

Weak Fano threefolds obtained by blowing-up a space curve and construction of Sarkisov links

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ABSTRACT

We characterize smooth curves in \mathbb{P}^3 whose blow-up produces a threefold with anticanonical divisor big and nef. These are curves C of degree d and genus g lying on a smooth quartic, such that (i) $4d - 30 \leq g \leq 14$ or $(g, d) = (19, 12)$, (ii) there is no 5-secant line, 9-secant conic nor 13-secant twisted cubic to C . This generalizes the classical similar situation for the blow-up of points in \mathbb{P}^2 .

We describe then Sarkisov links constructed from these blow-ups, and are able to prove the existence of Sarkisov links which were previously only known as numerical possibilities.

1. Introduction

A classical result in the theory of algebraic surfaces is that the blow-up of d points in \mathbb{P}^2 gives a *del Pezzo surface* (that is, a surface X with $-K_X$ ample) if and only if the points satisfy the following conditions.

- (i) The number d of points is at most eight.
- (ii) No three are collinear, no six are on the same conic and no eight belong to the same cubic being singular at one of the eight points.

More generally, the blow-up of the d points gives a *weak del Pezzo surface* (that is, a surface X with $-K_X$ nef. and big) if and only if the points satisfy the following conditions (see [5, III.2, Theorem 1]).

- (i) The number d of points is at most 8.
- (ii) No four are collinear, and no seven are on the same conic.

Moreover, in both cases conditions (ii) are open conditions on the set of d -uples of points of \mathbb{P}^2 for $1 \leq d \leq 8$.

Note that these statements can easily be extended to the blow-up of points in higher dimension: we shall derive in § 2.7 a similar characterization of weak Fano threefolds arising from the blow-up of d points in \mathbb{P}^3 .

In this paper, we study the similar but more difficult problem for the blow-up of a curve in \mathbb{P}^3 . We prove, in particular, that the blow-up of a smooth irreducible curve C of genus g and degree d in \mathbb{P}^3 is a weak-Fano threefold if and only if:

- (i) $4d - 30 \leq g \leq 14$ or $(g, d) = (19, 12)$;
- (ii) there is no 5-secant line, 9-secant conic nor 13-secant twisted cubic to C , and C is contained in a smooth quartic.

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We obtain in fact a more precise result, which is our main theorem below. Here, we denote by $\mathcal{H}_{g,d}^S$ the Hilbert scheme of smooth irreducible curves of genus g and degree d in \mathbb{P}^3 . We always work over the base field \mathbb{C} of complex numbers.

THEOREM 1.1 (see Table 1). *Let $C \in \mathcal{H}_{g,d}^S$ be a curve of genus g and degree d . We denote by X the blow-up of \mathbb{P}^3 along C and by $-K_X$ its anticanonical divisor. Consider the sets*

$$\begin{aligned}\mathcal{A}_0 &= \{(6, 5), (10, 6), (8, 8), (12, 9)\}; \\ \mathcal{A}_1 &= \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 3), (1, 4), (2, 5), (4, 6)\}; \\ \mathcal{A}_2 &= \{(1, 5), (3, 6), (5, 7), (10, 9)\}; \\ \mathcal{A}_3 &= \{(0, 5), (0, 6), (1, 6), (2, 6), (3, 4), \\ &\quad (3, 7), (4, 7), (6, 7), (6, 8), (7, 8), (9, 8), (9, 9), (12, 10), (19, 12)\}; \\ \mathcal{A}_4 &= \{(0, 7), (1, 7), (2, 7), (2, 8), (3, 8), \\ &\quad (4, 8), (5, 8), (6, 9), (7, 9), (8, 9), (10, 10), (11, 10), (14, 11)\}.\end{aligned}$$

These pairs form a partition of all (g, d) corresponding to non-empty $\mathcal{H}_{g,d}^S$ with $4d - 30 \leq g \leq 14$ or $(g, d) = (19, 12)$, and we have the following characterizations.

(i) *The variety X is Fano, that is, $-K_X$ is ample, if and only if $(g, d) \in \mathcal{A}_1$, or $(g, d) \in \mathcal{A}_2$ and there is no 4-secant line to C ;*

(ii) *The variety X is weak Fano, that is, $-K_X$ is big and nef., if and only if one of the following condition holds:*

- (a) $(g, d) \in \mathcal{A}_1 \cup \{(3, 6)\}$;
- (b) $(g, d) \in \mathcal{A}_2 \setminus \{(3, 6)\}$, and there is no 4-secant line to C ;
- (c) $(g, d) \in \mathcal{A}_3$, there is no 5-secant line to C , and C is contained in a smooth quartic;
- (d) $(g, d) \in \mathcal{A}_4$, there is no 5-secant line, 9-secant conic, nor 13-secant twisted cubic to C , and C is contained in a smooth quartic.

Moreover, Condition (i) corresponds to a non-empty open subset of $\mathcal{H}_{g,d}^S$ if $(g, d) \in \mathcal{A}_2$, and Condition (ii) corresponds to a non-empty open subset of $\mathcal{H}_{g,d}^S$ if $(g, d) \in \mathcal{A}_3 \cup \mathcal{A}_4$. Furthermore, for a general curve in these sets, there are finitely many irreducible curves intersecting trivially K_X , except for $(g, d) \in \{(3, 4), (6, 7), (9, 8), (12, 10), (19, 12)\}$.

Another aspect of our work is the construction of Sarkisov links. Recently, several articles appeared which contain lists of numerical possibilities for Sarkisov links between threefolds. In a series of two papers [16, 17], Jahnke, Peternell and Radloff embark on the classification of smooth threefolds X with big and nef. (but not ample) anticanonical divisor, and Picard number equal to 2. With these assumptions there are two contractions on X : a Mori contraction (of fibring type or divisorial type) and another one given by the linear system $|-mK_X|$ for $m \gg 0$, which we call the *anticanonical morphism*. If the latter one is a small contraction, which is the case treated in [17], then one can perform a flop to obtain another weak Fano threefold X' , with another Mori contraction. The whole picture is a Sarkisov link.

There are several cases depending on the type of the Mori contractions on X and X' . In [17] are classified all possible links where at least one of the contractions is of fibring type (conic bundle or del Pezzo fibration). In the paper by Cutrone and Marshburn [4], the classification is completed with a list of all possible links where both contractions are divisorial. We should also mention the paper by Kaloghiros [18] where she produces similar tables with a different motivation in mind. Note that in many cases the links appearing in these lists are only numerical possibilities, and the actual geometrical existence is left as an open problem.

TABLE 1. Pairs (g, d) with $(g, d) = (19, 12)$ or $4d - 30 \leq g \leq 14$ such that there exists a smooth curve $C \subset \mathbb{P}^3$ of genus g and degree d .

g	Plane	Quadric	Cubic	Quartic
0	(0, 1)(0, 2)	(0, 3)(0, 4)	(0, 5)(0, 6)	(0, 7)
1	(1, 3)	(1, 4)	(1, 5)(1, 6)	(1, 7)
2		(2, 5)	(2, 6)	(2, 7)(2, 8)
3	(3, 4)*		(3, 6)(3, 7)	(3, 8)
4		(4, 6)	(4, 7)	(4, 8)
5			(5, 7)	(5, 8)
6	(6, 5)	(6, 7)*	(6, 8)	(6, 9)
7			(7, 8)	(7, 9)
8		(8, 8)		(8, 9)
9		(9, 8)*	(9, 9)	
10	(10, 6)		(10, 9)	(10, 10)
11				(11, 10)
12		(12, 9)	(12, 10)*	
14				(14, 11)
...				
19			(19, 12)*	

The columns correspond to the minimal degree of a surface containing a general curve of type (g, d) . Furthermore, the blow-up X of such a general curve $C \subset \mathbb{P}^3$ has an anticanonical divisor $-K_X$ which is big; it is ample for bold pairs (\mathbf{g}, \mathbf{d}) . For the other cases, it is nef. except in the four cases crossed out, and the anticanonical morphism is a divisorial contraction if there is a star, and it is a small contraction otherwise.

TABLE 2. Pairs (g, d) such that a general curve in $\mathcal{H}_{g,d}^S$ does not lie on a cubic, and gives rise to a weak Fano threefold X with small anticanonical morphism, hence a Sarkisov link.

g	d	Link to	Reference(s)
0	7	$(0, 7) \subset \mathbb{P}^3$	[4, 90]
1	7	$(1, 7) \subset X_{22}$	[4, 98]
2	7	$(0, 5) \subset V_4$	[4, 103]
2	8	$(2, 8) \subset \mathbb{P}^3$	[4, 49]
3	8	$(3, 8) \subset \mathbb{P}^3$	[4, 75]
4	8	$(4, 10) \subset V_5$	[4, 89]
5	8	$(5, 8) \subset \mathbb{P}^3$	[4, 99]
6	9	$(6, 9) \subset \mathbb{P}^3$	[4, 50]
7	9	$(0, 3) \subset X_{12}$	[15, p. 103]
8	9	dP_5	[17, Proposition 6.5(25)]
10	10	$(10, 10) \subset \mathbb{P}^3$	[4, 51]
11	10	$(11, 10) \subset \mathbb{P}^3$	[4, 76]
14	11	$(14, 11) \subset \mathbb{P}^3$	[4, 52]

We also indicate the end product of the link, and a reference where the numerical possibility was announced.

As a consequence of our study, we obtain a proof of the existence of some previously hypothetical links. We gather in Table 2 the Sarkisov links, involving a flop and starting with the blow-up of a smooth space curve, whose existence was left open: they correspond to the set \mathcal{A}_4 , which is the fourth column in Table 1, and their existence is proved in § 5.4.

As a final remark, note that the truly new result in this paper lies in the last five lines in Theorem 1.1, about the generic existence of these curves. Indeed, it would be possible to recover the possible (g, d) from the papers [4, 17] cited above. However, we shall derive the bound on g and d starting from scratch, and it is interesting to keep in mind that both conditions (i) and (ii) are improvements on the following easy estimates. Once we know (by Riemann–Roch’s formula, see § 5.4) that the curve C is contained in many (in particular, two distinct) quartic surfaces, we have $d \leq 16$. Then condition (i) is really about bringing down this condition to $d \leq 12$. Similarly, a curve Γ of degree n and at least $4n + 1$ secant to C would forbid $-K_X$

to be nef.; but since by Bezout's Theorem such a bad curve must be contained in all quartic surfaces containing C , we have the crude estimate $n \leq 16 - d$. Then condition (ii) tells us that it is sufficient to check for the absence of bad curve in degree $n \leq 3$, and only among smooth rational ones (and in fact in many cases $n \leq 2$ or even 1 is sufficient, see Proposition 5.8 for details).

The organization of the paper is as follows.

After gathering in § 2 some preliminary results, we study the threefolds obtained by blowing-up a smooth curve contained in a surface $S \subset \mathbb{P}^3$ of small degree. This is motivated by the fact that if the blow-up X of a smooth curve $C \subset \mathbb{P}^3$ is a weak Fano threefold, then C is contained in a (possibly reducible) quartic.

The case $\deg(S) \leq 2$ in § 3 serves as a warm-up. This is essentially a nice exercise, and also the only case where we can do an exhaustive study: we do not make any assumption on the singularities of S or about the generality of C .

The case of a Sarkisov link starting with the blow-up of a curve in a smooth cubic was already found in the literature. In § 4, we give a fairly complete account of this case since we shall use a similar construction in the following harder case, and also generalize it to smooth curves contained in a normal rational cubic.

In § 5, we study the case when C is contained in a quartic surface but not in a cubic, where lies the principal difficulty of the paper. The proof of Theorem 1.1 is given in § 5.3, as a direct consequence of the previous study.

Since our motivation comes from the study of the Cremona group of rank 3 (remark that eight of the cases in Table 2 correspond to birational self-maps of \mathbb{P}^3), we focused the discussion on the already substantial case of Sarkisov links starting with the blow-up of a smooth curve in \mathbb{P}^3 . However, one could probably obtain in a similar way the existence of Sarkisov links starting from other Fano threefolds[†].

2. Preliminaries

2.1. Sarkisov links

Let $C \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d and let $\pi: X \rightarrow \mathbb{P}^3$ be its blow-up. We denote by $E = \pi^{-1}(C)$ the exceptional divisor, and f an exceptional curve. The Picard group $\text{Pic}(X)$ is generated by H and E , where H is the pull-back of a hyperplane. Moreover, $N_1(X)$ is generated by f and by l , the pull-back of a line. The intersection form on X is given by

$$H \cdot l = 1, \quad E \cdot f = -1, \quad H \cdot f = E \cdot l = 0.$$

The pseudo-effective cone $\overline{\text{Eff}}(X)$ is generated by E and R , where R should correspond to the strict transform of surfaces of degree n and passing with multiplicity m through C , with m/n maximal. It is not clear if the supremum is realized, so R might not be represented by an effective divisor.

Here are some situations where R is realized by an effective divisor, and where the contraction of R gives rise to a Sarkisov link.

If $-K_X$ is ample, then R is an extremal ray, and since X is smooth the contraction $X \rightarrow Y$ of R is either a divisorial contraction (Y is a terminal Fano threefold) or a Mori fibration (conic bundle or del Pezzo fibration).

[†]After this work was completed, Amad, Cutrone and Marshburn produced a follow-up to the paper [4] where they obtained the existence of many Sarkisov links. Interestingly their method is quite different from ours: see [1].

