INTRODUCTION TO BIRATIONAL GEOMETRY OF SURFACES
(PRELIMINARY VERSION)

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(Very) quick introduction

Let us recall some classical notions of algebraic geometry that we will need. We recommend to the reader to read the first chapter of Hartshorne [Har77] or another book of introduction to algebraic geometry, like [Rei88] or [Sha94].

We fix a ground field $\mathbf{k}$. All results of the first 4 sections work over any algebraically closed field, and a few also on non-closed fields. In the last section, the case of all perfect fields will be discussed. For our purpose, the characteristic is not important.

0.1. **Affine varieties.** The affine n-space $\mathbb{A}^n_{\mathbf{k}}$, or simply $\mathbb{A}^n$, is the set of n-tuples of elements of $\mathbf{k}$. Any point $x \in \mathbb{A}^n$ can be written as $x = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in \mathbf{k}$ are the coordinates of x.

An *algebraic set* $X \subset \mathbb{A}^n$ is the locus of points satisfying a set of polynomial equations:

$$X = \left\{(x_1, \ldots, x_n) \in \mathbb{A}^n \;\middle|\; f_1(x_1, \ldots, x_n) = \cdots = f_k(x_1, \ldots, x_n) = 0\right\}$$

where each $f_i \in \mathbf{k}[x_1, \ldots, x_n]$. We denote by $I(X) \subset \mathbf{k}[x_1, \ldots, x_n]$ the set of polynomials vanishing along X, it is an ideal of $\mathbf{k}[x_1, \ldots, x_n]$, generated by the f_i. We denote by $k[X]$ or $\mathcal{O}(X)$ the set of algebraic functions $X \to \mathbf{k}$, which is equal to $\mathbf{k}[x_1, \ldots, x_n]/I(X)$.

An algebraic set X is said to be *irreducible* if any writing $X = Y \cup Z$ where Y, Z are two algebraic sets implies that $Y = X$ or $Z = X$. Note that X is irreducible $\Leftrightarrow I(X) \subset \mathbf{k}[x_1, \ldots, x_n]$ is a prime ideal $\Leftrightarrow \mathbf{k}[X]$ is integral. If X is irreducible, we denote by $\mathbf{k}(X)$ the set of *rational functions* $X \dashrightarrow \mathbf{k}$, which is the field of fractions of $\mathbf{k}[X]$.

There is a topology on $\mathbb{A}^n$ whose closed subsets are the algebraic sets, it is called the *Zariski topology.*

An *affine algebraic variety* (or simply affine variety) is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is a *quasi-affine variety.*

0.2. **Projective varieties.** The projective n-space $\mathbb{P}^n_{\mathbf{k}}$, or simply $\mathbb{P}^n$, is the set of $n{+}1$-tuples $(x_0, \ldots, x_n) \in \mathbf{k}^{n+1} \setminus \{0\}$ modulo the equivalence relation $(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$ for any $\lambda \in \mathbf{k}^*$. Any point $x \in \mathbb{P}^n$ (or equivalence class) can be written as $x = (x_0 : \cdots : x_n)$, where $x_0, \ldots, x_n \in \mathbf{k}$ are the homogeneous coordinates of x.

An *algebraic set* $X \subset \mathbb{P}^n$ is the locus of points satisfying a set of polynomial equations:

$$X = \left\{(x_0 : \cdots : x_n) \in \mathbb{P}^n \;\middle|\; f_1(x_0, \ldots, x_n) = 0, \ldots, f_k(x_0, \ldots, x_n) = 0\right\}.$$

where each $f_i \in \mathbf{k}[x_0, \ldots, x_n]$ is an *homogeneous* polynomial. Again, X is said to be *irreducible* if any writing $X = Y \cup Z$ where Y, Z are two algebraic sets implies that $Y = X$ or $Z = X$. The Zariski topology on $\mathbb{P}^n$ is the topology whose closed subsets are the algebraic sets.

A *projective algebraic variety* (or simply projective variety) is an irreducible closed subset of $\mathbb{P}^n$. An open subset of a projective variety is a *quasi-projective variety*. Note that since $\mathbb{A}^n$ is open in $\mathbb{P}^n$, affine varieties are quasi-projective.

0.3. **Regular and rational functions.** We already defined the ring $\mathbf{k}[X]$ of functions of X for an affine algebraic variety, and its quotient field $\mathbf{k}(X)$, which is the field of rational functions on X. Let us extend this definition to quasi-projective varieties.

When $X \subset \mathbb{P}^n$ is a quasi-projective variety, we define $\mathbf{k}[X]$ as the set of functions $f\colon X \to \mathbf{k}$ such that for any point $x \in X$ there is an open subset $U \subset X$ which contains x, and two homogeneous polynomials $P, Q \in \mathbf{k}[x_0, \ldots, x_n]$ with Q not vanishing at any point of U, such that $f = P/Q$ on U. This is clearly a $\mathbf{k}$-algebra. Note that $\mathbf{k}[X]$ is also often written $\mathcal{O}(X)$.

We define $\mathbf{k}(X)$ as the set of *rational functions* $X \dashrightarrow \mathbf{k}$, which is a set of equivalence classes of pairs (U, f) where $U \subset X$ is an open subset and $f \in \mathbf{k}[U]$. Two pairs (U, f) and (V, g) are equivalent if $f = g$ on $U \cap V$. Note that $\mathbf{k}(X)$ is a field, which is the field of fractions of $\mathbf{k}[X]$ when X is affine but not in general. For instance $\mathcal{O}(\mathbb{P}^n) = \mathbf{k}[\mathbb{P}^n] = \mathbf{k}$, but $\mathbf{k}(\mathbb{P}^n) = \mathbf{k}(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$.

Let X be a quasi-projective variety and let $x \in X$ be a point. We define $\mathcal{O}_{p,X}$ as the set of equivalence classes of pairs (U, f), where $U \subset X$ is an open subset $x \in U$, and $f \in \mathbf{k}[U]$.

1. Divisors, Linear systems, Blow-ups and Intersection theory

1.1. **Divisors.** Let X be an algebraic variety. A *prime divisor on* X is simply an irreducible closed subset of X of codimension 1.

(1) If X is a curve and $\mathbf{k}$ is algebraically closed, the prime divisors of X corresponds to the points of X. In the case $\mathbf{k} = \mathbb{R}$, any prime divisor on a curve X can be a real point or the union of two imaginary conjugate points.
(2) If X is a surface, the prime divisors of X are the irreducible curves that lie on it.
(3) If $X = \mathbb{P}^n$, prime divisors are given by an irreducible homogeneous polynomial of some fixed degree, and have dimension $n-1$.

For instance, on a a surface the prime divisors are the irreducible curves that lie on the surface.

A *Weil divisor* on X is a formal finite sum of prime divisor with integer coefficients. The set of all Weil divisors is written $\mathrm{Div}(X)$, and is a discrete abelian group.

$$\mathrm{Div}(X) = \left\{ \sum_{i=1}^{m} a_i D_i \;\middle|\; m \in \mathbb{N}, a_i \in \mathbb{Z}, D_i \text{ is a prime divisor of } X \text{ for } i = 1, \ldots, m \right\}.$$

We say that a divisor is *effective* if all coefficients are positive. It corresponds to say that it is the divisor of an hypersurface, in general reducible. We say that two effective divisors D, D' *have a common component* if there exists a prime divisor P such that $D - P$ and $D' - P$ are effective. This means that the coefficient in P is positive for both D, D', and is also equivalent to say that the two hypersurfaces defined by D, D' have a common component.

We associate to any rational function $f \in \mathbf{k}(X)^*$ a divisor $\mathrm{div}(f) \in \mathrm{Div}(X)$. It is equal to

$$\mathrm{div}(f) = \sum \nu_f(D)$$

where the sum is taken over all prime divisors, and where $\nu_f(D)$ is the multiplicity of f at D. This multiplicity is $k > 0$ if f vanishes on D at the order k, is $-k < 0$ if f has a pole of order k on D, and is 0 otherwise. Note that $\mathrm{div}(f) \in \mathrm{Div}(X)$ since $\nu_f(D)$ is equal to 0 for all D except a finite number. We say that divisors obtained like this are *principal divisors*.

Observe that $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ for any $f, g \in \mathbf{k}(X)^*$, so the set of principal divisors is a subgroup of $\mathrm{Div}(X)$.

1.2. **Equivalence of divisors.** Let X be an algebraic variety. We say that two divisors D, D' on X are *linearly equivalent* if $D - D'$ is a principal divisor, i.e. equal to $\mathrm{div}(f)$ for some $f \in \mathbf{k}(X)^*$. This is an equivalence class, which corresponds to take the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors. Usually, this quotient is called $\mathrm{Cl}(X)$. When X is smooth, it canonically isomorphic to $\mathrm{Pic}(X)$, so we will often make an abuse of notation and speak of $\mathrm{Pic}(X)$ instead of $\mathrm{Cl}(X)$.

Lemma 1.1. *Let $D, D' \in \mathrm{Div}(X)$ be two effective divisors which have no common component. The following conditions are equivalent:*
(i) D and D' are linearly equivalent.

(ii) there exists a rational map $\pi\colon X \dashrightarrow \mathbb{P}^1$ *such that* D *and* D' *correspond respectively to the preimage of* $(0:1)$ *and* $(1:0)$.

