

EMBEDDINGS OF AFFINE SPACES INTO QUADRICS

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ABSTRACT. This article provides, over any field, infinitely many algebraic embeddings of the affine spaces \mathbb{A}^1 and \mathbb{A}^2 into smooth quadrics of dimension two and three, respectively, which are pairwise non-equivalent under automorphisms of the smooth quadric. Our main tools are the study of the birational morphism $\mathrm{SL}_2 \rightarrow \mathbb{A}^3$ and the fibration $\mathrm{SL}_2 \rightarrow \mathbb{A}^3 \rightarrow \mathbb{A}^1$ obtained by projections, as well as degenerations of variables of polynomial rings, and families of \mathbb{A}^1 -fibrations.

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1. INTRODUCTION

In what follows we denote by \mathbf{k} the ground field of our algebraic varieties. Given two affine algebraic varieties X, Y , we say that two closed embeddings $\rho, \rho': X \hookrightarrow Y$ are *equivalent* if there exists an automorphism $\varphi \in \mathrm{Aut}(Y)$ such that $\rho' = \varphi \circ \rho$. Similarly, we say that two closed subvarieties $X, X' \subset Y$ are *equivalent* if there exists an automorphism $\varphi \in \mathrm{Aut}(Y)$ such that $X' = \varphi(X)$. If two closed embeddings are equivalent, then their images are equivalent, but the converse is not always true and is related to the extension of automorphisms.

In the 1970s, Abhyankar and Sathaye conjectured that every closed embedding $\mathbb{A}_{\mathbf{k}}^{n-1} \hookrightarrow \mathbb{A}_{\mathbf{k}}^n$ is equivalent to a linear embedding (see for instance [vdE00, §3, p. 103]). This was the starting point for studying embedding problems in affine algebraic geometry. In the Bourbaki Seminar *Challenging problems on affine n -space* [Kra96], Kraft gives a list of eight fundamental problems related to the affine n -spaces. The third one is the following generalisation of the Abhyankar-Sathaye conjecture.

Embedding Problem. *Is every closed embedding $\mathbb{A}_{\mathbf{k}}^m \hookrightarrow \mathbb{A}_{\mathbf{k}}^n$ equivalent to the standard embedding $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$?*

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This question, asked over the ground field $\mathbf{k} = \mathbb{C}$ in [Kra96], has until now no negative answer. For $\mathbf{k} = \mathbb{R}$, it is easy to find counterexamples for $m = 1$ and $n = 3$, by taking embeddings which are not topologically trivial (non-trivial knots); see for instance the example of [Sha92], reproduced below in Example 6.1. In positive characteristic, there are counterexamples when $m = n - 1$ (see Proposition 3.17). The embedding problem has however a positive answer in the following cases:

- (1) $m = 1$, $n = 2$, $\text{char}(\mathbf{k}) = 0$ (Abhyankar-Moh-Suzuki Theorem) [AM75, Suz74], [vdE00, Theorem 2.3.5];
- (2) $n \geq 2m + 2$, \mathbf{k} infinite (Theorem of Kaliman, Nori and Srinivas [Kal91, Sri91]).

The case of hypersurfaces ($m = n - 1$), corresponding to the Abhyankar-Sathaye conjecture, is of particular interest. In this case, the image is given by the zero set of an irreducible polynomial equation $f \in \mathbf{k}[A^n]$. One necessary condition for an embedding to be equivalent to the standard embedding consists of asking that the other fibres of $f: A_{\mathbf{k}}^n \rightarrow A_{\mathbf{k}}^1$ are affine spaces. In fact, for any field \mathbf{k} and any $n \geq 1$, there is no known example of a hypersurface $X \subset A_{\mathbf{k}}^n$ isomorphic to $A_{\mathbf{k}}^{n-1}$ and given by $f = 0$, $f \in \mathbf{k}[A^n]$ irreducible such that another fibre $f = \lambda$ is not isomorphic to $A_{\mathbf{k}}^{n-1}$. The non-existence of such examples (at least when $\text{char}(\mathbf{k}) = 0$), is the Sathaye conjecture [vdE00, §3, p. 103], which is implied by the Abhyankar-Sathaye conjecture. Even this weaker conjecture is quite strong and seems “unlikely” (as Arno van den Essen says in [vdE00, §3, p. 103]). Moreover, for $n = 3$ and $\text{char}(\mathbf{k}) = 0$, the fact that infinitely many fibres $f = \lambda$ are isomorphic to $A_{\mathbf{k}}^2$ implies that the fibration is equivalent to the standard one, and in particular that all fibres are isomorphic to $A_{\mathbf{k}}^2$ [KZ01, Kal02, DK09]. The positive characteristic version of the Sathaye conjecture has until now no counterexample, and is open even in dimension $n = 2$ (this latter case corresponds to a question of Abhyankar; see [Gan11, Question 1.1]).

In this paper, we replace the affine space at the target by some analogue varieties, namely affine smooth quadrics. This simplifies the question in such a way that one can actually give an answer. Moreover, it also gives some idea on what kind of behaviour one could expect in a general situation.

In dimension $n = 2$, the most natural quadric is

$$Q_2 = \text{Spec}(\mathbf{k}[x, y, z]/(xy - z(z + 1))) \subset A_{\mathbf{k}}^3.$$

In fact, if \mathbf{k} is an algebraically closed field, then every smooth quadric hypersurface $Q \subset A_{\mathbf{k}}^3$ is isomorphic to $A_{\mathbf{k}}^2$, $(A_{\mathbf{k}}^1 \setminus \{0\}) \times A_{\mathbf{k}}^1$ or Q_2 , as one can see using the classification of quadratic forms. As all embeddings of $A_{\mathbf{k}}^1$ into $(A_{\mathbf{k}}^1 \setminus \{0\}) \times A_{\mathbf{k}}^1$ are constant on the first factor, they are all equivalent. Over any field, the group of automorphisms of Q_2 is similar to the one of $A_{\mathbf{k}}^2$, as it is an amalgamated product of two factors, corresponding to affine maps and triangular maps (see [DG77] or [BD11, Theorem 5.4.5(7)(a)]). This is also the case for the affine surface $\mathbb{P}_{\mathbf{k}}^2 \setminus \Gamma$, where $\Gamma \subset \mathbb{P}_{\mathbf{k}}^2$ is any smooth conic having a \mathbf{k} -point (see for instance [DD16, Theorem 2]). If $\text{char}(\mathbf{k}) = 0$, there is exactly one (respectively, two) closed curve $C \subset A_{\mathbf{k}}^2$ (respectively, $C \subset \mathbb{P}_{\mathbf{k}}^2 \setminus \Gamma$) isomorphic to $A_{\mathbf{k}}^1$, up to automorphism of the surface. This follows from the Abhyankar-Moh-Suzuki Theorem for $A_{\mathbf{k}}^2$ and from [DD16] for $\mathbb{P}_{\mathbf{k}}^2 \setminus \Gamma$. In particular, all automorphisms of the corresponding curves extend to automorphisms of $A_{\mathbf{k}}^2$ or $\mathbb{P}_{\mathbf{k}}^2 \setminus \Gamma$. Similarly, a complex toric affine surface

admits only finitely many embeddings of $\mathbb{A}_{\mathbb{C}}^1$, up to equivalence [AZ13]. By contrast, we prove the following result.

Theorem 1. *For each field \mathbf{k} , there is an infinite set of closed curves*

$$C_i \subset Q_2 = \text{Spec}(\mathbf{k}[x, y, z]/(xy - z(z + 1))), \quad i \in I,$$

which are pairwise non-equivalent up to automorphism of Q_2 , such that each C_i is isomorphic to $\mathbb{A}_{\mathbf{k}}^1$ and such that the identity is the only automorphism of C_i that extends to an automorphism of Q_2 . Moreover, if \mathbf{k} is uncountable, then I can be chosen uncountable as well.

In dimension $n = 3$, the most natural quadric is

$$\text{SL}_2 = \text{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1)) \subset \mathbb{A}_{\mathbf{k}}^4.$$

Similarly as in dimension two, over an algebraically closed field \mathbf{k} , every quadric hypersurface in $\mathbb{A}_{\mathbf{k}}^4$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^3$, $(\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}) \times \mathbb{A}_{\mathbf{k}}^2$, $Q_2 \times \mathbb{A}_{\mathbf{k}}^1$ or SL_2 . Moreover, the quotient of SL_2 by its maximal torus yields a morphism $\text{SL}_2 \rightarrow \text{SL}_2/T \simeq Q_2$, which is the “universal torsor” (also called the Cox quotient presentation or the characteristic space); see [ADHL15, Examples 4.5.13–4.5.14].

We consider the quadric hypersurface SL_2 more closely. Its automorphism group shares similar properties with the one of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^3)$ (see [LV13, BFL14, Mar15]). Both are known to be complicated, as they contain “wild” automorphisms [LV13], and do not preserve any fibration, as is the case for other varieties being topologically closer to $\mathbb{A}_{\mathbf{k}}^3$, like the Koras-Russell threefold. However, in contrast to the quadric Q_2 , the quadric SL_2 is closer to a contractible affine variety in the sense that the ring of regular functions on SL_2 is a unique factorisation domain (see [Pop74, Proposition 1] or Lemma 4.4). The first difference concerning embeddings of affine spaces with the surfaces $Q_2, \mathbb{A}_{\mathbf{k}}^2, \mathbb{P}_{\mathbf{k}}^2 \setminus \Gamma$ and with $\mathbb{A}_{\mathbf{k}}^3$ is that the “simplest embedding” $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$ is more rigid in the following sense.

Theorem 2. *Let \mathbf{k} be any field and let*

$$\begin{aligned} \rho_1: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 \\ (s, t) &\mapsto \begin{pmatrix} 1 & t \\ s & 1 + st \end{pmatrix} \end{aligned}$$

be the “standard” embedding. Then, an automorphism $(s, t) \mapsto (f(s, t), g(s, t))$ of $\mathbb{A}_{\mathbf{k}}^2$ extends to an automorphism of SL_2 , via ρ_1 , if and only if it has Jacobian determinant $\frac{\partial f}{\partial s} \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} \frac{\partial g}{\partial s} \in \mathbf{k}^*$ equal to ± 1 . In particular, the following holds:

- (1) every embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$ with image $\rho_1(\mathbb{A}_{\mathbf{k}}^2)$ is equivalent to an embedding

$$\begin{aligned} \rho_{\lambda}: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 \\ (s, t) &\mapsto \begin{pmatrix} 1 & t \\ \lambda s & 1 + \lambda st \end{pmatrix} \end{aligned}$$

for a certain $\lambda \in \mathbf{k}^*$. Moreover, ρ_{λ} and $\rho_{\lambda'}$ are equivalent if and only if $\lambda' = \pm \lambda$;

- (2) if \mathbf{k} has at least four elements, then not all automorphisms of $\mathbb{A}_{\mathbf{k}}^2$ extend to SL_2 via ρ_1 .

Remark 1.1. Let us make some comments on Theorem 2:

- (1) Over the field of complex numbers $\mathbf{k} = \mathbb{C}$, we show that all algebraic automorphisms of $\mathbb{A}_{\mathbf{k}}^2$ extend via the standard embedding ρ_1 to holomorphic automorphisms of SL_2 ; see Remark 4.7.

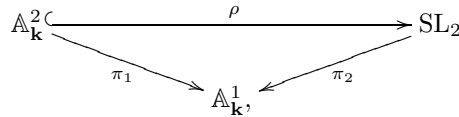
- (2) Let $f: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ be a closed embedding and denote by $\iota: \mathrm{SL}_2 \hookrightarrow \mathbb{A}_{\mathbf{k}}^4$ the standard embedding. If all coordinate functions of $\iota \circ f: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2 \subset \mathbb{A}_{\mathbf{k}}^4$ are polynomials of degree ≤ 2 , then f is equivalent to ρ_λ for a certain $\lambda \in \mathbf{k}^*$ (Proposition 5.19).

Next, we focus on the closed embeddings $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ that are compatible with the simplest \mathbb{A}^2 -fibration of SL_2 . More precisely we have the following.

Definition 1.2. A closed embedding $\rho: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ is said to be a *fibred embedding* if it is of the form

$$(\diamond) \quad \begin{array}{ccc} \rho: & \mathbb{A}_{\mathbf{k}}^2 & \hookrightarrow & \mathrm{SL}_2 \\ & (s, t) & \mapsto & \begin{pmatrix} p(s, t) & t \\ r(s, t) & q(s, t) \end{pmatrix} \end{array}$$

for some $p, q, r \in \mathbf{k}[s, t]$. This corresponds to the commutativity of the diagram



where $\pi_1: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^1$, $\pi_2: \mathrm{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^1$ are, respectively, given by $(s, t) \mapsto t$ and $\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto t$.

As we will show, there are a lot of fibred embeddings (i.e., embeddings of the form (\diamond)).

Theorem 3. *Let \mathbf{k} be any field, let $P \in \mathbf{k}[t, x, y]$ be a polynomial that is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$ (which means that P is the image of x by some automorphism of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$), and let*

$$H_P \subset \mathrm{SL}_2 = \mathrm{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$$

and $Z_P \subset \mathbb{A}_{\mathbf{k}}^3 = \mathrm{Spec}(\mathbf{k}[t, x, y])$ be the hypersurfaces given by $P = 0$.

- (1) The following conditions are equivalent:
 - (a) The hypersurface $H_P \subset \mathrm{SL}_2$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$.
 - (b) The hypersurface $H_P \subset \mathrm{SL}_2$ is the image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$.
 - (c) The fibre of $Z_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$, $(t, x, y) \mapsto t$ over every closed point of $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$ is isomorphic to \mathbb{A}^1 and the polynomial $P(0, x, y) \in \mathbf{k}[x, y]$ is of the form $\mu x^m(x - \lambda)$ or $\mu y^m(y - \lambda)$ for some $\mu, \lambda \in \mathbf{k}^*$ and some $m \geq 0$.
- (2) If $P, Q \in \mathbf{k}[t, x, y]$ are two polynomials of the above form satisfying the conditions (a) – (b) – (c), such that $H_P, H_Q \subset \mathrm{SL}_2$ are equivalent under an automorphism of SL_2 , then $Z_P, Z_Q \subset \mathbb{A}_{\mathbf{k}}^3$ are equivalent under an automorphism of $\mathbb{A}_{\mathbf{k}}^3$.
- (3) There are infinitely many fibred embeddings $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ having pairwise non-equivalent images in SL_2 . If \mathbf{k} is uncountable, we can moreover choose uncountably many such embeddings.

Remark 1.3. Let us make some comments on Theorem 3:

- (1) It is possible that H_P, H_Q are non-equivalent, even if Z_P, Z_Q are equivalent (Lemma 5.11).
- (2) If $\mathrm{char}(\mathbf{k}) = 0$, then every image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ is of the form H_P as above (Lemma 5.2(2)). This is false if $\mathrm{char}(\mathbf{k}) > 0$ (Lemma 5.3).

Let us make the following comment concerning embeddings of \mathbb{A}_k^1 into the smooth quadric SL_2 over the field $k = \mathbb{C}$. Although there are infinitely many non-equivalent embeddings of $\mathbb{A}_{\mathbb{C}}^2$ into SL_2 , it is not known whether all embeddings of $\mathbb{A}_{\mathbb{C}}^1$ into SL_2 are equivalent under an algebraic automorphism. It seems that this question is as difficult as the question of non-equivalent embeddings $\mathbb{A}_{\mathbb{C}}^1 \hookrightarrow \mathbb{A}_{\mathbb{C}}^3$. However, up to holomorphic automorphisms, all embeddings of $\mathbb{A}_{\mathbb{C}}^1$ into $\mathbb{A}_{\mathbb{C}}^3$ and into SL_2 are equivalent; see [Kal92, Sta15].

In the last section (Lemma 6.2), we give an example of an embedding $\mathbb{A}_{\mathbb{R}}^1 \hookrightarrow SL_2$ which is non-equivalent to the standard embedding.

2. THE SMOOTH QUADRIC OF DIMENSION TWO AND THE PROOF OF THEOREM 1

2.1. **The isomorphism with the complement of the diagonal in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$.** In this section, we study the smooth quadric $Q_2 \subset \mathbb{A}_k^3$ given by

$$Q_2 = \text{Spec}(k[x, y, z]/(xy - z(z + 1))),$$

and more particularly closed embeddings $\mathbb{A}_k^1 \hookrightarrow Q_2$. Since the closure of Q_2 in \mathbb{P}_k^3 is a smooth quadric, isomorphic to $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, we get the following classical isomorphism.

Lemma 2.1. *The morphism*

$$\rho: \quad Q_2 \quad \rightarrow \quad \mathbb{P}_k^1 \times \mathbb{P}_k^1$$

$$(x, y, z) \mapsto \begin{cases} \left(\begin{matrix} [y : z] \\ [z + 1 : x] \end{matrix}, \begin{matrix} [z : x] \\ [y : z + 1] \end{matrix} \right) & \text{if } z \neq 0, \\ \left(\begin{matrix} [y : z] \\ [z + 1 : x] \end{matrix}, \begin{matrix} [z : x] \\ [y : z + 1] \end{matrix} \right) & \text{if } z \neq -1, \end{cases}$$

yields an isomorphism $Q_2 \xrightarrow{\cong} (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta$, where $\Delta \subset \mathbb{P}_k^1 \times \mathbb{P}_k^1$ is the diagonal, with an inverse given by

$$\psi: \quad (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta \quad \rightarrow \quad Q_2$$

$$([u_0 : u_1], [v_0 : v_1]) \mapsto \left(\frac{u_1 v_1}{u_0 v_1 - u_1 v_0}, \frac{u_0 v_0}{u_0 v_1 - u_1 v_0}, \frac{u_1 v_0}{u_0 v_1 - u_1 v_0} \right).$$

Proof. We first check that $\rho((x, y, z)) \in (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta$ for each $(x, y, z) \in Q_2$. If $z \neq 0$, then $[y : z] \neq [z : x]$, since $xy - z^2 = z \neq 0$. If $z = 0$, then $xy = 0$, whence $[z + 1 : x] = [1 : x] \neq [y : 1] = [y : z + 1]$.

It remains then to check that $\rho \circ \psi = \text{id}_{(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta}$ and $\psi \circ \rho = \text{id}_{Q_2}$, which follows from a straightforward calculation. □

2.2. **Families of embeddings.** The following result is the key step in the proof of Theorem 1.

Lemma 2.2.

(1) For each polynomial $p \in k[t]$, the morphism $\nu_p: \mathbb{A}_k^1 \hookrightarrow Q_2$ given by

$$\nu_p: \quad \mathbb{A}_k^1 \quad \rightarrow \quad Q_2$$

$$t \mapsto (t(1 + tp(t)), p(t), tp(t))$$

is a closed embedding.

(2) If $p, q \in k[t]$ are polynomials of degree ≥ 3 such that $\alpha \nu_p = \nu_q \beta$ for some $\beta \in \text{Aut}(\mathbb{A}_k^1)$ and $\alpha \in \text{Aut}(Q_2)$, then there exist $\mu \in k$ and $\lambda \in k^*$ such that

$$p(t) = \lambda q(\lambda t + \mu), \quad \beta(t) = \lambda t + \mu, \quad \alpha(x, y, z) = \left(\lambda x + \frac{\mu^2}{\lambda} y + 2\mu z + \mu, \frac{y}{\lambda}, z + \frac{\mu}{\lambda} y \right).$$

Proof. Using the isomorphism $\rho: Q_2 \xrightarrow{\cong} (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta$ of Lemma 2.1, we obtain that $\rho \circ \nu_p: \mathbb{A}_k^1 \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ is given by $t \mapsto ([1 : t], [p(t) : 1 + tp(t)])$, which is the restriction of the closed embedding

$$\hat{\nu}_p: \mathbb{P}_k^1 \hookrightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1, [u : v] \mapsto ([u : v], [uP(u, v) : u^{d+1} + vP(u, v)]),$$

where $d = \deg(p)$ and $P(u, v) = p(\frac{v}{u})u^d$ is the homogenisation of p . This implies that $\Gamma_p = \hat{\nu}_p(\mathbb{P}_k^1) \subset \mathbb{P}_k^1 \times \mathbb{P}_k^1$ is a smooth closed curve (isomorphic to \mathbb{P}_k^1), and since $\Gamma_p \cap \Delta$ is given by $u(u^{d+1} + vP(u, v)) - vuP(u, v) = 0$, i.e., $u^{d+2} = 0$, this shows that ν_p is a closed embedding, and thus yields (1).

It remains to prove Assertion (2). We fix two polynomials $p, q \in k[t]$ of degree ≥ 3 such that $\alpha\nu_p = \nu_q\beta$ for some $\beta \in \text{Aut}(\mathbb{A}_k^1)$ and $\alpha \in \text{Aut}(Q_2)$. This implies in particular that the automorphism $\alpha' = \rho^{-1}\alpha\rho \in \text{Aut}((\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta)$ sends $\Gamma_p \setminus \Delta$ onto $\Gamma_q \setminus \Delta$.