If $-K_X$ is not ample, but big and nef., and if the anticanonical morphism given by $|-mK_X|$, $m \gg 0$, corresponds to a small birational map, then R corresponds to finitely many curves on X and we can consider the flop $X \dashrightarrow X'$ of these curves. Then X' admits an extremal contraction which again can be of divisorial or fibring type.

In any case, we have a Sarkisov link of type II (from \mathbb{P}^3 to Y) or I (from \mathbb{P}^3 to X') of the form

$$\begin{array}{ccc}
 & X \dashrightarrow X' & \\
 \swarrow & & \searrow \\
 \mathbb{P}^3 & & Y
 \end{array} \tag{1}$$

where $X \dashrightarrow X'$ is a flop or an isomorphism.

Note that we would have a similar discussion replacing \mathbb{P}^3 by any smooth prime Fano threefold. More generally, we have:

PROPOSITION 2.1. *Assume that X is a smooth threefold with Picard number 2, big and nef. anticanonical divisor, and small anticanonical morphism. Then the two contractions on X yield a Sarkisov link (with or without a flop depending if $-K_X$ is ample).*

This was the point of view adopted in [17].

The effective cone of curves $\overline{NE}(X) \subset N_1(X)$ is generated by f and r , where r should correspond to the strict transforms of curves of degree m which intersect C in n points, with n/m maximal. Again, it is not clear if r is represented by an effective curve.

We shall use repetitively the following simple observation, which follows from the identity $-K_X = 4H - E$:

LEMMA 2.2. *The anticanonical divisor $-K_X$ is nef. if and only if there is no irreducible curve in $NE(X)$ which is equivalent to $ml - nf$ with $n > 4m$.*

Similarly, if $-K_X$ is ample, then there is no irreducible curve in $NE(X)$ which is equivalent to $ml - nf$ with $n \geq 4m$.

We will see these problematic curves either on X , as curves equivalent to $ml - nf$, or on \mathbb{P}^3 , as curves of degree m passing n times (counted with multiplicity) through C .

Note that if C is contained in a quartic surface S , then all curves corresponding to $n > 4m$ must lie in S . Furthermore, the curves we are interested in must lie on quartics, as we shall see in §2.6.

2.2. Linear system of surfaces containing a curve

We recall the following standard consequence of the Riemann–Roch formula for curves:

LEMMA 2.3. *Let $C \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d .*

For any integer $n > (-2 + 2g)/d$ the projective dimension of the linear system of hypersurfaces of \mathbb{P}^3 of degree n that contain C is at least equal to

$$\frac{(n + 1)(n + 2)(n + 3)}{6} - nd - 2 + g.$$

In particular, we have the following.

If $2g - 2 < d < 3 + g$, C is contained in a plane.

If $(2g - 2)/2 < d < (9 + g)/2$, C is contained in a quadric surface.

If $(2g - 2)/3 < d < (19 + g)/3$, C is contained in a cubic surface.

Proof. Let $n \geq 1$ be an integer, and denote by $\phi_n: \mathbb{P}^3 \rightarrow \mathbb{P}(V_n)$ the n th Veronese embedding, where V_n is the vector space of polynomials of degree n in four variables, of dimension $N = (n + 1)(n + 2)(n + 3)/6$. Note that $N = 4, 10, 20$ for $n = 1, 2, 3$.

We compose the embedding $C \hookrightarrow \mathbb{P}^3$, which is given by a linear system of degree d , with ϕ_n and obtain an embedding $\phi: C \hookrightarrow \mathbb{P}^{N-1}$ of degree dn . Denote by D a hyperplane section with respect to this embedding. If the projective dimension of the complete linear system $|D|$ of all effective divisors of degree dn is strictly smaller than $N - 1$, then $\phi(C)$ is contained in a hyperplane, which amounts to say that $C \subset \mathbb{P}^3$ is contained in a hypersurface of degree d .

Applying Riemann–Roch to the system on the curve, we obtain

$$l(D) - l(K - D) = \deg D + 1 - g,$$

where $l(D) = 1 + \dim|D|$ is the vectorial dimension of $|D|$.

Since we assume $n > (-2 + 2g)/d$, we have $\deg(K - D) = -2 + 2g - nd < 0$. So D is non-special and we have $\dim|D| = nd - g$. In consequence, if $nd - g < N - 1$ (or equivalently $d < (N - 1 + g)/n$), we find a hypersurface of degree d which contains the curve. More generally, the vectorial dimension of the linear system of hypersurfaces of degree d which contain the curve is at least equal to $N - nd - 1 + g$. □

2.3. Cube of the anticanonical divisor

LEMMA 2.4. *Let $C \subset Y$ be a smooth curve of genus g in a smooth threefold, and let $\pi: X \rightarrow Y$ be the blow-up of C . Then*

$$\begin{aligned} K_X^2 \cdot E &= -K_Y \cdot C + 2 - 2g; \\ (-K_X)^3 &= (-K_Y)^3 + 2K_Y \cdot C - 2 + 2g. \end{aligned}$$

In particular, if $Y = \mathbb{P}^3$ and $C \in \mathcal{H}_{g,d}^S$, then

$$\begin{aligned} K_X^2 \cdot E &= 2 + 4d - 2g; \\ (-K_X)^3 &= 62 - 8d + 2g; \end{aligned}$$

and $(-K_X)^3 > 0 \iff 4d - 30 \leq g$.

Proof. Denoting by E the exceptional divisor, we have $K_X = \pi^*K_Y + E$; hence

$$\begin{aligned} K_X^2 \cdot E &= \pi^*K_Y \cdot (\pi^*K_Y \cdot E) + 2(\pi^*K_Y \cdot E) \cdot E + E^3; \\ K_X^3 &= K_Y^3 + 3\pi^*K_Y \cdot (\pi^*K_Y \cdot E) + 3(\pi^*K_Y \cdot E) \cdot E + E^3. \end{aligned}$$

Furthermore, denoting by f a fibre of the ruled surface E we have (see [15, Lemma 2.2.14])

$$\begin{aligned} \pi^*K_Y \cdot E &= (K_Y \cdot C)f; & E \cdot f &= -1; \\ \pi^*K_Y \cdot f &= 0; & E^3 &= K_Y \cdot C + 2 - 2g. \end{aligned}$$

Hence

$$\begin{aligned} \pi^*K_Y \cdot (\pi^*K_Y \cdot E) &= 0; \\ (\pi^*K_Y \cdot E) \cdot E &= -K_Y \cdot C; \end{aligned}$$

and the result follows. □

2.4. A result of Mori

We shall use the following result from [23].

PROPOSITION 2.5. *Let $g \geq 0$ and $d \geq 1$ be two integers such that $8g < d^2$. Then there exists a smooth quartic surface $S \subset \mathbb{P}^3$ containing a smooth irreducible curve of genus g and degree d .*

Moreover, Mori also gives a converse statement that we shall need in the slightly more general setting of a reducible curve.

If $C = \bigcup C_i$ is a projective (possibly reducible and singular) curve, recall that the Hilbert polynomial $P_C(T)$ has the form $P_C(T) = dT + 1 - g$, where d is the degree of C and where by definition g is the *arithmetic genus* of C . If furthermore C lies inside a smooth surface S , the arithmetic genus g of C satisfies the adjunction formula $2g - 2 = (K_S + C) \cdot C$. In particular, if S is a K3 surface, $2g - 2 = C^2$.

The proof of the following result is taken almost verbatim from Mori [23], where the case of an irreducible curve is treated. We reproduce the argument for the convenience of the reader.

PROPOSITION 2.6. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface, and let $C = \bigcup C_i \subset S$ be a curve of degree d and genus g . Assume that C is not a complete intersection of S with another surface. Then $8g < d^2$.*

Proof. Take H a hyperplane section on S . We have

$$\begin{aligned} C \cdot H &= d, \\ H \cdot H &= 4, \\ H \cdot (dH - 4C) &= 0. \end{aligned}$$

Since C is not a complete intersection, $dH - 4C \neq 0$. Hence, by the Hodge index Theorem, we have

$$0 > (dH - 4C) \cdot (dH - 4C) = 4d^2 - 8d^2 + 16(2g - 2).$$

This gives $d^2 - 8g > -8$. To show that $d^2 - 8g > 0$, we have to exclude the cases $d^2 - 8g = -7, -4$ or 0 (other values are not square modulo 8).

If $d^2 - 8g = 0$, then $d = 4d'$ for some $d' \in \mathbb{Z}$. Let $E = d'H - C$. Then

$$\begin{aligned} E \cdot H &= (d'H - C) \cdot H = 4d' - d = 0, \\ E^2 &= d'^2 H^2 + C^2 - 2d'H \cdot C = 4 \frac{d^2}{16} + 2g - 2 - 2 \frac{d^2}{4} = -2. \end{aligned}$$

Since a quartic surface has trivial canonical divisor and arithmetic genus 1, the Riemann–Roch formula on S gives:

$$l(E) + l(-E) \geq \frac{E^2}{2} + 2 = 1.$$

Hence, E or $-E$ is an effective curve, but then $H \cdot E = 0$ contradicts H ample.

If $d^2 - 8g = -7$, then $d = 2d' - 1$ for some $d' \in \mathbb{Z}$. Let $E = d'H - 2C$. Then

$$\begin{aligned} E \cdot H &= (d'H - 2C) \cdot H = 4d' - 2d = 2, \\ E^2 &= d'^2 H^2 + 4C^2 - 4d'H \cdot C = 4 \frac{(d+1)^2}{4} + 8g - 8 - 2d(d+1) = 0. \end{aligned}$$

The Riemann–Roch formula on S gives:

$$l(E) + l(-E) \geq \frac{E^2}{2} + 2 = 2.$$

Since $E \cdot H = 2$, E is effective and has at most two irreducible components. So E is a line or a (possibly degenerate) conic. But this contradicts $E^2 = 0$ (by adjunction, a smooth rational curve on S has square -2).

Finally, if $d^2 - 8g = -4$, then $d = 4d' - 1$ for some $d' \in \mathbb{Z}$. Let $E = d'H - C$. Then again $E \cdot H = 2$, $E^2 = 0$ and we can repeat the previous argument. □

2.5. *Linkage*

Let C and C' be (possibly singular or reducible) curves in \mathbb{P}^3 , of respective arithmetic genus and degree (g, d) and (g', d') . We say that there is a *linkage* of type $[n_1, n_2]$ between C and C' , if $C \cup C' \in \mathbb{P}^3$ is the complete intersection of two surfaces of respective degrees n_1 and n_2 . Then

$$2g(C \cup C') - 2 = n_1 n_2 (n_1 + n_2 - 4),$$

and we have the following basic relation (see [28, Proposition 3.1(vi)]) between the genus and degree of C and C' :

$$g - g' = \frac{n_1 + n_2 - 4}{2} (d - d').$$

Remark that Peskine and Szpiro focus on the case of curves which are arithmetically Cohen–Macaulay, but the formula above does not need this assumption (see [25, (2.1)]).

Assume further that C and C' are contained in a smooth surface S . Using adjunction formula on S , we have

$$2g(C \cup C') - 2 = (C + C')(C + C' + K_S) = C(C + K_S) + C'(C' + K_S) + 2CC'$$

which yields

$$g(C \cup C') = g + g' - 1 + (C \cdot C')_S.$$

Note that the intersection does not depend on the choice of the smooth surface S .

An application of linkage theory is the study of the irreducible components of the Hilbert scheme $\mathcal{H}_{g,d}^S$, as illustrated by the proof of the following fact that we shall need in the proof of the existence of the Sarkisov links listed in Table 2.

PROPOSITION 2.7. *Let (g, d) be one of the thirteen cases in Table 2. Then $\mathcal{H}_{g,d}^S$ is irreducible, except in the case $(14, 11)$ where there are exactly two irreducible components, one of them corresponding to curves of bidegree $(3, 8)$ on a smooth quadric surface.*

Proof. It is known that $\mathcal{H}_{g,d}^S$ is irreducible if $g + 3 \leq d$ (Ein), or if $g + 9 \leq 2d \leq 22$ (see [10] and references therein). This gives the result for the first twelve cases.

For $(14, 11)$, we now sketch a proof, which was explained to us by H. Nasu, using similar techniques as in [25].

First assume that $C \in \mathcal{H}_{14,11}^S$ is contained in a cubic surface S . Then:

- (i) S is not a cone: otherwise by [25, Lemma 2.10] we would have $d \equiv 0$ or $1 \pmod 3$, and in our case $d = 11 \equiv 2 \pmod 3$;

(ii) S is normal: otherwise the identity (see [25, (2.7)])

$$g = \frac{(k-1)(2d-3k-2)}{2}$$

should be satisfied for some $k \geq 0$, which is not the case for $(g, d) = (14, 11)$.

Then the curve C is a specialization of a curve in a smooth cubic surface [25, Theorem 2.7], and the family of such curves has dimension $g + d + 18 = 43$ [25, Proposition 2.5]. But it is a standard fact that any irreducible component of $\mathcal{H}_{g,d}^S$ has dimension greater than $4d$ (see [25, Remark 2.6]); in particular, any irreducible component of $\mathcal{H}_{14,11}^S$ must have dimension at least 44.

Assume now that C is contained in a quadric. If C was contained in a quadric cone, we would have $g = (a-1)(a+e-1)$ and $d = 2a+e$ for some $a, e \in \mathbb{N}$ (see the last part of the proof of Proposition 3.1); but there is no such relation for $(g, d) = (14, 11)$. Thus, C must be a curve of bidegree $(3, 8)$ on a smooth quadric. The dimension of such curves is $g + 2d + 8 = 44$, and they form an irreducible component of $\mathcal{H}_{14,11}^S$.