Proof. Suppose first that π exists. Denoting by g the rational function $\mathbb{P}^1 \dashrightarrow \mathbf{k}$ given by $(x:y) \dashrightarrow x/y$, the rational function $f = f \circ \pi$ has poles at D' and zeros at D. So $D - D' = \mathrm{div}(f)$, which shows that D and D' are linearly equivalent.

Suppose now that D and D' are linearly equivalent, which means that there exists $h \in \mathbf{k}(X)^*$ such that $\mathrm{div}(f) = D - D'$. Since D and D' have no common component, the zeros of f are D and the poles are D'. □

Lemma 1.2. $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}H$, *where* H *is the divisor of an hyperplane. Any hypersurface of degree* d *is linearly equivalent to* dH.

Proof. We define an homorphism of groups $\mathrm{Div}(\mathbb{P}^n) \to \mathbb{Z}$, by sending all prime divisors of degree d on the integer d.

Let $D = \sum a_i P_i$ be a divisor in the kernel, where each P_i is a prime divisor given by an homogeneous polynomial $F_i \in \mathbf{k}[x_0, \ldots, x_n]$ of some degree d_i. We define $f = \prod F_i^{a_i}$, which belongs to $\mathbf{k}(\mathbb{P}^n)^*$ since $\sum a_i d_i = 0$. By construction, $\mathrm{div}(f) = D$, so D is a principal divisor.

Conversely, any principal divisor is equal to $\mathrm{div}(f)$, where $f = F/G$ for some homogeneous polynomials F, G of the same degree. In consequence, any principal belongs to the kernel.

Since $\mathrm{Pic}(\mathbb{P}^n)$ is the quotient of $\mathrm{Div}(\mathbb{P}^n)$ by the subgroup of principal divisors, we naturally get an isomorphism $\mathrm{Pic}(\mathbb{P}^n) \to \mathbb{Z}$ by restricting the homorphism above to the quotient. An hyperplane being sent on 1, we get the result. □

1.3. **Linear systems.** Let X be a variety and let D be a divisor on X. We define

$$\mathcal{L}(D) = \{f \in \mathbf{k}(X)^* \mid \mathrm{div}(f) + D \text{ is effective}\} \subset \mathbf{k}(X).$$

Note that if $\mathcal{L}(D)$ is not empty, then $\mathcal{L}(D) \cup \{0\}$ is a $\mathbf{k}$-vector space. Moreover, it has finite dimension (see [Sha94, p.173] or the notes of S. Lamy), so we can identify it with $\mathbf{k}^{n+1}$ for some integer n. We define $|D|$ to be the set of effective divisors which are linearly equivalent to D. Clearly, $|D|$ is non-empty if and only $\mathcal{L}(D)$ is not empty. In this case, there is a surjection $\mathcal{L}_D \to |D|$ which sends f onto $\mathrm{div}(f) + D$. Moreover, two functions f, g have the same image if and only $f/g \in \mathbf{k}^*$. This shows that $|D|$ is parametrised by $\mathbb{P}^n$. We say that $|D|$ is a *complete linear system.* In general, a linear system is a linear subsystem of a complete linear system. This means that here we take a linear subspace $\mathbb{P}^m \subset \mathbb{P}^m$ and take the image of it by the parametrising map. A point $x \in X$ is said to be a base-point of a linear system if all members of the linear system pass through this point. We can moreover associate to this point a number, which is *the multiplicity of the linear system at* x. This is the least multiplicity of all members of the system at x, or equivalently the multiplicity of a general member of the linear system. A linear system of dimension 1 is called a *pencil.* We say that a linear system has *a fixed component* if there is an hypersurface contained in all members of the system.

1.4. **Rational maps associated to a linear system.** We can also associate to $|D|$ a rational map $X \overset{|D|}{\dashrightarrow} \mathbb{P}^m$, by choosing a parametrisation $\mathbb{P}^m \xrightarrow{\sim} |D|$. To explain this simple construction, recall that any hyperplane of $\mathbb{P}^m$ is given by an equation of the form

$$a_0 x_0 + a_1 x_1 + \cdots + a_m x_m = 0,$$

where the a_i belong to $\mathbf{k}$ and all cannot vanish at the same time. Moreover, multiplying all a_i by the same element of $\mathbf{k}^*$ does not change the hyperplane. In consequence, we can parametrise all hyperplanes by the points $(a_0 : \cdots : a_m)$ of $\mathbb{P}^m$, which is classically called the dual of the $\mathbb{P}^m$ we started with. We will use the parametrisation $\mathbb{P}^m \stackrel{\sim}{\to} |D|$, and identify this $\mathbb{P}^m$ with the set of hyperplanes of $\mathbb{P}^m$ to obtain the map $X \stackrel{|D|}{\dashrightarrow} \mathbb{P}^m$:

Take a point $x \in X$ which is not a base-point of $|D|$. The set of all members of $|D|$ which pass through x have dimension $n-1$; with the parametrisation, we find a linear system of hyperplanes of dimension $n-1$. These intersect into a single point, which is the image of x by the rational map $X \dashrightarrow \mathbb{P}^m$.

Note that another choice of the parametrisation would yield another map to $\mathbb{P}^m$, which differs from the first one only by the composition by an automorphism of $\mathbb{P}^m$ (i.e. an element of $\mathrm{PGL}(m+1, \mathbf{k})$).

1.5. **Very ample divisors.** We will say that a divisor D on X is *very ample* if $|D|$ induces an embedding $X \stackrel{|D|}{\to} \mathbb{P}^m$.

Lemma 1.3. [Har77, II, Proposition 7.3 and Remark 7.8.2] *Let D be a divisor on a smooth variety X, and let $\varphi\colon X \stackrel{|D|}{\dashrightarrow} \mathbb{P}^m$ be the rational map induced.*

A point $x \in X$ is a base-point of $|D|$ if and only if it is a base-point of φ.

The map φ is an embedding (i.e. D is very ample) if and only if the following occur:

(i) $|D|$ has no base-point.

(ii) $|D|$ separates the points (i.e. for any $x, y \in X$ there exists a member of the system which passes through x and not through y).

(iii) $|D|$ separates the tangent vectors (i.e. for any $x \in X$ and any tangent direction at x, there exists a member of the system which passes through x and has not the given direction at x).

1.6. **Hypersurfaces of $\mathbb{P}^n$ of some fixed degree.** The hypersurfaces of $\mathbb{P}^n$ of degree d being all linearly equivalent, they form a linear system; we can write any such hypersurface as

$$\sum a_{i_0,\ldots,i_0} x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n},$$

where the sum is taken over all n-uplets $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + i_1 + \ldots i_n = d$. We usually call this linear system $\mathcal{O}_{\mathbb{P}^n}(d)$.

For instance, the dimension of $\mathcal{O}_{\mathbb{P}^1}(d)$ is d, since any element is of the form

$$\sum_{i=0}^{d} a_0 x_0^i x_1^{d-i}.$$

Similarly, the dimension of $\mathcal{O}_{\mathbb{P}^2}(d)$ is $\frac{(d+1)(d+2)}{2} - 1 = \frac{d(d+3)}{2}$, since any element is of the form

$$\sum_{i=0}^{d} \left(\sum_{j=0}^{i} a_0 x_0^i x_1^j x_2^{d-i-j} \right).$$

Exercise 1.4. *Show that the dimension of $\mathcal{O}_{\mathbb{P}^3}(d)$ is equal to*

$$\frac{(d+1)(d+2)(d+3)}{3}-1=\frac{1}{6}(d^2+6d+11).$$

More generally, prove that the dimension of $\mathcal{O}_{\mathbb{P}^n}(d)$ is equal to

$$\frac{(d+1)(d+2)\dots(d+n)}{n!}-1=\left(\begin{array}{c} d+n \\ n \end{array}\right)-1.$$

Hint: Proceed by induction on the dimension and/or the degree.

Example 1.5. *When $X=\mathbb{P}^n$, the linear system $\mathcal{O}_{\mathbb{P}^n}(d)$ is the strict pull-back of the d-th Veronese embedding $\mathbb{P}^n \to \mathbb{P}^N$ with $N=\left(\begin{array}{c} d+n \\ n \end{array}\right)-1$. For instance, when $n=2$ and $d=2$, the Veronese embedding is*

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ (x:y:z) & \mapsto & (x^2:xy:xz:y^2:yz:z^2). \end{array}$$

Example 1.6. *Take $X=\mathbb{P}^2$ and $X\to\mathbb{P}^2$ given by*

$$(x:y:z)\to(x^2:y^2:z^2).$$

The linear system induced has no base-point, but the map is not an embedding.

Example 1.7. *Take X to be the Fermat cubic surface*

$$X=\{(w:x:y:z)\in\mathbb{P}^3 \mid w^3+x^3+y^3+z^3=0\},$$

and let $D\subset X$ be the line of equation $w+x=0$, $y+z=0$. The morphism $X\overset{|D|}{\to}\mathbb{P}^1$ given by

$$(w:x:y:z)\mapsto\left\{\begin{array}{ll} (w+x:y+z) & \textit{if } w\neq x \textit{ or } y\neq z \\ (w^2-wx+x^2:y^2-yz+z^2) & \begin{array}{l}\textit{if } w^2+wx+x^2\neq 0\\ \textit{or } y^2+yz+z^2\neq 0\end{array} \end{array}\right.$$

gives a linear system on X whose general fibre is a conic (we usually call it a conic bundle).