We first prove that $\alpha' \in \text{Aut}((\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta)$ extends to an automorphism $\hat{\alpha} \in \text{Aut}(\mathbb{P}_k^1 \times \mathbb{P}_k^1)$. Assume for contradiction that this is not the case. The map α' would then extend to a birational map $\hat{\alpha}: \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$, which is not an automorphism. We consider the minimal resolution of $\hat{\alpha}$, which is

$$\begin{array}{ccc} & Z & \\ \chi_1 \swarrow & & \searrow \chi_2 \\ \mathbb{P}_k^1 \times \mathbb{P}_k^1 & \xrightarrow{\hat{\alpha}} & \mathbb{P}_k^1 \times \mathbb{P}_k^1 \\ \uparrow & & \uparrow \\ (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta & \xrightarrow{\cong} & (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta \end{array}$$

where χ_1, χ_2 are birational morphisms. The resolution being minimal, every (-1) -curve $E \subset Z$ contracted by χ_2 is not contracted by χ_1 , so $\chi_1(E) \subset \mathbb{P}_k^1 \times \mathbb{P}_k^1$ is contracted by $\hat{\alpha}$. There is thus a unique (-1) -curve contracted by χ_2 , which is the strict transform $\tilde{\Delta}$ of Δ , and satisfies $\chi_1(\tilde{\Delta}) = \Delta$. As $\Delta^2 = 2$ and $(\tilde{\Delta})^2 = -1$, there are exactly three base-points of χ_1^{-1} that lie on the curve Δ (as proper point or infinitely near points). Since Γ_p is smooth of bidegree $(1, 1 + \deg p)$, we get $\Gamma_p \cdot \Delta = 2 + \deg p \geq 5$, which implies that the strict transforms of Γ_p and Δ on Z satisfy $\tilde{\Gamma}_p \cdot \tilde{\Delta} \geq 2$ (as only three points belonging to Δ have been blown-up). As the curve $\tilde{\Delta}$ is contracted by χ_2 , the curve $\chi_2(\tilde{\Gamma}_p)$ is singular. This contradicts the equality $\chi_2(\tilde{\Gamma}_p) = \Gamma_q$, which follows from the fact that $\hat{\alpha}(\Gamma_p \setminus \Delta) = \Gamma_q \setminus \Delta$.

We have shown that the extension of $\alpha' = \rho^{-1}\alpha\rho \in \text{Aut}((\mathbb{P}_k^1 \times \mathbb{P}_k^1) \setminus \Delta)$ is an automorphism $\hat{\alpha} \in \text{Aut}(\mathbb{P}_k^1 \times \mathbb{P}_k^1)$, which satisfies therefore $\hat{\alpha}(\Delta) = \Delta$ and $\hat{\alpha}(\Gamma_p) = \Gamma_q$. The curves Γ_p and Γ_q being of bidegree $(1, 1 + \deg p)$ and $(1, 1 + \deg q)$, we get $\deg p = \deg q$ and we obtain that $\hat{\alpha}$ does not exchange the two factors of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. Moreover, as the point $([0 : 1], [0 : 1]) = \Delta \cap \Gamma_p = \Delta \cap \Gamma_q$ is fixed, and as the diagonal Δ is invariant, we can write $\hat{\alpha}$ as

$$\hat{\alpha}([u_0 : u_1], [v_0 : v_1]) = ([u_0 : \lambda u_1 + \mu u_0], [v_0 : \lambda v_1 + \mu v_0])$$

for some $\lambda \in k^*, \mu \in k$.

The equality $\alpha\nu_p = \nu_q\beta$ implies that $\hat{\alpha}\hat{\nu}_p = \hat{\nu}_q\hat{\beta}$, for some automorphism $\hat{\beta} \in \text{Aut}(\mathbb{P}_k^1)$, which is the extension of β and therefore it is of the form $[u : v] \mapsto [u : \lambda v + \mu u]$. We then compute

$$\begin{aligned} \hat{\alpha}\hat{\nu}_p([u : v]) &= ([u : \lambda v + \mu u], [uP(u, v) : \lambda u^{d+1} + \lambda vP(u, v) + \mu uP(u, v)]), \\ \hat{\nu}_q\hat{\beta}([u : v]) &= ([u : \lambda v + \mu u], [uQ(u, \lambda v + \mu u) : u^{d+1} + (\lambda v + \mu u)Q(u, \lambda v + \mu u)]), \end{aligned}$$

and obtain that $P(u, v) = \lambda Q(u, \lambda v + \mu u)$. Remembering that $P(u, v) = p(\frac{v}{u})u^d$ and $Q(u, v) = q(\frac{v}{u})u^d$ we obtain that $p(t) = \lambda q(\lambda t + \mu)$. We then compute the explicit form of α by conjugating $\hat{\alpha}$ with ρ^{-1} . □

Example 2.3. For each $n \geq 1$, let $p_n(t) = t^n(t + 1)^{n+1}$. The closed curve $C_n = \nu_{p_n}(\mathbb{A}_{\mathbf{k}}^1) \subset Q_2$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^1$, via

$$\begin{aligned} \nu_{p_n}: \quad \mathbb{A}_{\mathbf{k}}^1 &\hookrightarrow Q_2 \\ t &\mapsto (t(1 + tp_n(t)), p_n(t), tp_n(t)). \end{aligned}$$

Then Lemma 2.2(2) shows that all curves C_n are non-equivalent for different $n \geq 1$, and that the identity is the only automorphism of C_n that extends to Q_2 .

The proof of Theorem 1 is now a consequence of Lemma 2.2.

Proof of Theorem 1. If \mathbf{k} is the field with two elements, then we conclude by Example 2.3. Hence we can assume that \mathbf{k} contains more than two elements. For each $n \geq 1$ and each $\lambda \in \mathbf{k}, \varepsilon \in \mathbf{k} \setminus \{0, 1\}$, one defines $p_{n,\varepsilon}(t) = t^n(t + 1)^{n+1}(t + \varepsilon)^{n+2} \in \mathbf{k}[t]$, and let $C_{n,\varepsilon} \subset Q_2$ be the closed curve given by $\nu_{p_{n,\varepsilon}}(\mathbb{A}_{\mathbf{k}}^1)$, which is isomorphic to $\mathbb{A}_{\mathbf{k}}^1$ (Lemma 2.2(1)).

Lemma 2.2(2) implies that the identity is the only automorphism of $C_{n,\varepsilon}$ that extends to an automorphism of Q_2 , since $\lambda p_{n,\varepsilon}(\lambda t + \mu) \neq p_{n,\varepsilon}(t)$ for $(\lambda, \mu) \in (\mathbf{k}^* \times \mathbf{k}) \setminus \{(1, 0)\}$.

Similarly, Lemma 2.2(2) shows that $C_{n,\varepsilon}$ is equivalent to $C_{n',\varepsilon'}$ if and only if $n = n'$ and $\varepsilon = \varepsilon'$. □

3. VARIABLES OF POLYNOMIAL RINGS

In this section, we give some results on variables of polynomial rings. Most of them are classical or belong to the folklore. We include them for self-containedness and for lack of precise references.

Definition 3.1. Let S be a ring and let $R \subset S$ be a subring. We denote by $\text{Aut}_R(S)$ the group of automorphisms of the R -algebra S . More precisely,

$$\text{Aut}_R(S) = \{ \text{automorphism of rings } f: S \rightarrow S \text{ such that } f|_R = \text{id}_R \}.$$

Definition 3.2. Let R be a domain and let S be a polynomial ring in $n \geq 1$ variables over R , i.e., $R \subset S$ and there exist $x_1, \dots, x_n \in S$ such that each element of S can be written in a unique way as $f(x_1, \dots, x_n)$, where f is a polynomial in the x_i with coefficients in R . An element $v \in S$ is called *variable of the R -algebra S* if there exists $f \in \text{Aut}_R(S)$ such that $f(v) = x_1$.

In what follows, we often denote by $R[t]$ or $R[x]$ the polynomial ring in one variable over R , by $R[x, y]$ the polynomial ring in two variables over R , and by $R[x_1, \dots, x_n]$ the polynomial ring in n variables over R .

Lemma 3.3. Let R be a domain, let $S = R[x_1, \dots, x_n]$ be the polynomial ring in n variables over R , and let $v \in S$. The following conditions are equivalent:

- (1) v is a variable of the R -algebra S ;
- (2) the $R[t]$ -algebra $S[t]/(v - t)$ is isomorphic to a polynomial ring in $n - 1$ variables over $R[t]$.

Proof. If v is a variable, then there exists $f \in \text{Aut}_R(S)$ such that $f(v) = x_1$. Using the natural inclusion $\text{Aut}_R(S) \hookrightarrow \text{Aut}_{R[t]}(S[t])$, we get isomorphisms of $R[t]$ -algebras

$$S[t]/(v - t) \xrightarrow{\cong} S[t]/(x_1 - t) \xrightarrow{\cong} R[x_2, \dots, x_n, t] \xrightarrow{\cong} R[t][x_2, \dots, x_n].$$

Conversely, suppose that the $R[t]$ -algebra $S[t]/(v - t)$ is isomorphic to a polynomial ring in $n - 1$ variables over $R[t]$. This yields an $R[t]$ -isomorphism $\psi: S[t]/(t - v) \xrightarrow{\cong} R[t][x_2, \dots, x_n]$. We then compose the isomorphisms of R -algebras

$$\begin{array}{ccc} S = R[x_1, \dots, x_n] & \xrightarrow{\cong} & S[t]/(t - v) & \xrightarrow{\psi} & R[t][x_2, \dots, x_n] \\ f & \mapsto & f + (t - v) \cdot S[t] & & \end{array}$$

and

$$\begin{array}{ccc} R[t][x_2, \dots, x_n] & \xrightarrow{\cong} & R[x_1, \dots, x_n] \\ f(t, x_2, \dots, x_n) & \mapsto & f(x_1, x_2, \dots, x_n) \end{array}$$

and obtain an element of $\text{Aut}_R(S)$ that sends v onto x_1 . □

Lemma 3.4. *Let \mathbf{k} be a field, let $\mathbf{k}[x_1, \dots, x_n]$ be the polynomial ring in $n \geq 1$ variables over \mathbf{k} , and let $w \in \mathbf{k}[x_1, \dots, x_n]$ be a variable of this \mathbf{k} -algebra.*

Then $\mathbf{k}[w]$ is factorially closed in $\mathbf{k}[x_1, \dots, x_n]$, i.e., for all $f, g \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$, and we have $fg \in \mathbf{k}[w] \Leftrightarrow f \in \mathbf{k}[w]$ and $g \in \mathbf{k}[w]$.

Proof. If $f \in \mathbf{k}[w]$ and $g \in \mathbf{k}[w]$, then $fg \in \mathbf{k}[w]$, since $\mathbf{k}[w]$ is a subring of $\mathbf{k}[x_1, \dots, x_n]$.

Conversely, suppose that $fg \in \mathbf{k}[w]$. Choose $\psi \in \text{Aut}_{\mathbf{k}}(\mathbf{k}[x_1, \dots, x_n])$ such that $\psi(w) = x_1$. Then, $\psi(f), \psi(g) \in \mathbf{k}[x_1, \dots, x_n]$ are two polynomials such that $\psi(f) \cdot \psi(g) \in \mathbf{k}[x_1]$. For each $i \geq 2$, the degree in x_i satisfies $\deg_{x_i}(\psi(f)) + \deg_{x_i}(\psi(g)) = \deg_{x_i}(\psi(f) \cdot \psi(g)) = 0$, so $\deg_{x_i}(\psi(f)) = \deg_{x_i}(\psi(g)) = 0$ since both elements are non-zero. Hence, $\psi(f), \psi(g) \in \mathbf{k}[x_1]$. Applying ψ^{-1} , we get $f \in \mathbf{k}[w]$ and $g \in \mathbf{k}[w]$. □

3.1. Variables of polynomial rings in two variables. We will need the following two technical lemmas.

Lemma 3.5. *Let \mathbf{k} be a field, let w be a variable of the \mathbf{k} -algebra $\mathbf{k}[x, y]$, and let $v \in \mathbf{k}[x, y]$ be a polynomial. The following conditions are equivalent:*

- (1) $v \in \mathbf{k}[w]$;
- (2) for each $u \in \mathbf{k}[w]$, the elements u and v are algebraically dependent over \mathbf{k} ;
- (3) there exists $u \in \mathbf{k}[w] \setminus \mathbf{k}$ such that u and v are algebraically dependent over \mathbf{k} .

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) being clear, we only need to prove (3) \Rightarrow (1). Replacing v and w with $f(v)$ and $f(w)$, for some $f \in \text{Aut}_{\mathbf{k}}(\mathbf{k}[x, y])$, we can assume that $w = x$. Denoting by $\overline{\mathbf{k}}$ the algebraic closure of \mathbf{k} , we have $\overline{\mathbf{k}}[x] \cap \mathbf{k}[x, y] = \mathbf{k}[x]$, so we can assume that $\mathbf{k} = \overline{\mathbf{k}}$.

We then consider the morphism $\tau: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^2$ given by $(x, y) \mapsto (u(x), v(x, y))$, which is dominant if and only if u, v are algebraically independent over \mathbf{k} . It remains then to see that τ is dominant if $u \in \mathbf{k}[x] \setminus \mathbf{k}$ and $v \notin \mathbf{k}[x]$. Let $v(x, y) = \sum_{i=0}^d v_i(x)y^i$, where $v_d \neq 0$ and $d > 0$. For a general $a \in \mathbf{k}$, $u(x) = a$ has a solution x_0 such that $v_d(x_0) \neq 0$, since \mathbf{k} is algebraically closed. Hence $v(x_0, y) = b$ has a solution for all $b \in \mathbf{k}$. This proves that τ is dominant. □

Lemma 3.6. *Let \mathbf{k} be a field, let $p \in \mathbf{k}[t]$ be an irreducible element, and let $\mathbf{k}_p = \mathbf{k}[t]/(p)$ be the corresponding residue field. Let $u, v \in \mathbf{k}[t][x, y]$ be elements such that $\mathbf{k}(t)[u, v] = \mathbf{k}(t)[x, y]$. Then, the classes $u_0, v_0 \in \mathbf{k}_p[x, y]$ of u, v satisfy one of the following properties, depending on the Jacobian determinant $\nu = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} \in \mathbf{k}[t] \setminus \{0\}$:*

- (1) *If p divides ν , then u_0, v_0 are algebraically dependent over \mathbf{k}_p .*
- (2) *If p does not divide ν , then $\mathbf{k}_p[u_0, v_0] = \mathbf{k}_p[x, y]$. In particular, both u_0 and v_0 are variables of the \mathbf{k}_p -algebra $\mathbf{k}_p[x, y]$.*

Proof. Since $\mathbf{k}(t)[u, v] = \mathbf{k}(t)[x, y]$, there are polynomials $P, Q \in \mathbf{k}(t)[X, Y]$ such that $P(u, v) = x, Q(u, v) = y$. Moreover, the polynomial $\nu = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} \in \mathbf{k}[t, x, y]$ belongs to $\mathbf{k}(t)^*$ and thus to $\mathbf{k}[t] \setminus \{0\}$. The element $\nu_0 = \frac{\partial u_0}{\partial x} \cdot \frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y} \cdot \frac{\partial v_0}{\partial x}$ is then the class of ν in \mathbf{k}_p .

We write $P = \frac{\tilde{P}}{\alpha}, Q = \frac{\tilde{Q}}{\beta}$, where $\tilde{P}, \tilde{Q} \in \mathbf{k}[t][X, Y], \alpha, \beta \in \mathbf{k}[t] \setminus \{0\}$ and such that p does not divide both α and \tilde{P} (and the same for β and \tilde{Q}). We then get

$$\tilde{P}_0(u_0, v_0) = \alpha_0 x, \quad \tilde{Q}_0(u_0, v_0) = \beta_0 y,$$

where $\tilde{P}_0, \tilde{Q}_0 \in \mathbf{k}_p[X, Y]$ are the classes of \tilde{P}, \tilde{Q} and $\alpha_0, \beta_0 \in \mathbf{k}_p$ are the classes of α, β .

If α_0 and β_0 are not equal to zero, then $\mathbf{k}_p[u_0, v_0] = \mathbf{k}_p[x, y]$. In particular, u_0 and v_0 are variables of the \mathbf{k}_p -algebra $\mathbf{k}_p[x, y]$ and $\nu_0 \in \mathbf{k}_p^*$, so p does not divide ν .

If $\alpha_0 = 0$, then $\tilde{P}_0 \neq 0$ and $\tilde{P}_0(u_0, v_0) = 0$ implies that u_0 and v_0 are algebraically dependent over \mathbf{k}_p . The same conclusion holds when $\beta_0 = 0$. In both cases, the Jacobian determinant ν_0 is equal to zero, so p divides ν .

This yields (1) and (2). □

We recall the following classical result, essentially equivalent to the Jung-van der Kulk Theorem.

Lemma 3.7. *Let \mathbf{k} be a field, let $\mathbf{k}[x, y]$ be the polynomial ring in two variables over \mathbf{k} , let $f \in \text{Aut}_{\mathbf{k}}(\mathbf{k}[x, y])$ and $u = f(x), v = f(y) \in \mathbf{k}[x, y]$. If $\deg(u) \geq \deg(v) > 1$, then there exists a polynomial P with coefficients in \mathbf{k} such that $\deg(u - P(v)) < \deg(u)$.*

Proof. By van der Kulk’s Theorem all automorphisms of $\mathbf{k}[x, y]$ are tame [Jun42, vdK53]. The statement is then a direct consequence of [vdE00, Corollary 5.1.6]. □

The following result is needed in what follows. When the characteristic of \mathbf{k} is zero, and $p = t$, it follows from [Fur02, Theorem 4]. We adapt here the proof of [Fur02] for our purpose.

Lemma 3.8. *Let \mathbf{k} be a field, let $p \in \mathbf{k}[t]$ be an irreducible element, and let $\mathbf{k}_p = \mathbf{k}[t]/(p)$ be the corresponding residue field. If $v \in \mathbf{k}[t, x, y]$ is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, then its class in $\mathbf{k}_p[x, y]$ is an element which belongs to $\mathbf{k}_p[w] \subset \mathbf{k}_p[x, y]$ for some variable w of the \mathbf{k}_p -algebra $\mathbf{k}_p[x, y]$.*

Proof. Let $f \in \text{Aut}_{\mathbf{k}(t)}(\mathbf{k}(t)[x, y])$ such that $f(x) = v$, and let us define $u = f(y)$. We denote by $v_0 \in \mathbf{k}_p[x, y]$ the class of v and will use the degree of polynomials in x, y with coefficients in $\mathbf{k}(t)$ or \mathbf{k}_p .

If $\deg(v) = 1$, then $\deg(v_0) \leq 1$. If $v_0 \in \mathbf{k}_p$ the result follows by taking any variable for w , for instance $w = x$. Otherwise, $v_0 = \alpha x + \beta y + \gamma$ for some $\alpha, \beta, \gamma \in \mathbf{k}_p$

with $(\alpha, \beta) \neq (0, 0)$. This implies that $w = \alpha x + \beta y$ is a variable, as it is the component of an element of $\text{GL}_2(\mathbf{k}_p)$, and the result follows.

We can thus assume that $\deg(v) > 1$ and prove the result by induction on the pair $(\deg(v), \deg(u))$, ordered lexicographically.

(i) If $\deg(u) \geq \deg(v)$, then there exists a polynomial $P \in \mathbf{k}(t)[X]$ such that $\deg(u - P(v)) < \deg(u)$ (Lemma 3.7). We can thus apply an induction hypothesis to $(u - P(v), v)$, since $\mathbf{k}(t)[u, v] = \mathbf{k}(t)[u - P(v), v]$, and obtain the result.

(ii) If $\deg(u) < \deg(v)$, we first replace u with $u - \lambda$ for some $\lambda \in \mathbf{k}(t)$ and assume that $u \in \mathbf{k}(t)[x, y]$ is a polynomial in x, y with no constant term. We then replace u with qu for some $q \in \mathbf{k}(t)^*$ and assume that $u \in \mathbf{k}[t][x, y]$ and the greatest common divisor in $\mathbf{k}[t]$ of the coefficients of u (as a polynomial in x, y) is equal to 1. One can then define the class $u_0 \in \mathbf{k}_p[x, y]$ of u , which is not equal to zero. Moreover, u_0 does not belong to \mathbf{k}_p , since u_0 has no constant term.

If v_0 is a variable of the \mathbf{k}_p -algebra $\mathbf{k}_p[x, y]$, then we are done. Otherwise, u_0, v_0 are algebraically dependent over \mathbf{k}_p (Lemma 3.6).

Since the pair $(\deg(v), \deg(u))$ is smaller than $(\deg(u), \deg(v))$, we can apply an induction hypothesis and get a variable $w \in \mathbf{k}_p[x, y]$ such that $u_0 \in \mathbf{k}_p[w]$. The fact that u_0 and v_0 are algebraically dependent over \mathbf{k}_p and that $u_0 \notin \mathbf{k}_p$ imply that $v_0 \in \mathbf{k}_p[w]$ (Lemma 3.5). □

We finish this section with several results relating variables and \mathbb{A}^1 -bundles.

Lemma 3.9. *Let \mathbf{k} be a field and let $P \in \mathbf{k}[x, y]$. Then, the following conditions are equivalent:*

- (1) *The polynomial P is a variable of the \mathbf{k} -algebra $\mathbf{k}[x, y]$.*
- (2) *The $\mathbf{k}[t]$ -algebra $\mathbf{k}[t, x, y]/(P - t)$ is a polynomial ring in one variable over $\mathbf{k}[t]$.*
- (3) *The $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]/(P - t)$ is a polynomial ring in one variable over $\mathbf{k}(t)$.*
- (4) *The morphism $\mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^1$ given by $(x, y) \mapsto P(x, y)$ is a trivial \mathbb{A}^1 -bundle.*
- (5) *The morphism $\mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^1$ given by $(x, y) \mapsto P(x, y)$ is a trivial \mathbb{A}^1 -bundle over some dense open subset $U \subset \mathbb{A}_{\mathbf{k}}^1$.*

Proof. (1) \Leftrightarrow (2) follows from Lemma 3.3.

(1) \Leftrightarrow (4): By definition, (1) is equivalent to the existence of $f \in \text{Aut}_{\mathbf{k}}(\mathbf{k}[x, y])$ such that $f(x) = P$. As $f = \varphi^*$ for some $\varphi \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$, this is equivalent to asking for $\varphi \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ that $\text{pr}_x \circ \varphi$ is the map $(x, y) \mapsto P(x, y)$, where $\text{pr}_x : \mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is given by $(x, y) \mapsto x$. This yields the equivalence (1) \Leftrightarrow (4).

(2) \Rightarrow (3) is trivially true.