Finally, we observe that we can construct C contained in an irreducible quartic by linkage. Indeed, start from a smooth curve $C' \in \mathcal{H}_{2,5}^S$ contained in a smooth quadric Q' : if we identify Q' to $\mathbb{P}^1 \times \mathbb{P}^1$, C' is a curve of bidegree $(2, 3)$. One can show that such a curve is 4-regular, and applying a theorem of Martin-Deschamps and Perrin [25, Theorem 3.4], we obtain that a general linkage of type $[4, 4]$ yields a smooth curve $C \in \mathcal{H}_{14,11}^S$. Denote by W the family of curves in $\mathcal{H}_{14,11}^S$ obtained by this process. Then by [25, Lemma 2.2], W has dimension $4(11-5) + 20 = 44$ (here, we use the fact that $\dim \mathcal{H}_{2,5}^S = 20$, which can be computed directly or alternatively by using linkages of type $[2, 4]$ to twisted cubics, whose parameter family has dimension 12). The closure of such curves is the second irreducible component of $\mathcal{H}_{14,11}^S$, and contains as a codimension 1 closed subset the curves lying in cubic surfaces. \square

2.6. Existence of smooth quartics containing C

PROPOSITION 2.8. *If the blow-up $\pi: X \rightarrow \mathbb{P}^3$ of a smooth curve $C \subset \mathbb{P}^3$ gives rise to a weak Fano threefold X , then C is contained in a pencil of quartics whose general member is smooth.*

Proof. By the Riemann–Roch formula on a threefold and Kawamata–Viehweg vanishing, $\dim |-K_X| = (-K_X)^3/2 + 2$.

Assume first that the linear system $|-K_X|$ is base point free, which implies that a general element $S \in |-K_X|$ is a smooth $K3$ surface. Then the restricted system $|-K_X|_{|E}$ is also base point-free. Since $-K_X \cdot f = 1$, where f is an exceptional curve for π , we deduce that $S|_E$ is a smooth irreducible curve which is a section of $\pi: E \rightarrow C$. In consequence $\pi(S)$, which is a general quartic through C , is smooth.

We now assume that the base locus of $|-K_X|$ is non-empty. By a result of Shin [31, Theorem (0.5)] (building on previous results of Shokurov and Reid), the base locus $|-K_X|$ is isomorphic to a smooth rational curve Γ , and the general member S of $|-K_X|$ is smooth away from Γ , hence globally smooth by Bertini’s Theorem (Shin assumes X with canonical singularities, but since in our setting X is smooth we obtain a more precise result). We do not repeat the argument of Shin here, but we want to point out that the main point is a result of Saint-Donat [29], which states that a big and nef. linear system on a smooth $K3$ surface which is not base point-free has the form $kM + \Gamma$, where $k > 0$, M is a smooth elliptic curve, Γ is a rational (-2) -curve and $M \cdot \Gamma = 1$. More specifically, Saint-Donat proved this for an ample linear system, and the same property holds in the big and nef. case by [31, Lemma (2.1)]. The result follows by applying Saint-Donat’s result to the linear system $|-K_X|_{|S}$.

We deduce that if a general quartic $\pi(S)$ through C is singular, then the singular locus is a unique ordinary double point $p = \pi(\Gamma)$, where $\Gamma \subset S$ is the base locus of $|-K_X|_S$, and is also a fibre of the exceptional divisor E . We now derive a contradiction from this situation.

By Lemma 2.4, we have $S \cdot S \cdot E = (-K_X)^2 \cdot E = 2 + 4d - 2g$. On the other hand, $S \cdot S \cdot E = (kM + \Gamma) \cdot E = kM \cdot E - 1$. Hence the relation

$$M \cdot E = \frac{3 + 4d - 2g}{k}.$$

We also have $1 = S \cdot M = -K_X \cdot M = -(\pi^*(K_{\mathbb{P}^3}) + E) \cdot M = 4 \deg \pi(M) - M \cdot E$. Since $\pi(M)$ is an elliptic curve, we have $\deg \pi(M) \geq 3$, and

$$M \cdot E \geq 11.$$

On the other hand, again by Lemma 2.4 we have $4d \leq g + 30$ hence $3 + 4d - 2g \leq 33 - g$. Thus, we have $3 + 4d - 2g < 33$, since if $g = 0$, then $4d \leq g + 30$ implies $4d \leq 28 < 30$. Since $3 + 4d - 2g$ is odd, we obtain $k = 1$. Moreover, we have a linkage of type $[4, 4]$ between $\pi(M)$ and C , hence $g - g(M) = 2(d - \deg(M))$ with $\deg(M) = 16 - d$ and $g(M) = 1$. So we obtain the contradiction $g = 2(2d - 16) + 1 = 4d - 31 < g$. □

2.7. Blow-up of points in \mathbb{P}^k

For the sake of completeness, we include here the case of blow-up of points in \mathbb{P}^k , $k \geq 3$, which is an easy but nice exercise. It is probably well known to specialists, but we did not find any reference in the literature. Note, however, that Geiser and Bertini like involutions associated with the blow-up of 6 or 7 points in \mathbb{P}^3 were classically known (see, for example, [6, 4.6 p. 91]).

PROPOSITION 2.9. *Let $k \geq 3$. The blow-up X of d points in \mathbb{P}^k is Fano if and only if $d = 1$. It is weak Fano if and only if $k = 3$, $d \leq 7$ and the following conditions hold:*

- (1) *no three points belong to a same line;*
- (2) *no five points belong to the same conic (or equivalently to the same plane);*
- (3) *no seven points belong to the same twisted cubic.*

Proof. The blow-up of one single point gives a Mori fibre space $X \rightarrow \mathbb{P}^{k-1}$ induced by the lines passing through the blown-up point. It is clearly a Fano variety.

Assume now $d \geq 2$. The strict transform l of any line passing through two points satisfies $K_X \cdot l = -k - 1 + 2(k - 1) = k - 3$. Hence $-K_X$ cannot be nef. if $k \geq 4$. We assume now $k = 3$. If X is weak-Fano, we have $(-K_X)^3 = 64 - 8d > 0$ so $d \leq 7$, and the three conditions above are clearly satisfied. It remains to prove that the conditions imply that X is weak-Fano.

Let Λ be the linear system of quadrics of \mathbb{P}^3 passing through the d blown-up points, which corresponds to the image of $|\frac{1}{2}K_X|$. The fact that $d \leq 7$ implies that $\dim \Lambda \geq 9 - d \geq 2$. Since no five of the points belong to the same plane, a general member of Λ is irreducible.

We can thus find two distinct irreducible quadrics Q_1 and Q_2 passing through the d points. If one irreducible curve of X intersects $-K_X$ negatively, it has to come from a curve $C \subset \mathbb{P}^3$ contained in Q_1 and Q_2 , which has thus degree $n \leq 4$. Writing m_1, \dots, m_d the multiplicities of C at the points, we have $\sum_{i=1}^d m_i > 2n$.

If $n = 1$, we find at least three points on C , impossible by hypothesis. We get a similar contradiction if C is a conic.

If $n = 3$, we have either seven points on the curve or six points whose 1 is singular. The curve cannot lie on a plane, otherwise six points would lie on a plane. And it cannot be a twisted cubic by hypothesis.

The last case is when $n = 4$ which implies that C is a complete intersection of two quadrics hence has arithmetic genus $g = 1$. In particular, C admits at most one double point and $\sum_{i=1}^d m_i \leq 8 = 2n$: contradiction. \square

3. Curves in a plane or a quadric

PROPOSITION 3.1. *Let $C \subset S \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d which is contained in a plane, a smooth quadric or a quadric cone S . Let $\pi: X \rightarrow \mathbb{P}^3$ be the blow-up of C . We note $\overline{\text{NE}}(X) = \langle f, r \rangle$, where f is a π -exceptional curve. Then $-K_X$ is big, $r = l - nf$ where l is the pull-back of a line of \mathbb{P}^3 and $n \in \mathbb{N}$, and one of the following occurs:*

- (i) $-K_X$ is ample; there exists a contraction $X \rightarrow Y$ which yields a Sarkisov link $\mathbb{P}^3 \dashrightarrow Y$ or $\mathbb{P}^3 \dashrightarrow X$. Namely we are in one of the following cases, the details of which are discussed in Examples 3.3(i)–(iii), 3.4(i)–(v) and 3.5(i)–(ii), respectively:
 - (a) S is a plane: $(g, d) = (0, 1), (0, 2)$ or $(1, 3)$;
 - (b) S is a smooth quadric, and C is not a plane curve: $(g, d) = (0, 3), (0, 4), (1, 4), (2, 5)$ or $(4, 6)$;
 - (c) S is a quadric cone, and C is not contained in any other quadric: $(g, d) = (2, 5)$ or $(4, 6)$;
- (ii) $-K_X$ is nef. but not ample, then the curves numerically equivalent to $r = 4l - f$ cover the surface S , the anticanonical morphism is a divisorial contraction and we are in one of the following cases:
 - (a) S is a plane: $(g, d) = (3, 4)$;
 - (b) S is a smooth quadric, and C is not a plane curve: $(g, d) = (0, 5), (3, 6), (6, 7)$ or $(9, 8)$.
 - (c) S is a quadric cone, and C is not contained in any other quadric: $(g, d) = (6, 7)$ or $(9, 8)$.
- (iii) $-K_X$ is not nef., then the curves numerically equivalent to $r = l - nf$ for $n \geq 5$ cover the surface S , $|-mK_X|$ has a fixed component for any m and we are in one of the following cases:
 - (a) S is a plane: $g = (d - 1)(d - 2)/2$ and $d \geq 5$; if $g \leq 14$, then $(g, d) = (6, 5)$ or $(10, 6)$;
 - (b) S is a smooth quadric, and C is not a plane curve; if $4d - 30 \leq g \leq 14$, then $(g, d) \in \{(0, 6), (0, 7), (4, 7), (5, 8), (6, 9), (8, 8), (10, 9), (12, 9), (12, 10), (14, 11)\}$.
 - (c) S is a quadric cone, C is not contained in any other quadric and $g \geq 16$.

Moreover, any of the possibilities given above occurs.

Proof. It is clear that $-K_X$ is big, since $|-K_X|$ contains the reducible linear system generated by the quartics containing S .

Suppose first that C is contained in a plane, which implies that $g = (d - 1)(d - 2)/2$ and that $r = l - df$, where l is the pull-back of a line of \mathbb{P}^3 . If $d \geq 2$, the curves numerically proportional to r cover the strict transform of the plane. If $d = 4$, these curves intersect trivially K_X , so $-K_X$ is nef. but the anticanonical morphism contracts the strict transform of the plane. If $d > 4$, these curves intersect negatively $-K_X$ so $|-mK_X|$ is not movable and not nef. If $d \leq 3$, these intersect positively $-K_X$ so X is Fano. The cases $d = 1, 2, 3$ yield $g = 0, 0, 1$ and the three Sarkisov links are described in Example 3.3.

Suppose now that C is contained in a smooth quadric $Q \subset \mathbb{P}^3$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. On this surface $C \sim af_1 + bf_2$, where f_1 and f_2 are the two fibres of the two projections, and the

restriction of a hyperplane section is $H = f_1 + f_2 = -\frac{1}{2}K_Q$. Thus, we find $d = H \cdot C = a + b$ and $-2 + 2g = C^2 - 2CH = 2ab - 2a - 2b$, so $g = (a - 1)(b - 1)$. We can assume that $a \geq b$ and that $a \geq 2$ (otherwise C is contained in a plane).

We now prove that $r = l - af$. Suppose for contradiction that $r = ml - nf$ with $n > am$ is the strict transform of a curve of degree m passing n times through C , its intersections with the quadric Q is $2m \leq am < n$, so the curve is contained in the quadric. On Q it is equivalent to $\alpha f_1 + \beta f_2$ with $\alpha + \beta = m$, and its intersection with C is $b\alpha + a\beta \leq a\alpha + a\beta = am < n$, a contradiction.

If $a \geq 4$, all curves equivalent to r cover the strict transform of Q , and $-K_X$ is not ample. Moreover, $-K_X$ is nef. if and only if $a = 4$, and in this case the anticanonical morphism contracts the strict transform of Q . The possibilities for $a = 4$ are $(a, b) = (4, 1), (4, 2), (4, 3)$ and $(4, 4)$, corresponding to $(g, d) = (0, 5), (3, 6), (6, 7)$ and $(9, 8)$. If $a \geq 5$, $-K_X$ is not nef. and thus $-mK_X$ is not movable; since $g = (a - 1)(b - 1)$ and $d = a + b$, the possibilities where $4d - 30 \leq g \leq 14$ are $(0, 6), (0, 7), (4, 7), (5, 8), (6, 9), (8, 8), (10, 9), (12, 9), (12, 10)$ and $(14, 11)$.

If $a \leq 3$, $-K_X$ is ample and the contraction of all curves equivalent to r yields a contraction $X \rightarrow Y$. The possibilities are $(a, b) = (2, 1), (2, 2), (3, 1), (3, 2)$ and $(3, 3)$, corresponding to $(g, d) = (0, 3), (1, 4), (0, 4), (2, 5)$ and $(4, 6)$: see Example 3.4.

Note that when $a = 3$ and $b \in \{1, 2, 3\}$, C is not contained in another quadric because $2H - C$ is not effective on Q .