1.7. **Definition of the blow-up of a point.** Given a point p in a smooth algebraic variety X of dimension n, we say that a morphism $\pi\colon Y\to X$ is a *blow-up of $x\in X$* if Y is a smooth variety, if π restricts to an isomorphism outside of p, i.e. to an isomorphism $Y\backslash\pi^{-1}(p)\to X\backslash\{p\}$, and if $\pi^{-1}(p)\cong\mathbb{P}^{n-1}$.

In fact, we have the following universal property: if $\pi\colon Y\to X$ is a blow-up of $p\in X$ and $\eta\colon Z\to X$ is a birational morphism satisfying that Z is smooth, η

In fact, if $\pi\colon Y\to X$ and $\pi'\colon Y'\to X$ are two blow-ups of the same point p, there exists an isomorphism $\theta\colon Y\to Y'$ such that $\pi=\pi'\theta$. We will thus talk about *the* blow-up of $x\in X$.

Remark 1.8. *Note that π is a birational map, since it is an isomorphism on open subsets. When $n=1$, it is an isomorphism, but when $n\geq 2$ it is not: it contracts the variety $E=\pi^{-1}(p)\cong\mathbb{P}^{n-1}$ onto a point.*

1.8. **The blow-up of a point in the affine plane.** Let us see what is the blow-up of $p=(0,0)$ in the affine plane $\mathbb{A}^2$. We define

$$W=\{\big((x,y),(u:v)\big)\in\mathbb{A}^2\times\mathbb{P}^1 \mid xv=yu\},$$

and observe that the morphism $\pi\colon W\to\mathbb{A}^2$ given by the projection on the first factor is the blow-up of $(0,0)$.

Firstly, $\pi^{-1}(p)$ is equal to $E=\{((0,0),(u:v)) \mid (u:v)\in\mathbb{P}^1\}$ and is thus isomorphic to $\mathbb{P}^1$. Secondly, if $q=(x,y)\in\mathbb{A}^2\backslash\{p\}$, $\pi^{-1}(q)=((x,y),(x:y))\in W\backslash E$, so the morphism π restricts to an isomorphism $W\backslash E\to\mathbb{A}^2\backslash\{p\}$ whose inverse is $(x,y)\mapsto((x,y),(x:y))$.

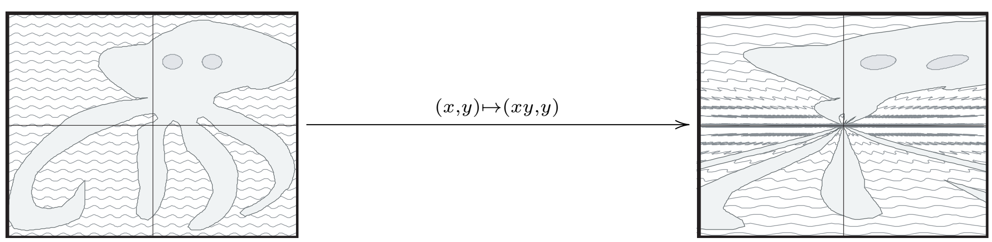
If $r\in\mathbb{A}^2$ is another point, its blow-up can be easily described from this construction. Take an automorphism φ of $\mathbb{A}^2$ which sends $(0,0)$ onto r. The morphism $W\to\mathbb{A}^2$ given by $\varphi\circ\pi$ is a blow-up of $r\in\mathbb{A}^2$.

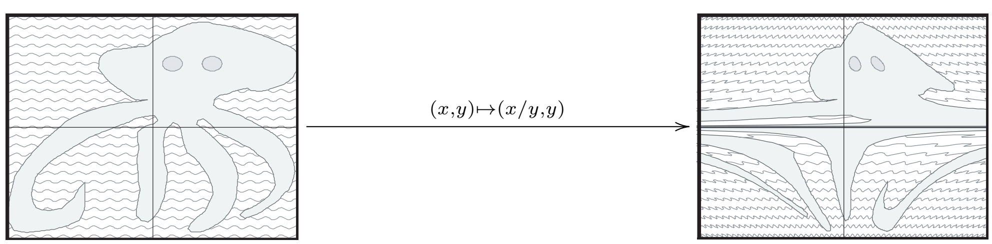
Let us describe the blow-up $\pi\colon W\to\mathbb{A}^2$ in affine charts. We choose two open subsets $U,V\subset W$ where respectively $v\neq 0$ and $u\neq 0$. These two open subsets are isomorphic to $\mathbb{A}^2$ via the maps

$$\begin{array}{ccc}\mathbb{A}^2 & \to & U\\ (y,u) & \mapsto & ((yu,y),(u:1))\end{array}\qquad\qquad \begin{array}{ccc}\mathbb{A}^2 & \to & V\\ (x,v) & \mapsto & ((x,xv),(1:v))\end{array}$$

In local coordinates, we can thus describe the blow-up by

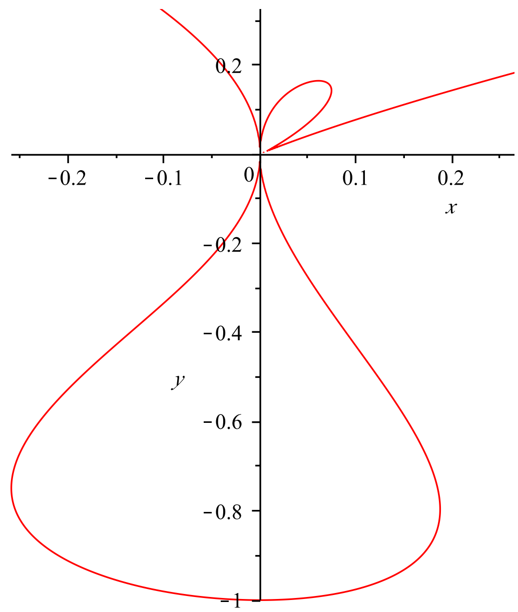
$$\begin{array}{ccc}\mathbb{A}^2 & \to & \mathbb{A}^2\\ (y,u) & \mapsto & (yu,y)\end{array}\qquad\qquad \begin{array}{ccc}\mathbb{A}^2 & \to & \mathbb{A}^2\\ (x,v) & \mapsto & (x,xv)\end{array}$$





Exercise 1.9. *Take the curves $x^{n-1}=y^n$ and $x^2-y^2+2x^3+y^3=0$ in $\mathbb{A}^2$ and blow-up the singular point. Make a picture of the situation, and see how the strict transform of the curve intersect the exceptional curve.*

Exercise 1.10. *Same question for the curve $y^6+x^2y^3+y^7-x^4+2yx^3-x^2y^2=0$, which should look like*



1.9. **The blow-up of a point in $\mathbb{P}^n$.** We define by W the following projective variety:

$W=\{\big((x_0:x_1:\cdots:x_n),(y_1:\cdots:y_n)\big)\in\mathbb{P}^n\times\mathbb{P}^{n-1}\mid x_iy_j=x_jy_i \text{ for } 0\leq i,j\leq n\}$, the map $\pi\colon W\to\mathbb{P}^n$ is the blow-up of $p=(1:0:\cdots:0)\in\mathbb{P}^n$. Indeed, $E=\pi^{-1}(p)=\{p\}\times\mathbb{P}^{n-1}$ and π restricts to an isomorphism $W\backslash E\to\mathbb{P}^n\backslash\{p\}$ whose inverse is $(x_0:\cdots:x_n)\mapsto\big((x_0:\cdots:x_n),(x_1:\cdots:x_n)\big)$.

Take now any smooth projective variety $X\subset\mathbb{P}^n$, which contains p and has dimension $d\leq n$. We define by $\tilde{X}\subset W$ the *strict transform* of X, which is the closure of $\pi^{-1}(X\backslash\{p\})$.

Exercise 1.11. $(\star)$ *Show that $\tilde{X}$ is irreducible and smooth, and that the restriction of π to $\tilde{X}$ is a birational morphism $\tilde{X}\to X$, which is the blow-up of $x\in X$.*

Using this result, we can see that if X is a smooth projective variety, the blow-up of any point in X is again a smooth projective variety. However, the blow-up of an affine variety is neither affine nor projective.

1.10. **Intersection of two curves on a surface.** From now on, all varieties that we will deal with are smooth projective surfaces, often called X,Y,Z, and curves inside, mostly called $C,D,E,\ldots$.

