$ be a K3 surface with a holomorphic involution ι. If ι has no fixed point the quotient is an ENRIQUES *surface*, otherwise it is a rational surface. As ENRIQUES *surfaces* are quotients of K3 surfaces by a group of order 2 acting without fixed points, their theory is similar to that of algebraic K3 surfaces.

***Theorem 3.4* ([9]).** — *Let $\mathcal{Z}$ be a complex compact surface. Assume that $\mathcal{Z}$ has an automorphism f with positive entropy. Then*
- *either f is conjugate to an automorphism on the unique minimal model of $\mathcal{Z}$ which is either a torus, or a K3 surface, or an* ENRIQUES *surface;*
- *or $\mathcal{Z}$ is rational, obtained from $\mathbb{P}^2(\mathbb{C})$ by blowing up $\mathbb{P}^2(\mathbb{C})$ in at least 10 points and f is birationally conjugate to a birational map of $\mathbb{P}^2(\mathbb{C})$.*

In particular $\mathcal{Z}$ is KÄHLER.

***Examples 3.5*.** —
- Set $\Lambda = \mathbb{Z}[\mathrm{i}]$ and $E = \mathbb{C}/\Lambda$. The group $\mathrm{SL}_2(\Lambda)$ acts linearly on $\mathbb{C}^2$ and preserves the lattice $\Lambda\times\Lambda$; then each element A of $\mathrm{SL}_2(\Lambda)$ induces an automorphism f_{A} on $E\times E$ which commutes with $\iota(x,y) = (\mathrm{i}x,\mathrm{i}y)$. Each automorphism f_{A} can be lifted to an automorphism $\widetilde{f_{\mathrm{A}}}$ on the desingularization of $(E\times E)/\iota$ which is a K3 surface. The entropy of $\widetilde{f_{\mathrm{A}}}$ is positive as soon as the modulus of one eigenvalue of A is strictly greater than 1.
- We have the following statement due to TORELLI.

  ***Theorem 3.6*.** — *Let $\mathcal{Z}$ be a K3 surface. The morphism*

  $$\mathrm{Aut}\,\mathcal{Z}\to\mathrm{GL}(\mathcal{H}^2(\mathcal{Z},\mathbb{Z})), \qquad\qquad f\mapsto f^*$$

  is injective.
  Conversely assume that F is an element of $\mathrm{GL}(\mathrm{H}^2(\mathcal{Z},\mathbb{Z}))$ which preserves the intersection form on $\mathrm{H}^2(\mathcal{Z},\mathbb{Z})$, the Hodge decomposition of $\mathrm{H}^2(\mathcal{Z},\mathbb{Z})$ and the KÄHLER *cone of $\mathrm{H}^2(\mathcal{Z},\mathbb{Z})$. Then there exists an automorphism f on $\mathcal{Z}$ such that $f^* = F$.*

The case of K3 surfaces has been studied by CANTAT, MCMULLEN, SILVERMAN, WANG and others (*see* for example [**10, 35, 41, 44**]). The context of rational surfaces produces much more examples (*see* for example [**36, 6, 7, 8, 12**]).

3.3. Case of rational surfaces. —

Exercise 3.1. — Let us consider the following statement due to NAGATA ([**37**], Theorem 5): let $\mathcal{Z}$ be a rational surface and let f be an automorphism on $\mathcal{Z}$ such that f_* is of infinite order; then there exists a sequence of holomorphic maps $\pi_{j+1}\colon \mathcal{Z}_{j+1}\to\mathcal{Z}_j$ such that $\mathcal{Z}_1=\mathbb{P}^2(\mathbb{C})$, $\mathcal{Z}_{N+1}=\mathcal{Z}$ and π_{j+1} is the blow up of $p_j\in\mathcal{Z}_j$.
If $\mathcal{Y}$ and $\mathcal{Z}$ are two projective surfaces, we say that $\mathcal{Y}$ *dominates* $\mathcal{Z}$ if there exists a surjective algebraic birational morphism from $\mathcal{Y}$ to $\mathcal{Z}$. A surface $\mathcal{Z}$ is *basic* if it can be obtained by successive blowups from the projective plane.

- Show that if $\mathcal{Z}$ is not $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ and if $\mathcal{Z}$ dominates $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ then $\mathcal{Z}$ dominates $\mathbb{P}^2(\mathbb{C})$.
- If $\mathcal{Z}$ is not $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ and if $\mathcal{Z}$ has two different rational fibrations, then $\mathcal{Z}$ is basic.
- If $\mathcal{Z}$ is not $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ and if $\mathcal{Z}$ is non basic, then $\mathcal{Z}$ contains a unique rational fibration and each automorphism of $\mathcal{Z}$ preserves this fibration.
- If $\mathcal{Z}$ is not $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ and $\mathcal{Z}$ is non basic then for each f in $\mathrm{Aut}(\mathcal{Z})$, the induced map f^* on $\mathrm{Pic}\mathcal{Z}$ is cyclic.

Remark that a surface obtained from $\mathbb{P}^2(\mathbb{C})$ via generic blow-ups has no nontrivial automorphism ([**28, 31**]). Moreover we have the following statement which can be found for example in [**14**].