Suppose finally that C is contained in a quadric cone $V \subset \mathbb{P}^3$. Blowing-up the singular point yields the Hirzebruch surface \mathbb{F}_2 , whose Picard group is generated by f_0 and s , where f_0 is a fibre and s the exceptional section of self-intersection -2 . On \mathbb{F}_2 , C is equivalent to $as + (2a + e)f_0$ for some $a, e \in \mathbb{N}$. Since C is smooth on V , its intersection $e = C \cdot s$ with s on \mathbb{F}_2 is equal to 0 or 1; moreover, $a = C \cdot f_0 \geq 0$. The restriction of a hyperplane section of \mathbb{P}^3 on \mathbb{F}_2 is $H = s + 2f_0 = -\frac{1}{2}K_{\mathbb{F}_2}$. This implies that $d = C \cdot H = 2a + e$ and that $-2 + 2g = C^2 - 2C \cdot H = ea + a(2a + e) - 2(2a + e)$, so $g = (a - 1)(a + e - 1)$.

We can assume that $C - 2H$ is not effective, since otherwise C would be contained in a plane or a smooth quadric. This means that either $(a, e) = (2, 1)$ or $a \geq 3$.

We now prove that $r = l - (a + e)f$, corresponding to the transform of a rule on V . Suppose for contradiction that $r = ml - nf$ with $n > (a + e)m$ is the strict transform of a curve C' of degree m passing n times through C . Its intersections with the quadric V is $2m \leq (a + e)m < n$, so the curve is contained in the quadric. On \mathbb{F}_2 , the curve C' is equivalent to $\alpha s + mf_0$, with $\alpha = C' \cdot f_0 \geq 0$ and its intersection with C is $e\alpha + am$. We have $C \cdot s = e$, $C' \cdot s = m - 2\alpha$, so the contraction of s produces $e(m - 2\alpha)$ new intersecting points. Hence the number of intersecting points of C and C' in \mathbb{P}^3 is $e\alpha + am + e(m - 2\alpha) \leq (a + e)m < n$, a contradiction.

If $a + e \geq 4$, all curves equivalent to r cover the strict transform of Q and $-K_X$ is not ample. Moreover, $-K_X$ is nef. if and only if $a + e = 4$, which corresponds to $(a, e) = (3, 1)$ and $(4, 0)$, and give, respectively, $(g, d) = (6, 7)$ and $(9, 8)$. In this case, the anticanonical morphism contracts the strict transform of Q .

If $a + e \geq 5$, $-K_X$ is not nef. and thus $|-mK_X|$ must have a fixed component for all m ; moreover, $g = (a - 1)(a + e - 1) \geq 16$. The possibilities where $4d - 30 \leq g \leq 14$ are $(0, 6), (0, 7), (4, 7), (5, 8), (6, 9), (8, 8), (10, 9), (12, 9), (12, 10), (14, 11)$.

If $a + e \leq 3$, $-K_X$ is ample and the possibilities for (a, e) are $(2, 1)$ and $(3, 0)$, which give, respectively, $(g, d) = (2, 5)$ and $(4, 6)$: see Example 3.5. □

COROLLARY 3.2. *Let $C \subset S \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d which is contained in a plane, a smooth quadric or a quadric cone S . Let $\pi: X \rightarrow \mathbb{P}^3$ be the blow-up of C .*

Then the following conditions are equivalent:

- (i) $-K_X$ is ample;
- (ii) there is no 4-secant line to C ;
- (iii) $(g, d) \in \mathcal{A}_1 = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 3), (1, 4), (2, 5), (4, 6)\}$.

The following are also equivalent:

- (i) $-K_X$ is big and nef;
- (ii) there is no 5-secant line to C ;
- (iii) $(g, d) \in \mathcal{A}_1 \cup \{(0, 5), (3, 4), (3, 6), (6, 7), (9, 8)\}$.

Proof. The proof is the same for a Fano or a weak Fano threefold. Implication (i) \Rightarrow (ii) follows from Lemma 2.2. Implications (ii) \Rightarrow (iii) \Rightarrow (i) follow from Proposition 3.1. \square

EXAMPLE 3.3. If $C \subset P \subset \mathbb{P}^3$ is a smooth plane curve of degree d , we have $g = (d-1)(d-2)/2$. The blow-up of C yields the following Sarkisov links.

- (i) If $d = 1$, the blow-up of the line C produces a link of type I. The variety X is a \mathbb{P}^2 -bundle over \mathbb{P}^1 , where the fibres correspond to planes through C .
- (ii) If $d = 2$, the blow-up of the conic C , followed by the contraction of the plane P on a smooth point, produces a link of type II. The variety Y is a quadric threefold.
- (iii) If $d = 3$, the blow-up of the plane cubic C , followed by the contraction of the plane P on a terminal point (contraction of type E5), produces a link of type II. The variety Y is a singular Fano threefold.

EXAMPLE 3.4. If $C \subset Q \subset \mathbb{P}^3$ is a smooth curve of bidegree (a, b) in a smooth quadric surface Q , we have $g = (a-1)(b-1)$ and $d = a + b$. The blow-up of C yields the following Sarkisov links.

- (i) If $(a, b) = (1, 2)$, $(g, d) = (0, 3)$. The curve C is a twisted cubic, and X is a \mathbb{P}^1 -bundle over \mathbb{P}^2 , where fibres correspond to bisecants to C . Link of type I.
- (ii) If $(a, b) = (2, 2)$, $(g, d) = (1, 4)$. The curve C is the base locus of a pencil of quadrics, and X is a del Pezzo fibration over \mathbb{P}^1 . Link of type I.
- (iii) If $(a, b) = (3, 1)$, $(g, d) = (0, 4)$. We can contract the transform of Q to a smooth rational curve to obtain Y a smooth prime Fano threefold. Link of type II.

We compute now $-K_Y^3$ to determine the nature of Y . Since the exceptional divisor of $X \rightarrow Y$ is $Q = \mathbb{P}^1 \times \mathbb{P}^1$, the normal bundle of Γ in Y has the form $\mathcal{N}_{\Gamma/Y} = \mathcal{O}(a) \oplus \mathcal{O}(a)$. We have $K_Y \cdot \Gamma = -2a - 2$, and by Lemma 2.4

$$(-K_Y)^3 = (-K_X)^3 - 2K_Y \cdot \Gamma - 2 = (-K_X)^3 + 4a + 6.$$

Now if $l \subset Q$ is a contracted fibre, and $s \subset S$ is a fibre of the other ruling, we have $Q \cdot l = -1$ and $Q \cdot s = a$. Thus

$$K_X \cdot s = \sigma^* K_Y \cdot s + Q \cdot s = K_Y \cdot \Gamma + a = -a - 2.$$

Since C was a curve of bidegree $(3, 1)$, we have $K_X \cdot s = -3$, hence $a = 1$. On the other hand, $(-K_X)^3 = 62 - 8 \cdot 4 = 30$, hence $(-K_Y)^3 = 30 + 4a + 6 = 40$. We conclude that Y is an index 2 Fano threefold $V_5 \subset \mathbb{P}^6$.

(iv) If $(a, b) = (3, 2)$, $(g, d) = (2, 5)$. We can contract the transform of S to a smooth rational curve to obtain Y a smooth prime Fano threefold. Link of type II.

A similar computation as in the previous example yields $(-K_Y)^3 = 32$. We conclude that Y is an index 2 Fano threefold $V_4 \subset \mathbb{P}^5$, smooth intersection of two quadrics.

(v) If $(a, b) = (3, 3)$, $(g, d) = (4, 6)$. We can contract the transform of S to a terminal point (contraction of type $E3$). Link of type II.

EXAMPLE 3.5. If $C \subset V \subset \mathbb{P}^3$ is a smooth curve in a quadric cone V , recall that blowing-up the singular point yields the Hirzebruch surface \mathbb{F}_2 , whose Picard group is generated by f_0 and s , where f_0 is a fibre and s the exceptional section of self-intersection -2 . On \mathbb{F}_2 , C is equivalent to $as + (2a + e)f_0$ for some $a, e \in \mathbb{N}$.

(i) If $(a, e) = (2, 1)$, $(g, d) = (2, 5)$, we can contract the transform of V to a smooth rational curve to obtain a prime Fano Y . Link of type II.

A similar computation as in Example 3.4(iv) yields $(-K_Y)^3 = 32$. So again Y is an index 2 Fano threefold $V_4 \subset \mathbb{P}^5$.

(ii) If $(a, e) = (3, 0)$, $(g, d) = (4, 6)$. We can contract the transform of V to a terminal point (contraction of type $E4$). Link of type II.

4. Curves in cubics

Recall that any smooth cubic surface of \mathbb{P}^3 is the blow-up of six points of \mathbb{P}^2 such that no three are collinear and no six are on a conic. More generally, if $\sigma: S \rightarrow \mathbb{P}^2$ is the blow-up of six points, maybe some infinitely near, such that $-K_S$ is nef. (or equivalently such that all curves of negative self-intersection are smooth rational (-1) -curves or (-2) -curves), the anticanonical map $\eta: S \rightarrow \mathbb{P}^3$ is a birational morphism to a normal cubic surface, which contracts all (-2) -curves of S (it is an isomorphism if and only if $-K_S$ is ample, which means that S is del Pezzo).

Conversely, all normal cubic surfaces except cones over smooth cubic curves are obtained in this way. Indeed such a surface \hat{S} admits only double points, is rational (project from a singularity) and satisfies $H^1(\hat{S}, \mathcal{O}_{\hat{S}}) = H^2(\hat{S}, \mathcal{O}_{\hat{S}}) = 0$ (use the exact sequence $0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\hat{S}} \rightarrow 0$); hence by [7, Proposition 8.1.8(ii)] \hat{S} admits only rational double points. Alternatively, one can use the classification of singularities on cubic surfaces in [2, §2], which does not rely on a cohomological argument. Then the minimal resolution $S \rightarrow \hat{S}$ is a weak del Pezzo surface and is the blow-up of six points of \mathbb{P}^2 by [7, Theorem 8.1.13]. For more details the reader can consult [5, 7, §8.1].

SET-UP 4.1. In this section, we consider $C \subset \hat{S} \subset \mathbb{P}^3$ a smooth curve of genus g and degree d which is contained in a normal rational cubic \hat{S} , but not in a quadric or a plane.

We denote by $\eta: S \rightarrow \hat{S}$ the minimal resolution of singularities of S , by $\sigma: S \rightarrow \mathbb{P}^2$ a birational morphism, which factorizes as $\sigma = \sigma_1 \circ \dots \circ \sigma_6$, where σ_i is the blow-up of a point p_i . We denote by $E_i \in \text{Pic}(S)$ the total transform of $\sigma_i^{-1}(p_i)$ on S , and by $L \in \text{Pic}(S)$ the pull-back of a general line, so that $\text{Pic}(S) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \dots \oplus \mathbb{Z}E_6$, and that the intersection form is the diagonal $(1, -1, \dots, -1)$.

Since C is not a line on \mathbb{P}^3 , we have $C \sim kL - \sum m_i E_i$ on S , where $k > 0$ is the degree of the image of C on \mathbb{P}^2 , and $m_1, \dots, m_6 \geq 0$ are the multiplicities at the points p_1, \dots, p_6 . The hyperplane section on \hat{S} is anti-canonical (the map $S \rightarrow \hat{S}$ is given by $|-K_S|$), so $d = C \cdot (-K_S)$. Since C is smooth on \mathbb{P}^3 , it is also smooth on S , and we have $-2 + 2g = C^2 + C \cdot K_S = C^2 - d$ on S . This gives the following equalities:

$$d = 3k - \sum_{i=1}^6 m_i,$$

$$g = \frac{(k-1)(k-2)}{2} - \sum_{i=1}^6 \frac{m_i(m_i-1)}{2}.$$

Since we assume that $C \subset \mathbb{P}^3$ is not contained in any quadric, we have that $-2K_S - C$ is not effective on S (recall that $-K_S$ is the trace of an hyperplane section, so $-2K_S - C$ is effective if and only if C is contained in a quadric). Up to reordering, we can also choose that $m_1 \geq m_2 \geq m_3 \geq m_4 \geq m_5 \geq m_6$, that $p_1 \in \mathbb{P}^2$ and that for $i \geq 2$ either $p_i \in \mathbb{P}^2$ or is infinitely near to p_{i-1} . We can, moreover, assume that $k \geq m_1 + m_2 + m_3$. Indeed, if $k < m_1 + m_2 + m_3$, the three points p_1, p_2 and p_3 are not collinear and there exists a quadratic map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ having base-points at p_1, p_2 and p_3 (this is because there is no (-3) -curve on \hat{S}); replacing $\sigma: S \rightarrow \mathbb{P}^2$ with its composition with the quadratic map replaces k with $2k - m_1 - m_2 - m_3$, which is strictly less than k . Reordering the points if needed and continuing this process again if $k < m_1 + m_2 + m_3$, we end up in a case where $k \geq m_1 + m_2 + m_3$.