Proposition 1.12. [Har77, Chapter V, Theorem 1.1, page 357] *Let X be a smooth projective surface. There exists an unique bilinear symmetric form (called* intersection form*)*

$$\begin{array}{ccc}\mathrm{Div}(X)\times\mathrm{Div}(X) & \to & \mathbb{Z}\\ (C,D) & \mapsto & C\cdot D\end{array}$$

having the following properties

(1) *If C and D are smooth curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of point of $C \cap D$*
(2) *If C, C' are linearly equivalent, then $C \cdot D = C' \cdot D$.*

In particular, this yields an intersection form

$$\begin{array}{ccc} \mathrm{Pic}(X) \times \mathrm{Pic}(X) & \to & \mathbb{Z} \\ (C, D) & \mapsto & C \cdot D. \end{array}$$

Example 1.13. *Take $X = \mathbb{P}^2$, the intersection form is given by the following: if C, D are two curves of degree m and n, $C \cdot D = m \cdot n$. Indeed, recall that* $\mathrm{Pic}(X) = \mathbb{Z}L$*, where L is the divisor of a line* (*Lemma* 1.2)*. The fact that $L \cdot L = 1$ follows from the fact that two distinct lines intersect into one point and are linearly equivalent. This shows that $C \cdot D = (mL) \cdot (nL) = mn$.*

Let us recall what is the *multiplicity of a curve at a point.* If $C \subset X$ is a curve and $p \in X$ is a point, we can define the multiplicity $m_p(C)$ of C at p. Taking a local equation f of C, it can be defined as the integer k such that $f \in \mathfrak{m}^k \backslash \mathfrak{m}^{k+1}$, where $\mathfrak{m}$ is the maximal ideal of the ring of functions $\mathcal{O}_{p,X}$ (see §0.3). If we can find an open neighbourhood U of p in X with $U \subset \mathbb{A}^2$ (which is equivalent to say that X is rational, which will be the case in all our examples), the point p can be choosed to $(0, 0)$ in this affine neighbourhood, and the equation of C is a polynomial

$$\sum_{i=0}^{r} P_i(x, y) = 0$$

where all P_i are homogeneous polynomials in two variables. The multiplicity $m_p(C)$ is equal to the lowest i such that P_i is not equal to 0. We always have the following:

(1) $m_p(C) \geq 0$;
(2) $m_p(C) = 0 \Leftrightarrow p \notin C$;
(3) $m_p(C) = 1 \Leftrightarrow p$ is a smooth point of C.

Moreover, if C, D are distinct curves with no common component, we define an integer $(C \cdot D)_p$ which counts the intersection of C and D at p. It is equal to 0 is either C or D does not pass through p. Otherwise, we take f, g the local equations of C, D in an neighbourhood of p, which belong to $\mathfrak{m}$, and define $(C \cdot D)_p$ to be the dimension of $\mathcal{O}_{p,X}/(f, g)$.

In fact, we have the following result

Proposition 1.14. [Har77, V, Proposition 1.4, page 360] *If C, D are distinct curves without any common component on a smooth surface, we have*

$$C \cdot D = \sum_{P \in C \cap D} (C \cdot D)_P.$$

In particular, $C \cdot D \geq 0$.

If $\pi \colon Y \to X$ is a blow-up of a point $p \in X$, we have a map $\pi^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$, which sends a curve $C \subset X$ into $\pi^{-1}(C)$. Moreover, if $C \subset X$ is an irreducible curve, the strict transform of C is obtained by taking the closure of $\pi^{-1}(C \backslash \{p\})$.

Lemma 1.15. *In* $\mathrm{Pic}(Y)$*, we have*

$$\pi^*(C) = \tilde{C} + m_p(C) E,$$

where $\tilde{C}$ is the strict transform of C, where $E = \pi^{-1}(p)$.

Proof. Take local coordinates x, y at p and write $k = m_p(C)$. The curve C is given by

$$p_k(x,y) + p_{k+1}(x,y) + \cdots + p_r(x,y),$$

where p_i are homogeneous polynomials of degree i. The blow-up can be viewed as $(u,v) \mapsto (uv, v)$. The pull-back of C becomes

$$v^k(p_k(u,1) + vp_{k+1}(u,1) + \cdots + v^{r-k}p_r(x,y)),$$

so it decomposes into k times the exceptional divisor E (here $v = 0$), and the strict transform. □

Proposition 1.16. [Har77, Chapter V, Proposition 3.2, page 386]

Let X be a smooth surface, let $x \in X$ be a point, and let $\pi\colon Y \to X$ be the blow-up of x. We denote by $E \subset Y$ the curve $\pi^{-1}(p)$, which is isomorphic to $\mathbb{P}^1$.

$$\mathrm{Pic}(Y) = \pi^*(\mathrm{Pic}(X)) \oplus \mathbb{Z}E$$

The intersection form on Y is induced by the intersection form on X via the following formulas:

$$\begin{array}{rcl} \pi^*(C)\cdot\pi^*(D) & = & C\cdot D \textit{ for any } C, D \in \mathrm{Pic}(X), \\ \pi^*(C)\cdot E & = & 0 \textit{ for any } C \in \mathrm{Pic}(X) \\ E\cdot E & = & -1 \end{array}$$

In particular, the curve obtained by blowing-up a point in a smooth surface is isomorphic to $\mathbb{P}^1$ and has self-intersection -1. We will say that it is a (-1)-*curve*. In fact, we have the following converse statement, due to G. Castelnuovo:

Proposition 1.17. [Har77, V, Theorem 5.7, page 414] *Let $E \subset X$ be a curve in a smooth projective surface. The following are equivalent:*

(i) There exists a morphism $\pi\colon X \to Y$ where Y is a smooth projective surface, which contracts E onto a point p and which is an isomorphism outside of E (π is the blow-up of $p \in Y$).

(ii) $E \cong \mathbb{P}^1$ and $E^2 = -1$ (i.e. E is a (-1)-curve).

2. Blow-ups of $\mathbb{P}^2$

Let us use Proposition 1.16 in a situation which will often appear in the sequel. We set $\mathbb{P}^2 = X_0$, take a point $p_1 \in X_0 = \mathbb{P}^2$, and denote by $\pi_1 \colon X_1 \to X_0$ the blow-up of this point. We then take a point $p_2 \in X_1$ and denote by $\pi_2 \colon X_2 \to X_1$ its blow-up. We continue this and stop at some point, obtaining $\pi_r \colon X_r \to X_{r-1}$. For $i = 1, \dots, r$, denote by $\mathcal{E}_i \subset X_i$ the (-1)-curve $\pi_i^{-1}(p_i)$. The morphism $\eta \colon X_r \to \mathbb{P}^2$ which is the composition $\eta = \pi_r \circ \cdots \circ \pi_1$ is often called *blow-up of* $p_1, \dots, p_r$.

Remark 2.1. *Note that* p_1 *is a point of* $\mathbb{P}^2$*, but* p_2 *is a point which does not lie on* $\mathbb{P}^2$*, it is a point of* X_0*. If* p_2 *does not belong to* $\mathcal{E}_1$*, the isomorphism* $\pi_1 \colon X_1 \backslash \mathcal{E}_1 \to \mathbb{P}^2 \backslash \{p_1\}$ *identifies* p_2 *with a point of* $\mathbb{P}^2$*, and we could exchange the roles of* p_1, p_2*, by blowing-up first* p_2 *and then* p_1*. If* $p_2 \in \mathcal{E}_1$*, this is not possible, the point* p_2 *does not correspond to a point of* $\mathbb{P}^2$ *but to a point* infinitely near *to* p_1*.*

The same holds for $i > 2$*. The point* p_i *can be infinitely near to* p_j *for some* $j < i$ *or corresponds to a point of* $\mathbb{P}^2$*.*

Recall that $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}L_{\mathbb{P}^2}$, where $L_{\mathbb{P}^2}$ is the divisor of a line of $\mathbb{P}^2$. For $i = 1, \dots, r$, write $E_i = (\pi_{i+1} \dots \pi_r)^*(\mathcal{E}_i) \in \mathrm{Pic}(X_r)$ (with $E_r = \mathcal{E}_r$), $L = (\pi_1 \dots \pi_r)^*(L_{\mathbb{P}^2}) \in \mathrm{Pic}(X_r)$. Applying r times Proposition 1.16 we get

$$\begin{array}{rcl}\mathrm{Pic}(X_r) & = & \mathbb{Z}(\pi_1\pi_2 \dots \pi_r)^*(L_{\mathbb{P}^2}) \oplus \mathbb{Z}(\pi_2 \dots \pi_r)^*(\mathcal{E}_1) \oplus \cdots \oplus \mathbb{Z}(\pi_r)^*(\mathcal{E}_{r-1}) \oplus \mathbb{Z}\mathcal{E}_r. \\ & = & \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_{r-1} \oplus \mathbb{Z}E_r.\end{array}$$

Moreover, all elements of the basis $\big((\pi_1\pi_2 \dots \pi_r)^*(L_{\mathbb{P}^2}), (\pi_2 \dots \pi_r)^*(\mathcal{E}_1), \dots, \mathcal{E}_r\big) = (L, E_1, \dots, E_r)$ are orthogonal and of self-intersection $(1, -1, \dots, -1)$.

Remark that $E_r = \mathcal{E}_r$ is a (-1)-curve, but the other E_i for $i < r$ maybe do not corresponds to (-1)-curves, in fact, we can be more precise, by making the following obvious observation:

Lemma 2.2. *For* $i = 1, \dots, r-1$*, the following conditions are equivalent:*

1. $(\pi_{i+1} \dots \pi_r)^{-1}(\mathcal{E}_i)$ *is a* (-1)*-curve of* X*;*
2. $(\pi_{i+1} \dots \pi_r)^{-1}(\mathcal{E}_i)$ *is irreducible;*
3. *for* $j = i+1, \dots, r$*, the point* $p_j \in X_{j-1}$ *does not belong to the strict transform of* $\mathcal{E}_i$*;*
4. *for* $j = i+1, \dots, r$*,* $(\pi_{i+1} \dots \pi_{j-1})(p_j) \notin \mathcal{E}_i$*;*
5. *for* $j = i+1, \dots, r$*,* $(\pi_i \dots \pi_{j-1})(p_j) \neq p_i$*.*

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) is because the total pull-back by a blow-up of a curve passing through the point blown-up is reducible. The equivalence of (3), (4) and (5) follows from the construction of the π_i. Finally, (3) implies that $(\pi_{i+1} \dots \pi_r)^{-1}$ restricts to an isomorphism on an open set which contains $\mathcal{E}_i$; since $\mathcal{E}_i$ is a (-1)-curve we get (1). □

The curve $\mathcal{E}_r$ corresponds to the element E_r of $\mathrm{Pic}(X_r)$, and the curves $(\pi_{i+1} \dots \pi_r)^{-1}(\mathcal{E}_i)$ correspond to elements E_i. Are there other curves equivalent to these divisors? The following lemma gives a negative answer:

Lemma 2.3. *For* $i = 1, \dots, r$*, the linear system* $|E_i|$ *is not empty but has dimension* 0*. Any effective divisor equivalent to* E_i *corresponds to the curve*

$$(\pi_{i+1} \dots \pi_r)^{-1}(\mathcal{E}_i)$$

(*which is irreducible if and only if the conditions of Lemma* 2.2 *are satisfied*).