Proposition 3.7. — *Let $\mathcal{Z}$ be a surface obtained from $\mathbb{P}^2(\mathbb{C})$ by blowing up $n\leq 9$ points. Let f be an automorphism on $\mathcal{Z}$. The topological entropy of f is zero.*
Moreover, if $n\leq 8$ then there exists an integer k such that f^k is birationally conjugate to an automorphism of the complex projective plane.

Exercise 3.2. — Prove the previous result.

Let f be an automorphism with positive entropy on a KÄHLER surface. The following statement gives properties on the eignevalues of f^*.

Theorem 3.8. — *Let f be an automorphism with positive entropy $\log\lambda(f)$ on a* KÄHLER *surface. The first dynamical degree $\lambda(f)$ is an eigenvalue of f^* with multiplicity 1 and this is the unique eigenvalue with modulus strictly greater than 1.*
If η is an eigenvalue of f^*, then either η belongs to $\{\lambda(f),\lambda(f)^{-1}\}$, or $|\eta|$ is equal to 1.

Exercise 3.3. — Prove the previous result.

Let χ_f denote the caracteristic polynomial of f^*. This is a monic polynomial whose constant term is ±1 (constant term is equal to the determinant of f^*). Let Ψ_f be the minimal polynomial of $\lambda(f)$. Except for $\lambda(f)$ and $\lambda(f)^{-1}$ all zeroes of χ_f (and thus of Ψ_f) lie on the unit circle. Such polynomial is a *Salem polynomial* and such a $\lambda(f)$ is a SALEM *number*.

4. Automorphisms with positive entropy on rational surfaces

4.1. The approach of McMULLEN ([36]). — In [**36**] McMULLEN, thanks to NAGATA's works and HARBOURNE's works, establishes a result similar to TORELLI's theorem for K3 surfaces: he constucts automorphisms on some rational surfaces prescribing the action of the automorphisms on cohomological groups of the surface. These surfaces are rational ones which own, up to multiplication by a constant, a unique meromorphic non-vanishing 2-form Ω. If f is an automorphism on $\mathcal{Z}$ obtained via this construction, $f^*\Omega$ is proportional to Ω and f preserves the poles of Ω. When we project $\mathcal{Z}$ on the complex projective plane, f induces a birational map preserving a cubic.

The relationship of the WEYL group to the birational geometry of the plane, used by McMULLEN, is discussed since 1895 in [**30**] and has been much developed since then ([**19, 37, 38, 11, 22, 32, 25, 33, 26, 39, 27, 17, 28, 46, 18**]).

4.1.1. WEYL *groups*. — Let $\mathcal{Z}$ be a surface obtained by blowing up the complex projective plane in a finite number of points. Let $\{e_0, \ldots, e_n\}$ be a basis of $\mathrm{H}^2(\mathcal{Z},\mathbb{Z})$; if

$$e_0\cdot e_0 = 1, \qquad e_j\cdot e_j = -1,\ \forall\, 1\leq j\leq k, \qquad e_i\cdot e_j = 0,\ \forall\, 0\leq i\neq j\leq n$$

then $\{e_0, \ldots, e_n\}$ is a *geometric basis*. Consider α in $\mathrm{H}^2(\mathcal{Z},\mathbb{Z})$ such that $\alpha\cdot\alpha=-2$, then $R_\alpha(x)=x+(x\cdot\alpha)\alpha$ sends α on $-\alpha$ and R_α fixes each element of $\alpha^\perp$; in other words R_α is a reflection in the direction α.
Consider the vectors given by

$$\alpha_0 = e_0-e_1-e_2-e_3, \qquad \alpha_j = e_{j+1}-e_j,\ 1\leq j\leq n-1.$$

Forall j in $\{0,\ldots,n-1\}$ we have $\alpha_j\cdot\alpha_j=-2$. When j is nonzero the reflection R_{α_j} induces a permutation on $\{e_j, e_{j+1}\}$. The subgroup generated by the R_{α_j}'s, with $1\leq j\leq\ n-\ 1$, is the set of permutations on the elements $\{e_1, \ldots, e_n\}$. Let $\mathrm{W}_n\subset\mathrm{O}(\mathbb{Z}^{1,n})$ denote the group $\langle R_{\alpha_j}\,|\,0\leq j\leq n-1\rangle$ which is called WEYL *group*. The WEYL groups are, for $3\leq n\leq 8$, isomorphic to the following finite groups

$$A_1\times A_2, \qquad A_4, \qquad D_5, \qquad E_6, \qquad E_7, \qquad E_8$$

and are associated to DEL PEZZO surfaces [(1)]. For $k\geq 9$ WEYL groups are infinite and for $k\geq 10$ WEYL groups contain elements with a spectral radius strictly greater than 1.
If f is an automorphism of $\mathcal{Z}$, by a theorem of NAGATA there exists a unique element w in W_n such that