PROPOSITION 4.2. *Let $C \subset \hat{S} \subset \mathbb{P}^3$ be a smooth curve of genus g and degree d which is contained in a rational normal cubic, but not in a quadric or a plane.*

Let $\pi: X \rightarrow \mathbb{P}^3$ be the blow-up of C . The divisor $-K_X$ is big, it is nef. if and only if C does not admit any 5-secant line. We note $\overline{NE}(X) = \langle f, r \rangle$, where f is a π -exceptional curve. Assuming Set-up 4.1, we are in one of the following cases:

- (i) $-K_X$ is ample; $r = l - 3f$ is generated by 3-secant lines and we have one of the four following cases:

g	d	k	(m_1, \dots, m_6)	$\dim -K_X \geq$
1	5	3	(1, 1, 1, 1, 0, 0)	14
3	6	4	(1, 1, 1, 1, 1, 1)	12
5	7	6	(2, 2, 2, 2, 2, 1)	10
10	9	9	(3, 3, 3, 3, 3, 3)	7

- (ii) $-K_X$ is nef. but not ample, and the anticanonical morphism is a small contraction; curves proportional to $r = l - 4f$ are combination of a finite number of 4-secant lines of C in \mathbb{P}^3 , all being in \hat{S} , and we have one of the nine following cases (the number of 4-secants is counted with multiplicity, if \hat{S} is not smooth, some 4-secants can count twice or more):

g	d	k	(m_1, \dots, m_6)	# of 4-secants	$\dim -K_X \geq$
0	5	2	(1, 0, 0, 0, 0, 0)	1	13
0	6	2	(0, 0, 0, 0, 0, 0)	6	9
1	6	3	(1, 1, 1, 0, 0, 0)	3	10
2	6	4	(2, 1, 1, 1, 1, 0)	1	11
3	7	4	(1, 1, 1, 1, 1, 0)	5	8
4	7	5	(2, 2, 1, 1, 1, 1)	2	9
6	8	6	(2, 2, 2, 2, 1, 1)	5	7
7	8	7	(3, 2, 2, 2, 2, 2)	1	8
9	9	7	(2, 2, 2, 2, 2, 2)	6	6

Moreover, if \hat{S} is smooth, the 4-secant lines correspond to the conic of \mathbb{P}^2 that passes through the last five points, to other conics through five points with similar multiplicities, and in the case $(g, d) = (6, 8)$ to the line through the two points of multiplicity 1.

- (iii) $-K_X$ is nef. but not ample and the anticanonical morphism is a divisorial contraction; $r = l - 4f$, the curves proportional to r cover the surface \hat{S} and we have one of the following three cases (same remark as in (ii) for the number of 4-secants):

g	d	k	(m_1, \dots, m_6)	# of 4-secants
5	8	6	(2,2,2,2,0)	10
12	10	9	(3,3,3,3,2)	10
19	12	12	(4,4,4,4,4)	27

- (iv) $-K_X$ is not nef., and there is a 5-secant line to C in \mathbb{P}^3 .

REMARK 4.3. Each case occurs in a smooth cubic: we leave the verification as an exercise to the reader (which can be done along the same lines as in Remark 4.4 below). Note that each case also occur on some singular cubics, depending on the singularity of the cubic, or equivalently on the position of the points blown-up by $\sigma: S \rightarrow \mathbb{P}^2$.

In case (ii), when we speak about the conic passing through the five last points, we speak about the divisor $2L - E_2 - E_3 - \dots - E_5$ on S , which is, in general, the strict transform of the conic passing through p_2, \dots, p_5 ; if this divisor corresponds to a reducible curve (which is possible when \hat{S} is singular), the image of this curve is also a 5-secant line in $\hat{S} \subset \mathbb{P}^3$.

Proof. We use the notation of Set-up 4.1.

As in the case of curves contained in a quadric, it is clear that $-K_X$ is big, since $|-K_X|$ gives a birational map (it contains the reducible linear system generated by the quartics containing S).

Suppose that one (effective) divisor $D \in \{E_1, L - E_5 - E_6, 2L - E_2 - E_3 - E_4 - E_5 - E_6\} \subset \text{Pic}(S)$ satisfies $D \cdot C \geq 5$ in S . If \hat{S} is smooth, D is a (-1) -curve of $S = \hat{S}$, which is a 5-secant of D , implying that $-K_X$ is not nef. If \hat{S} is not smooth, D is either a (-1) -curve, or a reducible curve consisting of one (-1) -curve connected to a chain of (-2) -curves. Since $D \cdot (-K_S) = 1$, the image of the reducible curve is a line of \mathbb{P}^3 , which intersects D into five points, which is a 5-secant, implying that $-K_X$ is not nef. This gives case (iv).

We can thus assume that $D \cdot C \leq 4$ for any $D \in \{E_1, L - E_5 - E_6, 2L - E_2 - E_3 - E_4 - E_5 - E_6\}$, which implies that $k - m_5 - m_6 \leq 4$, $2k - m_2 - m_3 - m_4 - m_5 - m_6 \leq 4$ and $m_1 \leq 4$. In particular, $m_i \leq 4$ for $i = 1, \dots, 6$. This gives finitely many possibilities for (k, m_1, \dots, m_6) , all listed in cases (i)–(iii).

We now show that these cases give in fact $-K_X$ nef. In all cases listed, Set-up 4.1 implies that either $L - E_5 - E_6, 2L - E_2 - E_3 - E_4 - E_5 - E_6$ or E_1 realizes the biggest intersection with C among all 27 divisors $E_i, L - E_i - E_j, 2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 + E_i$. Recall that each of these 27 divisors D gives a line in \mathbb{P}^3 , intersecting C into $C \cdot D$ points; the 27 lines are distinct if \hat{S} is smooth, but not if \hat{S} is singular. An easy check shows that the line with the biggest intersection is equivalent to $l - af$ on X , where $a = 3$ in case (i) and $a = 4$ in cases (ii), (iii).

We now prove that it is the extremal ray r . This will imply that $-K_X$ is ample in case (i) and nef. but not ample in cases (ii) and (iii). Suppose for contradiction that $r = ml - nf$ with $n > am$ is the strict transform of a curve Γ of degree m passing n times through C , its intersections with the cubic \hat{S} is $3m \leq am < n$, so the curve is contained in the cubic. On S the curve Γ is equivalent to a sum of m divisors among the 27 divisors $E_i, L - E_i - E_j, 2L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 + E_i$. This is a consequence of Mori’s Cone Theorem

(see [21, Theorem 3.7]), since $-K_X \cdot \Gamma < 0$ and since the extremal rays of the cone of curves of S negative against $-K_S$ correspond exactly to the lines in \hat{S} . Each of the 27 divisors intersecting at most a times C , we get the contradiction. \square

REMARK 4.4. There is one particularly interesting example that falls under case (iv) of Proposition 4.2, namely the case (15, 11). By Lemma 2.3, we know that a curve of type (15, 11) is contained in a cubic surface, which must be unique since the degree of the curve is greater than 9. It is easy to construct such a curve C on a smooth cubic: consider the transform of a curve of degree 10 on \mathbb{P}^2 with multiplicities (4, 3, 3, 3, 3, 3) at the six blown-up points. Note that such irreducible curves do exist: the linear system $|C|$ contains the transform of many reducible members, such as five conics (four passing through four points and one through three points), in particular, these reducible members are connected and do not admit a common base point, hence by Bertini's Theorem the general member C is smooth and irreducible. Observe that the conic through the five points of multiplicities 3 corresponds to a 5-secant line to C : we recover the fact that $-K_X$ is not nef.

More generally, if C is any smooth curve of type (15, 11), by intersecting the (possibly singular) cubic containing C with one of the irreducible quartics containing C (which does exist by a dimension count, using Lemma 2.3), we obtain a residual line L which must be a 5-secant to C , since $g(C \cup L) = 19$ (see formulas in §2.5). In particular, even for C a general element of $\mathcal{H}_{15,11}^S$, the anticanonical divisor of the blow-up X of C is never nef.

Note that as a consequence the construction in [18, Example 7 (1) and Table 1, Case 28] given as a candidate to be a Sarkisov link from \mathbb{P}^3 to $X_{10} \subset \mathbb{P}^7$ in fact never happens. Precisely, the starting assumption ‘Let $C \subset \mathbb{P}^3$ be a nonsingular curve that is an intersection of nonsingular quartic surfaces with $(p_a(C), \deg C) = (15, 11)$ ’ is never fulfilled, since by Bezout's Theorem any quartic containing C must also contain the 5-secant line to C .

REMARK 4.5. We can make a similar discussion about case (12, 10). By Lemma 2.3 such a curve is contained in a cubic surface, which must be unique. By taking the residual curves of the intersection of this cubic with a pencil of general quartics containing C , we obtain a pencil of 8-secant conics to C , hence the anticanonical morphism always correspond to a divisorial contraction.

This contrasts with the case (5, 8): we will see that a general curve in $\mathcal{H}_{5,8}^S$ does not lie on a cubic, and yields a weak Fano X with a small anticanonical morphism.

REMARK 4.6. Note that the case (19, 12) always corresponds to the complete intersection of a cubic and a quartic. To see that such a curve C is contained in a cubic, we slightly modify the argument of the proof of Lemma 2.3 as follows. We have $\deg K = 2g - 2 = 36 = \deg D$, so $l(K - D) = 1$ and $\dim |D| = 3d - g + 1 = 18$. Since $N - 1 = 19$, we find a (necessarily unique) cubic surface containing C .

Then by a dimension count, using Lemma 2.3, we see that C is also contained in an irreducible quartic. As a consequence, any curve of degree n in the cubic surface containing C is $4n$ -secant to C .

EXAMPLE 4.7. If \hat{S} is smooth, the four cases in Proposition 4.2(i) correspond to the following Sarkisov links.

(i) Case $(g, d) = (1, 5)$ is a link of type II to the quadric $Q \subset \mathbb{P}^4$ (blow-down to a curve (1, 5) in Q): see [24, p. 117] or [27, Example 3 (L.4)].

(ii) Case $(g, d) = (3, 6)$ corresponds to the classical general cubo-cubic transformation of \mathbb{P}^3 . This is the only example of a link of type II from \mathbb{P}^3 to \mathbb{P}^3 where the isomorphism in codimension 1 is in fact an isomorphism: see [19]. The transformation contracts a surface of degree 8 which is triple along C , and which is swept out by the 3-secant lines to C .

(iii) Case $(g, d) = (5, 7)$ corresponds to a link of type I, with a resulting conic bundle whose fibres correspond to 6-secant conics to C : see [14, Remark 2(ii); 26, Example 6(a)] or [3, Appendix A]. Note that by this last reference there is no need for a genericity assumption on C .

(iv) Case $(g, d) = (10, 9)$ corresponds to the complete intersection of two cubic surfaces, and so X is a fibration in del Pezzo surfaces of degree 3.

EXAMPLE 4.8. By Proposition 2.1, the nine cases in Proposition 4.2(ii) correspond to Sarkisov links involving a flop. When \hat{S} is smooth, all have already been described in the literature.

(i) In cases $(g, d) = (1, 6), (4, 7), (7, 8)$ we remark that d plus the number of 4-secant lines is equal to 9. So we have a pencil of cubics with base locus equal to the union of C and the 4-secant lines. After blow-up and flop, we obtain a del Pezzo fibration of degree 6, 5, 4, respectively. See [17, Proposition 6.5(27–28–29)].

(ii) In case $(g, d) = (2, 6)$, after blow-up and flop of the unique 4-secant line, we obtain a conic bundle: see [17, Theorem 7.14(16)]. One can verify, considering the residual curves of a pencil of cubics containing C (hence also the unique 4-secant line), that the fibres correspond to 6-secant conics to C : This case is a degenerate version of Example 4.7(iii).

(iii) Case $(g, d) = (0, 5)$ corresponds to a link from \mathbb{P}^3 to itself with flop of the unique 4-secant line: see [4, Proposition 2.9]. This case was known classically, see, for instance [30, first case in Table p. 185]. This is a degenerate version of Example 4.7(ii), and again the contracted divisor corresponds to the surface swept out by the 3-secant lines to C .

(iv) In case $(g, d) = (0, 6)$, after the flop of the six 4-secant lines we can contract the transform of S , isomorphic to \mathbb{P}^2 , on a smooth point, obtaining a Fano threefold Y_{22} of genus 12. See [32, (2.8.1)].

(v) In case $(g, d) = (3, 7)$, after the flop of the five 4-secant lines we can contract the transform of S , isomorphic to \mathbb{F}_1 , on a line in a Fano threefold Y_{16} of genus 9. See [13, (6.1)] or [20, § 2].

(vi) In case $(g, d) = (6, 8)$, after the flop of the five 4-secant lines we can contract the transform of S , isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, on a terminal point (contraction of type $E3$). We have $(-K_X)^3 = 10$ by Lemma 2.4, hence after the $E3$ contraction we obtain a terminal Fano threefold Z with $(-K_Z)^3 = 12$. See [4, 3.2, Case 4].

(vii) In case $(g, d) = (9, 9)$, after the flop of the six 4-secant lines we can contract the transform of S , isomorphic to \mathbb{P}^2 , on a terminal point (contraction of type $E5$); the terminal Fano Z that we obtain is such that $(-K_Z)^3 = \frac{21}{2}$. See [4, 3.3, Case 3].

EXAMPLE 4.9. All cases of Proposition 2.1 exist on singular cubics. We describe one among many other interesting cases. *** In case $(g, d) = (6, 8)$, pick $p_1, \dots, p_5 \in \mathbb{P}^2$, p_6 infinitely near to p_5 , such that no three of the p_i are collinear, and no six are on a conic. Then \hat{S} has an unique double point, corresponding to the image of the effective divisor $E_5 - E_6 \in S$. After the flop of the five 4-secant lines, we can contract the transform of \hat{S} , isomorphic to a quadric cone, on a terminal point (contraction of type $E4$). This was a case left open in [4, 3.2]. Note that this case is numerically (but not geometrically) equivalent to Example 4.8(vi); in particular, the cube of the anti-canonical divisor is again 12.