Proof. Suppose that C is an effective divisor equivalent to E_i. This implies that $(\pi_{i+1}\ldots\pi_r)(C)$ is linearly equivalent to $\mathcal{E}_i$ on X_i. In particular $(\pi_{i+1}\ldots\pi_r)(C)\cdot\mathcal{E}_i=(\mathcal{E}_i)^2=-1$, so the two curves have a common component, which means that $(\pi_{i+1}\ldots\pi_r)(C)=\mathcal{E}_i$, since $\mathcal{E}_i$ is irreducible.

is linearly equivalent to $\mathcal{E}_i$

Since $(E_i)^2=-1$, we have $C\cdot(\pi_{i+1}\ldots\pi_r)^{-1}(\mathcal{E}_i)=-1$, so C and $(\pi_{i+1}\ldots\pi_r)^{-1}(\mathcal{E}_i)$ have a common component. □

Lemma 2.4. *If $C\subset X_r$ is an irreducible curve, one of the following situations occurs:*

(i) C is equal to the strict transform of $\mathcal{E}_i\subset X_i$ for some $i\in\{1,\ldots,r\}$.

(ii) C is equal to the strict transform of a curve $D\subset\mathbb{P}^2$ of some degree $d>0$. In this case, C is linearly equivalent to

$$dL-\sum m_iE_i,$$

where $m_i\geq 0$ is the multiplicity of D (or its strict transform $\tilde{D}\subset X_{i-1}$) at $p_i\in X_{i-1}$.

Proof. If C is contracted by $\pi_{i+1}\ldots\pi_r$ onto a point, it has to be the strict transform of $\mathcal{E}_i\subset X_i$ for some $i\in\{1,\ldots,r\}$. Otherwise, $\pi_{i+1}\ldots\pi_r(C)$ is a curve $D\subset\mathbb{P}^2$ of some degree $d>0$. Moreover, the strict transform of this curve on X_r is contained in C, and since C is irreducible, it has to be equal to the strict transform of D on X_r. We apply r times Lemma 1.15 to get the result. □

The above lemma shows that elements of $\mathrm{Pic}(X_r)$ of the form $dL-\sum m_iE_i$ are really important, they correspond to curves of degree d passing through the points $p_1,\ldots,p_r$ with multiplicity $m_1,\ldots,m_r$, let us study the set of curves on X_r equivalent to such a divisor:

Lemma 2.5. *Let $D=dL-\sum m_iE_i\in\mathrm{Pic}(X_r)$ for some integers $m_i\geq 0$, and $d>0$. The linear system $|D|$ has the following properties:*

(i) The linear system $|D|$ has dimension

$$\dim|D|\geq\frac{(d+1)(d+2)}{2}-\sum_{i=1}^{r}\frac{m_i(m_i+1)}{2}-1.$$

This number is called the expected dimension *of $|D|$.*

(ii) Let $C\subset X_r$ be an irreducible member of $|D|$. The geometric genus $g(C)$ of C satisfies

$$g\leq\frac{(d-1)(d-2)}{2}-\sum_{i=1}^{r}\frac{m_i(m_i-1)}{2},$$

and equality holds if and only if C is smooth.

The integer $\frac{(d-1)(d-2)}{2}-\sum_{i=1}^{r}\frac{m_i(m_i-1)}{2}$ is usually called arithmetic genus *of C.*

(iii) $D^2=d^2-\sum m_i^2$.

Proof. Any element of the linear system $|D|$ corresponds to curves of degree d passing through the points $p_1,\ldots,p_r$ with multiplicity $m_1,\ldots,m_r$. The linear system of curves of $\mathbb{P}^2$ of degree d has dimension $\frac{(d+1)(d+2)}{2}-1$. Adding the condition to pass through p_i with multiplicity m_i yields $\frac{(m_i+1)(m_i+2)}{2}$ linear conditions on the coefficients of the linear system. If all these conditions are independent, we get a dimension equal to $\frac{(d+1)(d+2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2} - 1$, and otherwise we could get more. This proves (i).

Condition (iii) is obviously given by the fact that $D = dL - \sum_{i=1}^{r} m_i E_i$ in $\mathrm{Pic}(X_r)$.

The condition (ii) is given by the adjunction formula. Suppose first that a member C is smooth. The adjunction formula yields $-2 + 2g(C) = K_{X_r} \cdot C + C^2$, which yields $g(C) = \frac{1}{2}(K_{X_r} \cdot C + C^2) + 1$. Since $K_{X_r} = -3L + \sum_{i=1}^{r} E_i$ and $C = dL - \sum_{i=1}^{r} m_i E_i$, we get

$$\begin{array}{rcl} g(C) & = & \frac{1}{2}(-3d + \sum_{i=1}^{r} m_i + d^2 - \sum_{i=1}^{r} (m_i)^2) + 1 \\ & = & \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i-1)}{2}. \end{array}$$

If C is not smooth, we can blow-up its singular points so that it becomes smooth. This decreases the arithmetic genus of the curve, and the inequality given in (iii) is thus strict. □

Exercise 2.6. *Compute the geometric genus of the following curves:*
(i) *a line in* $\mathbb{P}^2$*;*
(ii) *a smooth conic in* $\mathbb{P}^2$*;*
(iii) *a smooth cubic in* $\mathbb{P}^2$*;*
(iv) *a sextic having ten double points. Do you think that such a curve can exist?*

Exercise 2.7. *What are the possible multiplicities that a quartic in* $\mathbb{P}^2$ *can have?*

Find the equation of a rational plane quartic (i.e. of genus 0) having only one singular point in $\mathbb{P}^2$

We now restrict ourselves to (-1)-curves.

Lemma 2.8. *If $C \subset X_r$ be a (-1)-curve, it is either equal to E_i, for some i such that no point p_j is infinitely near to p_i, or equal to $dL - \sum a_i E_i$ for some integers $d > 0, a_1, \ldots, a_r \geq 0$ with $\sum a_i^2 = d^2 + 1$, $\sum a_i = 3d - 1$.*

Moreover, the linear system $|C|$ only contains C.

Proof. Applying Lemma 2.4, the curve C is either the strict transform of $\mathcal{E}_i \subset X_i$ for some $i \in \{1, \ldots, r\}$ or is linearly equivalent to $dL - \sum m_i E_i$ for some $d > 0$, $m_1, \ldots, m_r \geq 0$. In the first case, the fact C is a (-1)-curve implies that no point p_j belongs to the strict transform of $\mathcal{E}_i$ for $j > i$, which means that no p_j is infinitely near to p_i. In the second case, the equalities $C^2 = -1$ and $CK_{X_r} = -1$ (given by adjunction formula) yield

$$d^2 - \sum (m_i)^2 = -1, \quad -3d + \sum m_i = -1,$$

which give the first assertion.

Let us prove the second assertion. If D is an effective divisors equivalent to C, we have $C \cdot D = -1$ so C and D have common component, which is C, so $D = C + E$ with E effective. Since C is equivalent to D, E is equivalent to 0 and effective, so trivial. □

3. Birational transformations of the plane (Cremona transformations)

In this section, we will study birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which are usually called *birational transformations of* $\mathbb{P}^2$, or also *Cremona transformations*. The set of all such maps is a group, where the group law is given by the composition, which is written $\mathrm{Bir}(\mathbb{P}^2)$ or sometimes $\mathrm{Cr}(2,\mathbf{k})$.

The group $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3,\mathbf{k})$ and by the *standard quadratic transformation* $\sigma\colon (x:y:z) \dashrightarrow (yz:xz:xy)$. This is the famous Noether-Castelnuovo theorem, announced first by Noether, whose proof was corrected by Castelnuovo in the beginning of the twentieth century. There is an extensive literature on this group. We refer for instance to [AC02] or [Dés09] for some introductive texts.

Any rational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is given by

$$(x:y:z) \dashrightarrow (P_1(x,y,z) : P_2(x,y,z) : P_3(x,y,z)),$$

for some polynomials P_i of the same degree d, without common factor. The linear system of φ is the preimage of the linear system of lines of $\mathbb{P}^2$ and is the system Λ_φ of curves given by

$$\sum a_i f_i = 0,$$

for $(a_0 : a_1 : a_2) \in \mathbb{P}^2$. The integer d is the degree of the polynomials P_i, the degree of the curves of the linear system, and is by definition the degree of φ.

