CREMONA
CONFERENCE
BASEL 2016

Exercises

1. Mini-Course 1 - Alberto CALABRI - Introduction to plane Cremona maps

Exercise 1. Consider the rational map $\alpha\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ given by

$$\alpha([x_0 : x_1 : x_2]) = [x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2]$$

and call S the image of this map.

(a) Determine the fundamental points of α.
(b) Choose your favourite line L_1 passing through $P = [1 : 0 : 0]$. Show that the image of L_1 via α is a line in $\mathbb{P}^4$.
(c) Do the same for each line passing through P.
(d) Choose your favourite line L_2 not passing through P. Show that the image of L_2 via α is a conic in $\mathbb{P}^4$.
(e) Do the same for each line not passing through P.
(f) Show that S is isomorphic to the blow up of $\mathbb{P}^2$ at P.
(g) Find the equations in $\mathbb{P}^4$ of the exceptional curve E of S.

Notation. Denote by σ the standard quadratic map $\sigma([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1]$.

Exercise 2. Let $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, $P_3 = [1 : 1 : 1]$ in $\mathbb{P}^2$.

(a) Define a quadratic plane Cremona map γ with fundamental points P_1, P_2, P_3.
(b) Find linear maps α, β such that $\gamma = \alpha \circ \sigma \circ \beta$.
(c) Find, it it exists, γ as above such that γ is an *involution*, i.e. $\gamma^{-1} = \gamma$.

One says that a quadratic Cremona map with three fundamental points is *of the first type*.

Exercise 3. Consider the rational map $\gamma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

$$\gamma([x_0 : x_1 : x_2]) = [x_2^2 : x_0x_1 : x_1x_2].$$

(a) Compute the fundamental points of γ.
(b) Show that γ is birational by computing its inverse γ^{-1}.
(c) Find an open subset of $\mathbb{P}^2$ where γ is an isomorphism.
(d) Describe γ as the composition of blowing-ups and blowing-downs.

(e) Find two quadratic maps σ_1, σ_2 of the first type such that $\gamma = \sigma_1 \circ \sigma_2$.
(f) Find three linear maps $\alpha_1, \alpha_2, \alpha_3$ such that $\gamma = \alpha_1 \circ \sigma \circ \alpha_2 \circ \sigma \circ \alpha_3$.

One says that γ is a quadratic plane Cremona map of the *second type.*

Exercise 4. Consider the birational map $\gamma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

$$\gamma([x_0 : x_1 : x_2]) = [x_1x_2 : x_2^2 - x_0x_1 : x_1^2].$$

(a) Compute the fundamental points of γ.
(b) Find an open subset of $\mathbb{P}^2$ where γ is an isomorphism.
(c) Describe γ as the composition of blowing-ups and blowing-downs.
(d) Find two quadratic maps σ_1, σ_2 of the second type such that $\gamma = \sigma_1 \circ \sigma_2$.
(e) Write γ as the composition of σ and linear maps. How many σ's do you use?

One says that γ is a quadratic plane Cremona map of the *third type.*

Exercise 5. Let $P_1 \in \mathbb{P}^2$ be a point and $\tau_1\colon S_1 \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at P_1, with exceptional curve E_1. Let $P_2 \in S_1$ be a point of E_1 and $\tau_2\colon S_2 \to S_1$ be the blowing up of S_1 at P_2, with exceptional curve E_2. Denote by $\tilde{E}_1$ the strict transform of E_1 via τ_2 and let P_3 be the point $E_2 \cap \tilde{E}_1$.

(a) Show that, for each point $P \in E_2$, $P \neq P_3$, there exists a plane conic C such that the strict transform of C via $\tau_1 \circ \tau_2$ passes through P.
(b) Show that there is no plane conic whose strict transform via $\tau_1 \circ \tau_2$ passes through P_3.

One says that P_3 is proximate to P_1 and P_3 is infinitely near of order 2 to P_1, so P_3 is an example of a *satellite* point (to P_1).

Notation. Let $\phi = \tau_1 \circ \cdots \circ \tau_n\colon S = S_n \to S_0 = \mathbb{P}^2$ be a sequence of blowing ups $\tau_k\colon S_k \to S_{k-1}$ at a single point $P_k \in S_{k-1}$. For each $k = 1, \dots, n$, denote by

- $E_k \subset S_k$ the exceptional curve of τ_k,
- $\tilde{E}_k$ the *strict* transform of E_k in S via $\tau_n \circ \cdots \circ \tau_{k+1}$,
- $\overline{E_k}$ the *total* transform of E_k in S via $\tau_n \circ \cdots \circ \tau_{k+1}$.
- L the total transform in S of a general line in $S_0 = \mathbb{P}^2$.

One says that P_h *is proximate to* P_k, and we write $P_h \to P_k$, if $h > k$ and P_h lies on the strict transform of E_k on S_{h-1} via $\tau_{h-1} \circ \cdots \circ \tau_{k+1}$. In particular $P_{k+1} \to P_k$ if $P_{k+1} \in E_k$.

Denote by Q the *proximity matrix*, namely the $n \times n$ matrix whose entries are

$$q_{ij} = \begin{cases} 1 & \text{if } P_j \to P_i \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6. With notation as above, show that $Q = (q_{ij})$ has the following properties:

(a) each column has at most two non-zero entries;
(b) if $q_{ij} = q_{hj} = 1$ and $h > i$, then $q_{ih} = 1$;
(c) two columns, each one with two non-zero entries, are not equal.

Exercise 7. With notation as above,

(a) show that $\{L, \overline{E_1}, \dots, \overline{E_n}\}$ is a set of generators of $\mathrm{Pic}(S) \cong \mathbb{Z}^{n+1}$,
(b) show that $\overline{E_i} \cdot \overline{E_j} = -\delta_{ij}$, for each $i, j = 1, \dots, n$;
(c) show that $L \cdot L = 1$ and that $L \cdot \overline{E_i} = 0$, for each $i = 1, \dots, n$;
(d) compute the $n \times n$ matrix $N = (n_{ij})$ such that

$$\tilde{E}_i = \sum_{j=1}^{n} n_{ij} \overline{E_k}$$

in terms of Q; show that N is invertible and compute its inverse in terms of Q;

(e) show that also $\{L, \tilde{E}_1, \ldots, \tilde{E}_n\}$ is a set of generators of $\mathrm{Pic}(S)$;
(f) compute the $n \times n$ matrix $(\tilde{E}_i \cdot \tilde{E}_j)$, $i, j = 1, \ldots, n$, in terms of Q.

Exercise 8. Let C be a plane curve of degree d and $\tilde{C}$ its strict transform in S.

(a) Show that there exist non-negative integers $m_1, \ldots, m_n$ such that

$$\tilde{C} \in |dL - m_1\overline{E_1} - \cdots - m_n\overline{E_n}|$$

where m_k is the multiplicity at P_k of the strict transform of C in S_{k-1}, $k = 1, \ldots, n$.
(b) Show that, for each $k = 1, \ldots, n$, one has

$$m_k \geq \sum_{j\colon P_j \to P_k} m_j, \tag{1}$$

that is called the *proximity inequality* at P_k.
(c) Find conditions such that the equality holds in (1).

Exercise 9. With notation as above, the *Enriques weighted graph* of $\tilde{C}$ is defined as the directed graph with vertices $P_1, \ldots, P_n$ and arrow from P_h to P_k if and only if $P_h \to P_k$, and such that m_k is the weight at P_k, $k = 1, \ldots, n$.

(a) Let $C_6\colon (x_1^2 + x_2^2)^3 - 4x_0^2x_1^2x_2^2 = 0$. Find P_1, P_2, P_3 such that $\tilde{C}_6$ is smooth and write its Enriques weighted graph.
(b) Let $C_3\colon x_0x_1^2 - x_2^3 = 0$. Find n and $P_1, \ldots, P_n$ such that $\tilde{C}_3$ is smooth and there is no exceptional curve E_k which does not meet transversely $\tilde{C}_3$. Write the Enriques weighted graph of $\tilde{C}_3$.
(c) Do the same for $C_5\colon x_0^3x_1^2 - x_2^5 = 0$.

Exercise 10. Consider the plane Cremona map $\gamma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$\gamma([x_0 : x_1 : x_2]) = [x_0x_1^2 : x_1^3 : x_0^3 + x_1^2x_2].$$

(a) Describe the base locus of γ, i.e. its base points including infinitely near ones.
(b) Show that there is no quadratic plane Cremona map ρ such that $\rho \circ \gamma$ is quadratic.
(c) Find three quadratic plane Cremona maps $\sigma_1, \sigma_2, \sigma_3$ such that $\gamma = \sigma_1 \circ \sigma_2 \circ \sigma_3$. Which types of quadratic transformations do you find?

Exercise 11. Compute all solutions $(d; m_1, \ldots, m_n)$ to Noether's equations

$$\sum_{k=1}^{n} m_k = 3(d-1), \qquad \sum_{k=1}^{n} m_k^2 = d^2 - 1,$$

for $2 \leq d \leq 7$. How many of them do not satisfy Hudson's test?

Exercise 12. Let γ be a plane De Jonquières map of degree $d > 2$ and let P be the base point of multiplicity $d-1$ of the homaloidal net $\mathcal{L}_\gamma$.

(a) Show that an infinitely near base point of $\mathcal{L}_\gamma$ can be satellite only to P.
(b) Show that the *simplicity* of γ is $(1, 2d-2, s)$ with $0 \leq s \leq d-2$.
(c) Show that the bounds for s are sharp by producing examples, at least for $d = 3, 4$.
(d) When $s = d-2$, describe the Enriques weighted graph of the general element of $\mathcal{L}_\gamma$, under the assumption that all base points of $\mathcal{L}_\gamma$ are infinitely near to P.
(e) Explain how to decompose De Jonquières transformations in quadratic ones.

2. Mini-Course 2 - Serge CANTAT - Examples of birational transformations

A.- Let f be a birational transformation of the projective plane and let $m_1, \dots, m_N$ be the multiplicities of its base points ($m_i \geq 1$, $\forall 1 \leq i \leq N$).

A.1. Assume that m_i is constant, i.e. $\exists\, m \geq 1$ such that $m_i = m$ $\forall\, 1 \leq i \leq N$. Show that the degree d of f and the multiplicity m satisfy

- $d = 2$, $m = 1$, $N = 3$

or
- $d = 5$, $m = 2$, $N = 6$

or
- $d = 8$, $m = 3$, $N = 7$

or
- $d = 17$, $m = 6$, $N = 8$.

A.2.- Give an example for $d = 2$, $m = 1$, $N = 3$.

A.3.- (i) Consider the system of quintic curves passing through 6 points $x_1, x_2, \dots, x_6$ of $\mathbb{P}^2$ with multiplicity 2. Show that this linear system has dimension $\geqslant 2$.

(ii) Assume that no line contains 3 of the x_i and that no conic contains all the x_i. Show that this linear system has no fixed component. Show that it determines a homaloïdal net, and that it determines a birational transformation with $d = 5$, $m = 2$ and $N = 6$.

B.- Consider the following birational transformation f of the affine plane $\mathbb{A}^2$: $\quad f(x,y) = (x+1, xy)$.

B.1.- Compute the n-th iterate f^n of f and its degree.

B.2.- Show that f contract $\{x=0\}$ onto the point $(1,0)$. Show that f^n contracts $\{x=-i\}$ onto $(n-i, 0)$ for every $i \in \{0, \dots, n-1\}$.

B.3.- Show that the curves $y=0$ and $y=\infty$ (in $\mathbb{P}^1 \times \mathbb{P}^1$) are f-invariant. List all f-invariant curves in $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e. f is viewed as a birational transformation of the compactification $\mathbb{P}^1 \times \mathbb{P}^1$ of $\mathbb{A}^2$).

B.4.- The transformation f permutes the lines $\{x = c^{st}\}$. Show that f does not preserve any other pencil of curves.

B.5.- Solve the equation

$$x \quad a(x+1) = (x+k)\, a(x)$$

for $a(x) \in \underline{k}(x)$ (here $\underline{k}$ is a field and $k \in \mathbb{Z}$).

B.6.- Assume that $g \in \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ commutes to f, i.e. $g \circ f = f \circ g$. From B4, deduce that

$$g(x,y) = \left(x + t,\ \frac{a(x)\, y + b(x)}{c(x)\, y + d(x)}\right)$$

C.– Consider the automorphism f_ε of the affine plane$/\mathbb{C}$ defined by

$$f_\varepsilon(x,y) = (y\ ,\ x + y^2 + \varepsilon)$$

with $\varepsilon = 0$ or 1.

C.1. Show that degree $(f_\varepsilon^n) = 2^n$.

C.2. Can you list the periodic points of f with coordinates in $\mathbb{R}$.

C.3.– Extend f_ε to a birational transformation of $\mathbb{P}^2$. Show that the point q with coordinates $[x:y:z] = [0:1:0]$ is a fixed point of f_ε. Show that this is a "super attracting" fixed point.

C.4 (Difficult).– Show that f_ε is not conjugate to a regular automorphism of a complex projective surface.

D.– Compute the degrees $d_i(\sigma)$, $0 \leq i \leq 4$, of the birational transformation $\sigma[x_0 : \dots : x_4] = [\frac{1}{x_0} : \dots : \frac{1}{x_4}]$ of $\mathbb{P}^4$. (recall that $d_i(\sigma)$ is the degree of the strict transform of a linear subspace of codimension i).

3. Mini-Course 3 - Stéphane LAMY - Polynomial automorphisms

Exercise 1. Let $n \geq 1$. Find a bijective polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is not a polynomial automorphism. Then meditate on that for a minute or two.

Exercise 2. Let $\mathbf{k}$ be the field with two elements. Your first instinct might be that $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ is a finite group. However, the exercise is to show that $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ contains a free group over two generators!

Exercise 3.
(1) Let T be a tree, and f, g two elliptic isometries of T without a common fixed point. Prove that $\langle f, g\rangle = \langle f\rangle * \langle g\rangle$.
(2) Let $\mathbf{k}$ be your favourite field. Find an example of two involutions $f, g \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ such that you can apply the previous question to the action on the Bass-Serre tree.

Exercise 4. Let $E = \{(x, y) \mapsto (x + P(y), y) \mid P \in \mathbf{k}[X]\}$ be the elementary group, and let E_λ be the conjugate of E by $a_\lambda \colon (x, y) \to (\lambda x + y, x)$, where $\lambda \in \mathbf{k}$. Prove that the subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ generated by the E_λ is a free product (and observe that if the field $\mathbf{k}$ is uncountable, this is a free product over uncountably many factors...):

$$\langle E_\lambda \mid \lambda \in \mathbf{k}\rangle = \mathop{\ast}_{\lambda\in k} E_\lambda \subset \mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Bir}(\mathbb{P}^2).$$

Exercise 5. Let $\mathbf{k}$ be your favourite field again. Give an explicit example of a polynomial automorphism $g \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ with exactly 8 base points.

Exercise 6. For this exercise we work over an algebraically closed field. We know by the classical theorem of Noether & Castelnuovo that any birational self-maps g of $\mathbb{P}^2$ can be decomposed as a product of quadratic maps, each of them with three proper base points (that is, no infinitely near base point). Moreover, any polynomial automorphism of $\mathbb{A}^2$ can be naturally extended as a birational map of $\mathbb{P}^2$. The question is: what is is the minimal number of such quadratic maps that you will need to factorize the polynomial automorphism $g \colon (x, y) \mapsto (x + y^3, y)$?