(3) \Rightarrow (5): Assertion (3) corresponds to say that the generic fibre of $(x, y) \mapsto P(x, y)$ is isomorphic to $\mathbb{A}_{\mathbf{k}(t)}^1$. This yields (5).

(5) \Rightarrow (1) : Assume that the subset U given in (5) contains a \mathbf{k} -rational point. Replacing P with $P + \lambda$, $\lambda \in \mathbf{k}$, one can assume that 0 belongs to the open subset U . One then observes that the curve $\Gamma \subset \mathbb{A}_{\mathbf{k}}^2$ given by $P = 0$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^1$ and equivalent to a line by a birational map of $\mathbb{A}_{\mathbf{k}}^2$, hence can be contracted by a birational map of $\mathbb{A}_{\mathbf{k}}^2$. By [BFH16, Proposition 2.29], there exists an automorphism $\varphi \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ which sends Γ onto the line given by $x = 0$. This implies that P is a variable of the \mathbf{k} -algebra $\mathbf{k}[x, y]$.

If U contains no \mathbf{k} -rational point, then \mathbf{k} is a finite field and thus it is perfect. For a finite Galois extension $\mathbf{k}' \supset \mathbf{k}$ the subset U contains a \mathbf{k}' -rational point. By

the argument above, P is a variable of the \mathbf{k}' -algebra $\mathbf{k}'[x, y]$ and hence $\mathbf{k}'[x, y] = \mathbf{k}'[P, Q]$ for some $Q \in \mathbf{k}'[x, y]$. Since P is a polynomial with coefficients in \mathbf{k} , it is fixed under the action of the Galois group $G = \text{Gal}(\mathbf{k}'/\mathbf{k})$ on $\mathbf{k}[x, y] = \mathbf{k}'[P, Q]$. For each $\sigma \in G$, there exists $(a_\sigma, b_\sigma) \in (\mathbf{k}')^* \times \mathbf{k}'[T]$ with

$$\sigma(Q) = a_\sigma Q + b_\sigma(P).$$

We can then find $d \geq 0$ such that $\{b_\sigma \mid \sigma \in G\}$ is contained in the finite dimensional \mathbf{k}' -vector subspace $V_d = \{f \in \mathbf{k}'[T] \mid \deg(P) \leq d\} \subset \mathbf{k}'[T]$. Thus $\sigma \mapsto (a_\sigma, b_\sigma)$ defines an element of $H^1(G, (\mathbf{k}')^* \times V_d)$. As $H^1(G, (\mathbf{k}')^*) = \{1\}$ and $H^1(G, V_d) = \{1\}$ [Ser68, Proposition 1, 2, Chp. X], we have $H^1(G, (\mathbf{k}')^* \times V_d) = \{1\}$. The fact that (a_σ, b_σ) is a trivial cocycle corresponds to the existence of a polynomial $Q_0 \in \mathbf{k}[x, y]$ such that $\mathbf{k}[P, Q_0] = \mathbf{k}[x, y]$. This implies that P is a variable of the \mathbf{k} -algebra $\mathbf{k}[x, y]$. □

We recall the following classical result.

Lemma 3.10. *Let \mathbf{k} be a field, let Z be an affine variety over \mathbf{k} , all of its irreducible components being surfaces, let $U \subseteq \mathbb{A}_{\mathbf{k}}^1$ be a dense open subset and let $\pi: Z \rightarrow U$ be a dominant morphism. Then, the following conditions are equivalent:*

- (1) *The morphism $\pi: Z \rightarrow U$ is a trivial \mathbb{A}^1 -bundle.*
- (2) *The morphism $\pi: Z \rightarrow U$ is a locally trivial \mathbb{A}^1 -bundle.*
- (3) *For each maximal ideal $\mathfrak{m} \subset \mathbf{k}[U]$, the fibre $\pi^{-1}(\mathfrak{m}) \subset Z$ is isomorphic to $\mathbb{A}_{\kappa(\mathfrak{m})}^1$ and the generic fibre of π is isomorphic to $\mathbb{A}_{\mathbf{k}(t)}^1$.*

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) are obvious. Assume (3) holds. Since each irreducible component of Z has dimension two, it follows that each of these irreducible components is mapped dominantly onto U via π . Thus π is flat. By [Asa87, Corollary 3.2] it follows now from (3) that $\mathbf{k}[Z]_{\mathfrak{m}}$ is a polynomial ring in one variable over $\mathbf{k}[U]_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset \mathbf{k}[U]$. Hence, by [BCW77], the morphism π is a vector bundle with respect to the Zariski topology and since $\mathbf{k}[U]$ is a principal ideal domain, π is a trivial \mathbb{A}^1 -bundle. □

Lemma 3.11. *Let \mathbf{k} be a field, let $P \in \mathbf{k}[t, x, y]$ be a polynomial which is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, let $U \subset \mathbb{A}_{\mathbf{k}}^1 = \text{Spec}(\mathbf{k}[t])$ be a dense open subset, let $Z \subset U \times \mathbb{A}_{\mathbf{k}}^2 = \text{Spec}(\mathbf{k}[U][x, y])$ be the hypersurface given by $P = 0$, and let $\pi: Z \rightarrow U$ be the morphism $(t, x, y) \mapsto t$. Then, the following conditions are equivalent:*

- (i) *P is a variable of the $\mathbf{k}[U]$ -algebra $\mathbf{k}[U][x, y]$;*
- (ii) *there is an isomorphism $\varphi: U \times \mathbb{A}_{\mathbf{k}}^1 \xrightarrow{\cong} Z$ such that $\pi\varphi$ is the projection $(t, x) \mapsto t$;*
- (iii) *the morphism $\pi: Z \rightarrow U$ is a trivial \mathbb{A}^1 -bundle;*
- (iv) *the morphism $\pi: Z \rightarrow U$ is a locally trivial \mathbb{A}^1 -bundle;*
- (v) *for each maximal ideal $\mathfrak{m} \subset \mathbf{k}[U]$, the fibre $\pi^{-1}(\mathfrak{m}) \subset Z$ is isomorphic to $\mathbb{A}_{\kappa(\mathfrak{m})}^1$.*

Proof. (i) \Rightarrow (ii): If P is a variable of the $\mathbf{k}[U]$ -algebra $\mathbf{k}[U][x, y]$, there exists $f \in \text{Aut}_{\mathbf{k}[U]}(\mathbf{k}[U][x, y])$ such that $f(x) = P$. The element f is then equal to ψ^* for some $\psi \in \text{Aut}(U \times \mathbb{A}_{\mathbf{k}}^2)$ such that $\pi\psi = \pi$. Hence, $\psi(Z)$ is the closed subset of $U \times \mathbb{A}_{\mathbf{k}}^2$ given by $x = 0$. Let $\theta: U \times \mathbb{A}_{\mathbf{k}}^2 \rightarrow U \times \mathbb{A}_{\mathbf{k}}^1$ be the projection given by $(t, x, y) \mapsto (t, y)$. The composition $\theta \circ \psi$ restricts to an isomorphism $Z \xrightarrow{\cong} U \times \mathbb{A}_{\mathbf{k}}^1$ that we denote by φ^{-1} . Thus $\text{pr}_1 \circ \varphi^{-1} = \pi$, where $\text{pr}_1: U \times \mathbb{A}_{\mathbf{k}}^1 \rightarrow U$ is the projection on the first factor. This yields (ii).

(ii) \Leftrightarrow (iii): is the definition of a trivial \mathbb{A}^1 -bundle.

(iii) \Rightarrow (iv) \Rightarrow (v) are obvious.

(v) \Rightarrow (i): This follows from the implication (3) \Rightarrow (1) of Lemma 3.10 (we use here the fact that the generic fibre of π is isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$, which is provided by the assumption that P is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$). \square

Remark 3.12. Lemma 3.11 is false if we do not assume P to be a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$. Let us choose for example \mathbf{k} to be algebraically closed of characteristic $p > 0$, $P = x + x^{p^q} + y^{p^2} - t$, and $U = \mathbb{A}^1$. Corollary 3.16 shows that (v) is satisfied but not (i). Moreover, the fact that P is not a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$ follows from Lemma 3.15.

Corollary 3.13. *Let \mathbf{k} be a field, let $P \in \mathbf{k}[t, x, y]$ be a polynomial which is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, let $Z \subset \mathbb{A}^3_{\mathbf{k}} = \text{Spec}(\mathbf{k}[t, x, y])$ be the hypersurface given by $P = 0$, and let $\pi: Z \rightarrow \mathbb{A}^1_{\mathbf{k}}$ be the morphism $(t, x, y) \mapsto t$. Then, the following conditions are equivalent:*

- (i) P is a variable of the $\mathbf{k}[t]$ -algebra $\mathbf{k}[t][x, y]$;
- (ii) there is an isomorphism $\varphi: \mathbb{A}^2_{\mathbf{k}} \xrightarrow{\sim} Z$ such that $\pi\varphi$ is the projection $(t, x) \mapsto t$;
- (iii) there is an isomorphism $\varphi: \mathbb{A}^2_{\mathbf{k}} \xrightarrow{\sim} Z$;
- (iv) the morphism $\pi: Z \rightarrow \mathbb{A}^1_{\mathbf{k}}$ is a trivial \mathbb{A}^1 -bundle;
- (v) the morphism $\pi: Z \rightarrow \mathbb{A}^1_{\mathbf{k}}$ is a locally trivial \mathbb{A}^1 -bundle;
- (vi) for each maximal ideal $\mathfrak{m} \subset \mathbf{k}[t]$, the fibre $\pi^{-1}(\mathfrak{m}) \subset Z$ is isomorphic to $\mathbb{A}^1_{\kappa(\mathfrak{m})}$.

Proof. Applying Lemma 3.11 with $U = \mathbb{A}^1_{\mathbf{k}}$, we obtain the equivalence between (i)-(ii)-(iv)-(v)-(vi).

We then observe that (ii) implies (iii). It remains then to prove (iii) \Rightarrow (iv). As P is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, the generic fibre of $\pi: Z \rightarrow \mathbb{A}^1_{\mathbf{k}}$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}(t)}$, so π is a trivial \mathbb{A}^1 -bundle over some dense open subset $U \subset \mathbb{A}^1_{\mathbf{k}}$. The fact that Z is isomorphic to $\mathbb{A}^2_{\mathbf{k}}$ implies then that π is a trivial $\mathbb{A}^1_{\mathbf{k}}$ -bundle (Implication (5) \Rightarrow (4) of Lemma 3.9). \square

3.2. Non-trivial embeddings in positive characteristic. In this paragraph, we recall the existence of non-trivial embeddings in positive characteristic. The family of examples that we give below seems classical (the case $\mathbb{A}^1_{\mathbf{k}} \hookrightarrow \mathbb{A}^2_{\mathbf{k}}$ with parameters equal to 1 corresponds in particular to [vdE00, Exercise 5(iii) in §5]). We give the (simple) proof here for a lack of a precise reference and for self-containedness.

Lemma 3.14. *For each field \mathbf{k} of characteristic $p > 0$, each $a \in \mathbf{k}$, $b \in \mathbf{k}^*$ and each integer $q \geq 0$, the morphism*

$$\begin{aligned} \rho: \mathbb{A}^1_{\mathbf{k}} &\hookrightarrow \mathbb{A}^2_{\mathbf{k}} \\ u &\mapsto (u^{p^2}, \frac{1}{b}(au^{p^q} + u)) \end{aligned}$$

is a closed embedding, with image being the closed curve of $\mathbb{A}^2_{\mathbf{k}} = \text{Spec}(\mathbf{k}[x, y])$ given by

$$x + a^{p^2} x^{p^q} - b^{p^2} y^{p^2} = 0.$$

Proof. We first compute the equality

$$b^p \left(\frac{1}{b}(au^{p^q} + u) \right)^p - a^p (u^{p^2})^q = (a^p u^{p^2 q} + u^p) - a^p u^{p^2 q} = u^p,$$

which shows that $\rho(\mathbb{A}_{\mathbf{k}}^1)$ is contained in the closed curve $\Gamma \subset \mathbb{A}_{\mathbf{k}}^2$ given by $P = 0$, where $P = x - (b^p y^p - a^p x^q)^p = x - b^{p^2} y^{p^2} + a^{p^2} x^{qp} \in \mathbf{k}[x, y]$. The equality also yields $u^p \in \mathbf{k}[u^{p^2}, \frac{1}{b}(au^{pq} + u)]$ and thus yields $\mathbf{k}[u^{p^2}, \frac{1}{b}(au^{pq} + u)] = \mathbf{k}[u]$, which implies that ρ is a closed embedding. It remains to see that the degree of ρ (maximum of the degree of both components) is equal to the degree of P , to obtain that P is irreducible and that it defines the irreducible curve $\rho(\mathbb{A}_{\mathbf{k}}^1)$. For $a = 0$, this follows, since $\deg(\rho) = p^2 = \deg(P)$. For $a \neq 0$, we have $\deg(\rho) = \max(p^2, pq) = \deg(P)$. \square

To show that the above embeddings are not equivalent to the standard one, when $q \geq 2$ is not a multiple of p and $a, b \neq 0$, one could make the argument on the degree of the components (no one divides the other) or can use the characterisation of variables given in Lemma 3.3 to show that $P = x + a^{p^2} x^{pq} - b^{p^2} y^{p^2} \in \mathbf{k}[x, y]$ is not a variable, by proving that $\mathbf{k}[x, y, t]/(P - t)$ is not a polynomial ring in one variable over $\mathbf{k}[t]$, as we do in Lemma 3.15 below. The second way has the advantage of giving examples in any dimension (see Proposition 3.17). This is related to the forms of the affine line over non-perfect fields (for more details on this subject, see [Rus70]).

Lemma 3.15. *For each field \mathbf{k} of characteristic $p > 0$, each $a, b \in \mathbf{k}^*$ and each integer $q \geq 2$, not a multiple of p , the curve*

$$\Gamma = \text{Spec} \left(\mathbf{k}(t)[x, y]/(x + a^{p^2} x^{pq} - b^{p^2} y^{p^2} - t) \right)$$

is not isomorphic to $\mathbb{A}_{\mathbf{k}(t)}^1$, but after the extension of scalars to $\mathbf{k}(t^{1/p})$ we have an isomorphism

$$\Gamma_{\mathbf{k}(t^{1/p})} \xrightarrow{\cong} \mathbb{A}_{\mathbf{k}(t^{1/p})}^1.$$

Proof. After extending the scalars to $\mathbf{k}(t^{1/p^2})$, the curve Γ becomes

$$\begin{aligned} \Gamma_{\mathbf{k}(t^{1/p^2})} &= \text{Spec} \left(\mathbf{k}(t^{1/p^2})[x, y]/(x + a^{p^2} x^{pq} - b^{p^2} y^{p^2} - t) \right) \\ &= \text{Spec} \left(\mathbf{k}(t^{1/p^2})[x, y]/\left(x + a^{p^2} x^{pq} - b^{p^2} \left(y + \frac{t^{1/p^2}}{b} \right)^{p^2} \right) \right). \end{aligned}$$

Replacing y with $y + \frac{t^{1/p^2}}{b}$ and applying Lemma 3.14 we obtain an isomorphism

$$\begin{aligned} \mathbb{A}_{\mathbf{k}(t^{1/p^2})}^1 &\xrightarrow{\cong} \Gamma_{\mathbf{k}(t^{1/p^2})} \\ u &\mapsto \left(u^{p^2}, \frac{1}{b} (au^{pq} + u - t^{1/p^2}) \right). \end{aligned}$$

Replacing then u with $u + t^{1/p^2}$ we get an isomorphism defined over $\mathbf{k}(t^{1/p})$:

$$\begin{aligned} \nu: \mathbb{A}_{\mathbf{k}(t^{1/p})}^1 &\xrightarrow{\cong} \Gamma_{\mathbf{k}(t^{1/p})} \\ u &\mapsto \left(u^{p^2} + t, \frac{1}{b} \left(a (u^p + t^{1/p})^q + u \right) \right). \end{aligned}$$

It remains that no isomorphism $\hat{\nu}: \mathbb{A}_{\mathbf{k}(t)}^1 \xrightarrow{\cong} \Gamma_{\mathbf{k}(t)}$ exists. If $\hat{\nu}$ exists, then $\nu^{-1}\hat{\nu} \in \text{Aut}(\mathbb{A}_{\mathbf{k}(t^{1/p})}^1)$ would be given by $u \mapsto \alpha u + \beta$, with $\alpha, \beta \in \mathbf{k}(t^{1/p})$, $\alpha \neq 0$. The second coordinate of $\hat{\nu}(u)$ would then be

$$\frac{1}{b} \left(a \left((\alpha u + \beta)^p + t^{1/p} \right)^q + (\alpha u + \beta) \right) \in \mathbf{k}(t)[u].$$

The coefficient of u being $\frac{\alpha}{b}$, we get $\alpha \in \mathbf{k}(t)$. Remembering that $q \geq 2$, the coefficient of $u^{p(q-1)}$ is equal to $\frac{\alpha}{b}q\alpha^{p(q-1)}(\beta^p + t^{1/p})$. As $\beta^p \in \mathbf{k}(t)$ we have $\beta^p + t^{1/p} \notin \mathbf{k}(t)$. Impossible, since q is not a multiple of p and $\alpha, a \neq 0$. \square

Corollary 3.16. *For each field \mathbf{k} of characteristic $p > 0$, each integer $q \geq 2$ which is not a multiple of p , each $\lambda, \mu \in \mathbf{k}^*$, and each integer $n \geq 2$, the polynomial*

$$f = x_1 + \lambda x_1^{pq} + \mu x_2^2 \in \mathbf{k}[x_1, x_2] \subset \mathbf{k}[x_1, \dots, x_n]$$

is not a variable of the \mathbf{k} -algebra $\mathbf{k}[x_1, \dots, x_n]$.

However, when \mathbf{k} is algebraically closed, the \mathbf{k} -algebra $\mathbf{k}[x_1, \dots, x_n]/(f - t)$ is isomorphic to $\mathbf{k}[x_1, \dots, x_{n-1}]$ for each $t \in \mathbf{k}$.

Proof. Showing that f is not a variable of $\mathbf{k}[x_1, \dots, x_n]$ is equivalent to asking that the $\mathbf{k}[t]$ -algebra $\mathbf{k}[x_1, x_2, \dots, x_n, t]/(f - t)$ not be a polynomial ring in $n - 1$ variables over $\mathbf{k}[t]$ (Lemma 3.3). It suffices then to show that $A_n = \mathbf{k}(t)[x_1, \dots, x_n]/(f - t)$ is not a polynomial ring in $n - 1$ variables over $\mathbf{k}(t)$.

We first prove the result for $n = 2$. By extending the scalars, we can assume that $\lambda = a^{p^2}$ and $\mu = -b^{p^2}$ for some $a, b \in \mathbf{k}^*$. Lemma 3.15 then shows that $A_2 = \mathbf{k}(t)[x_1, x_2]/(f - t)$ is not a polynomial ring in one variable.

As $A_n = A_2[x_3, \dots, x_n]$, the positive answer to the cancellation problem for curves [AHE72] implies that A_n is not a polynomial ring in $n - 1$ variables over $\mathbf{k}(t)$ for each $n \geq 2$.

It remains to assume that \mathbf{k} is algebraically closed and to show that the \mathbf{k} -algebra $\mathbf{k}[x_1, \dots, x_n]/(f - t)$ is isomorphic to $\mathbf{k}[x_1, \dots, x_{n-1}]$ for each $t \in \mathbf{k}$. Replacing x_2 with $x_2 + \nu$ for some $\nu \in \mathbf{k}$, we only need to consider the case $t = 0$, which follows from Lemma 3.14. \square

Proposition 3.17. *For each field \mathbf{k} of characteristic $p > 0$, each integer $q \geq 2$ which is not a multiple of p , each $a \in \mathbf{k}^*$ and each $n \geq 1$, the morphism*

$$\begin{aligned} \rho: \quad \mathbb{A}^n &\hookrightarrow \mathbb{A}^{n+1} \\ (x_1, \dots, x_n) &\mapsto (x_1^{p^2}, ax_1^{pq} + x_1, x_2, \dots, x_n) \end{aligned}$$

is a closed embedding, which is not equivalent to the standard one.

Proof. It follows from Lemma 3.14 that ρ is a closed embedding and that its image is given by the hypersurface with equation $f = 0$, where

$$f = x_1 + a^{p^2} x_1^{pq} - x_2^2 \in \mathbf{k}[x_1, x_2] \subset \mathbf{k}[x_1, \dots, x_n].$$

It remains to show that f is not a variable of $\mathbf{k}[x_1, \dots, x_n]$, which follows from Corollary 3.16. \square

4. LIFTINGS OF AUTOMORPHISMS AND THE PROOF OF THEOREM 2

4.1. Lifting of automorphisms of $\mathbb{A}_{\mathbf{k}}^3$ to affine modifications.

Proposition 4.1. *Let \mathbf{k} be a field and let*

$$R = \mathbf{k}[t, u, x, y]/(t^n u - h(t, x, y)),$$

where $n \geq 1$ and $h \in \mathbf{k}[t, x, y]$ is a polynomial such that $h_0 = h(0, x, y) \in \mathbf{k}[x, y]$ does not belong to $\mathbf{k}[w]$ for each variable $w \in \mathbf{k}[x, y]$.

(1) Every element of $R \setminus \mathbf{k}[t, x, y]$ can be written as

$$s + \sum_{i=1}^m f_i u^i$$

where $s \in \mathbf{k}[t, x, y]$, $m \geq 1$, $f_1, \dots, f_m \in \mathbf{k}[t, x, y]$ are polynomials of degree $< n$ in t , and $f_m \neq 0$.

(2) If $f \in R \setminus \mathbf{k}[t, x, y]$ is written as in (1) and $d = \nu(f_m)$ is the valuation of f_m in t , then $0 \leq d < n$ and $t^{mn-d}f = g(t, x, y) \in \mathbf{k}[t, x, y]$ satisfies $g(0, x, y) \in h_0 \cdot \mathbf{k}[x, y] \setminus \{0\}$.