Suppose now that φ is birational. If φ has some base-points (points where φ is not defined), we choose one, that we call $p_1 \in \mathbb{P}^2$ and denote by $\pi_1\colon X_1 \to \mathbb{P}^2$ the blow-up of this point. The map $\varphi_1 = \varphi \circ \pi_1$ is a birational map $X_1 \dashrightarrow \mathbb{P}^2$. If this one has at least one base-point, we again choose one that we call $p_2 \in X_1$ and denote by $\pi_2\colon X_2 \to X_1$ its blow-up. Again, the map $\varphi_2 = \varphi_1 \circ \pi_2$ is a birational map $X_2 \dashrightarrow \mathbb{P}^2$. We continue like this and stop when φ_r becomes a morphism. We will se afterwards that r really exists (i.e. is finite).

$$\begin{tikzcd}
X_r \arrow[d, "\pi_r"'] \arrow[ddddr, dashed, "\varphi_r"] & \\
\vdots & \\
X_2 \arrow[d, "\pi_2"'] \arrow[ddr, dashed, "\varphi_2"] & \\
X_1 \arrow[d, "\pi_1"'] \arrow[dr, dashed, "\varphi_1"] & \\
\mathbb{P}^2 \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2.
\end{tikzcd}$$

Recall (Section 2) how we can describe $\mathrm{Pic}(X_r)$. First, $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}L$ where L is the divisor of a line. Denote by $\mathcal{E}_i \subset X_i$ the (-1)-curve $\pi_i^{-1}(p_i)$, and write $E_i = (\pi_{i+1}\dots\pi_r)^*(\mathcal{E}_i) \in \mathrm{Pic}(X_r)$ (with $E_r = \mathcal{E}_r$), $l = (\pi_1 \dots \pi_r)^*(L) \in \mathrm{Pic}(X_r)$. Applying r times Proposition 1.16 we get

$$\begin{aligned}
\mathrm{Pic}(X_r) &= \mathbb{Z}(\pi_1\pi_2\dots\pi_r)^*(L) \oplus \mathbb{Z}(\pi_2\dots\pi_r)^*(\mathcal{E}_1) \oplus \dots \oplus \mathbb{Z}(\pi_r)^*(\mathcal{E}_{r-1}) \oplus \mathbb{Z}\mathcal{E}_r. \\
&= \mathbb{Z}l \oplus \mathbb{Z}E_1 \oplus \dots \oplus \mathbb{Z}E_{r-1} \oplus \mathbb{Z}E_r.
\end{aligned}$$

Moreover, all elements of the basis $(L, E_1, \dots, E_r)$ are orthogonal and of self-intersection $(1, -1, \dots, -1)$.

The linear system of φ (that we called Λ_φ) consists of curves of degree d, all passing through the p_i with some multiplicities m_i. Applying r times Lemma 1.15 the members of the linear system of φ_r, that we will write Λ_{φ_r}, are equivalent to

$$dL - \sum_{i=1}^{r} m_i E_i.$$

These curves have thus self-intersection $d^2 - \sum_{i=1}^r m_i^2$. Since all members of a linear system are equivalent, and because the dimension of this system is 2, the self-intersection has to be non-negative. This shows that the number r exists, so that there are finitely many base-points of φ, even when we count all *infinitely near* base-points (see Remark 2.1).

Denote by $\eta\colon X_r \to \mathbb{P}^2$ the composition $\eta = \pi_r \circ \cdots \circ \pi_1$, which is the blow-up of all points $p_1, \dots, p_r$, the base-points of φ (here we take also infinitely near points, not only point of $\mathbb{P}^2$ where φ is not defined). By construction, the map $\nu = \varphi_r$ is a birational morphism from X_r to $\mathbb{P}^2$. In fact, any birational morphism between smooth projective surfaces is a sequence of blow-ups. Thus ν can thus be written as $\nu = \nu_r \circ \cdots \circ \nu_1$, where ν_i is the blow-up of a point $q_i \in Y_i$, with $Y_0 = \mathbb{P}^2$ and $Y_r = X_r$ (the fact that the number of points blown-up is the same for ν and η follows from the computation of the rank of the Picard group).

In fact, the map $\nu = \varphi_r = \varphi \circ \eta$ is a birational morphism $\nu = \varphi_r\colon X \to \mathbb{P}^2$ which is the blow-up of the base-points of φ^{-1} and we have the following commutative diagram

(1)
$$\begin{tikzcd}
 & X_r = Y_r \arrow[dl, "\eta"'] \arrow[dr, "\nu=\varphi_r"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "\varphi"'] & & \mathbb{P}^2.
\end{tikzcd}$$

The linear system Λ_φ of φ corresponds to the strict pull-back of the system $\mathcal{O}_{\mathbb{P}^2}(1)$ of lines of $\mathbb{P}^2$ by φ, its image on X_r, the system Λ_{φ_r} is the strict pull-back by ν of the system $\mathcal{O}_{\mathbb{P}^2}(1)$. Let L be a general line of $\mathbb{P}^2$, which does not pass through any point blown-up by η; its pull-back $\nu^{-1}(L)$ corresponds to a smooth curve on X_r which has self-intersection -1 and genus 0. This shows that $(\nu^{-1}(L))^2 = 1$ and $\nu^{-1}(L) \cdot K_{X_r} = -3$, by adjunction formula. Remembering that the members of the system Λ_{φ_r} are equivalent to

$$dL - \sum_{i=1}^{r} m_i E_i,$$

and that $K_{X_r} = -3L + \sum_{i=1}^r E_i$, we see that $d^2 - \sum_{i=1}^r (m_i)^2 = 1$ and $3d - \sum_{i=1}^r m_i = 3$. Reordering these, we get the famous *equations of condition* (see [AC02, equations (2.15) and (2.16), page 51]

(2)
$$\begin{array}{lcl} \sum_{i=1}^r m_i & = & 3(d-1), \\ \sum_{i=1}^r (m_i)^2 & = & d^2 - 1. \end{array}$$

These equalities imply that the equivalence between $d = 1$ and the fact that there is no base-point. In particular, all automorphisms of $\mathbb{P}^2$ are linear (a fact which can be also proved directly, in any dimension). If $d = 2$, we see that $r = 3$

and that the multiplicities are $(1,1,1)$. We give here the list of possible solutions of the equalities for degrees $d \leq 4$:

d	$m_1, \ldots, m_r$
2	$1,1,1$
3	$2,1,1,1,1$
4	$3,1,1,1,1,1,1$
4	$2,2,2,1,1,1$

Exercise 3.1. *Find birational maps of degree $d \leq 5$ with the above multiplicities.*

Exercise 3.2. *Find the list of possible multiplicities of linear systems of degree* 5 *and* 6.

Exercise 3.3. *Find the list of possible multiplicities of linear systems with at most* 8 *base-points.*

These two equations imply that $\frac{(d+1)(d+2)}{2} - \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2} - 1 = 2$, so the expected dimension of the linear system is 2 (see 2.5), which is exactly the dimension of the linear system.

3.1. **How to give a Cremona transformation.** There are three standard ways of describing a Cremona transformation:

- Give explicitly the map $(x:y:z) \dashrightarrow (P_1(x,y,z) : P_2(x,y,z) : P_3(x,y,z))$ (and check that it is invertible for instance by giving its inverse).
- Give the base-points $p_1, \ldots, p_r$, the degree d of the map and the multiplicities $d_1, \ldots, d_r$. This uniquely determines the map (if it exists) up to an automorphism of $\mathbb{P}^2$ at the end.
- Give the base-points $p_1, \ldots, p_r$ blown-up by the map η and the curves contracted by the map φ_r (see Diagram (1)). As in (2), this uniquely determines the map (if it exists) up to an automorphism of $\mathbb{P}^2$ at the end.

Before giving some examples and exercises on how to construct maps, we define the notion of Jacobian of a map. If $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational map $(x:y:z) \dashrightarrow (P_1(x,y,z) : P_2(x,y,z) : P_3(x,y,z))$, the *Jacobian of* φ is a reducible curve whose equation is, if $\mathrm{char}(\mathbf{k}) = 0$, the determinant of the Jacobian matrix

$$\begin{bmatrix} \frac{\partial P_1}{\partial x} & \frac{\partial P_2}{\partial x} & \frac{\partial P_3}{\partial x} \\ \frac{\partial P_1}{\partial y} & \frac{\partial P_2}{\partial y} & \frac{\partial P_3}{\partial y} \\ \frac{\partial P_1}{\partial z} & \frac{\partial P_2}{\partial z} & \frac{\partial P_3}{\partial z} \end{bmatrix}.$$

It has degree $3d-3$, where d is the degree of the curve, and contains all curves of $\mathbb{P}^2$ which are contracted by φ. The map φ restricts to an isomorphism outside this curve.

Example 3.4. *The* standard quadratic transformation $\sigma \colon (x:y:z) \dashrightarrow (yz : xz : xy)$ *is an involution* ($\sigma^2 = 1$)*; it has exactly three base-points, which are* $p_1 = (1:0:0)$*,* $p_2 = (0:1:0)$*,* $p_3 = (0:0:1)$*, and contracts exactly three lines on these three points, which are* $x=0$*,* $y=0$*,* $z=0$*.*

If $\mathrm{char}(\mathbf{k}) = 0$*, its Jacobian is given by the determinant of*

$$\begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix},$$

which is $2xyz$.

