

Topologies and structures of the Cremona groups

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Abstract

We study the algebraic structure of the n -dimensional Cremona group and show that it is not an algebraic group of infinite dimension (ind-group) if $n \geq 2$. We describe the obstruction to this, which is of a topological nature. By contrast, we show the existence of a Euclidean topology on the Cremona group which extends that of its classical subgroups and makes it a topological group.

1. Introduction

In [Sha66, §3], I. R. Shafarevich asked

“Can one introduce a universal structure of an infinite-dimensional group in the group of all automorphisms (resp. all birational automorphisms) of arbitrary algebraic variety?”

In an open problem session held at the international congress (see [Mum76]), D. Mumford suggested

“Let $G = \text{Aut}_{\mathbb{C}}\mathbb{C}(x_0, x_1)$ be the Cremona group [...]. The problem is to topologize G and associate to it a Lie algebra consisting, roughly, of those meromorphic vector fields D on $\mathbb{P}^2(\mathbb{C})$ which “integrate” into an analytic family of Cremona transformations.”

In 2010, in the question session of the workshop “Subgroups of the Cremona group” in Edinburgh, J-P. Serre asked the following question (see also the introduction of [Fav10]):

“Is it possible to introduce such topology on $\text{Cr}_2(\mathbb{C})$ that is compatible with $\text{PGL}(3, \mathbb{C})$ and $\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$?”

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Let k be a fixed field, and let n be a positive integer. The n -dimensional Cremona group over k , written $\mathrm{Cr}_n(k)$, is the group of all birational transformations of the space of dimension n over k (affine or projective). Algebraically, it corresponds to the group of k -automorphisms of the field $k(x_1, \dots, x_n)$. We are interested in the possible structures one can put on it. For instance, is $\mathrm{Cr}_n(k)$ an algebraic group (of infinite dimension) or a topological group?

In the algebraic geometric setting, we would like to see $\mathrm{Cr}_n(k)$ as an algebraic group of infinite dimension. We will show that this is not possible when $n \geq 2$, and we shall describe the obstructions to this. For any algebraic variety A defined over k , there is a natural notion of families of elements of $\mathrm{Cr}_n(k)$ parametrised by A , or equivalently of elements of $\mathrm{Cr}_n(A)$, which we will recall (Definition 2.1). These are maps $A(k) \rightarrow \mathrm{Cr}_n(k)$ compatible with the structures of algebraic varieties.

When $n = 1$, $\mathrm{Cr}_n(k)$ is isomorphic to the algebraic group $\mathrm{PGL}(2, k)$, and families $A \rightarrow \mathrm{Cr}_1(k)$ correspond to morphisms of algebraic varieties.

For $n \geq 2$, the situation is different. Denoting by $\mathrm{Cr}_n(k)_d \subseteq \mathrm{Cr}_n(k)$ the set of birational maps of degree d (i.e., maps $f: \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^n$ of degree d), one can verify that $\mathrm{Cr}_n(k)_d$ has the structure of an algebraic variety defined over k such that families $A \rightarrow \mathrm{Cr}_n(k)_d$ correspond to morphisms of algebraic varieties (Proposition 2.15). One can thus decompose $\mathrm{Cr}_n(k)$ into a disjoint (infinite) union of algebraic varieties, having unbounded dimension. However, the structure of $\mathrm{Cr}_n(k)$ is more complicated; putting on $\mathrm{Cr}_n(k)$ the topology induced by all families, the group is connected [Bla10].

What we could expect is to have a structure of algebraic variety of infinite dimension (ind-algebraic variety) on $\mathrm{Cr}_n(k)$ so that families $A \rightarrow \mathrm{Cr}_n(k)$ would correspond to morphisms of algebraic varieties. We will show that this is not the case.

THEOREM 1. *For any $n \geq 2$, there is no structure of algebraic variety of infinite dimension on $\mathrm{Cr}_n(k)$ such that families $A \rightarrow \mathrm{Cr}_n(k)$ correspond to morphisms of algebraic varieties.*

The same result holds if we replace varieties by schemes, algebraic spaces or stacks. In fact, the problem does not arise because of high dimension; we will see that the set $\mathrm{Cr}_n(k)_{\leq d}$ of maps of degree $\leq d$ is not an algebraic variety (neither an algebraic space nor a stack) when $d, n \geq 2$ (Proposition 3.4), even if the $\mathrm{Cr}_n(k)_i$ are algebraic varieties for $i = 1, \dots, d$. The bad structure comes from the degeneration of maps of degree d into maps of smaller degree.

We can even show that the obstruction comes only from the topology.

THEOREM 2. *If k is algebraically closed, there is no k -algebraic variety of infinite dimension that is homeomorphic to $\mathrm{Cr}_n(k)$.*

The problem is, roughly speaking, that any point $\varphi \in \text{Cr}_n(\mathbf{k})$ is contained in a closed subset F of dimension 2 such that all curves of F pass through φ . See Section 4 for more details on the topological obstruction.

The question of J-P. Serre is whether $\text{Cr}_2(\mathbb{C})$ is a topological group, with a “natural topology,” i.e., with a topology that induces on $\text{Aut}(\mathbb{P}^2) = \text{PGL}(3, \mathbb{C})$ and $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)^0 = \text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$ the Euclidean topology.¹ We will give a positive answer to this question, after constructing in Section 5 a “Euclidean topology” on $\text{Cr}_n(\mathbf{k})$ for any local field \mathbf{k} .

THEOREM 3. *For any $n \geq 1$ and any (locally compact) local field \mathbf{k} , there is a natural topology on $\text{Cr}_n(\mathbf{k})$, called the Euclidean topology, which makes it a Hausdorff topological group, and whose restriction to algebraic subgroups (in particular, to $\text{Aut}(\mathbb{P}^n) = \text{PGL}(n+1, \mathbf{k})$ and $\text{Aut}((\mathbb{P}^1)^n)^0 = \text{PGL}(2, \mathbf{k})^n$) is the classical Euclidean topology.*

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2. The algebraic structure and Zariski topology of $\text{Cr}_n(\mathbf{k})$

2.1. Definition of the topology via families and morphisms. In this section, all our varieties will be algebraic varieties defined over some fixed algebraically closed field \mathbf{k} . In fact, all constructions made here are field-independent, and the generalisation to arbitrary fields can be carried out in the usual way. The topology on the varieties will only be the Zariski topology.

We recall the notion of families of birational maps, introduced by M. Demazure in [Dem70] (see also [Ser10], [Bla10]).

Definition 2.1. Let A, X be irreducible algebraic varieties, and let f be a A -birational map of the A -variety $A \times X$, inducing an isomorphism $U \rightarrow V$, where U, V are open subsets of $A \times X$, whose projections on A are surjective.

The rational map f is given by $(a, x) \dashrightarrow (a, p_2(f(a, x)))$, where p_2 is the second projection, and for each \mathbf{k} -point $a \in A$, the birational map $x \dashrightarrow p_2(f(a, x))$ corresponds to an element $f_a \in \text{Bir}(X)$. The map $a \mapsto f_a$ represents a map from A (more precisely from the $A(\mathbf{k})$ -points of A) to $\text{Bir}(X)$ and will be called a *morphism* from A to $\text{Bir}(X)$.

These notions yield the natural Zariski topology on $\text{Bir}(X)$, introduced by M. Demazure [Dem70] and J-P. Serre [Ser10].

¹Of course, here it cannot be the Zariski topology, since $\text{PGL}(3, \mathbb{C})$ and $\text{PGL}(2, \mathbb{C})$ are not topological groups when endowed with it.

Definition 2.2. A subset $F \subseteq \text{Bir}(X)$ is closed in the Zariski topology if for any algebraic variety A and any morphism $A \rightarrow \text{Bir}(X)$, the preimage of F is closed.

Moreover, any birational map $X \dashrightarrow Y$ yields a homeomorphism between $\text{Bir}(X)$ and $\text{Bir}(Y)$, and we can make the following observations:

- (1) For any $\varphi \in \text{Bir}(X)$, the maps $\text{Bir}(X) \rightarrow \text{Bir}(X)$ given by $\psi \mapsto \psi \circ \varphi$, $\psi \mapsto \varphi \circ \psi$ and $\psi \mapsto \psi^{-1}$ are homeomorphisms.
- (2) It is equivalent to working with $\text{Bir}(\mathbb{A}_{\mathbf{k}}^n)$ or $\text{Bir}(\mathbb{P}_{\mathbf{k}}^n)$; we thus obtain the Zariski topology on the Cremona group $\text{Cr}_n(\mathbf{k}) = \text{Bir}(\mathbb{A}_{\mathbf{k}}^n)$.

The remainder of this section consists of a description of the topology of $\text{Cr}_n(\mathbf{k})$ and its algebraic structure.

2.2. Bounded degree subsets. Recall that a birational transformation f of $\mathbb{P}^n = \mathbb{P}_{\mathbf{k}}^n$ is given by

$$h: (x_0 : \cdots : x_n) \dashrightarrow (h_0(x_0, \dots, x_n) : \cdots : h_n(x_0, \dots, x_n)),$$

where the h_i are homogeneous polynomials of the same degree. Choosing the h_i without common component, the degree of h is the degree of the h_i .

We denote by $\text{Bir}(\mathbb{P}^n)_{\leq d}$ (respectively $\text{Bir}(\mathbb{P}^n)_d$) the set of elements of $\text{Bir}(\mathbb{P}^n)$ of degree $\leq d$ (respectively of degree d), and we have an increasing sequence

$$\text{Aut}(\mathbb{P}^n) = \text{Bir}(\mathbb{P}^n)_{\leq 1} \subseteq \text{Bir}(\mathbb{P}^n)_{\leq 2} \subseteq \text{Bir}(\mathbb{P}^n)_{\leq 3} \subseteq \cdots$$

whose union gives the group $\text{Bir}(\mathbb{P}^n)$. As we will see, each $\text{Bir}(\mathbb{P}^n)_{\leq d}$ is closed in $\text{Bir}(\mathbb{P}^n)$ and the topology of $\text{Bir}(\mathbb{P}^n)$ is the inductive topology induced by the above sequence. It then suffices to describe the topology of $\text{Bir}(\mathbb{P}^n)_{\leq d}$ in order to understand the topology of $\text{Bir}(\mathbb{P}^n)$.

Definition 2.3. Let d be a positive integer.