(3) Writing $I \subset \mathbf{k}[t, x, y]$ for the ideal (t^n, h) , we have $t^n R \cap \mathbf{k}[t, x, y] = I$.

(4) Every element of $\text{Aut}_{\mathbf{k}[t]}(R)$ preserves the sets $\mathbf{k}[t, x, y]$ and I .

Proof. (1): We first prove that every element of $R \setminus \mathbf{k}[t, x, y]$ has the desired form. Every element of R can be written as $\sum_{i=0}^m f_i u^i$ for some polynomials $f_i \in \mathbf{k}[t, x, y]$. We denote by r the largest integer such that $\deg_t(f_r) \geq n$. If $r = 0$ or if no such integer exists, we are done. Otherwise, we write $f_r = t^n A + B$ for some $A, B \in \mathbf{k}[t, x, y]$ with $\deg_t(B) < n$. Then, replacing $f_{r-1}u^{r-1} + f_r u^r = f_{r-1}u^{r-1} + (t^n A + B)u^r$ in $\sum_{i=0}^m f_i u^i$ with $(f_{r-1} + h(t, x, y)A)u^{r-1} + Bu^r$ decreases the integer r . After finitely many such steps, we obtain the desired form.

(2): We write $f = s + \sum_{i=1}^m f_i u^i = s + \sum_{i=1}^m f_i \frac{h^i}{t^{mi}}$ as in (1), we write $d = \nu(f_m)$, which satisfies $0 \leq d < n$ (since $f_m \neq 0$ and $\deg_t(f_m) < n$), and obtain $g = t^{mn-d}f = st^{mn-d} + \sum_{i=1}^m f_i h^i t^{mn-d-ni}$. In particular, $g \in \mathbf{k}[t, x, y]$ and it satisfies $g(0, x, y) = (h_0)^m \cdot r$, where $r \in \mathbf{k}[x, y]$ is obtained by replacing $t = 0$ in $\frac{f_m}{t^d} \in \mathbf{k}[t, x, y]$. From $\{r, h_0\} \subset \mathbf{k}[x, y] \setminus \{0\}$, we deduce $g(0, x, y) \neq 0$.

(3): The inclusion $I \subset t^n R$ follows from $\{t^n, h\} = \{t^n \cdot 1, t^n \cdot u\} \subset t^n R$. To show that $t^n R \cap \mathbf{k}[t, x, y] \subset I$, we take $f \in R$ such that $t^n f \in \mathbf{k}[t, x, y]$ and show that $t^n f \in I$. If $f \in \mathbf{k}[t, x, y]$, then $t^n f \in t^n \mathbf{k}[t, x, y] \subset I$. Otherwise, we write $f = s + \sum_{i=1}^m f_i u^i$ as in (1) and use (2) to obtain that $g = t^{mn-d}f \in \mathbf{k}[t, x, y]$ with $0 \leq d = \nu(f_m) < n$, and we get $g(0, x, y) \neq 0$. The fact that $t^n f \in \mathbf{k}[t, x, y]$ implies then that $n > mn - d$, whence $n > d > (m - 1)n$, so $m = 1$. Hence $t^n f = t^n(s + f_1 u) = t^n s + h f_1 \in I$.

(4): Using (3) it suffices to show that every $\psi \in \text{Aut}_{\mathbf{k}[t]}(R)$ preserves $\mathbf{k}[t, x, y]$. The algebra R is canonically isomorphic to $\mathbf{k}[t, x, y][\frac{h}{t^n}] \subset \mathbf{k}(t)[x, y]$. Since $\mathbf{k}(t)[x, y]$ is the localisation of $\mathbf{k}[t, x, y][\frac{h}{t^n}]$ in the multiplicative system $\mathbf{k}[t] \setminus \{0\}$, we get a natural inclusion $\text{Aut}_{\mathbf{k}[t]}(R) \subset \text{Aut}_K K[x, y]$, with $K = \mathbf{k}(t)$.

Suppose for contradiction that some $\psi \in \text{Aut}_{\mathbf{k}[t]}(R)$ satisfies $\psi(\mathbf{k}[t, x, y]) \not\subset \mathbf{k}[t, x, y]$. This implies that $\psi(x) \notin \mathbf{k}[t, x, y]$ or $\psi(y) \notin \mathbf{k}[t, x, y]$. We assume that $\psi(x) \notin \mathbf{k}[t, x, y]$ (the case $\psi(y) \notin \mathbf{k}[t, x, y]$ being similar) and use (2) to obtain an integer $l > 0$ such that $g = t^l \psi(x) \in \mathbf{k}[t, x, y]$ satisfies $g(0, x, y) \in h_0 \cdot \mathbf{k}[x, y] \setminus \{0\}$. Since $\psi \in \text{Aut}_{\mathbf{k}[t]}(R) \subset \text{Aut}_K K[x, y]$, the element $\psi(x)$ is a variable of $K[x, y]$ and the same holds for $g(t, x, y) = t^l \psi(x)$. By Lemma 3.8, $g(0, x, y)$ belongs to $\mathbf{k}[w]$ for some variable $w \in \mathbf{k}[x, y]$. The fact that $g(0, x, y) \in h_0 \cdot \mathbf{k}[x, y] \setminus \{0\}$ implies then that $h_0 \in \mathbf{k}[w]$ (Lemma 3.4), contradicting the hypothesis. \square

Corollary 4.2. Let \mathbf{k} be a field and

$$R = \mathbf{k}[t, u, x, y]/(t^n u - h(t, x, y)),$$

where $n \geq 1$ and $h \in \mathbf{k}[t, x, y]$ is a polynomial such that $h_0 = h(0, x, y) \in \mathbf{k}[x, y]$ does not belong to $\mathbf{k}[w]$ for each variable $w \in \mathbf{k}[x, y]$. Writing I the ideal $(t^n, h) \subset$

$\mathbf{k}[t, x, y]$, we obtain a group isomorphism

$$\begin{array}{ccc} \text{Aut}_{\mathbf{k}[t]}(R) & \xrightarrow{\cong} & \{\psi \in \text{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t, x, y]) \mid \psi(I) = I\} \\ \varphi & \mapsto & \varphi|_{\mathbf{k}[t, x, y]}. \end{array}$$

Proof. According to Proposition 4.1(4), every element $\varphi \in \text{Aut}_{\mathbf{k}[t]}(R)$ preserves $\mathbf{k}[t, x, y]$ and I , and thus restricts to an element $\psi \in \text{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t, x, y])$ that preserves I .

Conversely, each automorphism $\psi \in \text{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t, x, y])$ that preserves I induces an automorphism of $R = \mathbf{k}[t, x, y][\frac{I}{t^n}] = \mathbf{k}[t, x, y][\frac{h}{t^n}]$. This latter is uniquely determined by ψ , since the morphism $\text{Spec}(R) \rightarrow \text{Spec}(\mathbf{k}[t, x, y])$ given by the inclusion $\mathbf{k}[t, x, y] \hookrightarrow R$ is birational. \square

Remark 4.3. According to [KZ99, Definition 1.1 and Proposition 1.1], $\text{Spec}(R)$ is the affine modification of $\mathbb{A}_{\mathbf{k}}^3 = \text{Spec}(\mathbf{k}[t, x, y])$ with locus (I, t^n) . It is thus natural that every automorphism of $\mathbb{A}_{\mathbf{k}}^3$ fixing the ideal and the fibration $(t, x, y) \mapsto t$ lifts to an automorphism of $\text{Spec}(R)$. In fact, this holds more generally for any affine modification; see [KZ99, Corollary 2.2 (a)]. The interesting part of Corollary 4.2 consists then in saying that all automorphisms of the $\mathbf{k}[t]$ -algebra R are obtained by a lift of an automorphism of this form.

4.2. Application of liftings to the case of SL_2 . We will apply Corollary 4.2 to the variety $\text{SL}_2 \subset \mathbb{A}^4$ given by

$$\text{SL}_2 = \left\{ \begin{pmatrix} x & t \\ u & y \end{pmatrix} \in \mathbb{A}^4 \mid xy - tu = 1 \right\},$$

and obtain Proposition 4.5 below. Before we give a proof, let us recall the following basic facts on the coordinate ring of the variety SL_2 .

Lemma 4.4. *Let R be the coordinate ring of SL_2 , i.e., $R = \mathbf{k}[t, u, x, y]/(xy - tu - 1)$. Then R is a unique factorisation domain and the units of R satisfy $R^* = \mathbf{k}^*$.*

Proof. Since the localisation $R_t = \mathbf{k}[t, \frac{1}{t}][x, y]$ is a unique factorisation domain, we only have to see that tR is a prime ideal of R , by [Mat89, Theorem 20.2]. This is the case, since $R/tR \simeq \mathbf{k}[u, x, y]/(xy - 1)$ is an integral domain. Moreover, we have $R^* \subseteq (R_t)^* = \{\mu t^n \mid \mu \in \mathbf{k}^*, n \in \mathbb{Z}\}$. Since t^n is invertible in R if and only if $n = 0$, it follows that $R^* = \mathbf{k}^*$. \square

In fact, by [Pop74, Proposition 1], the ring of regular functions of any simply connected algebraic group is a unique factorisation domain.

Proposition 4.5. *We consider the morphisms*

$$\begin{array}{ccccc} \text{SL}_2 = \text{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1)) & \xrightarrow{\eta} & \mathbb{A}_{\mathbf{k}}^3 & \xrightarrow{\pi} & \mathbb{A}_{\mathbf{k}}^1 \\ & & (t, u, x, y) & \mapsto & t \end{array}$$

and denote by $X \subset \text{SL}_2$ the hypersurface given by $t = 1$ and by $\Gamma \subset \mathbb{A}_{\mathbf{k}}^3$ the closed curve given by $t = xy - 1 = 0$.

Then, the birational morphism $\eta: \text{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ yields a group isomorphism

$$\begin{array}{ccc} \{g \in \text{Aut}(\text{SL}_2) \mid g(X) = X\} & \xrightarrow{\cong} & \{g \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^3) \mid \pi g = \pi, g(\Gamma) = \Gamma\} \\ g & \mapsto & \eta g \eta^{-1}. \end{array}$$

We moreover have

$$\begin{aligned} \{g \in \text{Aut}(\text{SL}_2) \mid g(X) = X\} &= \{g \in \text{Aut}(\text{SL}_2) \mid \pi \eta g = \pi \eta\} \\ &= \{g \in \text{Aut}(\text{SL}_2) \mid g^*(t) = t\}. \end{aligned}$$

Proof. Every automorphism g of $\mathbb{A}_{\mathbf{k}}^3 = \text{Spec}(\mathbf{k}[t, x, y])$ yields an automorphism $g^* \in \text{Aut}_{\mathbf{k}}(\mathbf{k}[t, x, y])$. Moreover, the condition $\pi g = g$ corresponds to $g^*(t) = t$, and the condition $g(\Gamma) = \Gamma$ to $g^*(I) = I$, where $I \subset \mathbf{k}[t, x, y]$ is the ideal of Γ , generated by t and $xy - 1$. The isomorphism $\text{Aut}(\mathbb{A}_{\mathbf{k}}^3) \rightarrow \text{Aut}_{\mathbf{k}}(\mathbf{k}[t, x, y])$ then yields a bijection

$$\begin{aligned} \{g \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^3) \mid \pi g = \pi, g(\Gamma) = \Gamma\} &\xrightarrow{\cong} \{\psi \in \text{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t, x, y]) \mid \psi(I) = I\} \\ g &\mapsto g^* . \end{aligned}$$

We then want to apply Corollary 4.2 with $n = 1$ and $h = xy - 1$. To check that it is possible, we need to see that h does not belong to $\mathbf{k}[w]$ for each variable $w \in \mathbf{k}[x, y]$. Indeed, $xy - 1 \in \mathbf{k}[w]$ would imply that $xy \in \mathbf{k}[w]$, and thus that $x, y \in \mathbf{k}[w]$, since $\mathbf{k}[w]$ is factorially closed (Lemma 3.4). This would yield $\mathbf{k}[w] = \mathbf{k}[x, y]$, a contradiction.

We then apply Corollary 4.2 and obtain a group isomorphism

$$\begin{aligned} \text{Aut}_{\mathbf{k}[t]}(R) &\xrightarrow{\cong} \{\psi \in \text{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t, x, y]) \mid \psi(I) = I\} \\ \varphi &\mapsto \varphi|_{\mathbf{k}[t, x, y]}, \end{aligned}$$

where $R = \mathbf{k}[t, u, x, y]/(tu - xy - 1)$. This yields then a group isomorphism

$$\begin{aligned} \{g \in \text{Aut}(\text{SL}_2) \mid \pi \eta g = \pi \eta\} &\xrightarrow{\cong} \{g \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^3) \mid \pi g = \pi, g(\Gamma) = \Gamma\} \\ g &\mapsto \eta g \eta^{-1}. \end{aligned}$$

It remains then to show that

$$\{g \in \text{Aut}(\text{SL}_2) \mid \pi \eta g = \pi \eta\} = \{g \in \text{Aut}(\text{SL}_2) \mid g(X) = X\}.$$

The inclusion “ \subset ” follows from the equality $X = (\pi \eta)^{-1}(1)$. It remains then to show the inclusion “ \supset ”.

To do this, we take $g \in \text{Aut}(\text{SL}_2)$ such that $g(X) = X$ and prove that $\pi \eta g = \pi \eta$. The element g corresponds to an element $g^* \in \text{Aut}_{\mathbf{k}}(R)$. The fact that $g(X) = X$ is then equivalent to asking if g^* sends the ideal generated by $t - 1$ onto itself. Since $R^* = \mathbf{k}^*$ by Lemma 4.4, so $t - 1$ is sent onto $\mu(t - 1)$ for some $\mu \in \mathbf{k}^*$. This implies that the restriction of g^* yields an automorphism of $\mathbf{k}[t]$, corresponding to an automorphism $\hat{g} \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^1)$ such that $\hat{g}\pi\eta = \pi\eta g$. As $(\pi\eta)^{-1}(0)$ is the only fibre of $\pi\eta$ that is not isomorphic to $\mathbb{A}_{\mathbf{k}}^2$, it has to be preserved under g . As the fibre $\pi^{-1}(1) = X$ is also preserved under g , we find that \hat{g} is the identity, so $\pi\eta g = \pi\eta$, as desired. □

Corollary 4.6. *The closed embedding*

$$\begin{aligned} \nu: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 \\ (x, y) &\mapsto \begin{pmatrix} x & 1 \\ xy - 1 & y \end{pmatrix} \end{aligned}$$

has the following property: an automorphism of $\mathbb{A}_{\mathbf{k}}^2$ extends to an automorphism of SL_2 , via ν , if and only if it has Jacobian determinant equal to ± 1 .

Proof. We denote by $X = \nu(\mathbb{A}_{\mathbf{k}}^2) \subset \text{SL}_2$ the closed hypersurface given by

$$X = \nu(\mathbb{A}_{\mathbf{k}}^2) = \left\{ \begin{pmatrix} x & 1 \\ xy - 1 & y \end{pmatrix} \mid (x, y) \in \mathbb{A}_{\mathbf{k}}^2 \right\} = \left\{ \begin{pmatrix} x & t \\ u & y \end{pmatrix} \in \text{SL}_2 \mid t = 1 \right\},$$

write $G = \{g \in \text{Aut}(\text{SL}_2) \mid g(X) = X\}$, and denote by $\tau: G \rightarrow \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ the group homomorphism such that $g \circ \nu = \nu \circ \tau(g)$ for each $g \in G$.

We first prove that the subgroup $H = \{h \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2) \mid \text{Jac}(h) \pm 1\}$ is contained in $\tau(G)$. The group H is generated by $(x, y) \mapsto (y, x)$, which is induced by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} y & t \\ u & x \end{pmatrix},$$

and by automorphisms of the form $(x, y) \mapsto (x, y + p(x))$, $p \in \mathbf{k}[x]$, induced by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ p(x) & 1 \end{pmatrix} \cdot \begin{pmatrix} x & t \\ u & y \end{pmatrix}.$$

It remains to take $g \in G$ and to prove that $\tau(g) \in H$. Proposition 4.5 implies that g can be written as

$$g \begin{pmatrix} x & t \\ u & y \end{pmatrix} = \begin{pmatrix} a(t, x, y) & t \\ s(t, u, x, y) & b(t, x, y) \end{pmatrix},$$

where $a, b \in \mathbf{k}[t, x, y]$, $s \in \mathbf{k}[t, u, x, y]$ and such that $\tilde{g}: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^2$ given by $\tilde{g}(t, x, y) = (t, a(t, x, y), b(t, x, y))$ is an automorphism of $\mathbb{A}_{\mathbf{k}}^3$ that preserves the curve Γ given by $t = xy - 1 = 0$. The Jacobian determinant of \tilde{g} is $\mu = \frac{\partial a}{\partial x} \cdot \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \cdot \frac{\partial b}{\partial x} \in \mathbf{k}^*$. Replacing with $t = 0$, we obtain the automorphism of $\mathbb{A}_{\mathbf{k}}^2$ given by $(x, y) \mapsto (a(0, x, y), b(0, x, y))$, which preserves the curve with equation $xy = 1$ and is thus of Jacobian ± 1 . Indeed, it is of the form $(x, y) \mapsto (\xi x, \xi^{-1}y)$ or $(x, y) \mapsto (\xi y, \xi^{-1}x)$ for some $\xi \in \mathbf{k}^*$ (see [BS15, Theorem 2 (iii)]). This shows that $\mu = \pm 1$. Replacing then $t = 1$ we get that the automorphism $\tau(g)$ which is given by $\tau(g)(x, y) = (a(1, x, y), b(1, x, y))$ has Jacobian ± 1 . \square

Proof of Theorem 2. We first observe that the embeddings $\rho_1, \bar{\nu}: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \text{SL}_2$ given by

$$\begin{aligned} \rho_1: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 & \bar{\nu}: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 \\ (a, b) &\mapsto \begin{pmatrix} 1 & b \\ a & ab + 1 \end{pmatrix}, & (a, b) &\mapsto \begin{pmatrix} a & 1 \\ -ab - 1 & -b \end{pmatrix}, \end{aligned}$$

are equivalent, under the map $\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} u & x \\ -y & -t \end{pmatrix}$. The embedding

$$\begin{aligned} \nu: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 \\ (x, y) &\mapsto \begin{pmatrix} x & 1 \\ xy - 1 & y \end{pmatrix} \end{aligned}$$

satisfies $\bar{\nu} = \nu\tau$, where $\tau \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ is the automorphism of Jacobian -1 given by $\tau: (x, y) \mapsto (x, -y)$. Corollary 4.6 then implies that there exists $\hat{\tau} \in \text{Aut}(\text{SL}_2)$ such that $\hat{\tau}\nu = \nu\tau = \bar{\nu}$, i.e., that the embeddings $\bar{\nu}$ and ν are equivalent, so ν and ρ_1 are equivalent. Corollary 4.6 implies then that an automorphism of $\mathbb{A}_{\mathbf{k}}^2$ extends to an automorphism of SL_2 , via ρ_1 , if and only if it has Jacobian determinant equal to ± 1 . It remains to prove Assertions (1) and (2) of Theorem 2.

Assertion (2) follows from the fact that the group homomorphism

$$\text{Jac}: \text{Aut}(\mathbb{A}_{\mathbf{k}}^2) \rightarrow \mathbf{k}^*$$

is surjective (taking for instance diagonal automorphisms), so there are automorphisms of Jacobian determinant in $\mathbf{k}^* \setminus \{\pm 1\}$ if and only if \mathbf{k} contains at least four elements.

To obtain Assertion (1), we observe that every closed embedding $\mathbb{A}_{\mathbf{k}}^2 \rightarrow \text{SL}_2$ having image in $\rho_1(\mathbb{A}^2)$ is of the form $\rho_1\nu$ for some $\nu \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$. Writing $d_\lambda \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ for the automorphism given by $(s, t) \mapsto (\lambda s, t)$, $\lambda \in \mathbf{k}^*$, we can write

$\nu = d_\lambda \nu_1$ for some $\lambda \in \mathbf{k}^*$ and some $\nu_1 \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$ of Jacobian determinant equal to 1. The result above implies that $\rho_1 \nu$ is equivalent to $\rho_1 d_\lambda = \rho_\lambda$.

It remains to observe that $\rho_{\lambda'} = \rho_\lambda d_{\lambda' \lambda^{-1}}$, so ρ_λ and $\rho_{\lambda'}$ are equivalent if and only if $\lambda' \lambda^{-1} \in \{\pm 1\}$, which corresponds to $\lambda' = \pm \lambda$. □

Remark 4.7. Over the field $\mathbf{k} = \mathbb{C}$ of complex numbers, all algebraic embeddings of \mathbb{C}^2 into $\text{SL}_2(\mathbb{C})$ with image equal to $\rho_1(\mathbb{C}^2)$ are equivalent under holomorphic automorphisms of $\text{SL}_2(\mathbb{C})$. Indeed, according to Theorem 2(1) it is enough to show that the embeddings

$$\begin{aligned} \rho_1: \quad \mathbb{C}^2 &\hookrightarrow \text{SL}_2 & \rho_\lambda: \quad \mathbb{C}^2 &\hookrightarrow \text{SL}_2 \\ (s, t) &\mapsto \begin{pmatrix} 1 & t \\ s & st + 1 \end{pmatrix}, & (s, t) &\mapsto \begin{pmatrix} 1 & t \\ \lambda s & \lambda st + 1 \end{pmatrix}, \end{aligned}$$

are equivalent under a holomorphic automorphism for all $\lambda \in \mathbb{C}^*$. Such a holomorphic automorphism of $\text{SL}_2(\mathbb{C})$ is given by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & t \\ \mu(x)u & y + \frac{\mu(x)-1}{x}tu \end{pmatrix},$$

where $\mu: \mathbb{C} \rightarrow \mathbb{C}^*$ is a holomorphic function with $\mu(1) = \lambda$ and $\mu(0) = 1$.