The blow-up $\eta\colon X_3 \to \mathbb{P}^2$ *of this three points yields a surface* X_3 *where the strict transform of the three lines* $x = 0$, $y = 0$, $z = 0$ *are linearly equivalent to*

$$L - E_2 - E_3, L - E_1 - E_3, L - E_1 - E_2.$$

The three curves are three skew (-1)*-curves, and the morphism* $\nu\colon X_3$ *here corresponds to the contraction of these three curves.*

Exercise 3.5. *For the following quadratic birational involutive maps of* $\mathbb{P}^2$*, described the resolution given by the blow-up of the base-points and then the contraction of curves* (*as in Diagram* 1).

$$\psi\colon (x : y : z) \dashrightarrow (xy : z^2 : yz)$$

$$\chi\colon (x : y : z) \dashrightarrow (xz : x^2 - yz : z^2)$$

Observe that any birational transformation is equal to the standard quadratic transformation or to one of these two, up to automorphisms of $\mathbb{P}^2$.

Example 3.6. *Take* 6 *distinct points* $p_1, \dots, p_6 \in \mathbb{P}^2$ *which do not belong to the same conic. Let* $\eta\colon X \to \mathbb{P}^2$ *be the blow-up of these* 6 *points. The strict transforms of the conics passing through* 5 *of the* 6 *points are* 6 *skew* (-1)*-curves on* X. *The blow-down of these* 6 *curves gives a birational morphism* $\nu\colon X \to \mathbb{P}^2$. *The map* $\varphi = \nu\eta^{-1}$ *is a birational map of* $\mathbb{P}^2$ *having* 6 *base-points.*

Exercise 3.7. *Find the degree and multiplicities of the birational map* φ *described in Example* 3.6.

$\star$ *If you have time... Compute it explicitly when the points are* $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(1 : 1 : 1)$, $(1 : \omega : \omega^2)$, $(1 : \omega^2 : \omega)$, *where* ω *is a* 3*-rd root of the unity.*

Hint: Choosing the coordinates at the end, three conics are contracted on $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$. *What does it imply on the polynomials* P_1, P_2, P_3 *which describe the map* $\varphi\colon (x : y : z) \dashrightarrow (P_1(x, y, z) : P_2(x, y, z) : P_3(x, y, z))$ *?*

3.2. **Composing maps and the Noether-Castelnuovo theorem.** Let $\varphi, \psi \in \mathrm{Bir}(\mathbb{P}^2)$ be maps of degree d_1, d_2, given by

$$\varphi\colon (x : y : z) \dashrightarrow (P_1(x, y, z) : P_2(x, y, z) : P_3(x, y, z))$$

$$\psi\colon (x : y : z) \dashrightarrow (Q_1(x, y, z) : Q_2(x, y, z) : Q_3(x, y, z))$$

for some homogeneous polynomials P_i and Q_i of degree d_1 and d_2 respectively.

The map $\varphi \circ \psi$ is given by the polynomials $P_1(Q_1, Q_2, Q_3)$, $P_2(Q_1, Q_2, Q_3)$ and $P_3(Q_1, Q_2, Q_3)$, which have degree $d_1 \cdot d_2$. If these have no common component, the map $\varphi \circ \psi$ has degree $d_1 \cdot d_2$, and otherwise it could have a degree strictly less. In fact, we will prove that this happens if and only if ψ^{-1} and φ have common base-points, and we will explicit how we can compute it explicitly.

Let us define the *free intersection* of two linear systems Λ, Λ' on a smooth surface X, both of dimension ≥ 1. Roughly speaking, it is the intersection of a general member of Λ with a general member of Λ' outside of the base-points. More precisely, we blow-up all base-points of Λ and Λ', and compute the intersection of the systems obtained on the blow-up. By construction, this is invariant after a blow-up, so is a birational invariant.

Lemma 3.8. *The degrees of* φ *and* φ^{-1} *are equal.*

Proof. The degree of φ is the degree of the linear system $\Lambda_\varphi = (\varphi)^{-1}(\mathcal{O}_{\mathbb{P}^2}(1))$, also equal to the free intersection of $\mathcal{O}_{\mathbb{P}^2}(1)$ with $(\varphi)^{-1}(\mathcal{O}_{\mathbb{P}^2}(1))$. Applying φ, it is equal to the free intersection of $\varphi(\mathcal{O}_{\mathbb{P}^2}(1))$ with $(\mathcal{O}_{\mathbb{P}^2}(1))$, equal to the degree of $\Lambda_{\varphi^{-1}}$, and thus to the degree of φ^{-1}. □

Proposition 3.9. *The degree of $\varphi \circ \psi$ is equal to the free intersection of Λ_φ and $\Lambda_{\psi^{-1}}$. It is equal to $\deg(\varphi)\cdot\deg(\psi)$ if φ and ψ^{-1} have no common base-points, and is strictly less otherwise.*

Proof. The degree of $\varphi\circ\psi$ is the degree of $(\varphi\circ\psi)^{-1}(\mathcal{O}_{\mathbb{P}^2}(1))$, or also the free intersection of $\mathcal{O}_{\mathbb{P}^2}(1)$ with $(\varphi\circ\psi)^{-1}(\mathcal{O}_{\mathbb{P}^2}(1))$. Applying ψ, this is the free intersection of $\psi(\mathcal{O}_{\mathbb{P}^2}(1)) = \Lambda_{\psi^{-1}}$ with $(\varphi)^{-1}(\mathcal{O}_{\mathbb{P}^2}(1)) = \Lambda_\varphi$.

if φ and ψ^{-1} have no common base-points, the free intersection is the intersection, equal to $\deg(\varphi)\cdot\deg(\psi^{-1}) = \deg(\varphi)\cdot\deg(\psi)$; otherwise, it is strictly less. □

Corollary 3.10. *Let φ be a quadratic birational map of $\mathbb{P}^2$, with base-points p_1, p_2, p_3. The degree of $\varphi\circ\psi$ is equal to $2m - k_1 - k_2 - k_3$, where m is the degree of ψ, and $k_i \geq 0$ is the multiplicity of ψ^{-1} at the point p_i.*

Proof. The degree of $\varphi\circ\psi$ is equal to the free intersection of Λ_φ and $\Lambda_{\psi^{-1}}$. Since Λ_φ consists of conics passing through p_1, p_2, p_3, this free intersection is equal to $2m - k_1 - k_2 - k_3$. □

Exercise 3.11. *Let φ, $\psi \in \mathrm{Bir}(\mathbb{P}^2)$. What are the possibilities for the degree and the multiplicities of $\varphi\circ\psi$?*

The idea of Noether to show that any birational map of the plane can be decomposed by quadratic and linear ones is to decrease the degree of a map by applying a quadratic transformation. Corollary 3.10 implies that we need to find three base-points having multiplicity whose sum is strictly bigger than the degree of the map. This was what Noether proved, obtaining the now famous Noether inequality, generalised by Iskovskikh in higher dimension to apply Sarkisov program.

Lemma 3.12 (Noether inequality)**.** *Let φ be a birational transformation of $\mathbb{P}^2$ of degree $d > 1$, with base-points $p_1, \ldots, p_r$ of multiplicities $m_1, \ldots, m_r$ (counting all points, including the infinitely near ones).*

Ordering the points such that $m_1 \geq m_2 \geq \ldots m_r$, the following inequality holds:

$$m_1 + m_2 + m_3 > d$$

Proof. Recall the two equations of condition (2)

$$\begin{array}{rcl} \sum_{i=1}^r m_i & = & 3(d-1), \\ \sum_{i=1}^r (m_i)^2 & = & d^2 - 1. \end{array}$$

Multiplying the first by m_3 and subtracting it from the second, we get

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) + \sum_{i=4}^{r} m_i(m_i - m_3) = (d-1)(d+1-3m_3).$$

This can be written

$$\begin{array}{rl} (d-1)(\sum_{i=1}^r m_i - (d+1)) & = (m_1 - m_3)(d-1-m_1) + (m_2 - m_3)(d-1-m_2) \\ & \quad + \sum_{i=4}^r m_i(m_3 - m_i) \end{array}$$

Note that $d \geq m_i + 1$ for each i. If $p_i \in \mathbb{P}^2$, this is because the pencil of lines passing through p_i has to intersect positively the linear system. If p_i is infinitely near to a point p_j, we have $m_i \leq m_j$ so $d \geq m_j + 1 \geq m_i + 1$.

In particular, we have $d - 1 \geq m_2$ and $d - 1 \geq m_3$, so the right hand side of the equality is non-negative, which implies that $m_1 + m_2 + m_3 \geq d + 1$. □

In particular, we see that if there exists a quadratic map ψ having p_1, p_2, p_3 as base-points, the map $\psi \circ \varphi^{-1}$ would have degree $2d - m_1 - m_2 - m_3 < d$. We can always change the ordering of the points so that p_1 is a point of $\mathbb{P}^2$, that p_2 is either a point of p_2 or infinitely near to p_1, and that p_3 is either a point of $\mathbb{P}^2$ or infinitely near to p_1 or p_2. Moreover, the three points p_1, p_2, p_3 are not collinear because of the Noether inequality. But it is possible that no quadratic map have these points as base-points. The proof of Castelnuovo used thus Jonquières maps (maps which preserve a pencil of lines) to decrease the degree and then showed that these map decompose into quadratic maps.