PROPOSITION 4.10. *Let $C \subset \mathcal{H}_{g,d}^S$ be a curve of genus g and degree d contained in a smooth quartic surface. Assume that (g, d) is one of the cases listed in Table 1, third column; so, in particular, C is contained in a cubic surface.*

Then the blow-up X of C has nef. anticanonical divisor if and only if there is no 5-secant line to C .

In particular, for any such pair (g, d) there exists a non-empty Zariski open set $U \subset \mathcal{H}_{g,d}^S$ such that the blow-up X of $C \in U$ is weak Fano.

Proof. Suppose that $-K_X$ is not nef., which implies the existence of an irreducible curve Γ of degree $n \geq 1$ which intersects at least $4n + 1$ times C . Note that Γ must be inside all quartics (and a fortiori cubics) containing C , so $\deg(\Gamma) \leq 12 - d$.

Assume first that $\deg(\Gamma) = 12 - d$, which means that $\Gamma \cup C$ is a complete intersection of a cubic and a smooth quartic S . Inside S we have $(C + \Gamma)^2 = 36$ and $C^2 = 2g - 2$, $\Gamma^2 = 2g(\Gamma) - 2$. By the formula from §2.5, we have $g - g(\Gamma) = \frac{3}{2}(d - \deg(\Gamma))$, hence

$$\begin{aligned} g(\Gamma) &= 18 + g - 3d; \\ g &= 18 + g(\Gamma) - 3d(\Gamma). \end{aligned}$$

Inside the quartic S we have:

$$\begin{aligned} C \cdot \Gamma &= \frac{36 - (2g - 2) - (2g(\Gamma) - 2)}{2} \\ &= 20 - g - g(\Gamma) \\ &= 3 \deg(\Gamma) + 2 - 2g(\Gamma) \\ &\leq 3 \deg(\Gamma) + 2. \end{aligned}$$

Since $12 - d = \deg(\Gamma) \geq 2$ in the cases under consideration, we obtain

$$C \cdot \Gamma \leq 4 \deg(\Gamma).$$

Suppose now that $n = \deg(\Gamma) < 12 - d$, which means that $\Gamma \cup C$ is contained in the smooth quartic S but is not a complete intersection of S with another surface (since $(g, d) \neq (9, 8)$ or $(6, 4)$). We can apply Proposition 2.6, and get $8g(\Gamma \cup C) < (n + d)^2$, where $g(\Gamma \cup C) \geq g(C) + g(\Gamma) + 4n$, which yields

$$8g(\Gamma) < (n + d)^2 - 8g - 32n = n^2 - 2(16 - d)n + d^2 - 8g.$$

We call $P_{g,d}$ the polynomial $n^2 - 2(16 - d)n + d^2 - 8g \in \mathbb{Z}[n]$. We find the following polynomials for the pairs (g, d) under consideration.

g	d	$P_{g,d}$	g	d	$P_{g,d}$
0	5	$(n - 2)(n - 20) - 15$	4	7	$(n - 1)(n - 17)$
0	6	$(n - 2)(n - 18)$	5	7	$(n - 1)(n - 17) - 8$
1	5	$(n - 1)(n - 21) - 4$	6	8	$(n - 2)(n - 14) - 12$
1	6	$(n - 2)(n - 18) - 8$	7	8	$(n - 1)(n - 15) - 7$
2	6	$(n - 2)(n - 18) - 16$	9	9	$(n - 1)(n - 13) - 4$
3	6	$(n - 1)(n - 19) - 7$	10	9	$(n - 1)(n - 13) - 12$
3	7	$(n - 2)(n - 16) - 7$	12	10	$(n - 1)(n - 12) - 8$

Since $8g(\Gamma) < P_{g,d}$ and $n < 12 - d$, we find that $n < 2$.

The last assertion follows from Proposition 4.2, which implies that the Zariski open set under consideration is non-empty (recall that if $-K_X$ is nef., then it is big since $(-K_X)^3 > 0$). \square

5. Curves in quartics

5.1. Curves contained in a singular rational quartic

The aim of this section is to produce examples of smooth curves that give rise to a nef anticanonical divisor after blow-up. We shall produce these curves by considering quartic surfaces with irrational singularities; such surfaces were classified in [12], in particular, the construction below corresponds to [12, § 2.4].

SET-UP 5.1. In this section, we consider a smooth cubic curve $\Gamma_0 \subset \mathbb{P}^2$, given by the equation $F_3(x, y, z) = 0$, and we choose a smooth quartic curve of equation $F_4(x, y, z)$ that intersects the curve Γ_0 into 12 distinct points p_1, \dots, p_{12} such that no three are collinear, no six on a conic, no eight on the same cubic singular at one of the eight points, and no nine are on the same cubic distinct from Γ_0 . Lemma 5.2 shows that this holds for a general quartic. The rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ given by

$$(x : y : z) \dashrightarrow (F_4(x, y, z) : xF_3(x, y, z) : yF_3(x, y, z) : zF_3(x, y, z))$$

induces a birational map $\psi: \mathbb{P}^2 \dashrightarrow Q$ where $Q \subset \mathbb{P}^3$ is the singular quartic

$$Q = \{(w : x : y : z) \in \mathbb{P}^3 \mid wF_3(x, y, z) = F_4(x, y, z)\}.$$

Denoting by $\sigma: S \rightarrow \mathbb{P}^2$ the blow-up of the 12 points p_1, \dots, p_{12} and by $\Gamma \subset S$ the strict transform of Γ_0 , the map ψ factorizes as $\eta\sigma^{-1}$, where $\eta: S \rightarrow Q$ is a birational morphism which contracts Γ onto the point $q = (1 : 0 : 0 : 0)$. We note $E_1, \dots, E_{12} \subset S$ the exceptional curves of S contracted by σ onto p_1, \dots, p_{12} and $L \in \text{Pic}(S)$ the transform of a general line of \mathbb{P}^2 . The curve $\Gamma \subset S$ is linearly equivalent to $3L - \sum E_i$, and an hyperplane section of Q corresponds to $H = 4L - \sum E_i \in \text{Pic}(S)$.

LEMMA 5.2. *Fixing Γ_0 , a general quartic curve intersects Γ_0 into 12 points such that no three are collinear, no six are on a conic, no eight are on the same cubic singular at one of the points, and no nine are on the same cubic distinct from Γ_0 . The set of 12-uples of points $p_1, \dots, p_{12} \in \Gamma_0$ obtained has dimension 11.*

Proof. Denoting by $l \in \text{Pic}(\Gamma_0)$ the restriction of an hyperplane section of \mathbb{P}^2 on Γ_0 , we have $|l| \simeq \mathbb{P}^2$, $|2l| \simeq \mathbb{P}^5$, $|3l| \simeq \mathbb{P}^8$ and $|4l| \simeq \mathbb{P}^{11}$. This follows from Riemann–Roch or simply from a direct computation by restricting the curves of \mathbb{P}^2 of degree 1, 2, 3, 4 to Γ_0 .

A general element of $|4l|$ yields 12 distinct points of Γ_0 . Any element of $|4l|$ which yields three collinear points or nine points on the same cubic distinct from Γ_0 corresponds to a sum $D_1 + D_3$ where $D_1 \in |4l|$ and $D_3 \in |3l|$: This is a consequence of the classical $AF + BG$ Theorem of Noether (see, for example, [8, Corollary p. 122]). The dimension of such elements is thus 10. Similarly, the elements which give six points on a conic also have dimension 10. If an element $D \in |4l|$ is the sum of 12 points p_1, \dots, p_{12} so that there exists a cubic C passing through p_1, \dots, p_8 and being singular at p_1 , the restriction of C to Γ_0 gives $2p_1 + \dots + p_8 \in |3l|$, so $D = (2p_1 + \dots + p_8) + (p_9 + \dots + p_{12} - p_1)$. Since $p_9 + \dots + p_{12} - p_1$ has degree 3, it is linearly equivalent to an effective divisor, and once again D decomposes as $D_1 + D_3$ as above. □

Let us point out the following easy fact:

LEMMA 5.3. *Let $C \subset S$ be an irreducible curve, its image $\eta(C) \subset Q \subset \mathbb{P}^3$ is a smooth curve of \mathbb{P}^3 if and only if C is smooth and $C \cdot \Gamma = 1$ in S .*

Proof. Denote by $\hat{\eta}: X \rightarrow \mathbb{P}^3$ the blow-up of $q = (1 : 0 : 0 : 0)$. It follows from Set-up 5.1 that the strict transform of Q on X is isomorphic to the smooth surface S , and that $\eta: S \rightarrow Q$ is the restriction of $\hat{\eta}$.

The curve $C \subset S$ is then the strict transform of $\hat{\eta}(C) \subset \mathbb{P}^3$. Denoting by $E \subset X$ the exceptional divisor, the curve $\eta(C)$ is smooth if and only if C is smooth and $C \cdot E = 1$ in X . Since Γ is the intersection of E and S , the intersection $C \cdot E$ on X is equal to the intersection $C \cdot \Gamma$ in S . \square

LEMMA 5.4. *There are exactly 12 lines contained in Q , all passing through q and corresponding to the image of E_i for some $i = 1, \dots, 12$.*

There are exactly 66 conics in Q , all passing through q , corresponding to the strict transforms of the lines of \mathbb{P}^2 passing through two of the p_i .

There are exactly 5544 twisted cubics (smooth cubics of genus 0) in Q , all passing through q , which correspond to the strict transforms of the conics of \mathbb{P}^2 passing through 5 of the p_i .

Proof. Each E_i is isomorphic to \mathbb{P}^1 on S , its intersection with Γ and H is 1, so its image in Q is again isomorphic to \mathbb{P}^1 , of degree 1 and passing through $\eta(\Gamma) = (1 : 0 : 0 : 0) = q$.

Let $C \subset S$ be a curve distinct from the E_i , isomorphic to \mathbb{P}^1 and whose image by η is again isomorphic to \mathbb{P}^1 , of degree d . It is linearly equivalent to $mL - \sum a_i E_i$, where $m > 0$ and $a_i \geq 0$ for $i = 1, \dots, 12$, and its intersection with E and H is, respectively, $\varepsilon = 3m - \sum a_i \in \{0, 1\}$ and $d = \varepsilon + m$.

If $d = 1$, we find $(m, \varepsilon) = (1, 0)$, which means that C is the strict transform of a line of \mathbb{P}^2 passing through three of the p_i , impossible.

If $d = 2$, we find $(2, 0)$ or $(1, 1)$ for (m, ε) . The first case is impossible since it would give conics through 6 of the p_i . The second case corresponds to the 66 lines passing through 2 of the p_i .

If $d = 3$, we find $(3, 0)$ or $(2, 1)$ for (m, ε) . The first case is again impossible, it corresponds to cubics passing through 8 of the p_i and being singular at one of them. The second case corresponds to the 5544 conics passing through 5 of the p_i . \square

COROLLARY 5.5. *Taking $D = kL - \sum m_i E_i$ where (k, m_1, \dots, m_{12}) are given in Table 3, if a curve $C \subset S$ is irreducible, smooth and equivalent to D , its image in \mathbb{P}^3 is a smooth irreducible curve of type (g, d) with no 5-secant, no 9-secant conic and no 13-secant twisted cubic.*

Proof. In each case, we have $\sum m_i = 3k - 1$, which means that $C \cdot \Gamma = 1$ on S , so the image of C in \mathbb{P}^3 is smooth by Lemma 5.3. The degree d of this curve in \mathbb{P}^3 is equal to $C \cdot H = 4k - \sum m_i = k + 1$. Its genus is $g = (k - 1)(k - 2)/2 - \sum (m_i(m_i - 1)/2)$, and is given in the table.

If $R \subset \mathbb{P}^3$ is a curve of degree $n \in \{1, 2, 3\}$ which intersects C in at least $4n + 1$ points, its intersection with Q is $4n < 4n + 1$, so it has to be contained in Q . By Lemma 5.4, any line of Q corresponds to one of the E_i ; since each a_i is at most 4, there is no 5-secant. Applying again Lemma 5.4, any conic of Q corresponds to a line of \mathbb{P}^2 passing through two points; a quick

check shows that $k - m_{11} - m_{12} \leq 8$ in each case, so there is no 9-secant conic. The twisted cubics correspond to conics by five points, and once again we can see that $2k - m_8 - m_9 - m_{10} - m_{11} - m_{12} \leq 12$. This finishes the proof. \square

PROPOSITION 5.6. *Any of the cases of Corollary 5.5 exists.*

Proof. Each of the case is given by (k, m_1, \dots, m_{12}) , where $\sum_{i=1}^{12} m_i = 3k - 1$.

We fix the smooth cubic curve $\Gamma_0 \subset \mathbb{P}^2$, and denote by $l \in \text{Pic}(\Gamma_0)$ the restriction of an hyperplane section, which is a divisor of degree 3 on Γ_0 . Recall that by Riemann–Roch’s formula on the elliptic curve Γ_0 , if D is a divisor of degree d on Γ_0 , then $|D|$ has dimension $d - 1$. In particular, for any set of twelve points p_1, \dots, p_{12} on the curve, since the divisor $kl - \sum m_i p_i$ has degree 1 there exists a unique point q on Γ_0 such that $q + \sum m_i p_i \sim kl$. We choose 12 distinct points p_1, \dots, p_{12} on the curve, so that that $p_1 + \dots + p_{12} \sim 4l$, no 3 of the p_i are collinear, no 6 are on the same conic, no 8 are on the same cubic singular at one of them and no 9 are on the same cubic distinct from Γ_0 (possible by Lemma 5.2); we can moreover assume that the point $q \in \Gamma_0$ equivalent to $kl - \sum m_i p_i$ is distinct from the p_i . This gives a particular case of Set-up 5.1.