5. FIBRED EMBEDDINGS OF $\mathbb{A}_{\mathbf{k}}^2$ INTO SL_2 AND THE PROOF OF THEOREM 3

In this section, we study fibred embeddings as in (\diamond) from Definition 1.2.

We will need the following simple description of the morphism $\eta: \text{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ already studied in Proposition 4.5.

Lemma 5.1. *Let $\Gamma \subset \mathbb{A}_{\mathbf{k}}^3 = \text{Spec}(\mathbf{k}[t, x, y])$ be the curve given by $t = xy - 1 = 0$ and let $\eta: \text{Bl}_\Gamma(\mathbb{A}_{\mathbf{k}}^3) \rightarrow \mathbb{A}_{\mathbf{k}}^3$ be the blow-up of Γ . We then have a natural open embedding $\text{SL}_2 \hookrightarrow \text{Bl}_\Gamma(\mathbb{A}_{\mathbf{k}}^3)$ such that the restriction of η corresponds to $(t, u, x, y) \mapsto (t, x, y)$.*

For completeness we insert a proof of this easy fact, which can in fact also be deduced from the more general statement [KZ99, Lemma 1.2], describing affine modifications as open subsets of blow-ups.

Proof. The blow-up of Γ can be seen as

$$\begin{aligned} \eta: \quad \text{Bl}_\Gamma(\mathbb{A}_{\mathbf{k}}^3) &= \{((t, x, y), [u : v]) \in \mathbb{A}_{\mathbf{k}}^3 \times \mathbb{P}_{\mathbf{k}}^1 \mid tu = (xy - 1)v\} &\rightarrow \mathbb{A}_{\mathbf{k}}^3 \\ &((t, x, y), [u : v]) &\mapsto (t, x, y). \end{aligned}$$

The open subset of $\text{Bl}_\Gamma(\mathbb{A}_{\mathbf{k}}^3)$ given by $v \neq 0$ is then naturally isomorphic to SL_2 , by identifying $((t, x, y), [u : 1])$ with $(t, u, x, y) \in \text{SL}_2$, and the birational morphism $\eta: \text{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ sends $((t, x, y), [u : 1])$ onto (t, x, y) . □

5.1. Polynomials associated to fibred embeddings. The following result associates to every fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$ a polynomial in $\mathbf{k}[t, x, y]$, and gives some basic properties of this polynomial (which will be studied in more detail after).

Lemma 5.2. *Let $\rho: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$ be a fibred embedding, and let $Z \subset \mathbb{A}_{\mathbf{k}}^3$ be the closure of $\eta(\rho(\mathbb{A}_{\mathbf{k}}^2))$, where*

$$\eta: \text{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3, \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto (t, x, y).$$

Then, Z is given by $P(t, x, y) = 0$, where $P \in \mathbf{k}[t, x, y]$ is a polynomial having the following properties:

- (1) The ring $\mathbf{k}[t, \frac{1}{t}][x, y]/(P)$ is a polynomial ring in one variable over $\mathbf{k}[t, \frac{1}{t}]$ (equivalently the morphism $\pi: Z \rightarrow \mathbb{A}_{\mathbf{k}}^1$ given by $(t, x, y) \mapsto t$ is a trivial \mathbb{A}^1 -bundle over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$).
- (2) If P is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$ (which is always true if $\text{char}(\mathbf{k}) = 0$ by (1) and the Abhyankar-Moh-Suzuki Theorem), then the polynomial $P(0, x, y) \in \mathbf{k}[x, y]$ is given by $\mu x^m(x - \lambda)$ or $\mu y^m(y - \lambda)$ for some $\mu, \lambda \in \mathbf{k}^*$ and some $m \geq 0$, and $\rho(\mathbb{A}_{\mathbf{k}}^2) \subset \text{SL}_2$ is the hypersurface given by $P = 0$.

Proof. We consider the morphisms

$$\begin{array}{ccc} \text{SL}_2 = \text{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1)) & \xrightarrow{\eta} & \mathbb{A}_{\mathbf{k}}^3 & \xrightarrow{\pi} & \mathbb{A}_{\mathbf{k}}^1 \\ & & (t, u, x, y) & \mapsto & t \end{array}$$

and observe that η yields an isomorphism between the two open subsets $(\text{SL}_2)_t \subset \text{SL}_2$ and $(\mathbb{A}_{\mathbf{k}}^3)_t \subset \mathbb{A}_{\mathbf{k}}^3$ given by $t \neq 0$. The morphism $\eta\rho: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ restricts thus to a closed embedding $(\mathbb{A}_{\mathbf{k}}^2)_t \hookrightarrow (\mathbb{A}_{\mathbf{k}}^3)_t$, where $(\mathbb{A}_{\mathbf{k}}^2)_t \subset \mathbb{A}_{\mathbf{k}}^2$ is the open subset where $t \neq 0$. This yields (1).

We now assume that $P \in \mathbf{k}[t, x, y]$ is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$. Applying Lemma 3.8 we obtain that $P_0 = P(0, x, y) \in \mathbf{k}[w]$ for some variable $w \in \mathbf{k}[x, y]$. In particular, $xy - 1$ does not divide P_0 (otherwise, by Lemma 3.4 we would have $xy - 1 \in \mathbf{k}[w]$ and then $x, y \in \mathbf{k}[w]$, impossible). This implies that $Z \cap \Gamma$ is a 0-dimensional scheme (which is a priori not reduced), where $\Gamma \subset \mathbb{A}_{\mathbf{k}}^3$ is the closed curve given by $t = xy - 1 = 0$. Recall that SL_2 is an open subset of $\text{Bl}_{\Gamma}(\mathbb{A}_{\mathbf{k}}^3)$ (Lemma 5.1) and that the exceptional divisor $E \subset \text{SL}_2$ is simply given by $t = xy - 1 = 0$ and is a trivial \mathbb{A}^1 -bundle over Γ . Since the pull-back $H \subset \text{SL}_2$ of Z on SL_2 , given by the equation $P = 0$ has all its irreducible components of pure codimension one we get $H = \rho(\mathbb{A}_{\mathbf{k}}^2) \simeq \mathbb{A}_{\mathbf{k}}^2$. As ρ is a fibred embedding, the morphism $H \rightarrow \mathbb{A}_{\mathbf{k}}^1$ given by the projection on t is a trivial \mathbb{A}^1 -bundle. This implies that $Z \cap \Gamma$ consists of a single reduced point, which is defined over \mathbf{k} and thus of the form $q = (0, \lambda, \frac{1}{\lambda}) \in \Gamma$ for some $\lambda \in \mathbf{k}^*$.

We can thus write $P_0 \in \mathbf{k}[w]$ as $P_0 = ab$ where $a, b \in \mathbf{k}[w]$ are such that $b(q) \neq 0$, a is irreducible, and $a(q) = 0$. This implies that a is a polynomial of degree 1 in w , so we can assume that $a = w$ (by replacing w with a).

We now show that $w = x - \lambda$ or $w = y - \frac{1}{\lambda}$ (after replacing w with μw , $\mu \in \mathbf{k}^*$), and can for this assume that \mathbf{k} is algebraically closed. As w is a variable in $\mathbf{k}[x, y]$, the curve $C \subset \mathbb{A}_{\mathbf{k}}^2$ defined by $w = 0$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^1$, and its closure in $\mathbb{P}_{\mathbf{k}}^2$ is a curve \overline{C} passing through exactly one point q_0 of the line at infinity $L_{\infty} = \mathbb{P}_{\mathbf{k}}^2 \setminus \mathbb{A}_{\mathbf{k}}^2$. Moreover, the tangent cone of \overline{C} in q_0 is supported on only one line L . The closure of Γ is then the curve $\overline{\Gamma} \subset \mathbb{P}_{\mathbf{k}}^2$ given by $xy - z^2 = 0$, and $\overline{\Gamma} \setminus \Gamma = \{[1 : 0 : 0], [0 : 1 : 0]\}$. We apply Bézout's Theorem and find

$$2 \deg(\overline{C}) = \overline{C} \cdot \overline{\Gamma} = (\overline{C} \cdot \overline{\Gamma})_q + (\overline{C} \cdot \overline{\Gamma})_{q_0} = 1 + (\overline{C} \cdot \overline{\Gamma})_{q_0}.$$

In particular, we get $q_0 \in \overline{\Gamma}$, so $q_0 \in \overline{\Gamma} \cap L_{\infty} = \{[1 : 0 : 0], [0 : 1 : 0]\}$. If \overline{C} is a line, it has to be given by $x - \lambda z = 0$ or $y - \frac{1}{\lambda} z = 0$, as we already know that \overline{C} passes through $[\lambda : \frac{1}{\lambda} : 1]$. This gives $w = x - \lambda$ or $w = y - \frac{1}{\lambda}$, as desired. It remains to derive a contradiction from $\deg(\overline{C}) > 1$. We denote by m the multiplicity of \overline{C} at q_0 and have $m < \deg(\overline{C})$, since \overline{C} is irreducible and $\deg(\overline{C}) > 1$. We apply

again Bézout’s Theorem to obtain $\deg(\overline{C}) = \overline{C} \cdot L_\infty = (\overline{C} \cdot L_\infty)_{q_0}$. The inequality $(\overline{C} \cdot L_\infty)_{q_0} > m$ implies that $L_\infty = L$, so the tangent cone of \overline{C} is supported on L_∞ . In particular, the tangent cones of \overline{C} and $\overline{\Gamma}$ at q_0 have distinct supports, because the conic $\overline{\Gamma}$ and the line L_∞ intersect transversally. This yields $m = (\overline{C} \cdot \overline{\Gamma})_{q_0}$, hence $2 \deg(\overline{C}) = 1 + (\overline{C} \cdot \overline{\Gamma})_{q_0} = 1 + m < 1 + \deg(\overline{C})$, and contradicts the assumption $\deg(\overline{C}) > 1$.

Now that $w = x - \lambda$ is proven (respectively, $w = y - \frac{1}{\lambda}$), we obtain $P_0 = wb$ for some $b \in \mathbf{k}[x]$ (respectively, $b \in \mathbf{k}[y]$) which does not vanish on any point of Γ . Hence, P_0 is equal to $x^m(x - \lambda)$ or $y^m(y - \frac{1}{\lambda})$ for some $m \geq 0$, after replacing P with μP , $\mu \in \mathbf{k}^*$. □

We now give an example which shows that the polynomial P given in Lemma 5.2 is not always a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$ (even if P is always such a variable when $\text{char}(\mathbf{k}) = 0$).

Lemma 5.3. *Let \mathbf{k} be a field of characteristic $p > 0$ and let $q \geq 2$ be an integer that does not divide p . Then, the polynomial*

$$P = (x - 1) - t^p(y^p - (x - 1)^q)^p \in \mathbf{k}[t, x, y]$$

has the following properties:

- (1) P is not a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$.
- (2) The hypersurface $Z_P \subset \mathbb{A}_{\mathbf{k}}^3 = \text{Spec}(\mathbf{k}[t, x, y])$ given by $P = 0$ satisfies that $Z_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$, $(t, x, y) \mapsto t$ is a trivial \mathbb{A}^1 -bundle (in particular, Z_P is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$).
- (3) The hypersurface $H_P \subset \text{SL}_2 = \text{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$ given by $P = 0$ is the image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$ (in particular, H_P is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$).

Proof. (1): Replacing x with $x + 1$, it suffices to show that $x - t^p(y^p - x^q)^p$ is not a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$. This follows from Corollary 3.16.

(2) We consider the morphisms

$$\begin{aligned} \tau: \quad \mathbb{A}_{\mathbf{k}}^2 &\rightarrow Z_P \\ (s, t) &\mapsto (t, t^p s^{p^2} + 1, t^q s^{pq} + s) \\ \chi: \quad Z_P &\rightarrow \mathbb{A}_{\mathbf{k}}^2 \\ (t, x, y) &\mapsto (y - t^q(y^p - (x - 1)^q)^q, t) \end{aligned}$$

and check that $\tau \circ \chi = \text{id}_{Z_P}$, $\chi \circ \tau = \text{id}_{\mathbb{A}_{\mathbf{k}}^2}$.

(3): The morphism $\eta: H_P \rightarrow Z_P$, $(t, u, x, y) \mapsto (t, x, y)$ being an isomorphism on the subsets given by $t \neq 0$, the morphism $\pi \circ \eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$, $(t, u, x, y) \mapsto t$ is a trivial \mathbb{A}^1 -bundle over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$. The zero fibre is moreover isomorphic to $\mathbb{A}_{\mathbf{k}}^1$ since $P(0, x, y) = x - 1$ and the line $\{x = 1\}$ intersects the conic $\{xy = 1\}$ transversally in one point (follows from Lemma 5.1). By Lemma 3.10 it follows that $\pi \circ \eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is a trivial \mathbb{A}^1 -bundle. Hence H_P is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$ and is the image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$. □

We now start from a polynomial $P \in \mathbf{k}[t, x, y]$ that is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$ and determine when this one comes from a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$, by the process determined in Lemma 5.2. This yields the following result, which corresponds to Part (1) of Theorem 3.

Proposition 5.4. *Let \mathbf{k} be any field, let $P \in \mathbf{k}[t, x, y]$ be a polynomial that is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, and let*

$$H_P \subset \mathrm{SL}_2 = \mathrm{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$$

and $Z_P \subset \mathbb{A}_{\mathbf{k}}^3 = \mathrm{Spec}(\mathbf{k}[t, x, y])$ be the hypersurfaces given by $P = 0$.

The following conditions are equivalent:

- (a) *The hypersurface $H_P \subset \mathrm{SL}_2$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$.*
- (b) *The hypersurface $H_P \subset \mathrm{SL}_2$ is the image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$.*
- (c) *The fibre of $Z_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$, $(t, x, y) \mapsto t$ over every closed point of $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$ is isomorphic to \mathbb{A}^1 and the polynomial $P(0, x, y) \in \mathbf{k}[x, y]$ is of the form $\mu x^m(x - \lambda)$ or $\mu y^m(y - \lambda)$ for some $\mu, \lambda \in \mathbf{k}^*$ and some $m \geq 0$.*

Proof. We will use the morphisms

$$\eta: \mathrm{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3, \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto (t, x, y), \quad \pi: \mathbb{A}_{\mathbf{k}}^3 \rightarrow \mathbb{A}_{\mathbf{k}}^1, \quad (t, x, y) \mapsto t.$$

(a) \Rightarrow (b): Proving that H_P is the image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ is equivalent to asking that $\pi \circ \eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ be a trivial \mathbb{A}^1 -bundle. Since P is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, it follows that the generic fibre of $\pi: Z_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is isomorphic to $\mathbb{A}_{\mathbf{k}(t)}^1$. Moreover, $\eta: \mathrm{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ is an isomorphism over $\{t \neq 0\}$, so the generic fibre of $\pi \circ \eta$ is also isomorphic to $\mathbb{A}_{\mathbf{k}(t)}^1$. The fact that H_P is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$ (which is the hypothesis (a)) implies that $\pi \circ \eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is a trivial \mathbb{A}^1 -bundle, by Lemma 3.9 ((3) \Rightarrow (4)).

(b) \Rightarrow (c): Follows from Lemma 5.2(1) and (2).

(c) \Rightarrow (a): Since $\eta: \mathrm{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ is an isomorphism over the open subset $\{t \neq 0\}$, it follows that all fibres of $\pi \circ \eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ over closed points of $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$ are isomorphic to \mathbb{A}^1 . Moreover, the fibre of $\pi \circ \eta$ over 0 is isomorphic to $\mathbb{A}_{\mathbf{k}}^1$, since the restriction $\eta|_{\{t=0\}}: \{t = 0\} \rightarrow \{t = xy - 1 = 0\} \subset \{0\} \times \mathbb{A}_{\mathbf{k}}^2$ is a trivial \mathbb{A}^1 -bundle over the curve $\{t = xy - 1 = 0\}$ and since $\{P(0, x, y) = 0\}$ intersects $\{xy = 1\}$ in exactly one point, transversally. The generic fibre of $\pi \circ \eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ being isomorphic to $\mathbb{A}_{\mathbf{k}(t)}^1$, it follows from Lemma 3.10 that $\pi \circ \eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is a trivial \mathbb{A}^1 -bundle and thus H_P is isomorphic to the affine plane $\mathbb{A}_{\mathbf{k}}^2$, which proves (a). \square

Example 5.5. For each $n \geq 1, m \geq 0, \mu \in \mathbf{k}^*$, and $q \in \mathbf{k}[t, x]$, the polynomial

$$P(t, x, y) = t^n y + \mu x^m(x - 1) + tq(t, x) \in \mathbf{k}[t, x, y]$$

defines a hypersurface $H_P \subset \mathrm{SL}_2$ which is the image of a fibred embedding. Indeed, since P has degree 1 in y with coefficient t^n , it is a variable of $\mathbf{k}[t, t^{-1}][x, y]$. We can thus apply Proposition 5.4 and only need to check that $P(0, x, y) = \mu x^m(x - 1)$ is of the desired form (as in Assertion (c)).

5.2. Determining when two fibred embeddings are equivalent. In this section, we consider embeddings satisfying the conditions of Proposition 5.4 (or equivalently of Theorem 3(1)) and determine when two of these are equivalent, by proving Theorem 3(2). We first characterise the case where the integer m of Proposition 5.4 (or equivalently of Lemma 5.2 or Theorem 3(1)) is equal to zero.

Lemma 5.6. *Let \mathbf{k} be any field and let $P \in \mathbf{k}[t, x, y]$ be a polynomial that is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, and let $H_P \subset \mathrm{SL}_2 = \mathrm{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$ and $Z_P \subset \mathbb{A}_{\mathbf{k}}^3 = \mathrm{Spec}(\mathbf{k}[t, x, y])$ be the hypersurfaces given by $P = 0$.*

Assume that H_P is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$, which implies that $P(0, x, y) \in \mathbf{k}[x, y]$ is of the form $\mu x^m(x - \lambda)$ or $\mu y^m(y - \lambda)$ for some $\mu, \lambda \in \mathbf{k}^*$ and some $m \geq 0$. Then, the following conditions are equivalent:

- (a) $m = 0$;
- (b) P is a variable of the $\mathbf{k}[t]$ -algebra $\mathbf{k}[t][x, y]$;
- (c) there is an isomorphism $\varphi: \mathbb{A}_{\mathbf{k}}^2 \xrightarrow{\cong} Z_P$ such that $\pi\varphi$ is the projection $(t, x) \mapsto t$;
- (d) there is an isomorphism $\varphi: \mathbb{A}_{\mathbf{k}}^2 \xrightarrow{\cong} Z_P$;
- (e) there exist $\varphi \in \text{Aut}(\text{SL}_2)$ such that $\varphi(H_P) = \rho_1(\mathbb{A}_{\mathbf{k}}^2)$, where ρ_1 is the standard embedding;
- (f) there exist $\varphi \in \text{Aut}(\text{SL}_2)$ such that $\varphi(H_P) = \rho_1(\mathbb{A}_{\mathbf{k}}^2)$ and $\varphi^*(t) = t$.

Proof. As before, we use the morphisms

$$\eta: \text{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3, \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto (t, x, y), \quad \pi: \mathbb{A}_{\mathbf{k}}^3 \rightarrow \mathbb{A}_{\mathbf{k}}^1, \quad (t, x, y) \mapsto t.$$

Proposition 5.4 says that $H_P \subset \text{SL}_2$ is the image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$, which corresponds to saying that $\pi\eta: H_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is a trivial \mathbb{A}^1 -bundle. Since $\eta: \text{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ is an isomorphism over the open subset $\{t \neq 0\}$, we obtain that $\pi: Z_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is a trivial \mathbb{A}^1 -bundle over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$.

We first prove (a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d), using Corollary 3.13. We observe that (b), (c), and (d) correspond, respectively, to the equivalent assertions (i), (ii), and (iii) of Corollary 3.13. Moreover, the condition $m = 0$ (which is (a)) corresponds to saying that the 0-fibre of $\pi: Z_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^1$. Since $\pi: Z_P \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is a trivial \mathbb{A}^1 -bundle over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$, Assertion (a) corresponds to Assertion (vi) of Corollary 3.13. Thus Corollary 3.13 yields

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d).$$

It remains to show that these are also equivalent to (e) and (f).

(b) \Rightarrow (f): Applying an automorphism of the form

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} \mu^{-1}x & t \\ u & \mu y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} \mu^{-1}y & t \\ u & \mu x \end{pmatrix}$$

for some $\mu \in \mathbf{k}^*$, we can assume that $P(0, x, y) = x - 1$. Since P is a variable of the $\mathbf{k}[t]$ -algebra $\mathbf{k}[t][x, y]$, there exists $f \in \text{Aut}_{\mathbf{k}[t]}(\mathbf{k}[t][x, y])$ such that $f(x - 1) = P$. The element $\psi \in \text{Aut}(\mathbb{A}_{\mathbf{k}}^3)$ satisfying $\psi^* = f$ is then such that $\pi\psi = \pi$ and sends Z_P onto the hypersurface of $\mathbb{A}_{\mathbf{k}}^3$ given by $x = 1$. The restriction of ψ to the hypersurface given by $t = 0$ is an automorphism of the form $(0, x, y) \mapsto (0, \nu(x, y), \rho(x, y))$ which preserves the curve given by $x - 1 = 0$. Replacing ψ with its composition with the inverse of $(t, x, y) \mapsto (t, \nu(x, y), \rho(x, y))$, we can assume that the restriction of ψ to the hypersurface $t = 0$ is the identity, so $\psi(\Gamma) = \Gamma$, where Γ is the curve given by $t = xy - 1 = 0$. Proposition 4.5 implies then that ψ lifts to an automorphism φ of SL_2 sending H_P onto $\rho_1(\mathbb{A}_{\mathbf{k}}^2)$. We moreover have $\varphi^*(t) = t$, since $\psi^*(t) = t$.