$$\begin{tikzcd} \mathbb{Z}^{1,n} \arrow[r, "w"] \arrow[d, "\varphi"'] & \mathbb{Z}^{1,n} \arrow[d, "\varphi"] \\ \mathrm{H}^2(\mathcal{Z},\mathbb{Z}) \arrow[r, "f_*"] & \mathrm{H}^2(\mathcal{Z},\mathbb{Z}) \end{tikzcd}$$

commutes; we said that w is *realized* by the automorphism f.
A product of generators R_{α_j} is a COXETER *element* of W_n. Note that all COXETER elements are conjugate so the spectral radius of a COXETER element is well defined.
The map σ is represented by the reflection $\kappa_{ijk}=R_{\alpha_{ijk}}$ where $\alpha_{ijk}=e_0-e_i-e_j-e_k$ and $i,\ j,\ k\geq 1$ are distinct elements; it acts as follows

$$e_0\to 2e_0-e_i-e_j-e_k, \qquad e_i\to e_0-e_j-e_k, \qquad e_j\to e_0-e_i-e_k$$

1. A DEL PEZZO surface is isomorphic either to $\mathbb{P}^2(\mathbb{C})$, or to $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$, or to $\mathbb{P}^2(\mathbb{C})$ blown up in $1\leq r\leq 8$ points in "generic position".

$$e_k \to e_0 - e_i - e_j, \qquad\qquad e_\ell \to e_\ell \text{ si } \ell \not\in \{0, i, j, k\}$$

When $n = 3$, we say that κ_{123} is the *standard element* of W_3. Consider the cyclic permutation

$$(123\dots n) = \kappa_{123} R_{\alpha_1} \dots R_{\alpha_{n-1}} \in \Sigma_n \subset W_n;$$

let us denote it π_n. For $n \geq 4$ we define the *standard element* w of W_n by $w = \pi_n \kappa_{123}$. It satisfies

$$w(e_0) = 2e_0 - e_2 - e_3 - e_4, \qquad w(e_1) = e_0 - e_3 - e_4, \qquad w(e_2) = e_0 - e_2 - e_4,$$

$$w(e_3) = e_0 - e_2 - e_3, \qquad w(e_j) = e_{j+1},\ 4 \leq j \leq n-2, \qquad w(e_{n-1}) = e_1.$$

4.1.2. Statements. — In [**36**] McMullen constructs examples of automorphisms with positive entropy "thanks to" elements of Weyl groups.

Theorem 4.1 **([36])**. — *For $n \geq 10$, each* Coxeter *element of* W_n *can be realizable by an automorphism f_n with positive entropy* $\log(\lambda_n)$ *of a rational surface $\mathcal{Z}_n$.*

More precisely the automorphism $f_n \colon \mathcal{Z}_n \to \mathcal{Z}_n$ can be chosen to have the following additional properties:

- $\mathcal{Z}_n$ is the projective plane blown up in n distinct points $p_1, \dots, p_n$ lying on a cuspidal cubic curve $\mathcal{C}$,
- there exists a nowhere vanishing meromorphic 2-form η on $\mathcal{Z}_n$ with a simple pole along the proper transform of $\mathcal{C}$,
- $f_n^*(\eta) = \lambda_n \cdot \eta$,
- $(\langle f_n \rangle, \mathcal{Z}_n)$ is minimal in the sense of Manin [(2)].

The first three properties determine f_n uniquely. The points p_i admit a simple description which leads to concrete formulas for f_n.

The smallest known Salem number is a root $\lambda_{\text{Lehmer}} \sim 1.17628081$ of Lehmer's polynom

$$L(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1.$$

Theorem 4.2 **([36])**. — *If f is an automorphism of a compact complex surface with positive entropy, then* $\mathrm{h}_{top}(f) \geq \log \lambda_{\text{Lehmer}}$.

Corollary 4.3 **([36])**. — *The map $f_{10} \colon \mathcal{Z}_{10} \to \mathcal{Z}_{10}$ is an automorphism of $\mathcal{Z}_{10}$ with the smallest possible positive entropy.*

Let us also mention a more recent work in this direction ([**43**]). Diller also find examples using plane cubics ([**14**]).

4.2. Another approach. —

4.2.1. Works of Bedford *and* Kim *(*[**6, 7, 8, 5**]*).* — A way to construct automorphisms on a rational surface $\mathcal{Z}$ is the following: starting with a birational map f of $\mathbb{P}^2(\mathbb{C})$, we try to find a sequence of blow-ups $\pi \colon \mathcal{Z} \to \mathbb{P}^2(\mathbb{C})$ such that the induced map $f_{\mathcal{Z}} = \pi f \pi^{-1}$ is an automorphism on $\mathcal{Z}$. The difficulty is to find such a sequence π... If f is an automorphism of the complex projective plane, f blows down a curve $\mathcal{C}_1$ to a point p_1; the first thing to do in order to obtain an automorphism from f is to blow up the point p_1 via $\pi_1 \colon \mathcal{Z}_1 \to \mathbb{P}^2(\mathbb{C})$. In the best case $f_{\mathcal{Z}_1} = \pi_1 f \pi_1^{-1}$ sends the transform of $\mathcal{C}_1$ on the exceptional divisor E_1. But if p_1 is not a point of indeterminacy $f_{\mathcal{Z}_1}$ blows down E_1 to $p_2 = f(p_1)$. In other words this process finishes only if f is not algebraically stable. Thanks to this method Bedford and Kim found some examples ([**6, 7, 8, 5**]); one of their statement is the following:

2. Let $\mathcal{Z}$ be a surface and G be a subgroup of $\mathrm{Aut}(\mathcal{Z})$. A birational map $f \colon \mathcal{Z} \dashrightarrow \widetilde{\mathcal{Z}}$ is G-equivariant if $\widetilde{G} = fGf^{-1} \subset \mathrm{Aut}(\widetilde{\mathcal{Z}})$. The pair $(G, \mathcal{Z})$ is minimal if every G-equivariant birational morphism is an isomorphism

Theorem 4.4 **([8])**. — *Consider two integers $n \geq 3$ and $k \geq 2$ such that n is odd and $(n,k) \neq (3,2)$. There exists a nonempty subset C_k of $\mathbb{R}$ such that, if $c \in C_k$ and $a = (a_2, a_4, \ldots, a_{n-3}) \in \mathbb{C}^{\frac{n-3}{2}}$, the map*

$$f_a\colon (x:y:z) \to \Big(xz^{n-1} : z^n : x^n - yz^{n-1} + cz^n + \sum_{\substack{\ell=2\\ \ell \text{ even}}}^{n-3} a_\ell x^{\ell+1} z^{n-\ell-1}\Big) \tag{4.1}$$

can be lifted to an automorphism of positive topological entropy of a rational surface X_a. The surfaces X_a are obtained by blowing up k infinitely near points of length $2n-1$ on the invariant line $\{x=0\}$ and form a holomorphic family over the parameter space given by the $a_j{}'s$.
If $k=2$ and $n \geq 5$ is odd, then there exists a neighborhood of 0 in $\mathbb{C}^{\frac{n-3}{2}}$ such that for all distinct elements a and a' in U with $a_{n-3} \neq 0$, X_a and $X_{a'}$ are not biholomorphic.

4.2.2. A "systematic" way ([**12**]*).* — Idea of the approach: this section is devoted to the construction of examples of rational surfaces with biholomorphisms of positive entropy. The strategy is the following: start with a birational map f of $\mathbb{P}^2(\mathbb{C})$. By the standard factorization theorem for birational maps on surfaces as a composition of blow-ups and blow-downs, there exist two sets of (possibly infinitely near) points $\widehat{P}_1$ and $\widehat{P}_2$ in $\mathbb{P}^2(\mathbb{C})$ such that f can be lifted to an automorphism between $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$. The data of $\widehat{P}_1$ and $\widehat{P}_2$ allow to get automorphisms of rational surfaces in the left $\mathrm{PGL}_3(\mathbb{C})$-orbit of f : assume that $k \in \mathbb{N}$ is fixed and let φ be an element of $\mathrm{PGL}_3(\mathbb{C})$ such that $\widehat{P}_1$, $\varphi\widehat{P}_2$, $(\varphi f)\varphi\widehat{P}_2$, $\ldots$, $(\varphi f)^{k-1}\varphi\widehat{P}_2$ have all distinct supports in $\mathbb{P}^2(\mathbb{C})$ and $(\varphi f)^k\varphi\widehat{P}_2 = \widehat{P}_1$. Then φf can be lifted to an automorphism of $\mathbb{P}^2(\mathbb{C})$ blown up at $\widehat{P}_1$, $\varphi\widehat{P}_2$, $(\varphi f)\varphi\widehat{P}_2$, $\ldots$, $(\varphi f)^{k-1}\varphi\widehat{P}_2$. Furthermore, if the conditions above are satisfied for a holomorphic family of φ, we get a holomorphic family of rational surfaces (whose dimension is at most eight). Therefore, we see that the problem of lifting an element in the $\mathrm{PGL}_3(\mathbb{C})$-orbit of f to an automorphism is strongly related to the equation $u(\widehat{P}_2) = \widehat{P}_1$, where u is a germ of biholomorphism of $\mathbb{P}^2(\mathbb{C})$ mapping the support of $\widehat{P}_2$ to the support of $\widehat{P}_1$. In concrete examples, when $\widehat{P}_1$ and $\widehat{P}_2$ are known, this equation can actually be solved and involves polynomial equations in the Taylor expansions of u at the various points of the support of $\widehat{P}_2$. It is worth pointing out that in the generic case, $\widehat{P}_1$ and $\widehat{P}_2$ consist of the same number d of distinct points in the projective plane, and the equation $u(\widehat{P}_2) = \widehat{P}_1$ gives $2d$ independent conditions on u (which is the maximum possible number if $\widehat{P}_1$ and $\widehat{P}_2$ have length d). Conversely, infinitely near points can considerably decrease the number of conditions on u as shown in our examples. This explains why holomorphic families of automorphisms of rational surfaces occur when multiple blow-ups are made.

Birational maps whose exceptional locus is a line
Let us consider the birational map defined by $\Phi_n = (xz^{n-1} + y^n : yz^{n-1} : z^n)$, with $n \geq 3$. The sequence $(\deg \Phi_n^k)_{k \in \mathbb{N}}$ is bounded (it's easy to see in the affine chart $z=1$), so Φ_n is conjugate to an automorphism on some rational surface $\mathcal{Z}$ and an iterate of Φ_n is conjugate to an automorphism isotopic to the identity ([**15**]). The map Φ_n blows up one point $P = (1:0:0)$ and blows down one curve $\Delta = \{z=0\}$.
Here we will assume that $n=3$ but the construction is similar for $n \geq 4$ (*see* [**12**]). We first construct two infinitely near points $\widehat{P}_1$ and $\widehat{P}_2$ such that Φ_3 induces an isomorphism between $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$. Then we give "theoric" conditions to produce automorphisms φ of $\mathbb{P}^2(\mathbb{C})$ such that $\varphi\Phi_3$ is conjugate to an automorphism on a surface obtained from $\mathbb{P}^2(\mathbb{C})$ by successive blow-ups.

First step: description of the sequence of blow-ups
i. First blow up the point P in the domain and in the range. Set $y = u_1$ and $z = u_1v_1$; remark that (u_1, v_1) are coordinates near $P_1 = (0,0)_{(u_1,v_1)}$, coordinates in which the exceptional divisor is given by $\mathrm{E} = \{u_1 = 0\}$ and the strict transform of Δ is given by $\Delta_1 = \{v_1 = 0\}$. Set $y = r_1s_1$ and $z = s_1$; note that (r_1, s_1) are coordinates

near $Q=(0,0)_{(r_1,s_1)}$, coordinates in which $\mathrm{E}=\{s_1=0\}$. One has

$$\begin{aligned}(u_1,v_1)&\to(u_1,u_1v_1)_{(y,z)}\to(v_1^2+u_1:v_1^2u_1:v_1^3u_1)\\&=\left(\frac{v_1^2u_1}{v_1^2+u_1},\frac{v_1^3u_1}{v_1^2+u_1}\right)_{(y,z)}\to\left(\frac{v_1^2u_1}{v_1^2+u_1},v_1\right)_{(u_1,v_1)}\end{aligned}$$

and

$$\begin{aligned}(r_1,s_1)&\to(r_1s_1,s_1)_{(y,z)}\to(1+r_1^3s_1:r_1s_1:s_1)\\&=\left(\frac{r_1s_1}{1+r_1^3s_1},\frac{s_1}{1+r_1^3s_1}\right)_{(y,z)}\to\left(r_1,\frac{s_1}{1+r_1^3s_1}\right)_{(r_1,s_1)};\end{aligned}$$

therefore P_1 is a point of indeterminacy, Δ_1 is blown down to P_1 and E is fixed.

ii. Let us blow up P_1 in the domain and in the range. Set $u_1=u_2$ and $v_1=u_2v_2$. Note that (u_2,v_2) are coordinates around $P_2=(0,0)_{(u_2,v_2)}$ in which $\Delta_2=\{v_2=0\}$ and $\mathrm{F}=\{u_2=0\}$. If we set $u_1=r_2s_2$ and $v_1=s_2$ then (r_2,s_2) are coordinates near $A=(0,0)_{(r_2,s_2)}$; in these coordinates $\mathrm{F}=\{s_2=0\}$. Moreover

$$(u_2,v_2)\to(u_2,u_2v_2)_{(u_1,v_1)}\to(1+u_2v_2^2:u_2^2v_2^2:u_2^3v_2^3)$$

and

$$(r_2,s_2)\to(r_2s_2,s_2)_{(r_1,s_1)}\to(r_2+s_2:r_2s_2^2:r_2s_2^3).$$

Remark that A is a point of indeterminacy. One also has

$$\begin{aligned}(u_2,v_2)&\to(u_2,u_2v_2)_{(u_1,v_1)}\to(1+u_2v_2^2:u_2^2v_2^2:u_2^3v_2^3)\to\left(\frac{u_2^2v_2^2}{1+u_2v_2^2},\frac{u_2^3v_2^3}{1+u_2v_2^2}\right)_{(y,z)}\\&\to\left(\frac{u_2^2v_2^2}{1+u_2v_2^2},u_2v_2\right)_{(u_1,v_1)}\to\left(\frac{u_2v_2}{1+u_2v_2^2},u_2v_2\right)_{(r_2,s_2)}\end{aligned}$$

so F and Δ_2 are blown down to A.

iii. Now let us blow up A in the domain and in the range. Set $r_2=u_3$ and $s_2=u_3v_3$; (u_3,v_3) are coordinates near $A_1=(0,0)_{(u_3,v_3)}$, coordinates in which $\mathrm{F}_1=\{v_3=0\}$ and $\mathrm{G}=\{u_3=0\}$. If $r_2=r_3s_3$ and $s_2=s_3$, then (r_3,s_3) is a system of coordinates in which $\mathrm{E}_2=\{r_3=0\}$ and $\mathrm{G}=\{s_3=0\}$. One has

$$(u_3,v_3)\to(u_3,u_3v_3)_{(r_2,s_2)}\to(1+v_3:u_3^2v_3^2:u_3^3v_3^3),$$

$$(r_3,s_3)\to(r_3s_3,s_3)_{(r_2,s_2)}\to(1+r_3:r_3s_3^2:r_3s_3^3).$$

The point $T=(-1,0)_{(r_3,s_3)}$ is a point of indeterminacy. Moreover

$$\begin{aligned}(u_3,v_3)&\to\left(\frac{u_3^2v_3^2}{1+v_3},\frac{u_3^3v_3^3}{1+v_3}\right)_{(y,z)}\to\left(\frac{u_3^2v_3^2}{1+v_3},u_3v_3\right)_{(u_1,v_1)}\\&\to\left(\frac{u_3v_3}{1+v_3},u_3v_3\right)_{(r_2,s_2)}\to\left(\frac{1}{1+v_3},u_3v_3\right)_{(r_3,s_3)};\end{aligned}$$

so G is fixed and F_1 is blown down to $S=(1,0)_{(r_3,s_3)}$.

$\mathfrak{iv}$. Let us blow up T in the domain and S in the range. Set $r_3=u_4-1$ and $s_3=u_4v_4$; in the system of coordinates (u_4,v_4) we have $\mathrm{G}_1=\{v_4=0\}$ and $\mathrm{H}=\{u_4=0\}$. Note that (r_4,s_4), where $r_3=r_4s_4-1$ and $s_3=s_4$, is a system of coordinates in which $\mathrm{H}=\{s_4=0\}$. On the one hand

$$\begin{aligned}(u_4,v_4)&\to(u_4-1,u_4v_4)_{(r_3,s_3)}\to((u_4-1)u_4v_4^2,(u_4-1)u_4^2v_4^3)_{(y,z)}\\ &\to((u_4-1)u_4v_4^2,u_4v_4)_{(u_1,v_1)}\to((u_4-1)v_4,u_4v_4)_{(r_2,s_2)}\to\left((u_4-1)v_4,\frac{u_4}{u_4-1}\right)_{(u_3,v_3)}\end{aligned}$$

so H is sent on F_2; on the other hand

$$(r_4,s_4)\to(r_4s_4-1,s_4)_{(r_3,s_3)}\to(r_4:(r_4s_4-1)s_4:(r_4s_4-1)s_4^2);$$

hence $B=(0,0)_{(r_4,s_4)}$ is a point of indeterminacy.

Set $r_3=a_4+1$, $s_3=a_4b_4$; (a_4,b_4) are coordinates in which $\mathrm{G}_1=\{b_4=0\}$ and $\mathrm{K}=\{a_4=0\}$. One can also set $r_3=c_4d_4+1$ and $s_3=d_4$; in the system of coordinates (c_4,d_4) the exceptional divisor K is given by $d_4=0$.
Note that

$$(u_3,v_3)\to\left(\frac{1}{1+v_3},u_3v_3\right)_{(r_3,s_3)}\to\left(-\frac{v_3}{1+v_3},-u_3(1+v_3)\right)_{(a_4,b_4)};$$

thus F_2 is sent on K.
We remark that

$$\begin{aligned}(u_1,v_1)&\to(v_1^2+u_1:u_1v_1^2:u_1v_1^3)=\left(\frac{u_1v_1^2}{u_1+v_1^2},\frac{u_1v_1^3}{u_1+v_1^2}\right)_{(y,z)}\to\left(\frac{u_1v_1^2}{u_1+v_1^2},v_1\right)_{(u_1,v_1)}\\ &\to\left(\frac{u_1v_1}{u_1+v_1^2},v_1\right)_{(r_2,s_2)}\to\left(\frac{u_1}{u_1+v_1^2},v_1\right)_{(r_3,s_3)}\to\left(-\frac{v_1}{u_1+v_1^2},v_1\right)_{(c_4,d_4)};\end{aligned}$$

so Δ_4 is blown down to $C=(0,0)_{(c_4,d_4)}$.

$\mathfrak{v}$. Now let us blown up B in the domain and C in the range. Set $r_4=u_5$, $s_4=u_5v_5$ and $r_4=r_5s_5$, $s_4=s_5$. Then (u_5,v_5) (resp. (r_5,s_5)) is a system of coordinates in which $\mathrm{L}=\{u_5=0\}$ (resp. $\mathrm{H}_1=\{v_5=0\}$ and $\mathrm{L}=\{s_5=0\}$). One notes that

$$(u_5,v_5)\to(u_5,u_5v_5)_{r_4,s_4}\to(1:v_5(u_5^2v_5-1):u_5v_5^2(u_5^2v_5-1))$$

and

$$r_5,s_5)\to(r_5s_5,s_5)_{r_4,s_4}\to(r_5:r_5s_5^2-1:s_5(r_5s_5^2-1)).$$

Therefore L is sent on Δ_5 and there is no point of indeterminacy.
Set $c_4=a_5$, $d_4=a_5b_5$ and $c_4=c_5d_5$, $d_4=d_5$. In the first (resp. second) system of coordinates the exceptional divisor M is given by $\{a_5=0\}$ (resp. $\{d_5=0\}$). One has

$$(u_1,v_1)\to\left(-\frac{v_1}{u_1+v_1^2},v_1\right)_{(c_4,d_4)}\to\left(-\frac{1}{u_1+v_1^2},v_1\right)_{(c_5,d_5)};$$

in particular Δ_5 is sent on M.

***Proposition 4.5* ([12]).** — *Let $\widehat{P}_1$ (resp. $\widehat{P}_2$) be the point infinitely near P obtained by blowing up $\mathbb{P}^2(\mathbb{C})$ at P, P_1, A, T and U (resp. P, P_1, A, S and U').*
The map Φ_3 induces an isomorphism between $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$.

The different components are swapped as follows

$$\Delta \to \mathrm{M}, \qquad \mathrm{E} \to \mathrm{E}, \qquad \mathrm{F} \to \mathrm{K}, \qquad \mathrm{G} \to \mathrm{G}, \qquad \mathrm{H} \to \mathrm{F}, \qquad \mathrm{L} \to \Delta.$$

Second step: gluing conditions

The gluing conditions reduce to the following problem: if u is a germ of biholomorphism in a neighborhood of P, find the conditions on u in order that $u(\widehat{P}_2) = \widehat{P}_1$.

***Proposition 4.6* ([12]).** — *Let* $u(y,z) = \left(\sum_{(i,j)\in\mathbb{N}^2} m_{i,j}y^iz^j, \sum_{(i,j)\in\mathbb{N}^2} n_{i,j}y^iz^j\right)$ *be a germ of biholomorphism at P. Then u can be lifted to a germ of biholomorphism between* $\mathrm{Bl}_{\widehat{P}_2}\mathbb{P}^2$ *and* $\mathrm{Bl}_{\widehat{P}_1}\mathbb{P}^2$ *if and only if*

$$m_{0,0} = n_{0,0} = n_{1,0} = m_{1,0}^3 + n_{0,1}^2 = 0, \qquad n_{2,0} = \frac{3m_{0,1}n_{0,1}}{2m_{1,0}}.$$

***Exercise 4.1*.** — Let σ be the birational map defined by $\sigma\colon (x:y:z) \dashrightarrow (yz:xz:xy)$; find P_1, P_2 and P_3 three points in $\mathbb{P}^2(\mathbb{C})$ such that σ induces an isomorphism of $\mathrm{Bl}_{P_1,P_2,P_3}$.
Find how the differents components are swapped.
Find the matrix of f_*.

***Exercise 4.2*.** — Let ρ be the birational map defined by $\rho\colon (x:y:z) \dashrightarrow (xy:z^2:yz)$; find P_1 and P_2 two points infinitely near $\mathbb{P}^2(\mathbb{C})$ or in $\mathbb{P}^2(\mathbb{C})$ such that ρ induces an isomorphism of Bl_{P_1,P_2}.
Find how the differents components are swapped.
Find the matrix of f_*.

Examples

In this section, we will use the two above steps to produce explicit examples of automorphisms of rational surfaces obtained from birational maps in the $\mathrm{PGL}_3(\mathbb{C})$-orbit of Φ_3. As we have to blow up $\mathbb{P}^2(\mathbb{C})$ at least ten times to have non zero-entropy, we want to find an automorphism φ of $\mathbb{P}^2(\mathbb{C})$ such that

$$(\varphi\Phi_3)^k\varphi(\widehat{P}_2) = \widehat{P}_1 \text{ with } (k+1)(2n-1) \geq 10 \text{ and } (\varphi\Phi_3)^i\varphi(P) \neq P \text{ for } 0 \leq i \leq k-1 \tag{4.2}$$

First of all let us introduce the following definition.

***Definition*.** — Let U be an open set of $\mathbb{C}^n$ and $\varphi\colon U \to \mathrm{PGL}_3(\mathbb{C})$ a holomorphic map. If f is a birational map of the projective plane, we say that the family of birational maps $(\varphi_{\alpha_1,\ldots,\alpha_n} f)_{(\alpha_1,\ldots,\alpha_n)\in U}$ is *holomorphically trivial* if for every $\alpha^0 = (\alpha_1^0, \ldots, \alpha_n^0)$ in U there exists a holomorphic map from a neighborhood U_{α^0} of α^0 to $\mathrm{PGL}_3(\mathbb{C})$ such that

- $M_{\alpha_1^0,\ldots,\alpha_n^0} = \mathrm{Id}$,
- $\forall\,(\alpha_1, \ldots, \alpha_n) \in U_{\alpha^0}$, $\varphi_{\alpha_1,\ldots,\alpha_n} f = M_{\alpha_1,\ldots,\alpha_n}(\varphi_{\alpha_1^0,\ldots,\alpha_n^0} f)M^{-1}_{\alpha_1,\ldots,\alpha_n}$.

***Theorem 4.7*.** — *Let* φ_α *be the automorphism of the complex projective plane given by*

$$\varphi_\alpha = \begin{bmatrix} \alpha & 2(1-\alpha) & (2+\alpha-\alpha^2) \\ -1 & 0 & (\alpha+1) \\ 1 & -2 & (1-\alpha) \end{bmatrix}, \qquad \alpha \in \mathbb{C}\setminus\{0, 1\}.$$

The map $\varphi_\alpha\Phi_3$ *is conjugate to an automorphism of* $\mathbb{P}^2(\mathbb{C})$ *blown up in* 15 *points.*
The first dynamical degree of $\varphi_\alpha\Phi_3$ *is* $\frac{3+\sqrt{5}}{2} > 1$.
The family $\varphi_\alpha\Phi_3$ *is holomorphically trivial.*

Proof. — The first assertion is given by Proposition 4.6.
The different components are swapped as follows (§4.2.2)

$$\begin{array}{llll}
\Delta \to \varphi_\alpha \mathrm{M}, & \mathrm{E} \to \varphi_\alpha \mathrm{E}, & \mathrm{F} \to \varphi_\alpha \mathrm{K}, & \mathrm{G} \to \varphi_\alpha \mathrm{G}, \\
\mathrm{H} \to \varphi_\alpha \mathrm{F}, & \mathrm{L} \to \varphi_\alpha \Delta, & \varphi_\alpha \mathrm{E} \to \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{E}, & \varphi_\alpha \mathrm{F} \to \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{F}, \\
\varphi_\alpha \mathrm{G} \to \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{G}, & \varphi_\alpha \mathrm{K} \to \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{K}, & \varphi_\alpha \mathrm{M} \to \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{M}, & \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{E} \to \mathrm{E}, \\
\varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{F} \to \mathrm{F}, & \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{G} \to \mathrm{G}, & \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{K} \to \mathrm{H}, & \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{M} \to \mathrm{L}.
\end{array}$$

So, in the basis

$$\{\Delta, \mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{L}, \varphi_\alpha \mathrm{E}, \varphi_\alpha \mathrm{F}, \varphi_\alpha \mathrm{G}, \varphi_\alpha \mathrm{K}, \varphi_\alpha \mathrm{M}\, \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{E}, \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{F}, \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{G}, \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{K}, \varphi_\alpha \Phi_3 \varphi_\alpha \mathrm{M}\},$$

the matrix of $(\varphi_\alpha \Phi_3)_*$ is

$$\begin{bmatrix}
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&3&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&3&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&3&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&-3&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&-3&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0
\end{bmatrix}$$

and its caracteristic polynomial is

$$(X^2-3X+1)(X^2-X+1)(X+1)^2(X^2+X+1)^3(X-1)^4.$$

Thus

$$\lambda(\varphi_\alpha \Phi_3) = \frac{3+\sqrt{5}}{2} > 1.$$

Fix a point α_0 in $\mathbb{C} \setminus \{0, 1\}$. We can find locally around α_0 a matrix M_α dépending holomorphically on α such that for all α near α_0 we have $\varphi_\alpha \Phi_3 = M_\alpha^{-1} \varphi_{\alpha_0} \Phi_3 M_\alpha$: if μ is a local holomorphic solution of the equation $\alpha = \mu^n \alpha_0$ such that $\mu_0 = 1$ we can take

$$M_\alpha = \begin{bmatrix} 1 & 0 & \alpha_0 - \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

A birational cubic map blowing down one conic and one line

Let f denote the following birational map

$$f = (y^2 z : x(xz+y^2) : y(xz+y^2));$$

it blows up two points and blown down two curves, more precisely

$$\mathrm{Ind}\, f=\{R=(1:0:0),P=(0:0:1)\},\qquad \mathrm{Exc}\, f=(\mathcal{C}=\{xz+y^2=0\})\cup(\Delta'=\{y=0\}).$$

One can verify that $f^{-1}=(y(z^2-xy):z(z^2-xy):xz^2)$ and

$$\mathrm{Ind}\, f^{-1}=\{Q=(0:1:0),R\},\qquad \mathrm{Exc}\, f^{-1}=(\mathcal{C}'=\{z^2-xy=0\})\cup(\Delta''=\{z=0\}).$$

Similar computations allow us to establish the following statement.

Theorem 4.8 **([12])**. — *Assume that* $f=(y^2z:x(xz+y^2):y(xz+y^2))$ *and that*

$$\varphi_\alpha=\left[\begin{array}{ccc}\frac{2\alpha^3}{343}(37\mathrm{i}\sqrt{3}+3) & \alpha & -\frac{2\alpha^2}{49}(5\mathrm{i}\sqrt{3}+11)\\ \frac{\alpha^2}{49}(-15+11\mathrm{i}\sqrt{3}) & 1 & -\frac{\alpha}{14}(5\mathrm{i}\sqrt{3}+11)\\ -\frac{\alpha}{7}(2\mathrm{i}\sqrt{3}+3) & 0 & 0\end{array}\right],\qquad \alpha\in\mathbb{C}^*.$$

The map $\varphi_\alpha f$ *is conjugate to an automorphism of* $\mathbb{P}^2(\mathbb{C})$ *blown up in* 15 *points.*
The first dynamical degree of $\varphi_\alpha f$ *is* $\lambda(\varphi_\alpha f)=\frac{3+\sqrt{5}}{2}$.
The family $\varphi_\alpha f$ *is holomorphically trivial.*

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