- (1) We define W_d to be the set of equivalence classes of nonzero $(n+1)$ -uples (h_0, \dots, h_n) of homogeneous polynomials $h_i \in \mathbf{k}[x_0, \dots, x_n]$ of degree d , where (h_0, \dots, h_n) is equivalent to $(\lambda h_0, \dots, \lambda h_n)$ for any $\lambda \in \mathbf{k}^*$. The equivalence class of (h_0, \dots, h_n) will be denoted by $(h_0 : \cdots : h_n)$.
- (2) We define $H_d \subseteq W_d$ to be the set of elements $h = (h_0 : \cdots : h_n) \in W_d$ such that the rational map $\psi_h: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by $(x_0 : \cdots : x_n) \dashrightarrow (h_0(x_0, \dots, x_n) : \cdots : h_n(x_0, \dots, x_n))$ is birational. We denote by π_d the map $H_d \rightarrow \text{Bir}(\mathbb{P}_{\mathbf{k}}^n)$ that sends h onto ψ_h .

LEMMA 2.4. *Let W_d, H_d be as in Definition 2.3. Then, the following hold:*

- (1) *The set W_d is isomorphic to \mathbb{P}^r , where $r = (n + 1) \cdot \binom{d+n}{d} - 1$.*
- (2) *The set H_d is locally closed in W_d and thus inherits from W_d the structure of an algebraic variety.*

- (3) The map $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)$ is a morphism (in the sense of Definition 2.1). Its image is the set $\text{Bir}(\mathbb{P}^n)_{\leq d}$ of birational transformations of degree $\leq d$.
- (4) For any $\varphi \in \text{Bir}(\mathbb{P}^n)_{\leq d}$, the set $(\pi_d)^{-1}(\varphi)$ is closed in W_d (hence in H_d).
- (5) If $F \subseteq H_m$ ($m \geq 1$) is closed, then $(\pi_d)^{-1}(\pi_m(F))$ is closed in H_d .

Proof. Assertion (1) follows from the fact that the set of homogeneous polynomials of degree d in $n + 1$ variables is equal to a \mathbf{k} -vector space of dimension $\binom{d+n}{d}$.

(2) We denote by $Y \subseteq W_{d^{n-1}} \times W_d$ the set consisting of elements (g, f) such that $h := (g_0(f_0, \dots, f_n), \dots, g_n(f_0, \dots, f_n))$ is a multiple (maybe 0) of the identity; i.e., $h_i x_j = h_j x_i$ for all i, j . This set of equalities yields the existence of a homogeneous polynomial a of degree $d^n - 1$ such that $h_i = ax_i$ for each i . If a is nonzero, then ψ_f and ψ_g are birational and inverses of each other. If a is zero, then ψ_f contracts the entire set \mathbb{P}^n onto a strict subvariety, which is included in the set where $g_i = 0$ for each i . In particular, for any $(g, f) \in Y$, the element ψ_f is birational if and only if its Jacobian is not zero. (The Jacobian is the determinant of the matrix obtained from the partial derivatives of the components of f .)

Since the inverse of every birational map of \mathbb{P}^n of degree d has degree $\leq d^{n-1}$ ([BCW82, Th. 1.5, p. 292]), any element $f \in H_d$ corresponds to (at least) one pair (g, f) in Y .

The description of Y shows that it is closed in $W_{d^{n-1}} \times W_d$. Since $W_{d^{n-1}}$ is a complete variety, the projection $p_2: W_{d^{n-1}} \times W_d \rightarrow W_d$ is a Zariski-closed morphism, so $p_2(Y)$ is closed in W_d . Denote by $U \subseteq W_d$ the open set of elements having a nonzero Jacobian. By construction, we have $H_d = U \cap p_2(Y)$. This implies that H_d is locally closed in W_d (and that it is closed in U).

- (3) Let $f: H_d \times \mathbb{P}^n \dashrightarrow H_d \times \mathbb{P}^n$ be the H_d -rational map given by

$$(h, x) \dashrightarrow (h, h(x)).$$

Let J be the polynomial that is the determinant of the matrix $(\partial h_i / \partial x_j)_{i,j=0}^n$, and let $V \subseteq H_d \times \mathbb{P}^n$ be the open set where J is not zero. We claim that the restriction of f to V is an open immersion, which will show that $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)$ is a morphism. To prove this claim, we need to see that f is radicial (i.e., universally injective) and étale [Gro67, Th. 17.9.1, p. 79]. For any extension K of \mathbf{k} , and any point $h \in H_d(K)$, there exists $h' \in H_m(K)$ without common factors such that the composition $h' \circ h$ is $(x_0 R : \dots : x_n R)$ for some nonzero polynomial R . The hypersurface $R = 0$ is contracted by h onto the set of base-points of h' , with codimension ≥ 2 , so R can be expressed as a product of some factors of the Jacobian of h . This shows that the extension of f to K restricts to an open immersion on $V(K) \cap (\{h\} \times \mathbb{P}^n)$. In particular,

f is radicial. The fact that the projection $H_d \times \mathbb{P}^n \rightarrow H_d$ is smooth and that the derivative with respect to x is injective at any point of V implies that f is étale on V [Gro67, Cor. 17.11.2, p. 84]. This completes the proof of the claim. It then follows from the construction of H_d that the image of π_d is the set of birational transformations of degree $\leq d$.

(4) Let φ be an element of $\text{Bir}(\mathbb{P}^n)_{\leq d}$. It corresponds to a birational map $\psi_h: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by $(x_0 : \dots : x_n) \dashrightarrow (h_0(x_0, \dots, x_n) : \dots : h_n(x_0, \dots, x_n))$, for some homogeneous polynomials of degree $k \leq d$, having no common divisor.

We observe that $(\pi_d)^{-1}(\varphi) \subseteq H_d$ is the set of elements $(g_0 : \dots : g_n)$ of W_d satisfying $g_i h_j = h_i g_j$ for all i, j . This set is therefore closed in W_d and hence in H_d .

(5) For any positive integer m and any closed subset $F \subseteq H_m$, we denote by Y_F the subset of $Y \times \overline{F}$ (where $Y \subseteq W_{d^{n-1}} \times W_d$ is as above and \overline{F} is the closure of F in W_m) consisting of elements $((g, f), h)$, where $f = (f_0 : \dots : f_n)$ and $h = (h_0 : \dots : h_n)$ yield the same map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. This corresponds to saying that $f_i h_j = f_j h_i$ for all i, j ; hence Y_F is closed in $Y \times \overline{F}$ and also in $W_{d^{n-1}} \times W_d \times W_m$. Let $p_2: W_{d^{n-1}} \times W_d \times W_m \rightarrow W_d$ be the second projection. The subset $p_2(Y_F)$ of W_d is closed in W_d and also in $p_2(Y)$. Intersecting with U , we see that $p_2(Y_F) \cap U$ is closed in $p_2(Y) \cap U = H_d$. By construction, $p_2(Y_F) \cap U = (\pi_d)^{-1}(\pi_m(F))$, which is thus closed in H_d . \square

Remark 2.5. Lemma 2.4 shows that W_d, H_d are naturally algebraic varieties. We will show that the same holds for the set $\text{Bir}(\mathbb{P}^n)_d$ of birational transformations of degree d but not for $\text{Bir}(\mathbb{P}^n)_{\leq d}$, which cannot be viewed as an algebraic variety for $d, n \geq 2$. However, the topology of $\text{Bir}(\mathbb{P}^n)_{\leq d}$ is given by the map $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$, in the sense that π_d is a topological quotient map.

2.3. *Liftings of morphisms from $\text{Bir}(\mathbb{P}^n)$ to the H_d and description of the Zariski topology.* The following technical lemma will be used to deduce three corollaries that describe the topology of $\text{Bir}(\mathbb{P}^n)$.

LEMMA 2.6. *Let A be an irreducible algebraic variety and $\rho: A \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism. There exists an open affine covering $(A_i)_{i \in I}$ of A such that for each i , there exists an integer d_i and a morphism $\rho_i: A_i \rightarrow H_{d_i}$ such that the restriction of ρ to A_i is equal to $\pi_{d_i} \circ \rho_i$.*

Proof. Let $\tau: A \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism given by a A -birational map $f: A \times \mathbb{P}^n \dashrightarrow A \times \mathbb{P}^n$, which restricts to an open immersion on an open set U . Let $a_0 \in A$ be some given point, and let $A_0 \subseteq A$ be an open affine set containing a_0 . We also fix an element $w_0 = (a_0, y) \in U$, and we fix homogeneous coordinates $(x_0 : \dots : x_n)$ on \mathbb{P}^n such that $y = (1 : 0 : \dots : 0)$

and that $f(w_0)$ does not belong to the plane $x_0 = 0$. We then denote by $\mathbb{A}^n \subseteq \mathbb{P}^n$ the affine set where $x_0 = 1$, which has natural affine coordinates $z_1 = \frac{x_1}{x_0}, \dots, z_n = \frac{x_n}{x_0}$. In these coordinates, f restricts to a rational map $A_0 \times \mathbb{A}^n \dashrightarrow \mathbb{A}^n$, which is defined at w_0 . Its composition with the projection on the i -th coordinate is a rational function on $A_0 \times \mathbb{A}^n$, which is defined at w_0 . We obtain that $f|_{A_0 \times \mathbb{A}^n}$ can be written, in a neighbourhood of w_0 , as $(a, (z_1, \dots, z_n)) \mapsto (\frac{P_1}{Q_1}, \dots, \frac{P_n}{Q_n})$ for some $P_i, Q_i \in \mathbf{k}[A_0][z_1, \dots, z_n]$ such that none of the Q_i vanish at w_0 . Homogenising the description, we see that f is given, in a neighbourhood of w_0 , by

$$(a, (x_0 : \dots : x_n)) \mapsto (h_0 : \dots : h_n),$$

where the $h_i \in \mathbf{k}[A_0][x_0, \dots, x_n]$ are homogeneous polynomials in x_0, \dots, x_n of the same degree d_0 such that not all vanish at w_0 . Let U_0 be the set of points of $(A_0 \times \mathbb{P}^n) \cap U$ where at least one of the h_i does not vanish. It is an open subset of $A \times \mathbb{P}^n$. Its projection $p_1(U_0)$ on A is an open subset of A_0 containing a_0 . Therefore, there exists an affine open subset $\tilde{A}_0 \subseteq p_1(U_0)$ containing a_0 . The n -uple (h_0, \dots, h_n) gives rise to a morphism $\rho_0: \tilde{A}_0 \rightarrow H_d$. By construction, the restriction of ρ to \tilde{A}_0 is equal to $\pi_d \circ \rho_0$. Repeating the process for each point of A , we obtain an open affine covering. \square

COROLLARY 2.7. *A set $F \subseteq \text{Bir}(\mathbb{P}^n)$ is closed if and only if $(\pi_d)^{-1}(F)$ is closed in H_d for any $d \geq 1$.*

Proof. By definition, F is closed in $\text{Bir}(\mathbb{P}^n)$ if and only if its preimage by any morphism is closed. Because each π_d is a morphism, we see that $(\pi_d)^{-1}(F)$ is closed in H_d if F is closed in $\text{Bir}(\mathbb{P}^n)$.

Suppose now that $(\pi_d)^{-1}(F)$ is closed in H_d for each d , and let us prove that $F \subseteq \text{Bir}(\mathbb{P}^n)$ is closed. We let $\rho: A \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism. We apply Lemma 2.6 and obtain an open affine covering $(A_i)_{i \in I}$ of A such that for each i , there exist an integer d_i and a morphism $\rho_i: A_i \rightarrow H_{d_i}$ such that the restriction of ρ to A_i is equal to $\pi_{d_i} \circ \rho_i$. Since $\pi_{d_i}^{-1}(F)$ is closed and $\rho^{-1}(F) \cap A_i$ is equal to $(\rho_i)^{-1}(\pi_{d_i}^{-1}(F))$, we obtain that $\rho^{-1}(F) \cap A_i$ is closed in A_i for each i . Consequently, $\rho^{-1}(F)$ is closed in A , as required. \square

COROLLARY 2.8. *For any d , the set $\text{Bir}(\mathbb{P}^n)_{\leq d}$ of birational transformations of degree $\leq d$ is closed in $\text{Bir}(\mathbb{P}^n)$.*

Proof. By Corollary 2.7, it suffices to see that

$$(\pi_m)^{-1}(\text{Bir}(\mathbb{P}^n)_{\leq d}) = (\pi_m)^{-1}(\pi_d(H_d))$$

is closed in H_m for any m . This follows from Lemma 2.4. \square

COROLLARY 2.9. *For any d , the map $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$ is surjective, continuous and closed. In particular, it is a topological quotient map.*

Proof. The surjectivity follows from the construction of H_d and π_d (see Lemma 2.4). The continuity of π_d follows from the fact that it is a morphism in the sense of Definition 2.1 (Lemma 2.4). Let $F \subseteq H_d$ be closed. By Lemma 2.4, $(\pi_m)^{-1}(\pi_d(F))$ is closed in H_m for any m , so $\pi_d(F)$ is closed in $\text{Bir}(\mathbb{P}^n)$ (Corollary 2.7). \square

PROPOSITION 2.10. *The topology of $\text{Bir}(\mathbb{P}^n)$ is the inductive limit topology given by the Zariski topologies of $\text{Bir}(\mathbb{P}^n)_{\leq d}$, $d \in \mathbb{N}$, which are the quotient topology of $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$, where H_d is endowed with its Zariski topology.*

Proof. This follows from Corollaries 2.7 and 2.9. \square

2.4. *More about the liftings.* Let us show that the liftings to H_d do not behave as simply as they might seem to.

Example 2.11. Let $C \subseteq \mathbb{P}^2$ be the nodal cubic given by $abc = a^3 + b^3$, where $(a : b : c)$ are homogeneous coordinates on \mathbb{P}^2 , and let $\rho_3: C \rightarrow H_3$ be given by

$$(a : b : c) \mapsto (x_0R : x_1S : x_2R : \cdots : x_nR),$$

$$R = ax_2^2 + cx_0x_2 + bx_0^2, \quad S = ax_2^2 + (b + c)x_0x_2 + (a + b)x_0^2.$$

LEMMA 2.12. *The morphism ρ_3 of Example 2.11 has the following properties:*

- (1) *The morphism $\rho = \pi_3 \circ \rho_3: C \rightarrow \text{Bir}(\mathbb{P}^n)$ satisfies $\rho(C) \subseteq \text{Bir}(\mathbb{P}^n)_{\leq 2}$.*
- (2) *There is no morphism $\rho_2: C \rightarrow H_2$ such that $\pi_2 \circ \rho_2 = \rho$.*

Proof. The image of $(0 : 0 : 1) \in C$ under ρ_3 is equal to $(x_0(x_0x_2) : x_1(x_0x_2) : \cdots : x_n(x_0x_2)) \in W_3$, which corresponds to the identity.

On the open subset $C \setminus \{(0 : 0 : 1)\}$ of C , neither a nor b is zero, so the restriction of ρ_3 to $C \setminus \{(0 : 0 : 1)\}$ corresponds to

$$(a : b : c) \mapsto (x_0R' : x_1S' : x_2R' : \cdots : x_nR'),$$

$$R' = abR = a^2bx_2^2 + abcx_0x_2 + ab^2x_0^2,$$

$$S' = abS = a^2bx_2^2 + (ab^2 + abc)x_0x_2 + ab(a + b)x_0^2.$$

We now observe that

$$R' = a^2bx_2^2 + (a^3 + b^3)x_0x_2 + ab^2x_0^2 = (a^2x_2 + b^2x_0)(bx_2 + ax_0)$$

and

$$S' = a^2bx_2^2 + (a^3 + b^3 + ab^2)x_0x_2 + ab(a + b)x_0^2 = (a^2x_2 + b(a + b)x_0)(bx_2 + ax_0).$$

This shows that $\pi_3(\rho_3(a : b : c))$ has degree 2 for any

$$(a : b : c) \in C \setminus \{(0 : 0 : 1)\}.$$

We have thus proved (1). To prove (2), assume the existence of a morphism $\rho_2: C \rightarrow H_2$ such that $\pi_2 \circ \rho_2 = \rho$. Since π_2 is bijective on $(\pi_2)^{-1}(\text{Bir}(\mathbb{P}^n)_2)$

and $\rho(C \setminus \{(0 : 0 : 1)\}) \subseteq \text{Bir}(\mathbb{P}^n)_2$, the restriction of ρ_2 to $C \setminus \{(0 : 0 : 1)\}$ is given by

$$(a : b : c) \mapsto (x_0R'' : x_1S'' : x_2R'' : \cdots : x_nR''),$$

$$R'' = a^2x_2 + b^2x_0, \quad S'' = a^2x_2 + b(a + b)x_0.$$

It remains to see that this morphism does not extend to C . Let $\varphi: \mathbb{P}^1 \rightarrow C$ be the birational morphism that sends $(u : v)$ onto $(u^2v : uv^2 : u^3 + v^3)$. The hypothetic morphism $\rho_2 \circ \varphi: \mathbb{P}^1 \rightarrow H_2$ sends $(u : v)$ onto

$$(x_0(u^2x_2 + v^2x_0) : x_1(u^2x_2 + v(u+v)x_0) : x_2(u^2x_2 + v^2x_0) : \cdots : x_n(u^2x_2 + v^2x_0)).$$

In particular, the points $\rho_2\varphi((1 : 0))$ and $\rho_2\varphi((0 : 1))$ are distinct, which is impossible, since $\varphi((1 : 0)) = \varphi((0 : 1))$. □

2.5. *Universal property with fixed degree.* Example 2.11 provides a morphism from an algebraic variety to $\text{Bir}(\mathbb{P}^n)_{\leq 2}$ that does not lift to H_2 . This phenomenon occurs because of the degeneration of elements of $\text{Bir}(\mathbb{P}^2)_2$ to elements of $\text{Bir}(\mathbb{P}^2)_1$. The situation is better if the degree is fixed.

LEMMA 2.13. *Let d_1, d_2 be integers with $1 \leq d_1 \leq d_2$. The set*

$$H_{d_1, d_2} = (\pi_{d_2})^{-1}(\text{Bir}(\mathbb{P}^n)_{d_1})$$

is locally closed in H_{d_2} and isomorphic to $H_{d_1, d_1} \times \mathbb{P}(\mathbf{k}[x_0, \dots, x_n]_{d_2-d_1})$, where $\mathbf{k}[x_0, \dots, x_n]_{d_2-d_1}$ denotes the \mathbf{k} -vector space of homogeneous polynomials of degree $d_2 - d_1$ in x_0, \dots, x_n . Moreover, the projection

$$\rho_{d_1, d_2}: H_{d_1, d_2} \rightarrow H_{d_1, d_1} \subseteq H_{d_1}$$

is such that $\pi_{d_1} \circ \rho_{d_1, d_2}$ and π_{d_2} coincide on H_{d_1, d_2} .

Proof. By Corollary 2.8, $(\pi_{d_2})^{-1}(\text{Bir}(\mathbb{P}^n)_{\leq d_1})$ and $(\pi_{d_2})^{-1}(\text{Bir}(\mathbb{P}^n)_{\leq d_1-1})$ are closed in H_{d_2} . This implies that H_{d_1, d_2} is locally closed in H_{d_2} and thus inherits from H_{d_2} a structure of algebraic variety. We have a natural bijective morphism

$$\begin{aligned} \tau: H_{d_1, d_1} \times \mathbb{P}(\mathbf{k}[x_0, \dots, x_n]_{d_2-d_1}) &\rightarrow H_{d_1, d_2} \\ ((f_0 : \cdots : f_n), h) &\rightarrow (f_0h : \cdots : f_nh), \end{aligned}$$

which is the restriction of a morphism

$$\begin{aligned} \hat{\tau}: W_{d_1} \times \mathbb{P}(\mathbf{k}[x_0, \dots, x_n]_{d_2-d_1}) &\rightarrow W_{d_2} \\ ((f_0 : \cdots : f_n), h) &\rightarrow (f_0h : \cdots : f_nh). \end{aligned}$$

Denoting by $U_{d_1} \subseteq W_{d_1}$ the open subset given by the elements $(f_0 : \cdots : f_n)$ such that the f_i have no common factor, it suffices to see that $\hat{\tau}$ restricts to an isomorphism of $V = U_{d_1} \times \mathbb{P}(\mathbf{k}[x_0, \dots, x_n]_{d_2-d_1})$ with its image in order to prove the result.

The morphism $\hat{\tau}$ is proper, since it is projective. Since $V = \hat{\tau}^{-1}(\hat{\tau}(V))$, the restrictions of $\hat{\tau}$ gives a proper morphism $V \rightarrow \hat{\tau}(V)$. It can be checked with differentials that it is unramified; it is, moreover, clearly universally bijective. Hence, it is an isomorphism. \square

COROLLARY 2.14. *Let A be an irreducible algebraic variety and $\rho: A \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism, whose image is contained in $\text{Bir}(\mathbb{P}^n)_d$ for some d . There exists a unique morphism $\tilde{\rho}: A \rightarrow H_d$ such that $\rho = \pi_d \circ \tilde{\rho}$.*

Proof. Lemma 2.6 yields an open affine covering $(A_i)_{i \in I}$ of A with the property that for each i , there exist an integer d_i and a morphism $\rho_i: A_i \rightarrow H_{d_i}$ such that the restriction of ρ to A_i is equal to $\pi_{d_i} \circ \rho_i$. Since $\rho(A)$ is contained in $\text{Bir}(\mathbb{P}^n)_d$, we have $d_i \geq d$ for each i and the image of ρ_i is contained in $H_{d,d_i} = (\pi_{d_i})^{-1}(\text{Bir}(\mathbb{P}^n)_d)$. We can thus replace ρ_i by $\rho_{d,d_i} \circ \rho_i$, where $\rho_{d,d_i}: H_{d,d_i} \rightarrow H_{d,d}$ is given in Lemma 2.13.

After this replacement, each d_i is equal to d . Because π_d restricts to a bijection $H_{d,d} \rightarrow \text{Bir}(\mathbb{P}^n)_d$, the maps ρ_i and ρ_j coincide on $A_i \cap A_j$. In particular, they yield a morphism $\tilde{\rho}: A \rightarrow H_d$ such that $\rho = \pi_d \circ \tilde{\rho}$. Its uniqueness also follows from the fact that π_d restricts to a bijection $H_{d,d} \rightarrow \text{Bir}(\mathbb{P}^n)_d$. \square

This yields the following result, also proved in [Ngu09, §3.4] (by other methods).

PROPOSITION 2.15. *Let $d \geq 1$ be an integer. The following hold:*

- (1) *The map $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$ restricts to a bijection*

$$H_{d,d} = (\pi_d)^{-1}(\text{Bir}(\mathbb{P}^n)_d) \rightarrow \text{Bir}(\mathbb{P}^n)_d,$$

where $H_{d,d}$ is open in H_d , and is thus an algebraic variety.

- (2) *Let A be an irreducible algebraic variety. The morphisms $A \rightarrow \text{Bir}(\mathbb{P}^n)$ whose image is in $\text{Bir}(\mathbb{P}^n)_d$ correspond, via π_d , to the morphisms of algebraic varieties $A \rightarrow H_{d,d}$. (This again shows that π_d restricts to a homeomorphism $H_{d,d} \rightarrow \text{Bir}(\mathbb{P}^n)_d$.)*

Proof. (1) Since $(\pi_d)^{-1}(\text{Bir}(\mathbb{P}^n)_{\leq d-1})$ is closed in H_d , $(\pi_d)^{-1}(\text{Bir}(\mathbb{P}^n)_d)$ is open in H_d and is thus an algebraic variety.

(2) If $\rho: A \rightarrow H_{d,d}$ is a morphism of algebraic varieties, then $\pi_d \circ \rho$ is a morphism $A \rightarrow \text{Bir}(\mathbb{P}^n)$ in the sense of Definition 2.1, having its image in $\text{Bir}(\mathbb{P}^n)_d$.

Conversely, let $\rho: A \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism having its image in $\text{Bir}(\mathbb{P}^n)_d$. It corresponds to a unique map $\tilde{\rho}: A \rightarrow H_{d,d}$ such that $\rho = \pi_d \circ \tilde{\rho}$. Corollary 2.14 implies that $\tilde{\rho}$ is a morphism of algebraic varieties. \square

Remark 2.16. One can see that $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)_d$ is irreducible if and only if $d \leq 3$ and is connected when $d \leq 6$ ([CD13], [Ngu09], [BCM]); the connectedness is open for large $d \geq 7$. The variety $\text{Bir}(\mathbb{P}_{\mathbb{C}}^3)_2$ has three irreducible components [PRV01, Prop. 2.4.1], but it is connected. The structure of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)_d$ for $d, n \geq 3$ is far from being understood.

Note that the identity is in the closure of $\text{Bir}(\mathbb{P}^n)_d$ for any d [Bla10, Lemma 3.3], so $\overline{\text{Bir}(\mathbb{P}^n)_2} = \text{Bir}(\mathbb{P}^n)_{\leq 2}$. For large d , $\overline{\text{Bir}(\mathbb{P}^n)_d} \subsetneq \text{Bir}(\mathbb{P}^n)_{\leq d}$ seems more plausible.

2.6. Algebraic subgroups.

PROPOSITION 2.17. *Let $G \subseteq \text{Bir}(\mathbb{P}^n)$ be a subgroup, which is closed and connected for the Zariski topology, and with $G \subseteq \text{Bir}(\mathbb{P}^n)_{\leq d}$ for some finite degree d .*

Choosing d minimal, the set $(\pi_d)^{-1}(G \cap \text{Bir}(\mathbb{P}^n)_d)$ is nonempty. Let K denote its closure in H_d . It has the following properties:

- (1) π_d induces a homeomorphism $K \rightarrow G$ (for the Zariski topology).
- (2) Let A be an irreducible algebraic variety. The morphisms $A \rightarrow \text{Bir}(\mathbb{P}^n)$ (in the sense of Definition 2.1) having image in G correspond, via π_d , to the morphisms of algebraic varieties $A \rightarrow K$.
- (3) The liftings to K of the product map $G \times G \rightarrow G$, $(f, g) \mapsto f \circ g$ and of the inverse map $G \rightarrow G$, $f \mapsto f^{-1}$ give rise to morphisms of algebraic varieties $K \times K \rightarrow K$ and $K \rightarrow K$.

This gives G a unique structure of algebraic group.

Proof. Since G is closed in $\text{Bir}(\mathbb{P}^n)_{\leq d}$, its pullback $(\pi_d)^{-1}(G)$ is closed in H_d and thus has a finite number of irreducible components I_1, \dots, I_r . By Corollary 2.9, the sets $\pi_d(I_1), \dots, \pi_d(I_r)$ are closed and irreducible and they cover G . Keeping only the maximal ones, we obtain the irreducible components of G , which we now call Z_1, \dots, Z_m .

For $i = 1, \dots, m$, we set $A_i = \{g \in G \mid gZ_1 = Z_i\}$, which is equal to $\{g \in G \mid gZ_1 \subseteq Z_i\} = \bigcap_{h \in Z_1} Z_i h^{-1}$ and is thus closed in G . Since G is the disjoint union of all A_i , each A_i is also open. There is thus an i with $A_i = G$, which implies that $GZ_1 = Z_i$. Since $GZ_1 = G$, we get $m = 1$ and see that G is irreducible.

Since G is irreducible, the open subset $G_d = \text{Bir}(\mathbb{P}^n)_d \cap G$ is irreducible, and dense in G . Writing $K_d = (\pi_d)^{-1}(G) \cap H_{d,d} = (\pi_d)^{-1}(G_d)$, the map π_d induces a homeomorphism $K_d \rightarrow H_d$ (Proposition 2.15), hence K_d is irreducible. Its closure $K = \overline{K_d}$ is contained in the closed set $(\pi_d)^{-1}(G)$, hence it satisfies $\pi_d(K) \subseteq G$. Furthermore, on the one hand, $\pi_d(K)$ is closed (by Corollary 2.9) and on the other hand, $\pi_d(K)$ contains the dense open set G_d of G . Therefore, $\pi_d(K) = G$.

Let us fix some element $f \in K$ and consider the map $K \rightarrow H_{d^2}$, $g \mapsto g \circ f$. We claim that there exists $p \in \mathbf{k}[x_0, \dots, x_n]_{d^2-d}$, a homogeneous polynomial of degree $d^2 - d$ that divides each component of each $g \circ f$, $g \in K$. Indeed, for any $g \in K$, the transformation $\pi_{d^2}(g \circ f)$ belongs to G and so has degree $\leq d$. If this degree is exactly d , there exist $p_g \in \mathbf{k}[x_0, \dots, x_n]_{d^2-d}$, $h_g \in H_d$ such that $g \circ f = p_g h_g$. Supposing that $\pi_d(g)$ has degree d , the hypersurface $p_g = 0$ is contracted by f onto the base-points of g , so p_g is a product of some factors of $\text{Jac } f$ (the Jacobian determinant of f). This gives, up to multiplicative constants, finitely many polynomials of degree $d^2 - d$, which we denote by p_1, \dots, p_s . For $i = 1, \dots, s$, let W_i be the set of elements $g \in K$ such that p_i divides each component of $g \circ f$. Each W_i is closed in K , and $W_1 \cup \dots \cup W_s$ contains the open subset of elements g such that $\pi_d(g)$ and $\pi_{d^2}(g \circ f)$ have degree d . Since $K = \overline{K_d}$ is irreducible, we get $K = W_i$ for some i and our claim is proved.

Consequently, there exists a morphism $\mu_f: K \rightarrow (\pi_d)^{-1}(G) \subseteq H_d$ such that for each $g \in K$, $g \circ f = p \mu_f(g)$. Since $U = (\mu_f)^{-1}(K_d) = (\mu_f)^{-1}(H_{d,d})$ is a dense open subset of K , we obtain $\mu_f(K) = \mu_f(\overline{U}) \subseteq \overline{\mu_f(U)} \subseteq K$.

Since $\pi_d(K) = G$, there exists $g \in K$ such that $\pi_{d^2}(f \circ g) = \pi_{d^2}(g \circ f) = \text{id}_{\mathbb{P}^n}$. As a consequence, the morphisms $m_g \circ m_f$ and $m_f \circ m_g: K \rightarrow K$ coincide with the identity morphism on K_d , which is dense in K , so they are equal to the identity on K . This shows that $m_f: K \rightarrow K$ is an isomorphism.

We are now ready to prove (1). Let us prove that if $\pi_d(g) = \pi_d(h)$ for some $g, h \in K$, then $g = h$. Choose $f \in K$ such $\pi_d(g) \circ \pi_d(f) \in G_d$. Since $\pi_d(m_f(g)) = \pi_d(m_f(h)) \in G_d$, this implies that $m_f(g) = m_f(h)$, whence $g = h$. We know that π_d induces a bijective morphism $K \rightarrow G$. By Corollary 2.9 this map is closed, so it is a homeomorphism.

(2) If $\rho: A \rightarrow K$ is a morphism of algebraic varieties, then $\pi_d \circ \rho$ is a morphism $A \rightarrow \text{Bir}(\mathbb{P}^n)$ having its image in G . Conversely, let $\varphi: A \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism having its image in G . It corresponds to a unique map $\rho: A \rightarrow K$ such that $\varphi = \pi_d \circ \rho$. By Proposition 2.15, ρ is a morphism of algebraic varieties on the open set $\varphi^{-1}(G_d)$. For any $f \in K$, we compose ρ and φ with m_f and the right-multiplication by $\pi_d(f)$, and we use Proposition 2.15 to see that ρ is a morphism on $\rho^{-1}((m_f)^{-1}(K_d))$. Since the open sets $(m_f)^{-1}(K_d)$ cover K , ρ is a morphism from A to K .