(f) \Rightarrow (e) being clear, it remains to show (e) \Rightarrow (a). For this implication, one can assume that \mathbf{k} is algebraically closed. Assertion (e) yields an automorphism $\varphi \in \text{Aut}(\text{SL}_2)$ such that $\varphi(H_P) = \rho_1(\mathbb{A}_{\mathbf{k}}^2)$. Hence the automorphism $\varphi^* \in \text{Aut}_{\mathbf{k}}(R)$, where $R = \text{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$, sends the ideal $(x - 1) \subset R$ onto the ideal $(P) \subset R$. It follows from Lemma 4.4 that $x - 1$ is sent onto μP , for some $\mu \in \mathbf{k}^*$.

In particular, for a general $a \in \mathbf{k}$, the variety $H_{P-a} \subset \mathrm{SL}_2$ given by $P - a = 0$ is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$. It remains to show that this implies that $m = 0$.

Since P is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, so is $P - a$. There exists then an open dense subset $U \subset \mathbb{A}_{\mathbf{k}}^1$ such that $U \times \mathbb{A}_{\mathbf{k}}^2 \rightarrow U \times \mathbb{A}_{\mathbf{k}}^1, (t, x, y) \mapsto (t, P(t, x, y) - a)$ is a trivial \mathbb{A}^1 -bundle. This implies that $q_a: H_{P-a} \rightarrow \mathbb{A}_{\mathbf{k}}^1, (t, u, x, y) \mapsto t$ is a trivial \mathbb{A}^1 -bundle over U . By Lemma 3.9, q_a is a trivial \mathbb{A}^1 -bundle (since $H_{P-a} \simeq \mathbb{A}_{\mathbf{k}}^2$), so the fibre $(q_a)^{-1}(\{0\})$ needs to be isomorphic to an affine line. Since $(q_a)^{-1}(\{0\})$ is given by the equations $xy - 1 = P(0, x, y) - a = t = 0$ in the affine 4-space $\mathbb{A}_{\mathbf{k}}^4 = \mathrm{Spec}(\mathbf{k}[t, u, x, y])$ and since $P(0, x, y) - a$ is equal to $\mu x^m(x - \lambda) - a$ or $\mu y^m(y - \lambda) - a$ and $a \in \mathbf{k}$ is general (\mathbf{k} is algebraically closed), this implies that $m = 0$ and yields (a) as desired. \square

Remark 5.7. Lemma 5.6 shows in particular that if $H_P, H_Q \subset \mathrm{SL}_2$ are two hypersurfaces given by two polynomials $P, Q \in \mathbf{k}[t, x, y]$ as in Theorem 3 (or as in the previous results), and if one of the two integers $m, m' \in \mathbb{N}$ associated to P, Q is equal to zero, then H_P, H_Q are equivalent if and only if $m = m' = 0$.

Proposition 5.8. *Let \mathbf{k} be any field, let $P, Q \in \mathbf{k}[t, x, y]$ be polynomials that are variables of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$, and let*

$$H_P, H_Q \subset \mathrm{SL}_2 = \mathrm{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$$

and $Z_P, Z_Q \subset \mathbb{A}_{\mathbf{k}}^3 = \mathrm{Spec}(\mathbf{k}[t, x, y])$ be the hypersurfaces given by $P = 0$ and $Q = 0$, respectively.

Suppose that H_P is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$ but that Z_P is not isomorphic to $\mathbb{A}_{\mathbf{k}}^2$, and that there exists $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ that sends H_P onto H_Q . Then, the following hold:

- (1) *There exists $\mu \in \mathbf{k}^*$ such that $\varphi^*(t) = \mu t$.*
- (2) *The birational map $\psi = \eta\varphi\eta^{-1}$ is an automorphism of $\mathbb{A}_{\mathbf{k}}^3$ which sends Z_P onto Z_Q , where $\eta: \mathrm{SL}_2 \rightarrow \mathbb{A}_{\mathbf{k}}^3$ is as before given by $(t, u, x, y) \mapsto (t, x, y)$.*
- (3) *There exists $m \geq 1$ such that $P(0, x, y)$ and $Q(0, x, y)$ are of the form $\mu x^m(x - \lambda)$ or $\mu y^m(y - \lambda)$ for some $\mu, \lambda \in \mathbf{k}^*$ (the integer m is the same for P, Q but μ, λ and the choice between x and y depend on P, Q).*

Proof. Since H_P is isomorphic to $\mathbb{A}_{\mathbf{k}}^2$, the same holds for H_Q . The hypersurfaces $H_P, H_Q \subset \mathrm{SL}_2$ are thus images of fibred embeddings $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ and there are thus integers $m, m' \geq 0$ and $\lambda, \lambda', \mu', \mu \in \mathbf{k}^*$ such that $P(0, x, y) \in \{\mu x^m(x - \lambda), \mu y^m(y - \lambda)\}$ and $Q(0, x, y) \in \{\mu' x^{m'}(x - \lambda'), \mu' y^{m'}(y - \lambda')\}$ (Proposition 5.4). Moreover, the fact that Z_P is not isomorphic to $\mathbb{A}_{\mathbf{k}}^2$ is equivalent to $m > 0$ and to the fact that H_P is not equivalent to the image $\rho_1(\mathbb{A}^2)$ of the standard embedding (Lemma 5.6). As H_P and H_Q are equivalent, the same holds for H_Q , so $m' > 0$.

The main part of the proof consists in proving (1). To do this, one can extend the scalars and assume \mathbf{k} to be algebraically closed. We moreover have $\varphi^*(Q) = \xi P$ for some $\xi \in \mathbf{k}^*$ (follows from Lemma 4.4). Replacing P with ξP , we can assume that $\varphi^*(Q) = P$. For each $a \in \mathbf{k}^*$, the element φ then sends H_{P-a} onto H_{Q-a} , where $H_{P-a}, H_{Q-a} \subset \mathrm{SL}_2$ are given by the polynomials $P - a, Q - a \in \mathbf{k}[t, x, y]$. Since P, Q are variables of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$ and because the t -projections $Z_P \rightarrow \mathbb{A}^1$ and $Z_Q \rightarrow \mathbb{A}^1$ are trivial \mathbb{A}^1 -bundles over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$ (Lemma 5.2(1)), the polynomials P, Q are also variables of the $\mathbf{k}[t, \frac{1}{t}]$ -algebra $\mathbf{k}[t, \frac{1}{t}][x, y]$ (follows from Lemma 3.11 with $U = \mathbb{A}^1 \setminus \{0\}$). Hence, the same holds for $P - a$ and $Q - a$. The morphisms $H_{P-a}, H_{Q-a}, Z_{P-a}, Z_{Q-a} \rightarrow \mathbb{A}_{\mathbf{k}}^1$ given by the projection on t are therefore trivial \mathbb{A}^1 -bundles over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$. We show now that the surfaces

$H_{P-a}, H_{Q-a}, Z_{P-a}, Z_{Q-a}$ are smooth for general $a \in \mathbf{k}^*$. The hypersurfaces Z_{P-a}, Z_{Q-a} are in fact isomorphic to $\mathbb{A}_{\mathbf{k}}^2$ for general $a \in \mathbf{k}^*$, as P, Q are variables of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$. Since H_{P-a} is an open subset of the strict transform of Z_{P-a} under the blow-up $\text{Bl}_{\Gamma}(\mathbb{A}_{\mathbf{k}}^3) \rightarrow \mathbb{A}_{\mathbf{k}}^3$ of Γ (follows from Lemma 5.1), it is enough to show that $\Gamma \cap Z_{P-a}$ and $\Gamma \cap Z_{Q-a}$ are reduced for general $a \in \mathbf{k}^*$. The corresponding ideal is given by $f(s) = \varepsilon_1 s^m (s - \varepsilon_2) - a$ in $\mathcal{O}(\Gamma) = \mathbf{k}[s, \frac{1}{s}]$ for $\varepsilon_1, \varepsilon_2 \in \mathbf{k}^*, m \geq 0$. This ideal is reduced for general $a \in \mathbf{k}^*$, since the derivative of f is a non-zero polynomial, not depending on a . We can thus see these varieties as open subsets of smooth projective surfaces $\overline{H_{P-a}}, \overline{H_{Q-a}}, \overline{Z_{P-a}}, \overline{Z_{Q-a}}$ obtained by blowing-up some Hirzebruch surfaces, so that the projection on t is the restriction of the morphism to $\mathbb{P}_{\mathbf{k}}^1$ given by a \mathbb{P}^1 -bundle of the Hirzebruch surface and having only one singular fibre. We can moreover assume that the boundary is a union of smooth rational curves of self-intersection 0 or ≤ -2 (in particular the projectivisation is minimal). Indeed, if a component of the singular fibre has self-intersection -1 and is in the boundary, we can contract it, and if the section has self-intersection -1 , then we blow-up a general point of the smooth fibre contained in the boundary and then contract the strict transform of this fibre to obtain a section of self-intersection 0. The 0-fibre of $Z_{P-a} \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is given by $t = \mu x^m (x - \lambda) - a = 0$ or $t = \mu y^m (y - \lambda) - a = 0$ and is thus a disjoint union $C \simeq \coprod_{i=1}^{m+1} \mathbb{A}_{\mathbf{k}}^1$ of $m + 1$ affine curves isomorphic to $\mathbb{A}_{\mathbf{k}}^1$. Similarly the 0-fibre of $Z_{Q-a} \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is a disjoint union $C' \simeq \coprod_{i=1}^{m'+1} \mathbb{A}_{\mathbf{k}}^1$ of $m' + 1$ affine curves isomorphic to $\mathbb{A}_{\mathbf{k}}^1$. The closure of C is contained in the singular fibre F_0 of $\overline{Z_{P-a}} \rightarrow \mathbb{P}_{\mathbf{k}}^1$, which is a tree of smooth rational curves of self-intersection ≤ -1 , being an SNC divisor. Hence, the closure of each component of C is a smooth rational curve of self-intersection ≤ -1 , which intersects the boundary into a component lying in F_0 . A similar description holds for C' .

The curves C, C' meet transversally the conic Γ given by $xy = 1$ (because of the form of $P(0, x, y) - a$ and $Q(0, x, y) - a$). The surfaces $\overline{H_{P-a}}, \overline{H_{Q-a}}$ are then obtained by blowing-up some points in each of the components of C, C' and removing these components, so we can choose the a minimal projectivisations of H_{P-a}, H_{Q-a} to be blowing-ups of the above points in $\overline{Z_{P-a}}, \overline{Z_{Q-a}}$ and get a dual graph of the boundary of these surfaces which is not a chain (or which is not “linear” or not a “zigzag”). This implies that the \mathbb{A}^1 -fibration given by the t -projection is unique up to automorphisms of the target (see [Giz71] or [Ber83, Théorème 1.8]). As the zero fibre of $H_{P-a}, H_{Q-a} \rightarrow \mathbb{A}_{\mathbf{k}}^1$ is the unique degenerate fibre, there exist $\mu_a \in \mathbf{k}^*$ and $q_a \in \mathbf{k}[t, u, x, y]$ such that $\varphi^*(t) = \mu_a t + q_a \cdot (P - a)$. Since this holds for a general a , we get $\varphi^*(t) = \mu t$ for some $\mu \in \mathbf{k}^*$. Indeed, replacing t with 0 in $\varphi^*(t)$ yields an element of $\mathbf{k}[u, x, y]/(xy - 1)$ which is divisible by $P - a$ for infinitely many a . This element is thus equal to zero.

We now show how Assertion (1) implies the two others. We write $\varphi = \varphi_1 \varphi_2$ where $(\varphi_1)^*(t) = t$ and φ_2 is given by

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & \mu t \\ \mu^{-1}u & y \end{pmatrix}.$$

The fact that $(\varphi_1)^*(t) = t$ implies that $\psi_1 = \eta \varphi_1 \eta^{-1}$ is an automorphism of $\mathbb{A}_{\mathbf{k}}^3$ (Proposition 4.5). Since $\psi_2 = \eta \varphi_2 \eta^{-1}$ is a diagonal automorphism of $\mathbb{A}_{\mathbf{k}}^3$, the element $\psi = \psi_1 \psi_2 = \eta \varphi \eta^{-1}$ is an automorphism of $\mathbb{A}_{\mathbf{k}}^3$. As φ sends H_P onto H_Q , the automorphism ψ sends Z_P onto Z_Q , which yields (2). As $\psi^*(t) = \mu t$, the

hyperplane $W \subset \mathbb{A}_k^3$ given by $t = 0$ is invariant, this implies that $m = m'$ and thus yields (3). □

Lemma 5.6 and Proposition 5.8 yield then the following result, which yields in particular Assertion (2) of Theorem 3.

Corollary 5.9. *If $P, Q \in \mathbf{k}[t, x, y]$ are polynomials which are variables of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$ and if the corresponding hypersurfaces $H_P, H_Q \subset \mathrm{SL}_2 = \mathrm{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$ are equivalent and isomorphic to \mathbb{A}_k^2 , the following hold:*

- (1) H_P, H_Q are the image of fibred embeddings $\mathbb{A}_k^2 \hookrightarrow \mathrm{SL}_2$.
- (2) There exists $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_P) = H_Q$ and $\varphi^*(t) = \mu t$ for some $\mu \in \mathbf{k}^*$. In particular, the element $\psi = \eta\varphi\eta^{-1} \in \mathrm{Aut}(\mathbb{A}_k^3)$ satisfies $\psi^*(t) = \mu t$, $\psi(Z_P) = Z_Q$ and $\psi(\Gamma) = \Gamma$, where $\eta: \mathrm{SL}_2 \rightarrow \mathbb{A}_k^3$ is the morphism $(t, u, x, y) \mapsto (t, x, y)$, $Z_P, Z_Q \subset \mathbb{A}_k^3$ are the two hypersurfaces given by $P = 0, Q = 0$, and $\Gamma \subset \mathbb{A}_k^3$ is the conic given by $t = xy - 1 = 0$.

Proof. Assertion (1) follows from Proposition 5.4. It remains then to show (2). We denote by $\varphi_0 \in \mathrm{Aut}(\mathrm{SL}_2)$ an element such that $\varphi_0(H_P) = H_Q$.

(i) If $\varphi_0^*(t) = \mu t$ for some $\mu \in \mathbf{k}^*$, we choose $\varphi = \varphi_0$ and denote by $\theta \in \mathrm{Aut}(\mathrm{SL}_2)$ the element

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & \mu^{-1}t \\ \mu u & y \end{pmatrix}$$

to obtain $(\varphi_0\theta)^*(t) = t$. Proposition 4.5 shows that $\hat{\psi} = \eta(\varphi_0\theta)\eta^{-1} \in \mathrm{Aut}(\mathbb{A}_k^3)$ and $\hat{\psi}(\Gamma) = \Gamma$. Since $\tilde{\theta} = \eta\theta\eta^{-1}$ is the automorphism of \mathbb{A}_k^3 given by $(t, x, y) \mapsto (\mu^{-1}t, x, y)$, we have $\psi = \eta\varphi_0\eta^{-1} \in \mathrm{Aut}(\mathbb{A}_k^3)$ and $\psi(\Gamma) = \Gamma$. The fact that $\varphi_0^*(t) = \mu t$ and $\varphi_0(H_P) = H_Q$ yields then $\psi^*(t) = \mu t$ and $\psi(Z_P) = Z_Q$.

(ii) If $\varphi_0^*(t) \notin \{\mu t \mid \mu \in \mathbf{k}^*\}$, then Proposition 5.8(1) does not hold, so Z_P is isomorphic to \mathbb{A}_k^2 . Applying the same argument to φ_0^{-1} shows that Z_Q is isomorphic to \mathbb{A}_k^2 . Lemma 5.6((d) \Rightarrow (f)) then shows that there exist $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi_1(H_P) = \varphi_2(H_Q) = \rho_1(\mathbb{A}_k^2)$ and $(\varphi_1)^*(t) = (\varphi_2)^*(t) = t$. We then choose $\varphi = (\varphi_2)^{-1}\varphi_1$ and apply case (i). □

5.3. Examples of non-equivalent embeddings.

Lemma 5.10. *To each polynomial $r \in \mathbf{k}[t]$, we associate the polynomial*

$$P_r = ty - (x - t)(x - 1 - t^2r(t)) \in \mathbf{k}[t, x, y]$$

and denote by $H_{P_r} \subset \mathrm{SL}_2 = \mathrm{Spec}(\mathbf{k}[t, u, x, y]/(xy - tu - 1))$ and $Z_{P_r} \subset \mathbb{A}_k^3 = \mathrm{Spec}(\mathbf{k}[t, x, y])$ the hypersurfaces given by $P_r = 0$. Then,

- (1) For each $r \in \mathbf{k}[t]$, the surface H_{P_r} is the image of a fibred embedding $\mathbb{A}_k^2 \hookrightarrow \mathrm{SL}_2$.
- (2) For each $r, s \in \mathbf{k}[t]$, the following are equivalent:
 - (i) There exists $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_{P_r}) = H_{P_s}$.
 - (ii) There exists $\varphi \in \mathrm{Aut}(\mathbb{A}_k^3)$ such that $\varphi(Z_{P_r}) = Z_{P_s}$.
 - (iii) The surfaces Z_{P_r} and Z_{P_s} are isomorphic.
 - (iv) $r = s$.

Proof. For each $r \in \mathbf{k}[t]$, we write $S_r(t, x) = (x - t)(x - 1 - t^2r(t)) \in \mathbf{k}[t, x]$ and observe that $P_r(t, x, y) = ty - S_r(t, x)$.

(1): Since P_r is of degree 1 in y , it is a variable of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$. Moreover, $P_r(0, x, y) = S_r(0, x) = x(x - 1)$ is of the form $\mu x^m(x - \lambda)$ (with $\mu, \lambda \in \mathbf{k}^*$ and $m \geq 0$). The coefficient of y in P_r being t , the morphism $Z_{P_r} \rightarrow \mathbb{A}_{\mathbf{k}}^1, (t, x, y) \mapsto t$ is a trivial \mathbb{A}^1 -bundle over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$. Proposition 5.4 ((c) \Rightarrow (b)) then implies that $H_{P_r} \subset \mathrm{SL}_2$ is the image of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$.

It remains to show that the assertions (i) – (ii) – (iii) – (iv) of (2) are equivalent.

The implications (iv) \Rightarrow (i) and (ii) \Rightarrow (iii) are trivial.

Lemma 5.6 implies that Z_{P_r} and Z_{P_s} are not isomorphic to $\mathbb{A}_{\mathbf{k}}^2$ (the integer m being here equal to 1). We can thus apply Proposition 5.8(2), which yields (i) \Rightarrow (ii).

It remains then to show (iii) \Rightarrow (iv). According to [DP09, Proposition 3.6], the surface Z_{P_r} and Z_{P_s} are isomorphic if and only if there exist $a, \mu \in \mathbf{k}^*, \tau \in \mathbf{k}[t]$, such that

$$S_r(at, x) = \mu^2 S_s(t, \mu^{-1}x + \tau(t)) \quad \text{inside } \mathbf{k}[t, x].$$

This corresponds to

$$(x - at)(x - 1 - a^2 t^2 r(at)) = (x + \mu(\tau(t) - t))(x + \mu(\tau(t) - 1 - t^2 s(t)))$$

and thus gives two possibilities:

(I): $at = \mu(t - \tau(t))$ and $1 + a^2 t^2 r(at) = \mu(1 + t^2 s(t) - \tau(t))$. The first equation yields $\tau(t) = (1 - \frac{a}{\mu})t$ and the second yields $\mu\tau(t) \equiv \mu - 1 \pmod{t^2}$, which gives $\tau = 0$ and then $\mu = 1$ and $a = 1$. The second equation thus yields $r(t) = s(t)$.

(II): $at = \mu(1 + t^2 s(t) - \tau(t))$ and $\mu(t - \tau(t)) = 1 + a^2 t^2 r(at)$. This yields

$$1 + a^2 t^2 r(at) - \mu t = -\mu\tau(t) = at - \mu(1 + t^2 s(t))$$

and thus $1 - \mu t \equiv -\mu + at \pmod{t^2}$, whence $\mu = -1$ and $a = 1$. Replacing in the equation above, we find $r(t) = s(t)$. □

The proof of Theorem 3 is now clear.

Proof of Theorem 3. Assertion (1) corresponds to Proposition 5.4.

Assertion (2) follows from Corollary 5.9.

Assertion (3) follows from Lemma 5.10, which yields hypersurfaces $H_{P_r} \subset \mathrm{SL}_2$ that are parametrised by $r \in \mathbf{k}[t]$, which are all images of fibred embeddings and are pairwise non-equivalent. □

We finish this subsection with two explicit examples.

Lemma 5.11. *Let us denote by $P, Q \in \mathbf{k}[t, x, y]$ the polynomials*

$$P = t^2 y - x(x + 1) \quad \text{and} \quad Q = t^2 y - x(x + 1 - t^2).$$

Then, the following hold:

- (1) *The hypersurfaces $Z_P, Z_Q \subset \mathbb{A}_{\mathbf{k}}^3$ given by $P = 0$ and $Q = 0$ are equivalent.*
- (2) *The hypersurfaces $H_P, H_Q \subset \mathrm{SL}_2$ given by $P = 0$ and $Q = 0$ are both images of fibred embeddings but are not equivalent.*

Proof. To get (1), it suffices to observe that the linear automorphism $\theta \in \mathrm{Aut}(\mathbb{A}_{\mathbf{k}}^3)$ given by $(t, x, y) \mapsto (t, x, y - x)$ satisfies $\theta^*(Q) = P$, so $\theta(Z_P) = Z_Q$.