Exercise 3.13. *Let $p \in \mathbb{P}^2$, and let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ be a map of degree $d \geq 0$ which sends the pencil of lines passing through p onto a pencil of lines. Show that the multiplicity of φ at p is equal to $d-1$, and find the multiplicities of the other points and how much there are.*

4. Blow-ups of $\mathbb{P}^2$ and del Pezzo surfaces

In this section, we fix r distinct points $p_1, \dots, p_r \in \mathbb{P}^2$, and denote by

$$\pi \colon X \to \mathbb{P}^2$$

the blow-up of these r points. We set

$$E_i = \pi^{-1}(p_i) \text{ for } i = 1, \dots, r,$$

and denote by $L \in \mathrm{Pic}(X)$ the divisor of the pull-back by π of a general line of $\mathbb{P}^2$.

As before, we have

$$\mathrm{Pic}(X) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r$$

and the intersection form on this is of type $(1, -1, \dots, -1)$, which means that

$$\begin{array}{lcrl} L \cdot L & = & 1, & \\ L \cdot E_i & = & 0, & \\ (E_i)^2 & = & -1, & \\ E_i \cdot E_j & = & 0 & \text{if } i \neq j. \end{array}$$

Moreover,

$$K_X = -3L + \sum_{i=1}^{r} E_i,$$

which implies that $(K_X)^2 = 9 - r$.

Proposition 4.1. *If $r \leq 8$, the number of (-1)-curves on X is finite. All such curves are equal to $E_1, \dots, E_r$, or to strict pull-back of curves of degree ≤ 6 if $r = 8$, of degree ≤ 3 if $r = 6, 7$ and of degree ≤ 2 if $r \leq 5$.*

Proof. Let $C \subset X$ be a (-1)-curve, distinct from the E_i. It has to be linearly equivalent to

$$dL - \sum m_i E_i$$

for some integers $d, m_1, \dots, m_r \geq 0$, $d > 0$. Moreover, there is no other (-1)-curve D equivalent to C, otherwise we would have $C \cdot D < 0$, which is impossible. To show that there are only finitely many (-1)-curves, it suffices thus to show that the number of possibilities for $(d, m_1, \dots, m_r)$ is finite. Since C is a (-1)-curve, it is smooth and of genus 0, so

$$\frac{(d-1)(d-2)}{2} - \sum_{i=1}^{r} \frac{a_i(a_i - 1)}{2} = 0.$$

Computing the self-intersection of C we find $-1 = d^2 - \sum (a_i)^2$. Combining this with the previous equality, we obtain

$$\begin{array}{lcl} \sum_{i=1}^{r} a_i & = & 3d - 1, \\ \sum_{i=1}^{r} (a_i)^2 & = & d^2 + 1, \end{array}$$

Using Cauchy-Schwarz inequality, we have $(\sum_{i=1}^{r} a_i)^2 \leq r \cdot \sum_{i=1}^{r} (a_i)^2$. This can also be viewed by taking the scalar product of $(1, \dots, 1)$ with $(a_1, \dots, a_r)$.

We find thus that $(3d-1)^2 \leq r(d^2+1)$, and obtain the following inequality:

$$(9 - r)d^2 - 6d + (1 - r) \leq 0.$$

In particular, d is bounded when $r \leq 8$. The possibilities for the a_i being also bounded, this shows the finiteness of the number of (-1)-curves. To obtain a more precise bound on the degree d, we compute the two roots of the quadratic polynomial $(9-r)d^2 - 6d + (1-r)$ and find that

$$d \leq \frac{3+\sqrt{10r-r^2}}{9-r}.$$

For $r = 8$, this implies that $d \leq 7$. The equality would imply that all a_i are equal (to obtain equality in Cauchy-Schwarz inequality), and would yield $8a_i = 20$, which is impossible. This shows that $d \leq 6$ and gives the result in this case.

For $r = 7$, we obtain $d \leq \frac{3+\sqrt{21}}{2} < 3.8$, so $d \leq 3$.

For $r = 5$, we obtain $d \leq 2$, and here the equality is possible (it corresponds to a conic passing through all points). □

A surface S is called *del Pezzo surface* if $-K_S$ is ample, which means that $-K_S \cdot C > 0$ for any irreducible curve $C \subset S$. We show now when this is the case for our blow-up X of r points in $\mathbb{P}^2$. In fact, any del Pezzo surface except $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained like this.

Proposition 4.2. *The following conditions are equivalent:*

1. $C \cdot (-K_X) > 0$ *for any irreducible curve* $C \subset X$ *(i.e.* $-K_X$ *is ample);*
2. $r \leq 8$, *no* 3 *of the points* p_i *are collinear, no* 6 *are on the same conic, no* 8 *lie on a cubic having a singular point at one of them;*
3. $(K_X)^2 > 0$ *and any irreducible curve* $C \subset X$ *has self-intersection* ≥ -1.

Proof. Suppose first that $r \geq 9$.

$(1 \Rightarrow 3)$ Let $C \subset X$ be an irreducible curve. The adjunction formula gives $C \cdot (C + K_X) \geq -2$, which implies that $C^2 \geq -2 + C \cdot (-K_X)$, which is bigger or equal to -1 because of (1).

It remains to show that $(K_X)^2 > 0$, which means that $r \leq 8$ since $(K_X)^2 = 9 - r$. If $r \geq 9$, we take 9 of the points p_i, and take a cubic passing through these 9 points (it exists but can maybe be reducible). The intersection of the strict transform of this cubic with $-K_X$ is negative, which contradicts (1).

$(3 \Rightarrow 2)$ Since $(K_X)^2 > 0$ and $(K_X)^2 = 9 - r$, we see that $r \leq 8$. If three points p_i where collinear, the strict transform of the line passing the three points would have self-intersection ≤ -2. The same occur for a conic passing through 6 points or a cubic passing through 8 points, being singular at one.

$(2 \Rightarrow 1)$ Let $C \subset X$ be an irreducible curve. If $C = E_i$ for some i, then $C \cdot (-K_X) = 1$. Otherwise, C is linearly equivalent to $dL - \sum_{i=1}^r a_i E_i$ with $d > 0$, $a_1, \ldots, a_r \geq 0$.

If $d = 1$, the curve is the strict transform of a line, which can only pass through 2 points by (2); this implies that $\sum_{i=1}^r a_i \leq 2$, whence $C \cdot (-K_X) \geq 1$. The cases $d = 2$ and $d = 3$ are similar, we have $\sum_{i=1}^r a_i \leq 5$ (respectively ≤ 8), so $C \cdot (-K_X) \geq 1$.

It remains to study the cases $d \geq 4$. Since $r \leq 9$, $-K_X$ is effective (it is linearly equivalent to the strict transform of a cubic passing through the r points), and because $d \geq 4$ and C is irreducible, C has no common component with $-K_X$ so $C \cdot (-K_X) \geq 0$. It remains to show that $C \cdot K_X = 0$ is not possible. Otherwise, by adjunction formula we would have $C^2 = -2 + 2g$ for some integer $g \geq 0$ (which is the arithmetic genus of C). We get thus

$$\sum_{i=1}^{r}(a_i)^2 = d^2+2-2g, \qquad\qquad \sum_{i=1}^{r} a_i = 3d.$$

Since $(\sum_{i=1}^{r} a_i)^2 \leq r\sum_{i=1}^{r}(a_i)^2 \leq 8\sum_{i=1}^{r}(a_i)^2$, we get $9d^2 \leq 8(d^2+2-2g)$ which implies that $d^2 \leq 8(2-2g)$. Since $d \geq 4$, the only possibility should be $d=4$, $g=0$. But the equality implies that $r=8$ and all a_i are equal, which contradicts the equality $\sum_{i=1}^{r} a_i = 3d$.

□

Proposition 4.3. *Suppose that X is a del Pezzo surface (i.e. the conditions of Proposition 4.2 are satisfied). Then, $|-K_X|$ is a linear system of dimension $d = 9-r$. Moreover, the following hold:*

(1) *The rational map $\psi\colon X \overset{|-K_X|}{\dashrightarrow} \mathbb{P}^d$ is a morphism if $d \geq 2$ and has one base-point when $d=1$.*
(2) *When $d \geq 3$, ψ induces an isomorphism $X \to \psi(X)$, where $\psi(X)$ is a smooth projective surface of degree d in $\mathbb{P}^d$.*
(3) *When $d=2$, $\psi\colon X \to \mathbb{P}^2$ is a double covering, ramified along a smooth quartic.*
(4) *When $d=1$, $\psi\colon X \dashrightarrow \mathbb{P}^1$ is an elliptic fibration with one base-point. Moreover $|-2K_X|$ has dimension 3 and $X \overset{|-2K_X|}{\dashrightarrow} \mathbb{P}^3$ is a double covering of a quadric cone, ramified over the singular point and a smooth sextic.*

References

[AC02] Maria Alberich-Carramiñana. *Geometry of the plane Cremona maps*, volume 1769 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[Dés09] Julie Déserti. Odyssée dans le groupe de Cremona. *Gaz. Math.*, (122):31–44, 2009.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Rei88] Miles Reid. *Undergraduate algebraic geometry*, volume 12 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988.

[Sha94] Igor R. Shafarevich. *Basic algebraic geometry. 1*. Springer-Verlag, Berlin, second edition, 1994. Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid.