Exercises

1. Mini-Course 1 - Alberto CALABRI - Introduction to plane Cremona maps

**Exercise 1.** Consider the rational map $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ given by
\[
\alpha([x_0 : x_1 : x_2]) = [x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2]
\]
and call $S$ the image of this map.

(a) Determine the fundamental points of $\alpha$.
(b) Choose your favourite line $L_1$ passing through $P = [1 : 0 : 0]$. Show that the image of $L_1$ via $\alpha$ is a line in $\mathbb{P}^4$.
(c) Do the same for each line passing through $P$.
(d) Choose your favourite line $L_2$ not passing through $P$. Show that the image of $L_2$ via $\alpha$ is a conic in $\mathbb{P}^4$.
(e) Do the same for each line not passing through $P$.
(f) Show that $S$ is isomorphic to the blow up of $\mathbb{P}^2$ at $P$.
(g) Find the equations in $\mathbb{P}^4$ of the exceptional curve $E$ of $S$.

**Notation.** Denote by $\sigma$ the standard quadratic map $\sigma([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1]$.

**Exercise 2.** Let $P_1 = [1 : 0 : 0], P_2 = [0 : 1 : 0], P_3 = [1 : 1 : 1]$ in $\mathbb{P}^2$.

(a) Define a quadratic plane Cremona map $\gamma$ with fundamental points $P_1, P_2, P_3$.
(b) Find linear maps $\alpha, \beta$ such that $\gamma = \alpha \circ \sigma \circ \beta$.
(c) Find, if it exists, $\gamma$ as above such that $\gamma$ is an involution, i.e. $\gamma^{-1} = \gamma$.

One says that a quadratic Cremona map with three fundamental points is of the first type.

**Exercise 3.** Consider the rational map $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$
\[
\gamma([x_0 : x_1 : x_2]) = [x_2^2 : x_0x_1 : x_1x_2].
\]

(a) Compute the fundamental points of $\gamma$.
(b) Show that $\gamma$ is birational by computing its inverse $\gamma^{-1}$.
(c) Find an open subset of $\mathbb{P}^2$ where $\gamma$ is an isomorphism.
(d) Describe $\gamma$ as the composition of blowing-ups and blowing-downs.
(e) Find two quadratic maps $\sigma_1, \sigma_2$ of the first type such that $\gamma = \sigma_1 \circ \sigma_2$.

(f) Find three linear maps $\alpha_1, \alpha_2, \alpha_3$ such that $\gamma = \alpha_1 \circ \sigma \circ \alpha_2 \circ \sigma \circ \alpha_3$.

One says that $\gamma$ is a quadratic plane Cremona map of the \textit{second type}.

\textbf{Exercise 4.} Consider the birational map $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

$$
\gamma([x_0 : x_1 : x_2]) = [x_1 x_2 : x_2^2 - x_0 x_1 : x_1^2].
$$

(a) Compute the fundamental points of $\gamma$.

(b) Find an open subset of $\mathbb{P}^2$ where $\gamma$ is an isomorphism.

(c) Describe $\gamma$ as the composition of blowing-ups and blowing-downs.

(d) Find two quadratic maps $\sigma_1, \sigma_2$ of the second type such that $\gamma = \sigma_1 \circ \sigma_2$.

(e) Write $\gamma$ as the composition of $\sigma$ and linear maps. How many $\sigma$’s do you use?

One says that $\gamma$ is a quadratic plane Cremona map of the \textit{third type}.

\textbf{Exercise 5.} Let $P_1 \in \mathbb{P}^2$ be a point and $\tau_1 : S_1 \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at $P_1$, with exceptional curve $E_1$. Let $P_2 \in S_1$ be a point of $E_1$ and $\tau_2 : S_2 \to S_1$ be the blowing up of $S_1$ at $P_2$, with exceptional curve $E_2$. Denote by $\tilde{E}_1$ the strict transform of $E_1$ via $\tau_2$ and let $P_3$ be the point $E_2 \cap \tilde{E}_1$.

(a) Show that, for each point $P \in E_2$, $P \neq P_3$, there exists a plane conic $C$ such that the strict transform of $C$ via $\tau_1 \circ \tau_2$ passes through $P$.

(b) Show that there is no plane conic whose strict transform via $\tau_1 \circ \tau_2$ passes through $P_3$.

One says that $P_3$ is proximate to $P_1$ and $P_3$ is infinitely near of order 2 to $P_1$, so $P_3$ is an example of a \textit{satellite point} (to $P_1$).

\textbf{Notation.} Let $\phi = \tau_1 \circ \cdots \circ \tau_n : S = S_n \to S_0 = \mathbb{P}^2$ be a sequence of blowing-ups $\tau_k : S_k \to S_{k-1}$ at a single point $P_k \in S_{k-1}$. For each $k = 1, \ldots, n$, denote by

- $E_k \subset S_k$ the exceptional curve of $\tau_k$,
- $\tilde{E}_k$ the \textit{strict} transform of $E_k$ in $S$ via $\tau_n \circ \cdots \circ \tau_{k+1}$,
- $E_k$ the \textit{total} transform of $E_k$ in $S$ via $\tau_n \circ \cdots \circ \tau_{k+1}$,
- $L$ the total transform in $S$ of a general line in $S_0 = \mathbb{P}^2$.

One says that $P_h$ is \textit{proximate} to $P_k$, and we write $P_h \to P_k$, if $h > k$ and $P_h$ lies on the strict transform of $E_k$ on $S_{k-1}$ via $\tau_{k-1} \circ \cdots \circ \tau_{k+1}$. In particular $P_{k+1} \to P_k$ if $P_{k+1} \in E_k$.

Denote by $Q$ the \textit{proximity matrix}, namely the $n \times n$ matrix whose entries are

$$
q_{ij} = \begin{cases} 
1 & \text{if } P_j \to P_i \\
0 & \text{otherwise.}
\end{cases}
$$

\textbf{Exercise 6.} With notation as above, show that $Q = (q_{ij})$ has the following properties:

(a) each column has at most two non-zero entries;

(b) if $q_{ij} = q_{hi} = 1$ and $h > i$, then $q_{ih} = 1$;

(c) two columns, each one with two non-zero entries, are not equal.

\textbf{Exercise 7.} With notation as above,

(a) show that $\{L, E_1, \ldots, E_n\}$ is a set of generators of $\text{Pic}(S) \cong \mathbb{Z}^{n+1}$,

(b) show that $E_i \cdot E_j = -\delta_{ij}$, for each $i, j = 1, \ldots, n$;

(c) show that $L \cdot L = 1$ and that $L \cdot E_i = 0$, for each $i = 1, \ldots, n$;

(d) compute the $n \times n$ matrix $N = (n_{ij})$ such that

$$
\tilde{E}_i = \sum_{j=1}^{n} n_{ij} E_k
$$

in terms of $Q$; show that $N$ is invertible and compute its inverse in terms of $Q$;
(e) show that also \( \{ L, \tilde{E}_1, \ldots, \tilde{E}_n \} \) is a set of generators of \( \text{Pic}(S) \);
(f) compute the \( n \times n \) matrix \( (\tilde{E}_i \cdot \tilde{E}_j), i, j = 1, \ldots, n, \) in terms of \( Q \).

**Exercise 8.** Let \( C \) be a plane curve of degree \( d \) and \( \tilde{C} \) its strict transform in \( S \).

(a) Show that there exist non-negative integers \( m_1, \ldots, m_n \) such that

\[
\tilde{C} \in |dL - m_1E_1 - \cdots - m_nE_n|
\]

where \( m_k \) is the multiplicity at \( P_k \) of the strict transform of \( C \) in \( S_{k-1}, k = 1, \ldots, n. \)

(b) Show that, for each \( k = 1, \ldots, n, \) one has

\[
m_k \geq \sum_{j: P_j \to P_k} m_j,
\]

that is called the proximity inequality at \( P_k. \)

(c) Find conditions such that the equality holds in (1).

**Exercise 9.** With notation as above, the Enriques weighted graph of \( \tilde{C} \) is defined as the directed graph with vertices \( P_1, \ldots, P_n \) and arrow from \( P_h \) to \( P_k \) if and only if \( P_h \to P_k \), and such that \( m_k \) is the weight at \( P_k, k = 1, \ldots, n. \)

(a) Let \( C_6: (x_1^2 + x_2^2)^3 - 4x_0x_1^2x_2^2 = 0. \) Find \( P_1, P_2, P_3 \) such that \( \tilde{C}_6 \) is smooth and write its Enriques weighted graph.

(b) Let \( C_3: x_0x_1^2 - x_3^3 = 0. \) Find \( n \) and \( P_1, \ldots, P_n \) such that \( \tilde{C}_3 \) is smooth and there is no exceptional curve \( E_k \) which does not meet transversely \( \tilde{C}_3. \) Write the Enriques weighted graph of \( \tilde{C}_3. \)

(c) Do the same for \( C_5: x_0^3x_1^2 - x_5^3 = 0. \)

**Exercise 10.** Consider the plane Cremona map \( \gamma: \mathbb{P}^2 \to \mathbb{P}^2 \) defined by

\[
\gamma([x_0 : x_1 : x_2]) = [x_0x_1^2 : x_1^3 : x_0^3 + x_1^2x_2].
\]

(a) Describe the base locus of \( \gamma, \) i.e. its base points including infinitely near ones.

(b) Show that there is no quadratic plane Cremona map \( \rho \) such that \( \rho \circ \gamma \) is quadratic.

(c) Find three quadratic plane Cremona maps \( \sigma_1, \sigma_2, \sigma_3 \) such that \( \gamma = \sigma_1 \circ \sigma_2 \circ \sigma_3. \) Which types of quadratic transformations do you find?

**Exercise 11.** Compute all solutions \( (d; m_1, \ldots, m_n) \) to Noether’s equations

\[
\sum_{k=1}^n m_k = 3(d-1), \quad \sum_{k=1}^n m_k^2 = d^2 - 1,
\]

for \( 2 \leq d \leq 7. \) How many of them do not satisfy Hudson’s test?

**Exercise 12.** Let \( \gamma \) be a plane De Jonquières map of degree \( d > 2 \) and let \( P \) be the base point of multiplicity \( d-1 \) of the homaloidal net \( \mathcal{L}_\gamma. \)

(a) Show that an infinitely near base point of \( \mathcal{L}_\gamma \) can be satellite only to \( P. \)

(b) Show that the simplicity of \( \gamma \) is \( (1, 2d-2, s) \) with \( 0 \leq s \leq d-2. \)

(c) Show that the bounds for \( s \) are sharp by producing examples, at least for \( d = 3, 4. \)

(d) When \( s = d-2, \) describe the Enriques weighted graph of the general element of \( \mathcal{L}_\gamma, \) under the assumption that all base points of \( \mathcal{L}_\gamma \) are infinitely near to \( P. \)

(e) Explain how to decompose De Jonquières transformations in quadratic ones.
A.- Let $f$ be a birational transformation of the projective plane and let $m_1, \ldots, m_N$ be the multiplicities of its base points ($m_i \geq 1 \, \forall i \in \mathbb{N}$).

A.1. Assume that $m_i$ is constant, i.e. $\exists m \geq 1$ such that $m_i = m \, \forall 1 \leq i \leq N$. Show that the degree $d$ of $f$ and the multiplicity $m$ satisfy

- $d = 2$, $m = 1$, $N = 3$
- $d = 5$, $m = 2$, $N = 6$
- $d = 8$, $m = 3$, $N = 7$
- $d = 17$, $m = 6$, $N = 8$.

A.2.- Give an example for $d = 2$, $m = 1$, $N = 3$.

A.3.- (i) Consider the system of quintic curves passing through 6 points $x_1, x_2, \ldots, x_6$ of $\mathbb{P}^2$ with multiplicity 2. Show that this linear system has dimension $\geq 2$.

(ii) Assume that no line contains 3 of the $x_i$ and that no conic contains all the $x_i$. Show that this linear system has no fixed component. Show that it determines a homaloidal net, and that it determines a birational transformation with $d = 5$, $m = 2$ and $N = 6$. 
B.- Consider the following birational transformation $f$ of the affine plane $\mathbb{A}^2$: $f(x, y) = (x + 1, xy)$.

B.1.- Compute the $n$-th iterate $f^n$ of $f$ and its degree.

B.2.- Show that $f$ contracts $\{x = 0\}$ onto the point $(1, 0)$. Show that $f^n$ contracts $\{x = -i\}$ onto $(n-i, 0)$ for every $i \in \{0, \ldots, n-1\}$.

B.3.- Show that the curves $y = 0$ and $y = \infty$ (in $\mathbb{P}^2 \times \mathbb{P}^1$) are $f$-invariant. List all $f$-invariant curves in $\mathbb{P}^2 \times \mathbb{P}^1$ (i.e., $f$ is viewed as a birational transformation of the compactification $\mathbb{P}^2 \times \mathbb{P}^1$ of $\mathbb{A}^2$).

B.4.- The transformation $f$ permutes the lines $\{x = c^t\}$. Show that $f$ does not preserve any other pencil of curves.

B.5.- Solve the equation

$$x \cdot a(x+1) = (x+b) \cdot a(x)$$

for $a(x) \in k(x)$ (where $k$ is a field and $k \in \mathbb{Z}$).

B.6.- Assume that $g \in \text{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ commutes to $f$, i.e., $g \circ f = f \circ g$. From B4, deduce that

$$g(x, y) = (x + t, \frac{a(x)y + b(x)}{c(x)y + d(x)})$$
C. - Consider the automorphism \( f_\varepsilon \) of the affine plane \( \mathbb{C} \) defined by
\[
  f_{\varepsilon_}(x, y) = (y, x + y^2 + \varepsilon)
\]
with \( \varepsilon = 0 \) or \( 1 \).

C.1. Show that degree \( (f_{\varepsilon}^m) = 2^m \).

C.2. Can you list the periodic points of \( f \) with coordinate in \( \mathbb{R} \)?

C.3. - Extend \( f_{\varepsilon} \) to a birational transformation of \( \mathbb{P}^2 \).
Show that the point \( q \) with coordinates
\[
  [x:y:z] = [0:1:0]
\]
is a fixed point of \( f_{\varepsilon} \).
Show that this is a "super attracting" fixed point.

C.4 (Difficult). - Show that \( f_{\varepsilon} \) is not conjugate to a regular automorphism of a complex projective surface.

D. - Compute the degrees \( d_i(\sigma) \), \( 0 \leq i \leq 4 \), of the birational transformation \( \sigma \) \( [x_0: \ldots: x_4] = [\frac{1}{x_0}: \ldots: \frac{1}{x_4}] \) of \( \mathbb{P}^4 \). (recall that \( d_i(\sigma) \) is the degree of the strict transform of a linear subspace of codimension \( i \)).
3. MINI-COURSE 3 - STÉPHANE LAMY - POLYNOMIAL AUTOMORPHISMS

Exercise 1. Let \( n \geq 1 \). Find a bijective polynomial map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) that is not a polynomial automorphism. Then meditate on that for a minute or two.

Exercise 2. Let \( k \) be the field with two elements. Your first instinct might be that \( \text{Aut}(\mathbb{A}_k^2) \) is a finite group. However, the exercise is to show that \( \text{Aut}(\mathbb{A}_k^2) \) contains a free group over two generators!

Exercise 3. (1) Let \( T \) be a tree, and \( f, g \) two elliptic isometries of \( T \) without a common fixed point. Prove that \( \langle f, g \rangle = \langle f \rangle \ast \langle g \rangle \).

(2) Let \( k \) be your favourite field. Find an example of two involutions \( f, g \in \text{Aut}(\mathbb{A}_k^2) \) such that you can apply the previous question to the action on the Bass-Serre tree.

Exercise 4. Let \( E = \{ (x, y) \mapsto (x + P(y), y) \mid P \in k[X] \} \) be the elementary group, and let \( E_\lambda \) be the conjugate of \( E \) by \( a_\lambda: (x, y) \mapsto (\lambda x + y, x) \), where \( \lambda \in k \). Prove that the subgroup of \( \text{Aut}(\mathbb{A}_2^2) \) generated by the \( E_\lambda \) is a free product (and observe that if the field \( k \) is uncountable, this is a free product over uncountably many factors...):

\[
\langle E_\lambda \mid \lambda \in k \rangle = \bigast_{\lambda \in k} E_\lambda \subset \text{Aut}(\mathbb{A}_2^2) \subset \text{Bir}(\mathbb{P}^2).
\]

Exercise 5. Let \( k \) be your favourite field again. Give an explicit example of a polynomial automorphism \( g \in \text{Aut}(\mathbb{A}_k^2) \) with exactly 8 base points.

Exercise 6. For this exercise we work over an algebraically closed field. We know by the classical theorem of Noether & Castelnuovo that any birational self-maps \( g \) of \( \mathbb{P}^2 \) can be decomposed as a product of quadratic maps, each of them with three proper base points (that is, no infinitely near base point). Moreover, any polynomial automorphism of \( \mathbb{A}^2 \) can be naturally extended as a birational map of \( \mathbb{P}^2 \). The question is: what is is the minimal number of such quadratic maps that you will need to factorize the polynomial automorphism \( g: (x, y) \mapsto (x + y^3, y) \)?