In the blow-up S of the twelve points, the divisor $D = kL - \sum m_i E_i$ satisfies $-3k + \sum m_i = D \cdot K_S = -1$. Moreover, the integer g given in the table is equal to $(k - 1)(k - 2)/2 - \sum(m_i(m_i - 1)/2)$ (and is the genus of an irreducible member of $|D|$ if such member exists, which is exactly what we want to prove). The system $|D|$ corresponds to curves of \mathbb{P}^2 of degree k having multiplicity m_i at each point p_i . Its (projective) dimension is thus at least equal to

$$\frac{(k + 1)(k + 2)}{2} - \sum \frac{m_i(m_i + 1)}{2} - 1 = g + 3k - \sum m_i - 1 = g. \tag{2}$$

Recall that we want to prove that $|D|$ contains a smooth irreducible member, for a general choice of the p_i as above.

The first case in the list (first line of Table 3) is special. The divisor D corresponds to a (-1) -curve on the blow-up of the points p_1, \dots, p_8 , which is a del Pezzo surface (because of the assumption on the points). There is thus a unique curve equivalent to D , which is irreducible, and smooth.

The second case of the list (second line of Table 3, where $g = 1$) can also be viewed directly. Denote by $\mu: S \rightarrow S_2$ the contraction of E_9, \dots, E_{12} and of the (-1) -curve equivalent to $3L -$

TABLE 3. *The thirteen cases in Corollary 5.5, which are special instances of the curves listed in Table 2.*

g	d	k	(m_1, \dots, m_{12})	$k - m_{11} - m_{12}$	$2k - \sum_{i=8}^{12} m_i$
0	7	6	(3,2,2,2,2,2,2,0,0,0,0)	6	10
1	7	6	(3,2,2,2,2,2,1,1,0,0,0)	6	10
2	7	6	(2,2,2,2,2,2,2,1,0,0,0)	6	9
2	8	7	(3,3,2,2,2,2,2,2,0,0,0)	7	10
3	8	7	(3,3,2,2,2,2,2,1,1,0,0)	7	10
4	8	7	(3,2,2,2,2,2,2,2,1,0,0)	7	9
5	8	7	(2,2,2,2,2,2,2,2,2,0,0)	7	8
6	9	8	(3,3,3,2,2,2,2,2,1,1,0)	7	10
7	9	8	(3,3,2,2,2,2,2,2,2,1,0)	7	9
8	9	8	(3,2,2,2,2,2,2,2,2,2,0)	6	8
10	10	9	(3,3,3,2,2,2,2,2,2,1,1)	7	10
11	10	9	(3,3,3,2,2,2,2,2,2,1)	6	9
14	11	10	(3,3,3,3,3,2,2,2,2,2,2)	6	10

$2E_1 - E_2 - \dots - E_7$. The surface S_2 is a del Pezzo surface of degree 2 since the blow-up of $p_1, \dots, p_8 \in \mathbb{P}^2$ is a del Pezzo surface. Moreover, $D = \mu^*(-K_{S_2}) - E_9$. Taking a general member of the subsystem of $|-K_{S_2}|$ of elements passing through $\mu(E_9)$, the strict transform will yield a smooth elliptic curve equivalent to D (note that the member will not pass through the points $\mu(E_{10}), \mu(E_{11}), \mu(3L - 2E_1 - E_2 - \dots - E_7)$).

It remains to study the cases where $g \geq 2$, which implies $\dim|D| \geq 2$. Recall that $\Gamma \subset S$, the strict transform of Γ_0 , has self-intersection -3 and is equivalent to $-K_S$. Any effective divisor C equivalent to D either contains Γ or intersect it into $C \cdot \Gamma = D \cdot (-K_S) = 1$ point. In particular, the restriction of the linear system D to Γ is equivalent to $kl - \sum m_i p_i \sim q$, so $C \cap \Gamma$ is either Γ or $\{q\}$. The system $|D|$ has (at least) one base-point, which is the point $q \in \Gamma$.

We need the following fact.

FACT 5.7. *The system*

$$l_D = \left| D + K_S - \sum_{m_i=0} E_i \right|$$

has no base-point outside of $\Gamma \cup \bigcup_{m_i=0} E_i$.

The proof of this fact proceeds by a case-by-case analysis.

In case (2, 7) (third line of Table 3), $l_D = |D - \Gamma - E_{10} - E_{11} - E_{12}| = |3L - E_1 - \dots - E_8|$, which has dimension at least 1 by formula (2). Remark that $\pi_*(l_D)$, the image of the system on \mathbb{P}^2 , corresponds to the linear system of cubics passing through the points p_1, \dots, p_8 . On the elliptic curve Γ_0 , the linear system $|3l - \sum_{i=1}^8 p_i|$ contains exactly one element (by Riemann–Roch’s formula), and thus this point q' of Γ_0 is a base point of $\pi_*(l_D)$. Because of Set-up 4.1, the point q' is distinct from the p_i (see Lemma 5.2). Observe that the system $\pi_*(l_D)$ has no reducible member: this would imply that there exists a line through three points or a conic through six points. Thus, the general member of l_D is the strict transform of an irreducible cubic through the points q', p_1, \dots, p_8 ; we conclude that the lift of q' on S , which lies on Γ , is the unique base-point of the system l_D .

In case (2, 8), a similar argument holds, applied to the mobile system $l_D = |4L - 2E_1 - 2E_2 - E_3 - \dots - E_9|$, which has also dimension at least 1. For a general choice of the p_i , we can assume that the point q' of Γ_0 corresponding to $|4l - 2p_1 - 2p_2 - \sum_{i=3}^9 p_i|$ is distinct from the p_i (for each i , the case $q' = p_i$ gives a strict closed subset of the 11-dimensional space, as in Lemma 5.2). The only reducible members of $\pi_*(l_D)$ are union of a line passing through p_1 and p_2 and a cubic by the points p_1, \dots, p_9 , this latter being only Γ_0 by Set-up 4.1. The dimension of l_D being at least 1, a general member comes from an irreducible quartic of \mathbb{P}^2 passing through p_1, \dots, p_9 , with multiplicity 2 at p_1 and p_2 . Since $(l_D)^2 = 1$, the lift on S of the point q' , which lies on Γ , is the unique base-point of $|l_D|$.

In cases (3, 8), $l_D = |4L - 2E_1 - 2E_2 - E_3 - \dots - E_8|$ has no base-point outside of $\Gamma \cup E_9$, because it contains the union of E_9 with the system of case (2, 8).

For (4, 8), $l_D = |4L - 2E_1 - E_2 - E_3 - \dots - E_8|$ and the result follows from the fact that this system contains the union of $|L - E_1|$, which has no base-point, with the system of case (2, 7).

In case (5, 8), $l_D = |4L - \sum_{i=1}^{10} E_i|$ has no base-point outside of $E_{11} \cup E_{12}$, since it contains the union of $E_{11} + E_{12}$ with the base-point free system $|4L - \sum_{i=1}^{12} E_i|$. Indeed the latter system corresponds to the morphism $S \rightarrow \mathbb{P}^3$, which is an isomorphism outside of Γ_0 .

The cases (6, 9), (7, 9) and (8, 9) give $|5L - 2E_1 - 2E_2 - 2E_3 - \sum_{i=4}^9 E_i|$, $|5L - 2E_1 - 2E_2 - \sum_{i=3}^{10} E_i|$ and $|5L - 2E_1 - \sum_{i=2}^{11} E_i|$ for l_D . The system contains the union of the unique (-1) -curve $C_1 \subset S$ equivalent to $3L - 2E_1 - \sum_{i=2}^7 E_i$ with a base-point-free system corresponding to conics of \mathbb{P}^2 by four of the p_i . Similarly it also contains the union of the curve C_2 equivalent

to $3L - 2E_1 - \sum_{i=3}^8 E_i$ with another base-point free system. Since $C_1 \cdot C_2 = 0$, the system l_D is base-point free.

The cases (10, 10), (11, 10) and (14, 11) yield $|6L - 2\sum_{i=1}^4 E_i - \sum_{i=5}^{10} E_i|$, $|6L - 2\sum_{i=1}^3 E_i - \sum_{i=4}^{11} E_i|$ and $|7L - 2\sum_{i=1}^5 E_i - 2E_5 - \sum_{i=6}^{12} E_i|$ for l_D . For $i \neq j$, we denote by $R_{ij} \subset S$ the (-1) -curve equivalent to $L - E_i - E_j$. For $(i, j) = \{(8, 9), (8, 10), (9, 10)\}$, the system l_D contains the union of $C_1 \cup R_{ij}$ with a base-point free system. This implies that l_D has no base-point except maybe on C_1 . Replacing C_1 by C_2 , we see that l_D has no base-point. This ends the proof of the Fact 5.7.

Now that we have proved that $l_D = |D + K_S - \sum_{m_i=0} E_i|$ has no base-point outside of $\Gamma \cup \bigcup_{m_i=0} E_i$, this implies that the same property holds for $|D|$. Indeed, taking a member $C \in |l_D|$, the reducible effective divisor $C + \Gamma + \sum_{m_i=0} E_i$ is a member of $|D|$.

This implies that D is nef. Indeed, if an irreducible curve intersects negatively D , it is a fixed component of $|D|$ and has to be Γ or E_i for some i . Each of these curves intersects non-negatively D , so D is nef. The fact that $D^2 > 0$ implies that D is also big, and this gives $h^i(D + K_S) = 0$ for $i = 1$ or 2 . Thus, $\dim|D + K_S| = (D + K_S) \cdot D/2 + 1 \leq \dim|D| - 1$. A general member C of $|D|$ does not contain $\Gamma \sim -K_S$ and thus $C \cap \Gamma = \{q\}$, so q is the only base-point of $|D|$ on Γ . Moreover, the image of C on \mathbb{P}^2 is a curve of degree k intersecting Γ_0 into $3k$ points (counted with multiplicity), which have to be $\sum m_i p_i + q$. If $m_i = 0$, the point $p_i \in \mathbb{P}^2$ is therefore not a point of $\pi(C)$, which means that $C \cdot E_i = 0$. This shows that q is the only base-point of $|D|$ on S .

By Bertini's theorem C is smooth outside of q , and since $C \cdot \Gamma = 1$ we also have C smooth at q . □

5.2. Curves in a smooth quartic

PROPOSITION 5.8. *Let $C \in \mathcal{H}_{g,d}^S$ be a curve of genus g and degree d . Assume that C is contained in a smooth quartic surface, and that (g, d) is one of the thirteen cases listed in Table 2.*

Then the blow-up X of C has nef. anticanonical divisor if and only if there is no $(4n + 1)$ -secant rational smooth curve of degree n for $n = 1, 2, 3$.

The 13-secant twisted cubics are possible only if $(g, d) \in \{(0, 7), (2, 8), (3, 8)\}$.

The 9-secant conics are possible only if $(g, d) \in \{(6, 9), (7, 9)\}$.

The 5-secant lines are possible in all cases except for $(g, d) = (14, 11)$.

Proof. Suppose that $-K_X$ is not nef., which implies the existence of an irreducible curve Γ of degree n which intersects at least $4n + 1$ times C . We want to prove that this curve is smooth, rational and has degree at most 3, and describe when it could exist. Note that C must be inside all quartics containing C , so $\deg(\Gamma) \leq 16 - d$.

Assume first that $\deg(\Gamma) = 16 - d$, which means that $\Gamma \cup C$ is a complete intersection of two quartics, which by assumption can be taken smooth. Inside one of the quartics, we have $(C + \Gamma)^2 = 64$ and $C^2 = 2g - 2, \Gamma^2 = 2g(\Gamma) - 2$. By the formula from § 2.5, we have $g - g(\Gamma) = 2(d - \deg(\Gamma))$, hence $g(\Gamma) = g - 4d + 32 \geq 2$ in all cases under consideration. The formula being symmetric, we find

$$g = 32 + g(\Gamma) - 4 \deg(\Gamma).$$

Hence (inside a quartic)

$$\begin{aligned} C \cdot \Gamma &= \frac{64 - (2g - 2) - (2g(\Gamma) - 2)}{2} \\ &= 34 - g - g(\Gamma) \end{aligned}$$

$$\begin{aligned}
 &= 4 \deg(\Gamma) + 2 - 2g(\Gamma) \\
 &< 4 \deg(\Gamma).
 \end{aligned}$$

For the last line we used $g(\Gamma) \geq 2$.

Now assume that $C \cup \Gamma$ is the complete intersection of a smooth quartic with a cubic. As in the first part of the proof of Proposition 4.10 we can prove that

$$C \cdot \Gamma \leq 3 \deg(\Gamma) + 2.$$

Note that we have $n \geq 12 - d \geq 2$, since in the case (14, 11) the residual curve would have genus -1 and degree 1, which is impossible. Hence, we have $C \cdot \Gamma \leq 4 \deg(\Gamma)$.

Suppose finally that $n = \deg(\Gamma) < 16 - d$, and that $\Gamma \cup C$ is not a complete intersection of S with another surface. We can apply Proposition 2.6, and get $8g(\Gamma \cup C) < (n + d)^2$, where $g(\Gamma \cup C) \geq g(C) + g(\Gamma) + 4n$, which yields

$$8g(\Gamma) < (n + d)^2 - 8g - 32n = n^2 - 2(16 - d)n + d^2 - 8g.$$

We call $P_{g,d}$ the polynomial $n^2 - 2(16 - d)n + d^2 - 8g \in \mathbb{Z}[n]$. For $(g, d) = (14, 11)$, we find $8g(\Gamma) < P_{14,11} = n^2 - 10n + 9 = (n - 1)(n - 9)$, since $1 \leq n \leq 5$, we have a contradiction. More generally, we find the following polynomials for the other pairs (g, d) .

g	d	$P_{g,d}$	g	d	$P_{g,d}$
0	7	$(n - 4)(n - 14) - 7$	5	8	$(n - 2)(n - 14) - 4$
1	7	$(n - 3)(n - 15) - 4$	6	9	$(n - 3)(n - 11)$
2	7	$(n - 3)(n - 15) - 12$	7	9	$(n - 3)(n - 11) - 8$
2	8	$(n - 4)(n - 12)$	8	9	$(n - 2)(n - 12) - 7$
3	8	$(n - 4)(n - 12) - 8$	10	10	$(n - 2)(n - 10)$
4	8	$(n - 3)(n - 13) - 7$	11	10	$(n - 2)(n - 10) - 8$

Since $8g(\Gamma) < P_{g,d}$ and $n < 16 - d$, we find that $n < 4$.

If $n = 3$, we have $(g, d) \in \{(0, 7), (2, 8), (3, 8)\}$; the curve Γ has to be smooth and rational otherwise it would be contained in a plane and could not intersect C at $13 > d$ points.

If $n = 2$, the curve is contained in a plane and intersects C at 9 points, so $d \geq 9$. The only cases where $P_{g,d}(n) \geq 1$ are $(g, d) \in \{(6, 9), (7, 9)\}$.

The case $n = 1$ is always possible, except for $(g, d) = (14, 11)$, treated above. □

5.3. Proof of Theorem 1.1

We consider the blow-up $X \rightarrow \mathbb{P}^3$ of a curve $C \in \mathcal{H}_{g,d}^S$ of genus g and degree d .

We first assume that $-K_X$ is big and nef. Then C does not admit any 5-secant line, 9-secant conic or 13-secant twisted cubic (Lemma 2.2).

If C is contained in a quadric or a plane, the exhaustive study in § 3, and in particular Corollary 3.2, implies that (g, d) is one of the cases in the first two columns of Table 1 which are not crossed.

We now assume that C is not contained in any quadric. By Riemann–Roch on a threefold and Kawamata–Viehweg vanishing, we have $\dim|-K_X| = \frac{1}{2}(-K_X^3) + 2 \geq 3$. Thus, C is contained in a pencil of quartics, whose general member can be taken smooth by Proposition 2.8. In particular, $d \leq 16$.

If C is the complete intersection of a smooth quartic with another surface F , then F has to be a cubic: This yields the case $(g, d) = (19, 12)$.

On the other hand, if C is not a complete intersection, by Proposition 2.6, we have

$$8g < d^2.$$

Since $(-K_X)^3 > 0$, by Lemma 2.4, we have

$$4d - 30 \leq g.$$

Putting together both inequalities yields $d^2 > 8(4d - 30)$, hence

$$d^2 - 32d + 240 = (d - 12)(d - 20) > 0.$$

Since we already know that $d \leq 16$, we obtain $d \leq 11$, hence $g \leq 15$. But $g = 15$ implies $d = 11$, and by Remark 4.4 curves of type $(15, 11)$ always admit a 5-secant line; so we obtain $g \leq 14$.

Since C is not in any quadric we have the Castelnuovo bound on the genus of C (see [11, Theorem IV.6.4]):

$$g < \left\lfloor \frac{d^2}{4} \right\rfloor - d + 1.$$

Pairs (g, d) which satisfy this inequality together with $8g < d^2$ and $4d - 30 \leq g \leq 14$ are in the last two columns of Table 1, or equal to $(0, 4)$. But curves of type $(0, 4)$ are contained in quadrics (Lemma 2.3).

We now assume that $-K_X$ is ample. It is clear by Lemma 2.2 that C admits no 4-secant line. Then we use a classical formula of Cayley, in the following precise formulation as proved by Le Barz [22].

PROPOSITION 5.9. *Let $C \in \mathcal{H}_{g,d}^S$. Assume that C does not admit infinitely many n -secant lines for any $n \geq 4$. Then the number of 4-secant lines to C , counted with multiplicity, is given by the formula*

$$\frac{(d - 2)(d - 3)^2(d - 4)}{12} - \frac{(d^2 - 7d + 13 - g)g}{2}.$$

In particular, among the (g, d) that could produce a weak Fano threefold, this formula gives 0 precisely in the cases listed in the subsets \mathcal{A}_1 and \mathcal{A}_2 in Theorem 1.1.

Conversely, assume that $C \in \mathcal{H}_{g,d}^S$ with $(g, d) \in \mathcal{A}_1$, or $(g, d) \in \mathcal{A}_2$ and C does not admit a 4-secant line. We want to prove that X is Fano. If C lies on a quadric, the result follows from Corollary 3.2. If C does not lie on a quadric, then (g, d) is in the set

$$\mathcal{A}_2 = \{(1, 5), (3, 6), (5, 7), (10, 9)\}.$$

In these cases Lemma 2.3 tells us that C is contained in a pencil of (irreducible) cubic surfaces. If Γ is an irreducible curve of degree n which is $4n$ -secant to C , then Γ must be in the base locus of this pencil, and $2 \leq n \leq 9 - d$. This rules out $(10, 9)$. Since the three other cases have degree $d \leq 7$, clearly they do not admit a 8-secant conic: the plane containing the conic has to meet C in d points only. By a similar argument $(1, 5)$ does not admit a 12-secant curve Γ of degree 3, by considering the plane or the quadric containing Γ . Finally, if $\Gamma \cup C$ is a complete intersection with C of type $(1, 5)$ (resp. $(3, 6)$), by the formulas in §2.5 we see that Γ is a 10-secant (resp. 8-secant) rational curve of degree 4 (resp. 3). We conclude that X is Fano.

Finally, assume that $C \in \mathcal{H}_{g,d}^S$ is contained in a smooth quartic, with $4d - 30 \leq g \leq 14$ or $(g, d) = (19, 12)$, and that there is no 5-secant line, 9-secant conic, nor 13-secant twisted cubic to C . If (g, d) is in \mathcal{A}_1 or \mathcal{A}_2 , we already know that X is Fano.

So we can assume that (g, d) is in the third or fourth column of Table 1.

In the former case, we conclude by Proposition 4.10, and in the latter one, by Proposition 5.8, that X is weak Fano.

The fact that curves of type $(3, 4), (6, 7), (9, 8), (12, 10)$ and $(19, 12)$ yield a divisorial anticanonical morphism was observed in Proposition 3.1(ii) and Remarks 4.5 and 4.6.

It remains to prove that Conditions (i) and (ii) in Theorem 1.1 correspond to non-empty open sets in relevant $\mathcal{H}_{g,d}^S$.

For (g, d) in $\mathcal{A}_2 \cup \mathcal{A}_3$, this follows from the explicit construction of examples on smooth cubic surfaces, as described in Proposition 4.2.

The case $(g, d) \in \mathcal{A}_4$ is more delicate. By Proposition 2.7 we know that $\mathcal{H}_{g,d}^S$ is irreducible in all cases, except for $(14, 11)$ where there are two components. We denote by $V_{14,11}$ the irreducible component whose general members lie on a quartic surface, and $V_{g,d} = \mathcal{H}_{g,d}^S$ in the other cases.

By Lemma 2.3, we know that the system of quartics containing any $C \in V_{g,d}$ has dimension at least 3. Consider W the set of pairs (C, Q) where $C \in V_{g,d}$, $Q \subset \mathbb{P}^3$ is a quartic surface and $C \subset Q$. By Proposition 2.5, we know that there exists a smooth quartic surface containing a smooth curve of genus g and degree d . Hence the open set $W_s \subset W$ of pairs (C, Q) such that Q is smooth is non-empty. Projecting on the first factor, we obtain $V_1 \subset V_{g,d}$ a non-empty open subset such that any $C \in V_1$ is contained in a smooth quartic surface.

From Proposition 5.6, we know that there exists a non-empty open set $V_2 \subset V_{g,d}$ such that if $C \in V_2$, then C does not admit any 5-secant line, 9-secant conic or 13-secant twisted cubic.

By irreducibility of $V_{g,d}$, we have $V_1 \cap V_2 \neq \emptyset$: this is the desired open set.

REMARK 5.10. The assumption that C is contained in a smooth quartic in Theorem 1.1 is probably superfluous. The point to remove it would be to prove that for $(g, d) \in \mathcal{A}_3 \cup \mathcal{A}_4$, if $C \in \mathcal{H}_{g,d}^S$ does not have a 5-secant line, then C is contained in a smooth quartic.

For instance, consider C lying on a smooth cubic surface S , coming from the transform of a curve of degree 9 in \mathbb{P}^2 with multiplicities $(3, 3, 3, 3, 2, 2)$ at the six blown-up points. Then $C \in \mathcal{H}_{14,11}^S$, and any quartic containing S breaks into two components, S and a plane. Note that C admits 5-secant lines, for instance, the transform of the line through the two points with multiplicity 2. More generally, any $C \in \mathcal{H}_{14,11}^S$ which lies on a cubic cannot lie on a irreducible quartic, otherwise C would be linked to a curve with genus -1 and degree 1: contradiction. But we have been unable to prove that if $C \in \mathcal{H}_{14,11}^S$ lies in a irreducible quartic, then the quartic can be taken to be smooth.

5.4. Existence of Sarkisov links

PROPOSITION 5.11. *If (g, d) is one of the thirteen cases listed in Table 2, and C is a general curve in $\mathcal{H}_{g,d}^S$, then the blow-up X of C is weak Fano, the anticanonical morphism is a small birational map and thus yields a Sarkisov link involving a flop (see Example 5.12 for details).*

Proof. By Theorem 1.1, there exists a non-empty open subset $V \subset \mathcal{H}_{g,d}^S$ such that for any $C \in V$, the blow-up X of C is weak Fano.

Now for such an X obtained from $C \in V$, suppose that the anticanonical morphism is a divisorial contraction. Then we would be in one of the 24 cases of [16, Theorem 4.9]. The only coincidence between the two lists are $(5, 8)$ and $(11, 10)$. In case $(5, 8)$ the exceptional divisor of the anticanonical map is the transform of a cubic surface through C . But curves of type $(5, 8)$ contained in a cubic surface form a codimension 1 closed subset in the 32-dimensional $\mathcal{H}_{5,8}^S$ (see [9]). The case $(11, 10)$ is more subtle, since even in the divisorial case the curve has no reason to lie on a cubic. Nevertheless, curves that correspond to a small anticanonical morphism form again a dense open set in $V_1 \cap V_2$: this follows from the construction given in [16, No. 19 p. 617], which involves taking the minors of a (4×4) matrix M with linear entries

in $\mathbb{C}[x_0, \dots, x_4]$. It is noted there that the anticanonical morphism is a divisorial contraction when M is symmetric, and is a small contraction otherwise.

In conclusion, we obtain in all cases a non-empty open subset of $\mathcal{H}_{g,d}^S$ with the desired properties. The fact that it yields a Sarkisov link was discussed in § 2.1, and the presence of a flop comes from the fact that X is never Fano in the cases under consideration. \square

EXAMPLE 5.12. The numerical possibilities listed in Table 2 were already found in the literature, but without proof of actual existence (except maybe for $(g, d) = (11, 10)$, see end of the proof of Proposition 5.11, and $(g, d) = (7, 9)$, see below).

Note that our proof gives the existence of these links, without knowing *a priori* the end result.

(i) Cases $(g, d) = (0, 7)$, $(2, 8)$, $(3, 8)$, $(5, 8)$, $(6, 9)$, $(10, 10)$, $(11, 10)$ and $(14, 11)$ must correspond to Sarkisov links from \mathbb{P}^3 to \mathbb{P}^3 : see [4], cases 90, 49, 75, 99, 50, 51, 76, 52 in their Tables 1–3. We plan to come back to the description of these interesting Cremona transformations in a future paper.

(ii) Case $(g, d) = (1, 7)$ corresponds to a Sarkisov link to the prime Fano X_{22} of genus 12, with a flop and contraction to another curve of genus 1 and degree 7: see [4, Case 98].

(iii) Case $(g, d) = (2, 7)$ corresponds to a Sarkisov link to the intersection of two quadrics $V_4 \subset \mathbb{P}^5$, with a flop and contraction to a curve of genus 0 and degree 5: see [4, Case 103].

(iv) Case $(g, d) = (4, 8)$ corresponds to a Sarkisov link to the Fano threefold $V_5 \subset \mathbb{P}^6$, with a flop and contraction to a curve of genus 4 and degree 10: see [4, Case 89].

(v) Case $(g, d) = (7, 9)$ corresponds to a Sarkisov link to the Fano threefold X_{12} of genus 7, with a flop and contraction to a curve of genus 0 and degree 3. The existence of this link is claimed in [15, p. 103], at least in the case of a curve lying in a special smooth quartic surface as constructed in the proof of Proposition 2.5, but the details of the argument are not given.

(vi) Case $(g, d) = (8, 9)$ corresponds to a Sarkisov link of type I to a fibration in del Pezzo surfaces of degree 5 (after one flop): see [17, Proposition 6.5(25)].

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