Since P, Q are of degree 1 in y , both are variables of the $\mathbf{k}(t)$ -algebra $\mathbf{k}(t)[x, y]$. Moreover, $P(0, x, y) = Q(0, x, y) = -x(x + 1)$ is of the form $\mu x^m(x - \lambda)$ (with $\mu, \lambda \in \mathbf{k}^*$ and $m = 1 \geq 0$). Since the coefficient of y in P and Q is t^2 , the morphisms $Z_P, Z_Q \rightarrow \mathbb{A}_{\mathbf{k}}^1, (t, x, y) \mapsto t$ are trivial \mathbb{A}^1 -bundles over $\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}$. Proposition 5.4

((c) \Rightarrow (b)) then implies that $H_P, H_Q \subset \mathrm{SL}_2$ are images of fibred embeddings $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$.

To get (2), we suppose that there is $\varphi \in \mathrm{Aut}(\mathrm{SL}_2)$ such that $\varphi(H_P) = H_Q$ and derive a contradiction. Corollary 5.9 yields an automorphism $\psi \in \mathrm{Aut}(\mathbb{A}_{\mathbf{k}}^3)$ such that $\psi^*(t) = \mu t$ for some $\mu \in \mathbf{k}^*$ and such that $\psi(Z_P) = Z_Q$ and $\psi(\Gamma) = \Gamma$, where $\Gamma \subset \mathbb{A}_{\mathbf{k}}^3$ is the conic given by $t = xy - 1 = 0$. The restriction of ψ to the hyperplane $H \subset \mathbb{A}_{\mathbf{k}}^3$ given by $t = 0$ then preserves Γ and also the curve $C = H \cap Z_P = H \cap Z_Q$, given by $t = x(x + 1) = 0$ (which is isomorphic to two copies of \mathbb{A}^1). The fact that C is preserved implies that $\psi|_H$ is of the form $(x, y) \mapsto (x, ay + p(x))$ or $(x, y) \mapsto (-1 - x, ay + p(x))$ for some $a \in \mathbf{k}^*$ and $p \in \mathbf{k}[x]$. The fact that Γ is preserved implies that $\psi|_H = \mathrm{id}$.

The element $\xi = \theta^{-1}\psi \in \mathrm{Aut}(\mathbb{A}_{\mathbf{k}}^3)$ then satisfies $\xi(Z_P) = Z_P$, $\xi^*(t) = \mu t$ and $\xi|_H$ is the automorphism $(x, y) \mapsto (x, x + y)$. To show that this is impossible, we use [DP09, Theorem 3.11] to see that every automorphism of Z_P preserves C and its action on C corresponds to an element of the subgroup $G = G_0 \cup G_1 \simeq G_0 \rtimes (\mathbb{Z}/2\mathbb{Z})$ of $\mathrm{Aut}(C)$ given by

$$\begin{aligned} G_0 &= \{(x, y) \mapsto (x, \alpha y + (2x + 1)\beta) \mid \alpha \in \mathbf{k}^*, \beta \in \mathbf{k}\} \\ G_1 &= \{(x, y) \mapsto (-1 - x, \alpha y + (2x + 1)\beta) \mid \alpha \in \mathbf{k}^*, \beta \in \mathbf{k}\}. \end{aligned} \quad \square$$

We then study an explicit example of a fibred embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ whose image is not equivalent to the standard embedding.

Example 5.12. According to the above study, the “simplest” example of a hyper-surface $E \subset \mathrm{SL}_2$ being the image of a fibred embedding but not being equivalent to the image of the standard embedding is given by

$$E = \{(t, u, x, y) \in \mathbb{A}_{\mathbf{k}}^4 \mid xy - tu = 1, ty = x(x - 1)\}.$$

Indeed, using the polynomial $P = ty - x(x - 1)$, which yields $P(0, x, y) = -x(x - 1)$, the surface E is the image of a fibred embedding $\rho: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ (Example 5.5) but is not equivalent to $\rho_1(\mathbb{A}_{\mathbf{k}}^2)$ (Lemma 5.6).

One can construct an explicit embedding $\rho: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ having image E in the following way. First, denoting by $E_t \subset E$ and $(\mathbb{A}_{\mathbf{k}}^2)_t \subset \mathbb{A}_{\mathbf{k}}^2 = \mathrm{Spec}(\mathbf{k}[x, t])$ the open subsets given by $t \neq 0$, we get isomorphisms

$$\begin{aligned} (\mathbb{A}_{\mathbf{k}}^2)_t &\xrightarrow{\simeq} E_t & \text{and} & \quad E_t &\xrightarrow{\simeq} (\mathbb{A}_{\mathbf{k}}^2)_t \\ (x, t) &\mapsto \begin{pmatrix} x & t \\ \frac{x^2(x-1)-t}{t^2} & \frac{x(x-1)}{t} \end{pmatrix} & & \begin{pmatrix} x & t \\ u & y \end{pmatrix} &\mapsto (x, t). \end{aligned}$$

To obtain a fibred embedding $\rho: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathrm{SL}_2$ having image equal to E , we need to remove the denominators of the isomorphism $(\mathbb{A}_{\mathbf{k}}^2)_t \xrightarrow{\simeq} E_t$. We then compose with the automorphism of $(\mathbb{A}_{\mathbf{k}}^2)_t$ given by $(x, t) \mapsto (t^2x + t + 1, t)$ and get isomorphisms

$$\begin{aligned} \mathbb{A}_{\mathbf{k}}^2 &\xrightarrow{\rho} E \\ (x, t) &\mapsto \begin{pmatrix} 1 + t + t^2x & t \\ \frac{(1+t+t^2x)^2(t+t^2x)-t}{t^2} & \frac{(1+t+t^2x)(t+t^2x)}{t} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} E &\xrightarrow{\rho^{-1}} \mathbb{A}_{\mathbf{k}}^2 \\ \begin{pmatrix} x & t \\ u & y \end{pmatrix} &\mapsto \left(\frac{x-t-1}{t^2}, t\right). \end{aligned}$$

We can observe that all components of ρ are indeed polynomials, and that $\frac{x-t-1}{t^2} \in \mathbf{k}[E]$. To show the latter, we compute $y = \frac{x(x-1)}{t} \in \mathbf{k}[E]$, $u = \frac{xy-1}{t} = \frac{x^2(x-1)-t}{t^2} \in \mathbf{k}[E]$, $y^2 - ux + u = \frac{x-1}{t} \in \mathbf{k}[E]$, which yields $\frac{x-t-1}{t^2} = u - (x+1) \left(\frac{x-1}{t}\right)^2 \in \mathbf{k}[E]$.

Writing

$$\tilde{\rho}(x, t) = A\rho(x, t)A \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \text{SL}_2,$$

we get an equivalent closed embedding $\tilde{\rho}: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \text{SL}_2$, which is an isomorphism

$$\begin{aligned} \tilde{\rho}: \quad \mathbb{A}_{\mathbf{k}}^2 &\xrightarrow{\cong} \tilde{E} \\ (x, t) &\mapsto \begin{pmatrix} 1+t^2x & t \\ x+3tx+2t^2x^2+2t^3x^2+t^4x^3 & 1+tx+2t^2x+t^3x^2 \end{pmatrix}, \end{aligned}$$

where $\tilde{E} = \{(t, u, x, y) \in \mathbb{A}_{\mathbf{k}}^4 \mid xy - tu = 1, t(y + t) = (x + t)(x + t - 1)\}$. The morphism $\tilde{\rho}$ corresponds to the closed embedding $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \mathbb{A}_{\mathbf{k}}^4$

$$\begin{aligned} \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \mathbb{A}_{\mathbf{k}}^4 \\ (x, t) &\mapsto (t, 1+t^2x, 1+tx+2t^2x+t^3x^2, x+3tx+2t^2x^2+2t^3x^2+t^4x^3) \end{aligned}$$

that we can simplify using elementary automorphisms of $\mathbb{A}_{\mathbf{k}}^4$ to the embedding

$$\begin{aligned} \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \mathbb{A}_{\mathbf{k}}^4 \\ (x, t) &\mapsto (t, t^2x, tx+t^3x^2, x+2t^2x^2-t^3x^2+t^4x^3) \\ &= (t, t^2x, tx(1+t^2x), x+t^2x^2(2-t+t^2x)). \end{aligned}$$

Question 5.13. Is the closed embedding

$$\begin{aligned} \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \mathbb{A}_{\mathbf{k}}^4 \\ (x, t) &\mapsto (t, t^2x, tx(1+t^2x), x+t^2x^2(2-t+t^2x)) \end{aligned}$$

equivalent to the standard one?

5.4. Embeddings of $\mathbb{A}_{\mathbf{k}}^2$ into SL_2 of small degree. Let $\iota: \text{SL}_2 \hookrightarrow \mathbb{A}_{\mathbf{k}}^4$ be the standard embedding and let $f: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$ be a closed embedding. This last subsection consists in showing the second part of Remark 1.1, which claims that if all coordinate functions $\iota \circ f: \mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2 \subset \mathbb{A}_{\mathbf{k}}^4$ are polynomials of degree ≤ 2 , then f is equivalent to ρ_λ for a certain $\lambda \in \mathbf{k}^*$. This will be done in Proposition 5.19 below, after a few lemmas.

We first make the following easy observation.

Lemma 5.14. *For each fibred embedding*

$$\begin{aligned} \rho: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 \\ (s, t) &\mapsto \begin{pmatrix} a(s, t) & t \\ c(s, t) & b(s, t) \end{pmatrix} \end{aligned}$$

(with $a, b, c \in \mathbf{k}[s, t]$) there is an automorphism $g \in \text{Aut}(\text{SL}_2)$ such that $g\rho$ is a fibred embedding given by

$$\begin{aligned} g\rho: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \text{SL}_2 \\ (s, t) &\mapsto \begin{pmatrix} 1+stp(s, t) & t \\ s(p(s, t)+q(s, t)+stp(s, t)q(s, t)) & 1+stq(s, t) \end{pmatrix} \end{aligned}$$

for some $p, q \in \mathbf{k}[s, t]$ such that $p(s, 0)+q(s, 0) \in \mathbf{k}^*$ and such that $\deg(1+stp(s, t)) \leq \deg(a)$, $\deg(1+stq(s, t)) \leq \deg(b)$ (where the degree is here the degree of polynomials in s, t).

Remark 5.15. The standard embedding ρ_1 is of the above form with $p = 0$ and $q = 1$. More generally, the embeddings $\{\rho_\lambda\}_{\lambda \in \mathbf{k}^*}$ of Theorem 2 are given by $p = 0$ and $q = \lambda$.

Proof. Replacing t with 0 yields two elements $a(s, 0), b(s, 0) \in \mathbf{k}[s]$ such that $a(s, 0) \cdot b(s, 0) = 1$. This implies that $a(s, 0), b(s, 0) \in \mathbf{k}^*$. Applying the automorphism

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} \mu x & t \\ u & \mu^{-1}y \end{pmatrix}$$

for some $\mu \in \mathbf{k}^*$, we can assume that $a(s, 0) = b(s, 0) = 1$. We then apply

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & t \\ u & y \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ d(t) & 1 \end{pmatrix}$$

for some $d \in \mathbf{k}[t]$ and replace $a(s, t)$ with $a(s, t) + td(t)$, so can assume that $a(0, t) = 1$. Applying similarly an automorphism of the form

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ e(t) & 1 \end{pmatrix} \cdot \begin{pmatrix} x & t \\ u & y \end{pmatrix},$$

we can assume that $b(0, t) = 1$. This yields $p, q \in \mathbf{k}[s, t]$ such that $a = 1 + stp$ and $b = 1 + stq$, which yields $c = s(p + q + stpq)$. Replacing t with 0 yields a closed embedding

$$\begin{array}{ccc} \mathbb{A}_{\mathbf{k}}^1 & \hookrightarrow & \mathrm{SL}_2 \\ s & \mapsto & \begin{pmatrix} 1 & 0 \\ s(p(s, 0) + q(s, 0)) & 1 \end{pmatrix}, \end{array}$$

whence $p(s, 0) + q(s, 0) \in \mathbf{k}^*$. □

Corollary 5.16. *Each fibred embedding*

$$\begin{array}{ccc} \rho: \mathbb{A}_{\mathbf{k}}^2 & \hookrightarrow & \mathrm{SL}_2 \\ (s, t) & \mapsto & \begin{pmatrix} a(s, t) & t \\ c(s, t) & b(s, t) \end{pmatrix} \end{array}$$

where $a, b, c \in \mathbf{k}[s, t]$ are such that $\deg a + \deg b \leq 4$ is equivalent to the embedding

$$\begin{array}{ccc} \rho_\lambda: \mathbb{A}_{\mathbf{k}}^2 & \hookrightarrow & \mathrm{SL}_2 \\ (s, t) & \mapsto & \begin{pmatrix} 1 & t \\ \lambda s & 1 + \lambda st \end{pmatrix} \end{array}$$

for some $\lambda \in \mathbf{k}^*$.

Proof. Applying Lemma 5.14, one can assume that $a = 1 + stp$, $b = 1 + stq$, $c = s(p + q + stpq)$ for some $p, q \in \mathbf{k}[s, t]$ with $p(s, 0) + q(s, 0) \in \mathbf{k}^*$. If $p = 0$, then $\rho(\mathbb{A}^2)$ is equal to $\rho_1(\mathbb{A}^2)$, so the result follows from Theorem 2(1). The same holds if $q = 0$ by applying the automorphism

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} y & t \\ u & x \end{pmatrix}.$$

To finish the proof, we assume that $pq \neq 0$ and derive a contradiction. The fact that $\deg a + \deg b \leq 4$ implies that $p, q \in \mathbf{k}^*$. Hence we have $\mathbf{k}[s, t] = \mathbf{k}[a, b, c, t] = \mathbf{k}[t, st, s(p + q + stpq)] = \mathbf{k}[t, st, s(st + \xi)]$ with $\xi = \frac{p+q}{pq} \neq 0$, and thus the morphism

$$\begin{array}{ccc} \mathbb{A}_{\mathbf{k}}^2 & \mapsto & \mathbb{A}_{\mathbf{k}}^3 \\ (s, t) & \mapsto & (t, st, s(st + \xi)) \end{array}$$

would be a closed embedding. This is false, since the image is properly contained in the irreducible hypersurface given by $\{(x, y, z) \in \mathbb{A}_{\mathbf{k}}^3 \mid xz = y(y + \xi)\}$ (the line given by $x = y + \xi = 0$ is missing). \square

It remains to generalise Corollary 5.16 to the case of embeddings $\mathbb{A}_{\mathbf{k}}^2 \hookrightarrow \text{SL}_2$ of small degree (which are fibred or not).

In the sequel we will use the following subgroups of $\text{Aut}(\mathbb{A}_{\mathbf{k}}^2)$.

Definition 5.17.

$$\begin{aligned} \text{Aff}_2(\mathbf{k}) &= \left\{ (s, t) \mapsto (as + bt + e, cs + dt + f) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbf{k}), e, f \in \mathbf{k} \right\}, \\ \text{GL}_2(\mathbf{k}) &= \left\{ (s, t) \mapsto (as + bt, cs + dt) \mid a, b, c, d \in \mathbf{k}, ad - bc \neq 0 \right\}. \end{aligned}$$

Lemma 5.18. *Let \mathbf{k} be an algebraically closed field and let $\rho: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^2 \setminus \{0\}$ be a morphism of the form*

$$(s, t) \mapsto (f(s, t), g(s, t))$$

such that f, g have degree 2 and that the homogeneous parts f_2 and g_2 of f, g of degree 2 are linearly independent. Then, there exist $\alpha \in \text{Aff}_2$ and $\beta \in \text{GL}_2$ such that

$$\beta\rho\alpha = (s, t) \mapsto (s^2, st + 1).$$

Proof. We first observe that replacing ρ with $\beta\rho\alpha$, where $\alpha \in \text{Aff}_2$ and $\beta \in \text{GL}_2$, does not change the degree of f, g or the fact that f_2 and g_2 are linearly independent. We then observe that we can assume that $f_2 = s^2$. If f_2 is a square, it suffices to replace f with $\rho\alpha$ for some $\alpha \in \text{GL}_2$. If f_2 is not a square, we choose $\xi \in \mathbf{k}$ such that $g_2 + \xi f_2$ is a square (this is possible since the discriminant of $g_2 + \xi f_2$ is a polynomial of degree 2 in ξ and \mathbf{k} is algebraically closed). We then apply an element of GL_2 at the target to replace f_2, g_2 with $g_2 + \xi f_2, f_2$, and then apply as before an element of GL_2 at the source, to obtain $f_2 = s^2$.

For each irreducible factor P of f , we denote by $C_P \subset \mathbb{A}_{\mathbf{k}}^2 = \text{Spec}(\mathbf{k}[s, t])$ the irreducible curve given by $P = 0$, and observe that g yields an invertible function on C_P .

(a) If f is a product of factors of degree 1, all belong to $\mathbf{k}[s]$, since $f_2 = s^2$. We can then write $f = \prod_{i=1}^2 (s - \lambda_i)$, for some $\lambda_i \in \mathbf{k}$. If $\lambda_1 = \lambda_2$, we replace s with $s - \lambda_1$ and get $f = s^2$, which yields $g = s(as + bt + c) + d$ for some $a, b, c \in \mathbf{k}, d \in \mathbf{k}^*$. The parts of degree 2 of f and g being linearly independent, we get $b \neq 0$. Replacing t with $\frac{t - as - c}{b}$, we replace g with $st + d$. We then apply diagonal elements of the form $(s, t) \mapsto (s, \mu t)$, $\mu \in \mathbf{k}^*$ at the source and target and replace d with 1, which yields the desired form. To finish case (a), it remains to see that $\lambda_1 \neq \lambda_2$ is impossible. To derive this contradiction, we apply an element of Aff_2 at the source and get $f = s(s - 1)$. This yields $g = sp(s, t) + \mu$, where $\mu \in \mathbf{k}^*$ and $p \in \mathbf{k}[s, t]$ is of degree 1. We moreover obtain $p(1, t) \in \mathbf{k} \setminus \{-\mu\}$, so $p(s, t) = (s - 1)\xi + \nu$ for some $\xi, \nu \in \mathbf{k}$. This yields $g \in \mathbf{k}[s]$, which is impossible since g_2 is not a multiple of $f_2 = s^2$.

(b) We can now assume that f is not a product of factors of degree 1, i.e., f is irreducible, and derive a contradiction. We observe that the curve $C_f \subset \mathbb{A}^2$ given by $f = 0$ is isomorphic to \mathbb{A}^1 . Indeed, the closure of C_f in $\mathbb{P}_{\mathbf{k}}^2$ is an irreducible and thus a smooth conic with one point at infinity since $f_2 = s^2$ (recall that \mathbf{k} is assumed to be algebraically closed). This implies that the restriction $g|_{C_f}$ is a non-zero constant and so $g = \mu + \xi f$ for some $\mu \in \mathbf{k}^*, \xi \in \mathbf{k}$. This contradicts the fact that f_2 and g_2 are linearly independent. \square

Proposition 5.19. *Each closed embedding*

$$\begin{aligned} \rho: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \mathrm{SL}_2 \\ (s, t) &\mapsto \begin{pmatrix} f_{11}(s, t) & f_{12}(s, t) \\ f_{21}(s, t) & f_{22}(s, t) \end{pmatrix} \end{aligned}$$

where $f_{11}, f_{12}, f_{21}, f_{22} \in \mathbf{k}[s, t]$ have at most degree 2 is equivalent to the embedding

$$\begin{aligned} \rho_\lambda: \quad \mathbb{A}_{\mathbf{k}}^2 &\hookrightarrow \mathrm{SL}_2 \\ (s, t) &\mapsto \begin{pmatrix} 1 & t \\ \lambda s & 1 + \lambda st \end{pmatrix} \end{aligned}$$

for some $\lambda \in \mathbf{k}^*$.

Proof. Applying Theorem 2, one only needs to show the existence of an automorphism of SL_2 that sends $\rho(\mathbb{A}_{\mathbf{k}}^2)$ onto $\rho_1(\mathbb{A}_{\mathbf{k}}^2)$. We distinguish the following cases:

(a) Suppose first that one of the polynomials f_{ij} is constant. One can assume that it is f_{11} by using permutation of coordinates (with signs). The case $f_{11} = 0$ is impossible, since the image would then be contained in

$$\left\{ \begin{pmatrix} x & t \\ u & y \end{pmatrix} \in \mathrm{SL}_2 \mid x = 0 \right\} \simeq (\mathbb{A}_{\mathbf{k}}^1 \setminus \{0\}) \times \mathbb{A}_{\mathbf{k}}^1.$$

We then have $f_{11} \neq 0$ and apply a diagonal automorphism of SL_2 to get $f_{11} = 1$, which corresponds to $\rho(\mathbb{A}_{\mathbf{k}}^2) = \rho_1(\mathbb{A}_{\mathbf{k}}^2)$.

(b) Suppose then that one of the f_{ij} has degree 1. Applying permutations one can assume that f_{12} has degree 1. Applying an element of Aff_2 at the source (see Definition 5.17), we do not change the degree of the f_{ij} 's and can assume that $f_{12} = t$. Since $\deg(f_{11}f_{22}) = \deg(f_{12}f_{21}) \leq 4$, the result follows from Corollary 5.16.

(c) It remains to study the case where $\deg(f_{ij}) = 2$ for each $i, j \in \{1, 2\}$. If the homogeneous parts of f_{11} and f_{12} of degree 2 are collinear, we apply

$$\begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} x & t \\ u & y \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & t + \mu x \\ u & y + \mu u \end{pmatrix}$$

for some $\mu \in \mathbf{k}$ and obtain $\deg(f_{12}) \leq 1$, which reduces to the cases (a), (b). To achieve the proof of (c), we now assume that the homogeneous parts of f_{11} and f_{12} of degree 2 are linearly independant and prove that this implies that $\mathbf{k}[f_{11}, f_{12}, f_{21}, f_{22}] \subsetneq \mathbf{k}[s, t]$ (which contradicts the fact that ρ is a closed embedding). To show this, one can extend the scalars and assume that \mathbf{k} is algebraically closed. We then apply Lemma 5.18 to the morphism $\nu: \mathbb{A}_{\mathbf{k}}^2 \rightarrow \mathbb{A}_{\mathbf{k}}^2 \setminus \{0\}$ given by $(s, t) \mapsto (f_{11}(s, t), f_{12}(s, t))$, and find $\alpha \in \mathrm{Aff}_2(\mathbf{k}), \beta \in \mathrm{GL}_2(\mathbf{k})$ such that $\beta\nu\alpha = (s, t) \mapsto (s^2, st + 1)$. We write $\mu = \det(\beta) \in \mathbf{k}^*$ and replace ρ with $\hat{\beta}\rho\alpha$, where $\hat{\beta} \in \mathrm{Aut}(\mathrm{SL}_2)$ is of the form

$$\hat{\beta}: \begin{pmatrix} x & t \\ u & y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix} \cdot \beta \cdot \begin{pmatrix} x & t \\ u & y \end{pmatrix}.$$

This change being made, we obtain $f_{11} = s^2, f_{12} = st + 1$. Since $1 = f_{11}f_{22} - f_{12}f_{21} = s^2f_{22} - (st + 1)f_{21}$, we obtain $f_{21} = st - 1 + g(s, t)s^2$ for some $g \in \mathbf{k}[s, t]$. This implies that $f_{11}, f_{12} - 1, f_{21} + 1$ all belong to the maximal ideal $(s, t)^2 \subset \mathbf{k}[s, t]$, which yields the desired contradiction

$$\mathbf{k}[f_{11}, f_{12}, f_{21}, f_{22}] = \mathbf{k}[f_{11}, f_{12} - 1, f_{21} + 1, f_{22}] \subsetneq \mathbf{k}[s, t].$$

□

6. A NON-TRIVIAL EMBEDDING OF \mathbb{A}^1 INTO SL_2 , OVER THE REALS

In this section, we provide over the field $\mathbf{k} = \mathbb{R}$ an explicit example of an algebraic embedding $\mathbb{A}_{\mathbb{R}}^1 \hookrightarrow SL_2$ which is not equivalent to the standard embedding

$$\begin{aligned} \tau_1: \mathbb{A}_{\mathbb{R}}^1 &\hookrightarrow SL_2 \\ t &\mapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \end{aligned}$$

Example 6.1. In [Sha92] the closed embedding

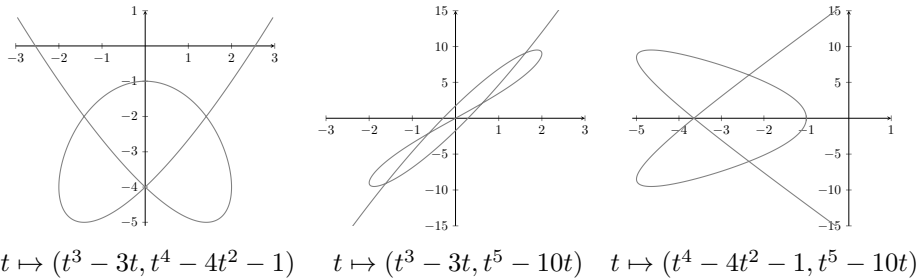
$$\begin{aligned} \gamma: \mathbb{A}^1 &\hookrightarrow \mathbb{A}^3 \\ t &\mapsto (t^3 - 3t, t^4 - 4t^2 - 1, t^5 - 10t) \end{aligned}$$

is given. This one is not equivalent to the standard embedding $\mathbb{A}^1 \hookrightarrow \mathbb{A}^3, t \mapsto (t, 0, 0)$, over the field \mathbb{R} of real numbers. The reason is that it corresponds, as an embedding $\mathbb{R} \hookrightarrow \mathbb{R}^3$, to the (open) trefoil knot.

The fact that γ is a closed embedding, over any field \mathbf{k} , can be shown as follows. Writing $\gamma_1 = t^3 - 3t, \gamma_2 = t^4 - 4t^2 - 1, \gamma_3 = t^5 - 10t \in \mathbf{k}[t]$, we get

$$t = 3\gamma_3 - 12\gamma_1 - 5\gamma_1\gamma_2 + \gamma_2\gamma_3 - \gamma_1^3.$$

The fact that $\gamma: \mathbb{R} \rightarrow \mathbb{R}^3$ corresponds to the open trefoil knot can be seen by looking at the three projections:



We now use Example 6.1 to provide a similar example in SL_2 .

Lemma 6.2.

(1) For each field \mathbf{k} of characteristic $\neq 2$, the morphism

$$\begin{aligned} \tau: \mathbb{A}^1 &\hookrightarrow SL_2 \\ t &\mapsto \begin{pmatrix} t^3 - 3t & t^4 - 4t^2 - 1 \\ 1 + \frac{t^2(17t^6 - 56t^4 - 137t^2 + 452)}{16} & \frac{t(17t^8 - 73t^6 - 149t^4 + 609t^2 + 172)}{16} \end{pmatrix} \end{aligned}$$

is a closed embedding.

(2) If $\mathbf{k} = \mathbb{R}$, then τ is not equivalent to the standard embedding, because the fundamental group $\pi_1(SL_2(\mathbb{R}) \setminus \tau(\mathbb{R}))$ is not isomorphic to the free group $\pi_1(SL_2(\mathbb{R}) \setminus \tau_1(\mathbb{R}))$.

Proof. (1): The fact that τ is a closed embedding can be done explicitly by giving a formula for t , but can also be shown by using the \mathbb{A}^1 -bundle

$$\begin{aligned} p: SL_2 &\rightarrow \mathbb{A}^2 \setminus \{(0, 0)\} \\ \begin{pmatrix} x & t \\ u & y \end{pmatrix} &\mapsto (x, t). \end{aligned}$$

Writing $\gamma_1 = t^3 - 3t, \gamma_2 = t^4 - 4t^2 - 1, \gamma_3 = t^5 - 10t \in \mathbf{k}[t]$ as in Example 6.1, we get $\gamma_1^2(\gamma_1^2 - 4) - \gamma_2(\gamma_2^2 + 9\gamma_2 + 24) = 16$ and thus get a birational morphism

$$\begin{aligned} \mathbb{A}^1 &\rightarrow \Gamma = \{(x, t) \in \mathbb{A}^2 \mid x^2(x^2 - 4) - t(t^2 + 9t + 24) = 16\} \\ t &\mapsto (\gamma_1(t), \gamma_2(t)) \end{aligned}$$

from \mathbb{A}^1 to the singular affine quartic curve $\Gamma \subset \mathbb{A}^2$. We then get a morphism

$$\begin{aligned} f: \Gamma &\rightarrow \mathrm{SL}_2 \\ (x, t) &\rightarrow \begin{pmatrix} x & t \\ \frac{t^2+9t+24}{16} & \frac{x(x^2-4)}{16} \end{pmatrix} \end{aligned}$$

which satisfies $p \circ f = \mathrm{id}_\Gamma$ and is thus a section of p over Γ . This implies that

$$\begin{aligned} \Gamma \times \mathbb{A}^1 &\hookrightarrow \mathrm{SL}_2 \\ ((x, t), a) &\mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} f(x, t) \end{aligned}$$

is a closed embedding. Since $\gamma = (\gamma_1, \gamma_2, \gamma_3): \mathbb{A}^1 \rightarrow \Gamma \times \mathbb{A}^1 \subset \mathbb{A}^3$ is a closed embedding, the morphism

$$\begin{aligned} \tau: \mathbb{A}^1 &\hookrightarrow \mathrm{SL}_2 \\ t &\mapsto \begin{pmatrix} 1 & 0 \\ t^5 - 10t & 1 \end{pmatrix} f(\gamma_1(t), \gamma_2(t)) \end{aligned}$$

is a closed embedding. Replacing γ_1 and γ_2 in the above formula yields the explicit form of the morphism given in the statement of the lemma.

(2): In the remaining part of the proof, we work over $\mathbf{k} = \mathbb{R}$ and use the Euclidean topology. The \mathbb{R} -bundle $p: \mathrm{SL}_2(\mathbb{R}) \rightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ is trivial, as it admits a (rational) continuous section given by

$$\begin{aligned} \xi: \mathbb{R}^2 \setminus \{(0, 0)\} &\rightarrow \mathrm{SL}_2(\mathbb{R}) \\ (x, t) &\mapsto \begin{pmatrix} x & t \\ -\frac{t}{x^2+t^2} & \frac{x}{x^2+t^2} \end{pmatrix}. \end{aligned}$$

This yields a birational diffeomorphism

$$\begin{aligned} \varphi: \mathbb{R}^2 \setminus \{(0, 0)\} \times \mathbb{R} &\rightarrow \mathrm{SL}_2(\mathbb{R}) \\ ((x, t), a) &\mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \xi(x, t). \end{aligned}$$

In particular, $\mathrm{SL}_2(\mathbb{R}) \setminus \tau_1(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2 \setminus \{(0, 0), (0, 1)\} \times \mathbb{R}$, which implies that the fundamental group $\pi_1(\mathrm{SL}_2(\mathbb{R}) \setminus \tau_1(\mathbb{R}))$ is a free group (over two generators). It remains to show that $\pi_1(\mathrm{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R}))$ is not a free group. This will imply that no diffeomorphism of $\mathrm{SL}_2(\mathbb{R})$ sends $\tau(\mathbb{R})$ onto $\tau_1(\mathbb{R})$, and in particular no algebraic automorphism defined over \mathbb{R} .

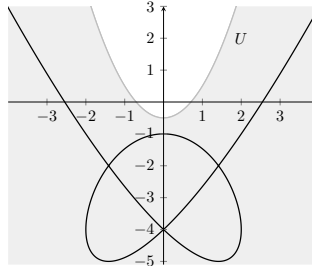
We extend $f: \Gamma(\mathbb{R}) \rightarrow \mathrm{SL}_2(\mathbb{R})$ to a global continuous section $\hat{f}: \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathrm{SL}_2(\mathbb{R})$ of p (which exists, since p is a trivial \mathbb{R} -bundle). This yields a rational diffeomorphism

$$\begin{aligned} g: \mathbb{R}^2 \setminus \{(0, 0)\} \times \mathbb{R} &\xrightarrow{\cong} \mathrm{SL}_2(\mathbb{R}) \\ ((x, t), a) &\mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \hat{f}(x, t) \end{aligned}$$

which maps $\gamma(\mathbb{R})$ onto $\tau(\mathbb{R})$.

We take an open subset $U \subset \mathbb{R}^2 \setminus \{(0, 0)\}$ (for the Euclidean topology) that contains the singular curve $\Gamma(\mathbb{R})$ and a homeomorphism $h: U \xrightarrow{\cong} \mathbb{R}^2$ which fixes

$\Gamma(\mathbb{R})$ pointwise, and which is homotopic to the inclusion $U \hookrightarrow \mathbb{R}^2$, via a homotopy that fixes $\Gamma(\mathbb{R})$ pointwise. We can for instance take $U = \{(x, t) \in \mathbb{R}^2 \mid t \leq x^2 - \frac{1}{2}\}$ and construct a homeomorphism and a homotopy which preserve the fibres of the projection $(x, t) \mapsto x$ as follows:



The homeomorphisms

$$(\star) \quad p^{-1}(U) \setminus \tau(\mathbb{R}) \xrightarrow{g^{-1}} (U \times \mathbb{R}) \setminus \gamma(\mathbb{R}) \xrightarrow{h \times \text{id}} \mathbb{R}^3 \setminus \gamma(\mathbb{R})$$

yield isomorphisms of the fundamental groups

$$\pi_1(p^{-1}(U) \setminus \tau(\mathbb{R})) \xrightarrow{\cong} \pi_1((U \times \mathbb{R}) \setminus \gamma(\mathbb{R})) \xrightarrow{\cong} \pi_1(\mathbb{R}^3 \setminus \gamma(\mathbb{R})).$$

Since $\gamma: \mathbb{R} \hookrightarrow \mathbb{R}^3$ is the (open) trefoil knot, it follows that $\pi_1(\mathbb{R}^3 \setminus \gamma(\mathbb{R}))$ is the braid group with three strands and thus $\pi_1(p^{-1}(U) \setminus \tau(\mathbb{R}))$ is not a free group. It remains then to see that the group homomorphism

$$\iota: \pi_1(p^{-1}(U) \setminus \tau(\mathbb{R})) \rightarrow \pi_1(\text{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R}))$$

induced by the inclusion $p^{-1}(U) \setminus \tau(\mathbb{R}) \hookrightarrow \text{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R})$ is injective (as a subgroup of a free group is free). Every element $\alpha \in \text{Ker}(\iota)$ lies in the kernel of the map $\iota': \pi_1(p^{-1}(U) \setminus \tau(\mathbb{R})) \rightarrow \pi_1(\mathbb{R}^3 \setminus \gamma(\mathbb{R}))$ induced by the composition

$$p^{-1}(U) \setminus \tau(\mathbb{R}) \hookrightarrow \text{SL}_2(\mathbb{R}) \setminus \tau(\mathbb{R}) \xrightarrow{g^{-1}} ((\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}) \setminus \gamma(\mathbb{R}) \hookrightarrow \mathbb{R}^3 \setminus \gamma(\mathbb{R}),$$

which corresponds simply to the composition

$$(\star\star) \quad p^{-1}(U) \setminus \tau(\mathbb{R}) \xrightarrow{g^{-1}} (U \times \mathbb{R}) \setminus \gamma(\mathbb{R}) \hookrightarrow \mathbb{R}^3 \setminus \gamma(\mathbb{R}).$$

Since $h: U \rightarrow \mathbb{R}^2$ is homotopic to the inclusion $U \hookrightarrow \mathbb{R}^2$ via a homotopy that fixes $\Gamma(\mathbb{R})$ pointwise, the two compositions $p^{-1}(U) \setminus \tau(\mathbb{R}) \rightarrow \mathbb{R}^3 \setminus \gamma(\mathbb{R})$ of (\star) and $(\star\star)$ are homotopic, so ι' is an isomorphism. This implies that ι is injective and achieves the proof. □

Question 6.3. Working over the field of complex numbers \mathbb{C} , is the algebraic embedding $\tau: \mathbb{A}^1 \rightarrow \text{SL}_2$ of Lemma 6.2(1) equivalent to the standard embedding $\tau_1: \mathbb{A}^1 \hookrightarrow \text{SL}_2$?

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REFERENCES

- [ADHL15] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. MR3307753
- [AHE72] Shreeram S. Abhyankar, William Heinzer, and Paul Eakin, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra **23** (1972), 310–342, DOI 10.1016/0021-8693(72)90134-2. MR0306173
- [AM75] Shreeram S. Abhyankar and Tzuong Tsieng Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148–166. MR0379502
- [Asa87] Teruo Asanuma, *Polynomial fibre rings of algebras over Noetherian rings*, Invent. Math. **87** (1987), no. 1, 101–127, DOI 10.1007/BF01389155. MR862714
- [AZ13] Ivan Arzhantsev and Mikhail Zaidenberg, *Acyclic curves and group actions on affine toric surfaces*, Affine algebraic geometry, World Sci. Publ., Hackensack, NJ, 2013, pp. 1–41, DOI 10.1142/9789814436700_0001. MR3089030
- [BCW77] H. Bass, E. H. Connell, and D. L. Wright, *Locally polynomial algebras are symmetric algebras*, Invent. Math. **38** (1976/77), no. 3, 279–299, DOI 10.1007/BF01403135. MR0432626
- [BD11] Jérémy Blanc and Adrien Dubouloz, *Automorphisms of \mathbb{A}^1 -fibered affine surfaces*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5887–5924, DOI 10.1090/S0002-9947-2011-05266-9. MR2817414
- [Ber83] José Bertin, *Pinceaux de droites et automorphismes des surfaces affines* (French), J. Reine Angew. Math. **341** (1983), 32–53, DOI 10.1515/crll.1983.341.32. MR697306
- [BFH16] Jérémy Blanc, Jean-Philippe Furter, and Mattias Hemmig, *Exceptional isomorphisms between complements of affine plane curves*, Duke Mathematical Journal, to appear. arXiv:1609.06682.
- [BFL14] Cinzia Bisi, Jean-Philippe Furter, and Stéphane Lamy, *The tame automorphism group of an affine quadric threefold acting on a square complex* (English, with English and French summaries), J. Éc. polytech. Math. **1** (2014), 161–223, DOI 10.5802/jep.8. MR3322787
- [BS15] Jérémy Blanc and Immanuel Stampfli, *Automorphisms of the plane preserving a curve*, Algebr. Geom. **2** (2015), no. 2, 193–213, DOI 10.14231/AG-2015-009. MR3350156
- [DG77] M. H. Gizatullin and V. I. Danilov, *Automorphisms of affine surfaces. II* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 1, 54–103, 231. MR0437545
- [DD16] Julie Decaup and Adrien Dubouloz, *Affine lines in the complement of a smooth plane conic*, Boll. Unione Mat. Ital. **11** (2018), no. 1, 39–54, DOI 10.1007/s40574-017-0119-z. MR3782690
- [DK09] Daniel Daigle and Shulim Kaliman, *A note on locally nilpotent derivations and variables of $k[X, Y, Z]$* , Canad. Math. Bull. **52** (2009), no. 4, 535–543, DOI 10.4153/CMB-2009-054-5. MR2567148
- [DP09] Adrien Dubouloz and Pierre-Marie Poloni, *On a class of Danielewski surfaces in affine 3-space*, J. Algebra **321** (2009), no. 7, 1797–1812, DOI 10.1016/j.jalgebra.2008.12.009. MR2494748
- [Fur02] Jean-Philippe Furter, *On the length of polynomial automorphisms of the affine plane*, Math. Ann. **322** (2002), no. 2, 401–411, DOI 10.1007/s002080100276. MR1893923
- [Gan11] Richard Ganong, *The pencil of translates of a line in the plane*, Affine algebraic geometry, CRM Proc. Lecture Notes, vol. 54, Amer. Math. Soc., Providence, RI, 2011, pp. 57–71. MR2768634
- [Giz71] M. H. Gizatullin, *Quasihomogeneous affine surfaces* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1047–1071. MR0286791
- [Jun42] Heinrich W. E. Jung, *Über ganze birationale Transformationen der Ebene* (German), J. Reine Angew. Math. **184** (1942), 161–174, DOI 10.1515/crll.1942.184.161. MR0008915
- [Kal91] Shulim Kaliman, *Extensions of isomorphisms between affine algebraic subvarieties of k^n to automorphisms of k^n* , Proc. Amer. Math. Soc. **113** (1991), no. 2, 325–334, DOI 10.2307/2048516. MR1076575
- [Kal92] Shulim Kaliman, *Isotopic embeddings of affine algebraic varieties into \mathbb{C}^n* , The Madison Symposium on Complex Analysis (Madison, WI, 1991), Contemp.

- Math., vol. 137, Amer. Math. Soc., Providence, RI, 1992, pp. 291–295, DOI 10.1090/conm/137/1190990. MR1190990
- [Kal02] Sh. Kaliman, *Polynomials with general \mathbf{C}^2 -fibers are variables*, Pacific J. Math. **203** (2002), no. 1, 161–190, DOI 10.2140/pjm.2002.203.161. MR1895930
- [Kra96] Hanspeter Kraft, *Challenging problems on affine n -space*, Astérisque **237** (1996), Exp. No. 802, 5, 295–317. Séminaire Bourbaki, Vol. 1994/95. MR1423629
- [KZ99] Sh. Kaliman and M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups **4** (1999), no. 1, 53–95, DOI 10.1007/BF01236662. MR1669174
- [KZ01] Shulim Kaliman and Mikhail Zaidenberg, *Families of affine planes: the existence of a cylinder*, Michigan Math. J. **49** (2001), no. 2, 353–367, DOI 10.1307/mmj/1008719778. MR1852308
- [LV13] Stéphane Lamy and Stéphane Vénéreau, *The tame and the wild automorphisms of an affine quadric threefold*, J. Math. Soc. Japan **65** (2013), no. 1, 299–320, DOI 10.2969/jmsj/06510299. MR3034406
- [Mar15] Alexandre Martin, *On the acylindrical hyperbolicity of the tame automorphism group of $\mathrm{SL}_2(\mathbf{C})$* , Bull. Lond. Math. Soc. **49** (2017), no. 5, 881–894, DOI 10.1112/blms.12071. MR3742454
- [Mat89] Hideyuki Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461
- [Pop74] V. L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector fiberings* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 294–322. MR0357399
- [Rus70] Peter Russell, *Forms of the affine line and its additive group*, Pacific J. Math. **32** (1970), 527–539. MR0265367
- [Ser68] Jean-Pierre Serre, *Corps locaux* (French), Hermann, Paris, 1968. Deuxième édition; Publications de l'Université de Nancago, No. VIII. MR0354618
- [Sha92] Anant R. Shastri, *Polynomial representations of knots*, Tohoku Math. J. (2) **44** (1992), no. 1, 11–17, DOI 10.2748/tmj/1178227371. MR1145717
- [Sri91] V. Srinivas, *On the embedding dimension of an affine variety*, Math. Ann. **289** (1991), no. 1, 125–132, DOI 10.1007/BF01446563. MR1087241
- [Sta15] Immanuel Stampfli, *Algebraic embeddings of \mathbf{C} into $\mathrm{SL}_n(\mathbf{C})$* , Transform. Groups **22** (2017), no. 2, 525–535, DOI 10.1007/s00031-015-9358-1. MR3649466
- [Suz74] Masakazu Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace \mathbf{C}^2* (French), J. Math. Soc. Japan **26** (1974), 241–257, DOI 10.2969/jmsj/02620241. MR0338423
- [vdE00] Arno van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, vol. 190, Birkhäuser Verlag, Basel, 2000. MR1790619
- [vdK53] W. van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wiskunde (3) **1** (1953), 33–41. MR0054574

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