(3) Since the map $\pi_d: K \rightarrow \text{Bir}(\mathbb{P}^n)$ is a morphism in the sense of Definition 2.1, so also is $I \circ \pi_d: K \rightarrow \text{Bir}(\mathbb{P}^n)$, where $I: \text{Bir}(\mathbb{P}^n) \rightarrow \text{Bir}(\mathbb{P}^n)$ is the map that sends an element on its inverse. By (2), the morphism $I \circ \pi_d: K \rightarrow G$ corresponds to a morphism of algebraic varieties $K \rightarrow K$, which is the inverse map of the group structure. In the same way, the composition of the morphism (of algebraic varieties) $K \times K \rightarrow H_{d^2}$, $(f, g) \mapsto f \circ g$ and of the morphism (in

the sense of Definition 2.1) $\pi_{d^2}: H_{d^2} \rightarrow \text{Bir}(\mathbb{P}^n)$ (see Lemma 2.4) gives a morphism $K \times K \rightarrow \text{Bir}(\mathbb{P}^n)$ having image in G . By (2), we obtain a morphism $K \times K \rightarrow K$ that is the product map of the group structure. \square

COROLLARY 2.18. *Let $G \subseteq \text{Bir}(\mathbb{P}^n)$ be a subgroup, closed for the Zariski topology and of bounded degree. There exist an algebraic group K , together with a morphism $K \rightarrow \text{Bir}(\mathbb{P}^n)$ inducing a homeomorphism $\pi: K \rightarrow G$ that is a group homomorphism, and such that for any irreducible algebraic variety A , the morphisms $A \rightarrow \text{Bir}(\mathbb{P}^n)$ (in the sense of Definition 2.1) having their image in G correspond, via π , to the morphisms of algebraic varieties $A \rightarrow K$.*

Proof. The fact that G has a finite number of irreducible components can be checked in the same way as at the beginning of the proof of Proposition 2.17. As for algebraic groups, one can successively establish the following points: (1) exactly one irreducible component of G passes through id ; (2) this irreducible component is a (closed) normal subgroup of finite index in G , whose cosets are the connected as well as irreducible components of G (see [Hum75, §7.3]). The result then follows from the connected case, treated in Proposition 2.17. \square

LEMMA 2.19. *Let A be an algebraic group, and let $\rho: A \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism (in the sense of Definition 2.1), which is also a group homomorphism. Then, the image G of A is a closed subgroup of $\text{Bir}(\mathbb{P}^n)$, which has bounded degree. Denoting by $\pi: K \rightarrow G$ the homeomorphism constructed in Corollary 2.18, there is a unique morphism of algebraic groups $\tilde{\rho}: A \rightarrow K$ such that $\rho = \pi \circ \tilde{\rho}$.*

Proof. By Lemma 2.6, the image $G = \rho(A)$ is of bounded degree. Denote by \overline{G} its closure, and observe that it is a subgroup of $\text{Bir}(\mathbb{P}^n)$. (See the proof of [Hum75, Prop. A, p. 54].)

By Corollary 2.18, we obtain a canonical homeomorphism $K \rightarrow \overline{G}$, where K is an algebraic group, and a lift $\tilde{\rho}: A \rightarrow K$ of the morphism $\rho: A \rightarrow \text{Bir}(\mathbb{P}^n)$ having image in \overline{G} . Since ρ is a group homomorphism, $\tilde{\rho}$ is a morphism of algebraic groups, hence its image is closed, so it is K itself. This shows that $\overline{G} = G$ and completes the proof. \square

Remark 2.20. In the literature, an algebraic subgroup G of $\text{Bir}(X)$ corresponds to taking an algebraic group G and a morphism $G \rightarrow \text{Bir}(X)$ (in the sense of Definition 2.1) that is a group homomorphism and whose schematic kernel is trivial (which is not the case of the action of the additive group on \mathbb{A}^1 ($a, x \mapsto (x + a^p)$ in characteristic p)).

Corollary 2.18 allows one to give a more intrinsic definition in the case of $X = \mathbb{P}^n$, which corresponds to taking closed subgroups of $\text{Bir}(\mathbb{P}^n)$ of bounded degree, and Lemma 2.19 shows that the two definitions agree.

Remark 2.21. One can show that any algebraic subgroup G of $\text{Bir}(\mathbb{P}^n)$ is linear, and this reduces to the connected case. By a classical result of Weil, G acts by automorphisms on some (smooth) rational variety X . Denote by $\alpha_X: X \rightarrow A(X)$ the Albanese morphism that is the universal morphism to an abelian variety. Then, the Nishi-Matsumura theorem asserts that the induced action of G on $A(X)$ factors through a morphism $A(G) \rightarrow A(X)$ with finite kernel (see [Bri10]). However, since X is rational, $A(X)$ and then $A(G)$ are trivial. Hence, G is affine by the structure theorem of Chevalley.

3. A crucial example and the proof of Theorem 1

The following example will be essential in the proofs of Theorems 1 and 2.

Example 3.1. Let \widehat{V} be $\mathbb{P}^2 \setminus \{(0 : 1 : 0), (0 : 0 : 1)\}$, and let $\tilde{\rho}: \widehat{V} \rightarrow H_2$ be the morphism that sends $(a : b : c)$ onto

$$(x_0(ax_2 + cx_0) : x_1(ax_2 + bx_0) : x_2(ax_2 + cx_0) : \cdots : x_n(ax_2 + cx_0)).$$

Define $V \subseteq \text{Bir}(\mathbb{P}^n)_{\leq 2}$ to be the image of the morphism $\rho = \pi_2 \circ \tilde{\rho}: \widehat{V} \rightarrow \text{Bir}(\mathbb{P}^n)$. The map $\rho: \widehat{V} \rightarrow V$ sends the line $L \subseteq \widehat{V}$ corresponding to $b = c$ to the identity and induces a bijection $\widehat{V} \setminus L \rightarrow V \setminus \{\text{id}\}$.

Remark 3.2. The above map corresponds in affine coordinates to

$$\begin{aligned} \widehat{V} \times \mathbb{A}^n &\dashrightarrow \mathbb{A}^n, \\ ((a : b : c), (x_1, \dots, x_n)) &\dashrightarrow (x_1 \cdot \frac{ax_2+b}{ax_2+c}, x_2, \dots, x_n). \end{aligned}$$

LEMMA 3.3. *The algebraic varieties and maps of Example 3.1 satisfy*

- (1) *The set $V \subseteq \text{Bir}(\mathbb{P}^n)$ is closed.*
- (2) *The map $\rho: \widehat{V} \rightarrow V$ is a topological quotient map.*

Proof. For (1), we show that $\tilde{\rho}(\widehat{V})$ is closed in H_2 . (Recall that π_2 is closed by Corollary 2.9.) Let $A \subseteq H_2$ be the closed set of elements $(h_0 : \cdots : h_n) \in H_2$ satisfying $h_0x_j = h_jx_0$ for $j = 2, \dots, n$. Each element of A is of the form $(x_0l_1 : l_2 : x_2l_1 : \cdots : x_nl_1)$, where l_i is a form of degree i . Imposing the conditions that l_1 depends only on x_0 and x_2 and that l_2 is a linear combination of x_1x_2 and x_0x_1 , we get

$$(x_0(ax_2 + cx_0) : x_1(dx_2 + bx_0) : x_2(ax_2 + cx_0) : \cdots : x_n(ax_2 + cx_0))$$

for some $a, b, c, d \in \mathbf{k}$ satisfying $(b, d) \neq (0, 0)$ and $(a, c) \neq (0, 0)$. Adding the condition that $a = d$ yields the closed set $\tilde{\rho}(\widehat{V})$.

(2) The description of $\tilde{\rho}(\widehat{V})$ shows that $\tilde{\rho}$ is a closed embedding. Since π_2 is closed and continuous (Corollary 2.9), so is ρ . □

The following result implies Theorem 1.

PROPOSITION 3.4. *For $n, d \geq 2$, the following hold:*

- (1) *There is no structure of algebraic variety (or ind-algebraic variety) on $\text{Bir}(\mathbb{P}^n)_{\leq d}$ such that algebraic A -families $A \rightarrow \text{Bir}(\mathbb{P}^n)$ having image in $\text{Bir}(\mathbb{P}^n)_{\leq d}$ correspond to morphisms of (ind)-algebraic varieties $A \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$.*
- (2) *There is no structure of ind-algebraic variety (or algebraic variety) on $\text{Bir}(\mathbb{P}^n)$, such that algebraic A -families $A \rightarrow \text{Bir}(\mathbb{P}^n)$ correspond to the morphisms of (ind)-algebraic varieties $A \rightarrow \text{Bir}(\mathbb{P}^n)$.*

Proof. Suppose, for contradiction, the existence of the ind-algebraic variety structure on $\text{Bir}(\mathbb{P}^n)_{\leq d}$ or $\text{Bir}(\mathbb{P}^n)$. Let $\rho: \widehat{V} \rightarrow \text{Bir}(\mathbb{P}^n)$ be the morphism given by Example 3.1. Its image is $V \subseteq \text{Bir}(\mathbb{P}^n)_{\leq 2} \subseteq \text{Bir}(\mathbb{P}^n)_{\leq d}$, which is closed in $\text{Bir}(\mathbb{P}^n)$ and hence in $\text{Bir}(\mathbb{P}^n)_{\leq d}$ (Lemma 3.3). The map ρ thus corresponds to a morphism of ind-algebraic varieties from \widehat{V} to $\text{Bir}(\mathbb{P}^n)_{\leq d}$ or $\text{Bir}(\mathbb{P}^n)$. Since \widehat{V} is an algebraic variety, the morphism factors through a morphism from \widehat{V} to a closed algebraic variety (of finite dimension). Since the image V of \widehat{V} is closed, the map $\widehat{V} \rightarrow V$ is then a morphism of algebraic varieties, so V is an irreducible variety of dimension 2. Lemma 3.3 asserts that this map is a topological quotient map. Hence, all closed sets of V correspond either to points, to V itself or to images of curves of \widehat{V} . In particular, all curves of V pass through the same point, which is impossible for an algebraic variety of dimension ≥ 2 . \square

Remark 3.5. In the above proof, the fact that $\rho: \widehat{V} \rightarrow V$ cannot be a morphism of algebraic varieties can be seen in another way. Extending to \mathbb{P}^2 , we would obtain a birational map $\mathbb{P}^2 \dashrightarrow V$, defined on the dense open subset \widehat{V} , which contracts the line $L \subseteq \widehat{V}$ onto a point. This is impossible, since the line has positive self-intersection and does not contain any base-point. It can also be verified that $\rho: \widehat{V} \rightarrow V$ is not a morphism of algebraic spaces or of algebraic stacks.

4. Topological obstructions to being an ind-algebraic group

To distinguish the topology on $\text{Cr}_n(\mathbf{k})$ from those of algebraic varieties, we define the following notion. (Note that the fact that \mathbf{k} is algebraically closed is important here.)

Definition 4.1. Let S be a topological space. We say that a closed point $p \in S$ is an *attractive point of S* if S contains an infinite proper closed subset, and if p is contained in every infinite closed set $F \subseteq S$.

Recall that an ind-algebraic variety Y corresponds to a sequence $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \dots$ of algebraic varieties, each closed in the next one. There are two classical ways of putting a topology on Y . The first, introduced by Shafarevich [Sha81] corresponds to saying that a set $F \subseteq Y$ is closed if $F \cap Y_i$

is closed in Y_i for each i . The second, introduced by Kambayashi [Kam96], [Kam03], is defined by taking zero sets of functions obtained by projective limits of functions on the Y_i , when all the Y_i are affine. (If the Y_i are not affine we can define the topology on sequences of affine subvarieties as is done in the usual way for varieties.) Following [Sta12], we call the first one the ind-topology and the second one the Zariski topology. Note that the two topologies are obviously the same on algebraic varieties; however, they differ on most ind-algebraic varieties (see [Sta12]).

LEMMA 4.2. *Let X be an ind-algebraic variety. Putting the Zariski or ind-topology on X , the set of attractive points of any closed irreducible subset of X is empty.*

Proof. We let $Y \subseteq X$ be an irreducible closed subset. We want to prove that the topological space given by the points of Y does not contain any attractive points.

Assume first that Y is an algebraic variety of dimension ≥ 2 . Then, for any point $p \in Y$, we can find a closed curve $\Gamma \subseteq Y$ that does not pass through p . Consequently, Y does not contain any attractive points. The same holds if there is a closed subset $Z \subseteq Y$ that is an algebraic variety of dimension ≥ 2 . If Y is a point or an irreducible curve, it does not contain any infinite closed proper subset and so does not contain any attractive points by definition.

It remains to study the case in which Y is an ind-variety, being the union $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \dots$, where all Y_i are algebraic varieties of dimension ≤ 1 and where $Y_i \neq Y_{i+1}$ for each i . We take a point $p \in Y$ and denote by k an integer such that $p \in Y_k$. To obtain the result, we construct a closed set $F \subseteq Y$ that is infinite and does not contain p .

If Y is endowed with the ind-topology, we choose $q_i \in Y_i \setminus Y_{i-1}$ for any $i \geq k+1$ and let F be the union of all q_i .

If Y is endowed with the Zariski topology, we take a sequence $Z_1 \subseteq Z_2 \subseteq \dots$ of affine varieties of dimension ≤ 1 , where Z_i is open and dense in Y_i for each i ; this corresponds to removing a finite number of points from each nonaffine curve in Y_i . We can choose $p \in Z_k$ and $Z_i \neq Z_{i+1}$ for each i . For $i = 1, \dots, k$, let $f_i \in \mathbf{k}[Z_i]$ be the constant function 1. Then, for each $i \geq k+1$, we choose a function $f_i \in \mathbf{k}[Z_i]$ that restricts to f_{i-1} on Z_{i-1} and vanishes at some point of $Z_i \setminus Z_{i-1}$. The vanishing set F' of the $(f_i)_{i \in \mathbb{N}}$ is infinite, closed in the union of the Z_i and does not contain p . The set F that we require can be chosen as the closure of F' in Y . \square

Remark 4.3. Let us note that the second case considered in the proof of Lemma 4.2 can actually occur, in the sense that Y might actually be irreducible! Indeed, consider the ind-variety Y given by the filtration $(Y_n)_{n \geq 1}$, where Y_n is

the zero set of $(x - 1)(x - 2) \cdots (x - n)(y - 1)(y - 2) \cdots (y - n)$ in the complex affine plane $\mathbb{A}_{\mathbb{C}}^2 = \text{Spec } \mathbb{C}[x, y]$; then Y is irreducible for the ind-topology (and hence for the Zariski topology, which is weaker).

The following result implies Theorem 2.

PROPOSITION 4.4. *If $n \geq 2$, any element $\varphi \in \text{Cr}_n(\mathbf{k})$ is an attractive point of an irreducible closed subset $Y \subseteq \text{Cr}_n(\mathbf{k})$.*

Proof. We may assume that φ is the identity, since the map $\text{Cr}_n(\mathbf{k}) \rightarrow \text{Cr}_n(\mathbf{k})$ given by $\varphi' \mapsto \varphi \circ \varphi'$ is a homeomorphism. We use the map $\rho: \widehat{V} \rightarrow V \subseteq \text{Bir}(\mathbb{P}^n) \cong \text{Cr}_n(\mathbf{k})$ of Example 3.1. By Lemma 3.3, V is closed in $\text{Cr}_n(\mathbf{k})$, and its topology is given by the quotient $\widehat{V} \rightarrow V$. The set \widehat{V} is irreducible, contains infinitely many infinite closed subsets (the irreducible ones corresponding to itself and curves) and each intersects $L = (\rho)^{-1}(\text{id})$. This shows that V is irreducible, contains infinitely many infinite closed subsets and that each contains the identity. \square

5. Euclidean topology

On the points of a real or complex algebraic variety, we can put the Euclidean (also called transcendental) topology, which is finer than the Zariski topology. This gives any algebraic group the structure of a topological group. We imitate this in this section, working over a local field k . We will assume that k is locally compact and nondiscrete. Particular cases are \mathbb{C} , \mathbb{R} , $\mathbb{F}_q((t))$ or a finite extension of \mathbb{Q}_p .

We fix some integer n , which will be the dimension, and we will prove that $\text{Cr}_n(k)$ is a topological group, endowed with the *Euclidean topology*, which we will define. Note that the set of \tilde{k} -points, where \tilde{k} is a subfield of k , will also inherit the structure of a topological group, such as for example $\text{Cr}_n(\mathbb{Q})$.

For $n = 1$, the group $\text{Cr}_n(k) = \text{Aut}(\mathbb{P}_k^1) = \text{PGL}(2, k)$ is obviously a topological group, so we will deal only with the cases in which $n \geq 2$.

In Section 5.1, we define the Euclidean topology on $\text{Bir}(\mathbb{P}^n)_{\leq d}$ and show that the natural inclusion $\text{Bir}(\mathbb{P}^n)_{\leq d} \hookrightarrow \text{Bir}(\mathbb{P}^n)_{\leq d+1}$ is a closed embedding. This allows us, in Section 5.2, to define the Euclidean topology on $\text{Bir}(\mathbb{P}^n)$ as the inductive limit topology induced by those of $\text{Bir}(\mathbb{P}^n)_{\leq d}$: a subset $F \subseteq \text{Bir}(\mathbb{P}^n)$ is closed if and only if $F \cap \text{Bir}(\mathbb{P}^n)_{\leq d}$ is closed in $\text{Bir}(\mathbb{P}^n)_{\leq d}$ for each d .

5.1. *The Euclidean topology on $\text{Bir}(\mathbb{P}^n)_{\leq d}$.* We use the notation W_d, H_d as in Definition 2.3; these varieties are defined over any field. By Lemma 2.4, W_d is a projective space and H_d is locally closed in W_d for the Zariski topology.

We can thus put the Euclidean topology on the projective space W_d . For example, we say that the distance between $(x_0 : \cdots : x_r)$ and $(y_0 : \cdots : y_r)$ is

equal to

$$\left(\sum_{i < j} |x_i y_j - y_j x_i|^2 \right) / \left(\left(\sum_i |x_i|^2 \right) \cdot \left(\sum_i |y_i|^2 \right) \right)$$

(see [Wey39]). We then put the induced topology on H_d . Because of the behaviour of the Zariski topology of $\text{Bir}(\mathbb{P}^n)$, it is natural to give the following definition.

Definition 5.1. The Euclidean topology on $\text{Bir}(\mathbb{P}^n)_{\leq d}$ will be the quotient topology induced by the surjective map $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$, where we put the Euclidean topology on H_d .

If $f: X \rightarrow Y$ is a quotient map between topological spaces and A is a subspace of X , note that the induced map $A \rightarrow f(A)$ is not always a quotient map. However, this becomes true if A is open and $A = f^{-1}(f(A))$ [Bou98, I, §3.6, Cor. 1]. Since $H_{d,d} = (\pi_d)^{-1}(\text{Bir}(\mathbb{P}^n)_d)$ is open in H_d for the Zariski topology (see Proposition 2.15) and hence also for the Euclidean topology, π_d restricts to a homeomorphism $(\pi_d)^{-1}(\text{Bir}(\mathbb{P}^n)_d) \rightarrow \text{Bir}(\mathbb{P}^n)_d$ for any $d \geq 1$.

LEMMA 5.2. *Let $d \geq 1$ be an integer. Endowed with the Euclidean topology, W_d and H_d are locally compact metric spaces. In particular, the sets $W_d, H_d, \text{Bir}(\mathbb{P}^n)_{\leq d}$ are sequential spaces: a subset F is closed if the limit of every convergent sequence with values in F belongs to F .*

Proof. By construction of the topology, W_d and H_d are metric spaces. Since W_d is compact and H_d is locally closed in W_d (Lemma 2.4), H_d is locally compact. It remains to recall that metric spaces are sequential spaces and that quotients of sequential spaces are sequential (see [Fra65]). \square

Remark 5.3. Following [Bou98], we say that a map $f: X \rightarrow Y$ between two topological spaces is *proper* if it is continuous and universally closed; i.e., for each topological space Z , the map $f \times \text{id}_Z: X \times Z \rightarrow Y \times Z$ is closed. We also say that a topological space is locally compact if it is Hausdorff and if each of its points has a compact neighbourhood.

LEMMA 5.4. *For any $d \geq 1$, the following hold:*

- (1) *the topological map $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$ is proper (and closed);*
- (2) *the topological space $\text{Bir}(\mathbb{P}^n)_{\leq d}$ is locally compact (and Hausdorff).*

Proof. We recall the following result of topology [Bou98, I, §10.4, Prop. 9]: if $f: X \rightarrow Y$ is a quotient map between topological spaces such that X is locally compact, then f is proper if and only if it is closed and the preimages of points are compact. This implies, moreover, that Y is locally compact.

The fact that $(\pi_d)^{-1}(\varphi)$ is compact for any $\varphi \in \text{Bir}(\mathbb{P}^n)_{\leq d}$ follows from Lemma 2.4 since $(\pi_d)^{-1}(\varphi)$ is closed in the compact space W_d . Since the topological space H_d is locally compact (Lemma 5.2), it suffices to see that π_d is closed.

We let $F \subseteq H_d$ be a closed subset. We want to prove that $\pi_d(F)$ is closed in $\text{Bir}(\mathbb{P}^n)_{\leq d}$, which amounts to showing that the saturated set $\widehat{F} = (\pi_d)^{-1}(\pi_d(F))$ is closed in H_d . For this, we take a sequence $(\varphi_i)_{i \in \mathbb{N}}$ of elements in \widehat{F} , which converges to $\varphi \in H_d$, and show that $\varphi \in \widehat{F}$.

Since the map π_d is continuous by construction, the sequence $(\pi_d(\varphi_i))_{i \in \mathbb{N}}$ converges to $\pi_d(\varphi)$ in $\text{Bir}(\mathbb{P}^n)_{\leq d}$. Replacing $(\varphi_i)_{i \in \mathbb{N}}$ by a subsequence, we may assume that the degree of all $(\pi_d(\varphi_i))_{i \in \mathbb{N}}$ is constant, equal to $m \leq d$. If $m = d$, then $(\pi_d)^{-1}(\pi_d(\varphi_i)) = \{\varphi_i\}$ for each i , so each φ_i belongs to F , which implies that $\varphi \in F \subseteq \widehat{F}$ and completes the proof. We may thus assume that $m < d$ (and hence that $d \geq 2$). We denote by k the difference $d - m \geq 1$. For each i , there exists a nonzero homogeneous polynomial $a_i \in k[x_0, \dots, x_n]$ of degree k such that

$$\varphi_i = (a_i f_{i,0} : \dots : a_i f_{i,n}),$$

and $(f_{i,0} : \dots : f_{i,n}) \in W_m$ corresponds to a birational map of degree $m < d$. Since each a_i is defined up to a constant, and since $\mathbb{P}(k[x_0, \dots, x_n])$ is compact, we can take a subsequence and assume that $(a_i)_{i \in \mathbb{N}}$ converges to a nonzero homogeneous polynomial $a \in k[x_0, \dots, x_n]$ of degree k .

Then, we can again take a subsequence and assume that $\{(f_{i,0} : \dots : f_{i,n})\}_{i \in \mathbb{N}}$ converges to an element $(f_0 : \dots : f_n)$ of the set W_m , which is also a projective space. Since $(\varphi_i)_{i \in \mathbb{N}}$ converges to φ , we get $\varphi = (af_0 : \dots : af_n)$ in H_d .

For each i , there exists $\varphi'_i \in F$ with $\pi_d(\varphi'_i) = \pi_d(\varphi_i)$ (because $\varphi_i \in \widehat{F} = (\pi_d)^{-1}(\pi_d(F))$). We then have $\varphi'_i = (b_i f_{i,0} : \dots : b_i f_{i,n})$ for some nonzero homogeneous polynomial $b_i \in k[x_0, \dots, x_n]$ of degree k . We may again assume that the sequence $(b_i)_{i \in \mathbb{N}}$ converges to a nonzero homogeneous polynomial $b \in k[x_0, \dots, x_n]$ of degree k . In particular, the sequence $(\varphi'_i)_{i \in \mathbb{N}}$ converges to $(bf_0 : \dots : bf_n)$, which is in F since F is closed. This implies that $\varphi = (af_0 : \dots : af_n)$ belongs to \widehat{F} . □

Remark 5.5. For any $d \geq 2$, the map $\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$ is not open. To see this, we define $f_m = (x_0 x_2^{d-1} : x_1(x_2^{d-1} + \frac{1}{m} x_0^{d-1}) : x_2^d : x_3 x_2^{d-1} : \dots : x_n x_2^{d-1}) \in H_d$ for $m \geq 1$. Since $\{f_m\}_{m \in \mathbb{N}}$ converges in H_d to $f_\infty \in \pi_d^{-1}(\text{id})$, we get a sequence $\{\pi_d(f_m)\}_{m \in \mathbb{N}}$ of elements of degree d converging to the identity. Let g be any element of $\pi_d^{-1}(\text{id})$ such that $g \neq f_\infty$, and let $U \subseteq H_d$ be an open neighbourhood of g not containing any of the f_m . Then, $\pi_d(U)$ is not open since $\{\pi_d(f_m)\}_{m \in \mathbb{N}}$ converges to $\text{id} \in \pi_d(U)$ but $\pi_d(U)$ does not contain any of the $\pi_d(f_m)$.

LEMMA 5.6. *For any positive integer d , the natural injection $\iota_d: \text{Bir}(\mathbb{P}^n)_{\leq d} \hookrightarrow \text{Bir}(\mathbb{P}^n)_{\leq d+1}$ is a closed embedding, i.e., a homeomorphism onto its image, which is closed in $\text{Bir}(\mathbb{P}^n)_{\leq d+1}$.*

Proof. We define a map $\widehat{\iota}_d: H_d \rightarrow H_{d+1}$ by $\widehat{\iota}_d((f_0 : \dots : f_n)) = (x_0 f_0 : \dots : x_0 f_n)$. It is a morphism of algebraic varieties, which is a closed immersion; it is thus continuous and closed with respect to the Euclidean topology. We have the following commutative diagram:

$$\begin{array}{ccc} H_d & \xrightarrow{\widehat{\iota}_d} & H_{d+1} \\ \downarrow \pi_d & & \downarrow \pi_{d+1} \\ \text{Bir}(\mathbb{P}^n)_{\leq d} & \xrightarrow{\iota_d} & \text{Bir}(\mathbb{P}^n)_{\leq d+1}. \end{array}$$

a) The continuity of $\widehat{\iota}_d$ directly implies the continuity of ι_d . Indeed, if U is an open subset of $\text{Bir}(\mathbb{P}^n)_{\leq d+1}$, the equality $(\pi_d)^{-1}((\iota_d)^{-1}(U)) = (\pi_{d+1}\widehat{\iota}_d)^{-1}(U)$ shows that $(\pi_d)^{-1}((\iota_d)^{-1}(U))$ is open in H_d , i.e., that $(\iota_d)^{-1}(U)$ is open in $\text{Bir}(\mathbb{P}^n)_{\leq d}$.

b) It is clear that ι_d is injective. We only need to prove that it is closed. Since π_{d+1} and $\widehat{\iota}_d$ are closed, so is $\pi_{d+1} \circ \widehat{\iota}_d = \iota_d \circ \pi_d$. Because π_d is continuous and surjective, this implies that ι_d is closed. \square

5.2. *The Euclidean topology on the Cremona group.* Using Lemma 5.6, one can put on $\text{Bir}(\mathbb{P}^n)$ the inductive limit topology given by the $\text{Bir}(\mathbb{P}^n)_{\leq d}$: a subset of $\text{Bir}(\mathbb{P}^n)$ is closed (respectively open) if and only if its intersection with each $\text{Bir}(\mathbb{P}^n)_{\leq d}$ is closed (respectively open). In particular, the injections $\text{Bir}(\mathbb{P}^n)_{\leq d} \hookrightarrow \text{Bir}(\mathbb{P}^n)$ are closed embeddings. As explained earlier, the topology defined here is called the *Euclidean topology of $\text{Bir}(\mathbb{P}^n)$* .

In this subsection we show that $\text{Bir}(\mathbb{P}^n)$, endowed with the Euclidean topology, is a topological group.

LEMMA 5.7. *For any $d \geq 1$, the map $I_d: \text{Bir}(\mathbb{P}^n)_{\leq d} \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d^{n-1}}$ that sends an element onto its inverse is continuous.*

Proof. As in the proof of Lemma 2.4, we define $Y \subseteq W_{d^{n-1}} \times W_d$ to be the set of elements (g, f) such that $(g_0(f_0, \dots, f_n), \dots, g_n(f_0, \dots, f_n))$ is a multiple (maybe 0) of the identity and let $U \subseteq W_d$ (resp. $U' \subseteq W_{d^{n-1}}$) be the set of elements having a nonzero Jacobian.

As we have already observed, Y is closed in $W_{d^{n-1}} \times W_d$ and U is open in W_d , so that $L := Y \cap (W_{d^{n-1}} \times U) = Y \cap (U' \times U)$ is locally closed in $W_{d^{n-1}} \times W_d$ (it is an algebraic variety).

The projection on the second factor induces a surjective morphism $\eta_2: L \rightarrow H_d$. The projection on the first factor is a morphism $\eta_1: L \rightarrow H_{d^{n-1}}$, which is not surjective in general. By construction, we have the following

commutative diagram:

$$\begin{array}{ccc}
 H_d & \xleftarrow{\eta_2} L & \xrightarrow{\eta_1} H_{d^{n-1}} \\
 \downarrow \pi_d & & \downarrow \pi_{d^{n-1}} \\
 \text{Bir}(\mathbb{P}^n)_{\leq d} & \xrightarrow{I_d} & \text{Bir}(\mathbb{P}^n)_{\leq d^{n-1}}.
 \end{array}$$

We claim that η_2 is a closed map for the Euclidean topology. Since $W_{d^{n-1}}$ is compact, the second projection $W_{d^{n-1}} \times W_d \rightarrow W_d$ is a closed map. Its restriction to the closed subset $Y \subseteq W_{d^{n-1}} \times W_d$ obviously yields a closed map $\eta'_2 : Y \rightarrow W_d$. Finally, let us recall that if $\varphi : A \rightarrow B$ is any continuous closed map between topological spaces and if C is any subset of B , then φ induces a continuous closed map $\varphi^{-1}(C) \rightarrow C$. Therefore, since $L = (\eta'_2)^{-1}(H_d)$, the claim is proved.

For any $F \subseteq \text{Bir}(\mathbb{P}^n)_{\leq d^{n-1}}$, we have $\eta_2((\pi_{d^{n-1}}\eta_1)^{-1}(F)) = (I_d\pi_d)^{-1}(F)$. Indeed, both sets correspond to elements $(f_0 : \dots : f_n) \in W_d$ such that the rational map ψ_f is the inverse of an element of F .

Assume that F is closed in $\text{Bir}(\mathbb{P}^n)_{\leq d^{n-1}}$. Since η_1 and $\pi_{d^{n-1}}$ are continuous for the Euclidean topology, the set $F_L = (\pi_{d^{n-1}}\eta_1)^{-1}(F)$ is closed in L . Therefore $\pi_d^{-1}(I_d^{-1}(F)) = (I_d\pi_d)^{-1}(F) = \eta_2(F_L)$ is closed in H_d , which implies that $I_d^{-1}(F)$ is closed in $\text{Bir}(\mathbb{P}^n)_{\leq d}$. \square

COROLLARY 5.8. *The map $I : \text{Bir}(\mathbb{P}^n) \rightarrow \text{Bir}(\mathbb{P}^n)$ that sends a map onto its inverse is a homeomorphism.*

Proof. Because I is its own inverse, we only need to prove that I is continuous. As we mentioned before, the degree of the inverse of a birational transformation of \mathbb{P}^n of degree d has degree at most d^{n-1} . Consequently, I restricts to an injective map $I_d : \text{Bir}(\mathbb{P}^n)_{\leq d} \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d^{n-1}}$ for any $d \geq 1$. Because of the definition of the topology of $\text{Bir}(\mathbb{P}^n)$, it suffices to prove that I_d is continuous for each d . This follows from Lemma 5.7. \square

LEMMA 5.9. *For any d, k , the map*

$$\chi_{d,k} : \text{Bir}(\mathbb{P}^n)_{\leq d} \times \text{Bir}(\mathbb{P}^n)_{\leq k} \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq dk}$$

that sends (φ_1, φ_2) onto $\varphi_1 \circ \varphi_2$ is continuous.

Proof. We use the following commutative diagram:

$$\begin{array}{ccc}
 H_d \times H_k & \xrightarrow{\widehat{\chi}_{d,k}} & H_{dk} \\
 \downarrow \pi_d \times \pi_k & & \downarrow \pi_{dk} \\
 \text{Bir}(\mathbb{P}^n)_{\leq d} \times \text{Bir}(\mathbb{P}^n)_{\leq k} & \xrightarrow{\chi_{d,k}} & \text{Bir}(\mathbb{P}^n)_{\leq dk},
 \end{array}$$

where $\widehat{\chi}_{d,k}$ sends $((f_0 : \cdots : f_n), (g_0 : \cdots : g_n))$ onto $(f_0(g_0, \dots, g_n) : \cdots : f_n(g_0, \dots, g_n))$. Since $\widehat{\chi}_{d,k}$ is a morphism of algebraic varieties, it is continuous (for the Euclidean topology). For any closed subset $F \subseteq \text{Bir}(\mathbb{P}^n)_{\leq dk}$, the set $(\pi_{dk}\widehat{\chi}_{d,k})^{-1}(F)$ is then closed in $H_d \times H_k$. Since the diagram is commutative, we have $(\pi_{dk}\widehat{\chi}_{d,k})^{-1}(F) = (\pi_d \times \pi_k)^{-1}(K)$, where $K = (\chi_{d,k})^{-1}(F)$. It remains to prove that $\pi_d \times \pi_k$ is a quotient map, which will imply that K is closed and thus give the continuity of $\chi_{d,k}$.

If $f : X \rightarrow Y$ is a quotient map between topological spaces and Z is locally compact, a theorem of Whitehead [Whi48, Lemma 4] asserts that $f \times \text{id} : X \times Z \rightarrow Y \times Z$ is a quotient map. More generally, if $f : X \rightarrow Y$ and $g : Z \rightarrow W$ are quotient maps and Y and Z are locally compact, then the product $f \times g : X \times Z \rightarrow Y \times W$ is a quotient map. (Use the Whitehead theorem twice, since $f \times g = (\text{id} \times g) \circ (f \times \text{id})$.)

Alternatively, one can avoid using Whitehead’s theorem by noting that the product of two proper maps is proper ([Bou98, I, §10.1, Prop. 4]), so that $\pi_d \times \pi_k$ is proper and hence closed. This implies that $\pi_d \times \pi_k$ is a quotient map. □

COROLLARY 5.10. *The map $P : \text{Bir}(\mathbb{P}^n) \times \text{Bir}(\mathbb{P}^n) \rightarrow \text{Bir}(\mathbb{P}^n)$ that sends (φ, φ') onto $\varphi \circ \varphi'$ is continuous (where we endow $\text{Bir}(\mathbb{P}^n) \times \text{Bir}(\mathbb{P}^n)$ with the product topology of the Euclidean topology of $\text{Bir}(\mathbb{P}^n)$).*

Proof. Because of the definition of the topology of $\text{Bir}(\mathbb{P}^n)$, it suffices to prove that the restriction $\chi_{d,k} : \text{Bir}(\mathbb{P}^n)_{\leq d} \times \text{Bir}(\mathbb{P}^n)_{\leq k} \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq dk}$ is continuous for each d, k . This follows from Lemma 5.9. □

Corollaries 5.8 and 5.10 complete the proof that $\text{Bir}(\mathbb{P}^n)$ is a topological group.

5.3. Restriction of the topology on algebraic subgroups. As we saw in Section 2.6, an algebraic subgroup of $\text{Bir}(\mathbb{P}^n)$ corresponds to a Zariski-closed subgroup $G \subseteq \text{Bir}(\mathbb{P}^n)$ of bounded degree; moreover, there exists an algebraic group K , together with a morphism $K \rightarrow \text{Bir}(\mathbb{P}^n)$ inducing a homeomorphism $\pi : K \rightarrow G$, which is a group homomorphism (Corollary 2.18).

PROPOSITION 5.11. *Let $G \subseteq \text{Bir}(\mathbb{P}^n)$ be a Zariski-closed subgroup of bounded degree. Let K be its associated algebraic group (as in Corollary 2.18). Putting on G the restriction of the Euclidean topology of $\text{Bir}(\mathbb{P}^n)$, we obtain the Euclidean topology of the algebraic group K , via the bijection $\pi : K \rightarrow G$, which becomes a homeomorphism.*

Proof. Via the action of G on itself by multiplication, we may restrict ourselves to the case where G is connected. In this case, we may assume that K is a Zariski-closed subset of H_d and that $\pi : K \rightarrow G$ is induced by

$\pi_d: H_d \rightarrow \text{Bir}(\mathbb{P}^n)_{\leq d}$ (see Proposition 2.17). Then, K is also closed for the Euclidean topology. The map π_d restricts to a bijection $K \rightarrow G$, which is closed and continuous for the Euclidean topology (Lemma 5.4) and is thus a homeomorphism. \square

5.4. *Properties of the Euclidean topology of $\text{Bir}(\mathbb{P}^n)$.*

LEMMA 5.12. *The topological group $\text{Bir}(\mathbb{P}^n)$ is Hausdorff.*

Proof. We recall that a topological group is Hausdorff if and only if the trivial one-element subgroup is closed (see [Bou98, III, §2.5, Prop. 13]). Since any point of $\text{Bir}(\mathbb{P}^n)$ is closed (being closed in some $\text{Bir}(\mathbb{P}^n)_{\leq d}$, by Lemma 2.4), this implies that $\text{Bir}(\mathbb{P}^n)$ is Hausdorff. \square

LEMMA 5.13. *Any compact subset of $\text{Bir}(\mathbb{P}^n)$ is contained in $\text{Bir}(\mathbb{P}^n)_{\leq d}$ for some d .*

Proof. Assume by contradiction that K is a compact subset of $\text{Bir}(\mathbb{P}^n)$ containing a sequence $(\varphi_i)_{i \in \mathbb{N}}$ with $\deg(\varphi_{i+1}) > \deg(\varphi_i)$ for each i . Since $K' := \{\varphi_i \mid i \in \mathbb{N}\}$ is a closed subset of the compact set K , it should be compact. However, the intersection of any subset of K' with $\text{Bir}(\mathbb{P}^n)_{\leq d}$ is closed, so K' is an infinite set endowed with the discrete topology; hence it cannot be compact. \square

COROLLARY 5.14. *Any convergent sequence of $\text{Bir}(\mathbb{P}^n)$ has bounded degree.*

Proof. Indeed, if the sequence $(\varphi_i)_{i \in \mathbb{N}}$ of $\text{Bir}(\mathbb{P}^n)$ converges to φ , then $K := \{\varphi_i \mid i \in \mathbb{N}\} \cup \{\varphi\}$ is compact. \square

LEMMA 5.15. *For $n \geq 2$, the topological space $\text{Bir}(\mathbb{P}^n)$ is not locally compact.*

Proof. Let $U \subseteq \text{Bir}(\mathbb{P}^n)$ be an open neighbourhood of the identity. Let us show that U is not contained in any compact subset of $\text{Bir}(\mathbb{P}^n)$. By Lemma 5.13, it suffices to show that U contains elements of arbitrarily large degree. For any integers $m, k \geq 1$, we consider the birational map $f_{m,k}$ of \mathbb{A}_k^n given by

$$f_{m,k}: (x_1, x_2, \dots, x_n) \dashrightarrow \left(x_1 + \frac{1}{k}x_2^m, x_2, x_3, \dots, x_n\right).$$

Fixing m , we observe that the sequence $\{f_{m,k}\}_{k \geq 1}$ converges to the identity. In particular, $f_{m,k}$ belongs to U when k is large enough. \square

LEMMA 5.16. *For $n \geq 2$, the topological space $\text{Bir}(\mathbb{P}^n)$ is not metrisable.*

Proof. We consider the set $k[X]$ of polynomials in one variable and let $k[X] \hookrightarrow \text{Aut}(\mathbb{A}^n) \subset \text{Bir}(\mathbb{P}^n)$ be the inclusion sending P to

$$(x_1, x_2, \dots, x_n) \dashrightarrow (x_1 + P(x_2), x_2, x_3, \dots, x_n).$$

Note that $k[X]$ is closed in $\text{Bir}(\mathbb{P}^n)$ and that for any d , the induced topology on $k[X]_{\leq d}$ is the topology as a vector space (or as an algebraic group). The induced topology on $k[X]$ is thus the inductive limit topology given by

$$k[X]_{\leq 1} \subseteq k[X]_{\leq 2} \subseteq \cdots .$$

For any sequence $l = (l_n)_{n \in \mathbb{N}}$ of positive integers, the set $U_l = \{\sum_{i=0}^d a_i X^i \mid |a_i| < 1/l_i\}$ is open in $k[X]$. This implies that $k[X]$ is not first countable and thus not metrisable. The same holds for $\text{Bir}(\mathbb{P}^n)$. \square

LEMMA 5.17. *The topological group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ is compactly generated if and only if $n \leq 2$.*

Proof. The group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^1) = \text{PGL}(2, \mathbb{C})$ is a linear algebraic group and hence is compactly generated. By the classical Noether-Castelnuovo theorem, the group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is generated by $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C})$ and by the standard quadratic transformation $\sigma = (x : y : z) \dashrightarrow (yz : xz : xy)$. Since the linear algebraic group $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C})$ is compactly generated, so is $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$.

For $n \geq 3$, the group $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ is not generated by $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)_{\leq d}$ for any integer d . This can be seen by showing that the birational type of the hypersurfaces that are contracted by some element of $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)_{\leq d}$ is bounded. (See [Pan99] for more details.) The fact that $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ is not compactly generated follows from Lemma 5.13. \square

Remark 5.18. In [Bla10, Th. 5.1], it is proved that $\text{Bir}(\mathbb{P}_{\mathbb{k}}^n)$ is connected for the Zariski topology when \mathbb{k} is algebraically closed. Looking at the proof, we observe that $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ and $\text{Bir}(\mathbb{P}_{\mathbb{R}}^n)$ are in fact path connected (and hence connected) for the Euclidean topology. The same holds for the proof of [Bla10, Prop. 4.1, Th. 4.2], which can be adapted to see that $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is a simple topological group for the Euclidean topology (although it is not a simple group [CL13]).

Remark 5.19. Having the natural Euclidean topology on $\text{Bir}(\mathbb{P}^n)_{\leq d}$, there are many ways of extending it to the union of all $\text{Bir}(\mathbb{P}^n)_{\leq d}$, which is the Cremona group $\text{Bir}(\mathbb{P}^n)$; the one that we have chosen is the finest one.

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