Dynamical degrees of affine-triangular automorphisms of affine spaces

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Abstract. We study the possible dynamical degrees of automorphisms of the affine space $\mathbb{A}^n$. In dimension $n = 3$, we determine all dynamical degrees arising from the composition of an affine automorphism with a triangular one. This generalizes the easier case of shift-like automorphisms which can be studied in any dimension. We also prove that each weak Perron number is the dynamical degree of an affine-triangular automorphism of the affine space $\mathbb{A}^n$ for some n, and we give the best possible n for quadratic integers, which is either 3 or 4.

Contents

1. *Introduction*

1.1. *Dynamical degrees of polynomial endomorphisms.*   In this text, we work over an arbitrary field **k**. For each $n \geq 1$, recall that an endomorphism $f \in \text{End}(\mathbb{A}^n)$ of $\mathbb{A}^n = \mathbb{A}^n_{\mathbf{k}}$ is given by

$$f : (x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)),$$

where $f_1, \ldots, f_n \in \mathbf{k}[x_1, \ldots, x_n]$. To simplify the notation, we often write $f = (f_1, \ldots, f_n)$ and thus identify $\mathrm{End}(\mathbb{A}^n)$ with $(\mathbf{k}[x_1, \ldots, x_n])^n$.

The degree of an endomorphism $f = (f_1, \ldots, f_n)$, denoted by $\deg(f)$, is defined to be $\deg(f) = \max(\deg(f_1), \ldots, \deg(f_n))$. The set $\mathrm{End}(\mathbb{A}^n)$ of endomorphisms of $\mathbb{A}^n$ is a monoid, for the composition law, and the subset of invertible elements is the group $\mathrm{Aut}(\mathbb{A}^n)$ of automorphisms of $\mathbb{A}^n$.

The dynamics of endomorphisms of $\mathbb{A}^n$, specially in the case of the ground field $\mathbf{k} = \mathbb{C}$, was studied intensively in the last decades, see for instance [**FsW98**, **Sib99**, **Mae00**, **BFs00**, **Mae01a**, **Mae01b**, **Gue02**, **GS02**, **Gue04**, **Ued04**, **FJ11**, **JW12**, **Xie17**, **DL18**]. For each dominant endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$, the (*first*) *dynamical degree* is defined as the real number

$$\lambda(f) = \lim_{r \to \infty} \deg(f^r)^{(1/r)} \in \mathbb{R}_{\geq 1}$$

(the limit exists by Fekete's subadditivity lemma, see Lemma 2.2.1). If $f \in \mathrm{End}(\mathbb{A}^1)$ or $f \in \mathrm{Aut}(\mathbb{A}^2)$, then $\lambda(f)$ is an integer, but in higher dimensions, it can be quite complicated to understand the possible dynamical degrees. In [**DF**, Corollary 3], the authors conjecture that $\lambda(f)$ is an algebraic integer of degree $\leq n$, and of degree $\leq n-1$ if $f \in \mathrm{Aut}(\mathbb{A}^n)$, a conjecture proven until now only for $n \leq 2$.

In this article, we study some particular family of automorphisms of $\mathbb{A}^n$ that we call *affine-triangular*. These are compositions consisting of one affine automorphism and one triangular automorphism (see Definition 2.1.1). Our two main results are Theorem 1 and Theorem 2.

THEOREM 1. *For each field $\mathbf{k}$ and each integer $d \geq 2$, the set of dynamical degrees of all affine-triangular automorphisms of $\mathbb{A}^3$ of degree $\leq d$ is equal to*

$$\left\{ \frac{a + \sqrt{a^2 + 4bc}}{2} \mid (a, b, c) \in \mathbb{N}^3, a + b \leq d, c \leq d \right\} \setminus \{0\}.$$

Moreover, for $a, b, c \in \mathbb{N}$ such that $\lambda = ((a + \sqrt{a^2 + 4bc})/2) \neq 0$, the dynamical degree λ is achieved by either of the automorphisms

$$(x_3 + x_1^a x_2^b, x_2 + x_1^c, x_1) \quad or \quad (x_3 + x_1^a x_2^{bc}, x_1, x_2).$$

Using Theorem 1, we prove in [**BvS**, Theorem 2] that the set of dynamical degrees of all automorphisms of degree 3 of $\mathbb{A}^3$ is equal to

$$\left\{ 1, \sqrt{2}, \frac{1 + \sqrt{5}}{2}, \sqrt{3}, 2, \frac{1 + \sqrt{13}}{2}, 1 + \sqrt{2}, \sqrt{6}, \frac{1 + \sqrt{17}}{2}, \frac{3 + \sqrt{5}}{2}, 1 + \sqrt{3}, 3 \right\}.$$

Note that $(3 + \sqrt{5})/2$ is the only number that does not belong to the list in Theorem 1 and so it is the dynamical degree of an automorphism of degree 3 of $\mathbb{A}^3$ that is not conjugate to an affine-triangular automorphism of any degree.

For the next theorem, we recall the definition of (weak) Perron numbers (see Theorem 3.2.4 for some equivalent characterizations).

Definition 1.1.1. A *Perron number* (respectively *weak Perron number*) is a real number $\lambda \geq 1$ that is an algebraic integer such that all other Galois conjugates $\mu \in \mathbb{C}$ satisfy $|\mu| < \lambda$ (respectively $|\mu| \leq \lambda$).

THEOREM 2. *Each weak Perron number λ is the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^n$ for some integer n. Moreover:*

(1) *if $\lambda > 1$ is an integer, the least n possible is 2;*
(2) *if λ is a quadratic integer and its conjugate is negative, the least possible n is 3;*
(3) *if λ is a quadratic integer and its conjugate is positive, the least possible n is 4.*

Note that statement (1) in Theorem 2 is well known, as $\{\lambda(f) \mid f \in \mathrm{Aut}(\mathbb{A}^2)\} = \mathbb{Z}_{\geq 1}$. We include it to emphasize the relation between the degree of the weak Perron numbers and the possible n. In view of the above theorems and of the techniques developed in this text, it is natural to ask the following.

Question 1.1.2. Is every dynamical degree of any element of $\mathrm{End}(\mathbb{A}^n)$ (respectively $\mathrm{Aut}(\mathbb{A}^n)$) equal to a weak Perron number of degree $\leq n$ (respectively of degree $\leq n - 1$)?

As already mentioned above, a positive answer to this question, where 'weak Perron number' is replaced by 'algebraic integer', was conjectured in the recent preprint [**DF**, Corollary 3] (that appeared after we asked the above question in a first version of this text). In [**DF**], it is also proven that the dynamical degree of any element in $\mathrm{Aut}(\mathbb{A}^3)$ is an algebraic number of degree at most six. More generally, they prove that the dynamical degree of any element of $\mathrm{End}(\mathbb{A}^n)$ is an algebraic number of degree at most n in case the square of the first dynamical degree is bigger than the second dynamical degree of f [**DF**, Theorem 2].

In particular, Theorem 1 shows that the dynamical degree of every affine-triangular automorphism of $\mathbb{A}^3$ is equal to the dynamical degree of a shift-like automorphism. However, for each $d \geq 3$, the set of dynamical degrees of all affine-triangular automorphisms of $\mathbb{A}^3$ of degree d strictly contains the set of dynamical degrees of all shift-like automorphisms of $\mathbb{A}^3$ of degree d. Indeed, the latter set of dynamical degrees consists of the numbers $(a + \sqrt{a^2 + 4d - 4a})/2$, where $0 \leq a \leq d$, and does not contain $(1 + \sqrt{1 + 4d})/2$, which is the dynamical degree of the affine-triangular automorphism $(x_3 + x_1 x_2, x_2 + x_1^d, x_1)$, see Corollary 4.3.7.

Note that $2\sqrt{2}$ and $\sqrt{3}$ appear as dynamical degrees of affine-triangular automorphisms in degree 4 and 3, respectively (and not smaller), even if $2\sqrt{2} < 3$ and $\sqrt{3} < 2$. Similarly, for each prime p, the number $\sqrt{p}$ is the dynamical degree of a shift-like automorphism of degree p, but it is not the dynamical degree of an affine-triangular automorphism of degree $< p$.

1.2. *Dynamical degrees of affine-triangular automorphisms in higher dimensions.* In dimension $n \geq 4$, we are not able to compute all dynamical degrees of all affine-triangular automorphisms, but can get some large families. The case of shift-like automorphisms is covered by our method, and we retrieve a proof of the result of Mattias Jonsson (Proposition 4.2.5), but we can also study wider classes. We give the dynamical degrees

TABLE 1. Dynamical degrees of low degree shift-like and affine-triangular automorphisms.

d	Dynamical degrees of shift-like automorphisms of $\mathbb{A}^3$ of degree d not appearing in degree $< d$	Dynamical degrees of affine-triangular automorphisms of $\mathbb{A}^3$ of degree d not appearing in degree $< d$
1	$\{1\}$	$\{1\}$
2	$\left\{\sqrt{2}, \dfrac{1+\sqrt{5}}{2}, 2\right\}$	$\left\{\sqrt{2}, \dfrac{1+\sqrt{5}}{2}, 2\right\}$
3	$\{\sqrt{3}, 1+\sqrt{2}, 3\}$	$\left\{\sqrt{3}, \dfrac{1+\sqrt{13}}{2}, 1+\sqrt{2}, \sqrt{6}, \dfrac{1+\sqrt{17}}{2}, 1+\sqrt{3}, 3\right\}$
4	$\left\{\dfrac{1+\sqrt{13}}{2}, 1+\sqrt{3}, \dfrac{3+\sqrt{13}}{2}, 4\right\}$	$\left\{2\sqrt{2}, 1+\sqrt{5}, \dfrac{3+\sqrt{13}}{2}, \dfrac{1+\sqrt{33}}{2}, 2\sqrt{3}, \dfrac{1+\sqrt{37}}{2}, \dfrac{3+\sqrt{17}}{2}, 1+\sqrt{7}, \dfrac{3+\sqrt{21}}{2}, 4\right\}$

of all permutation-elementary automorphisms (a family that strictly includes the shift-like automorphisms) in §4.2 (especially Proposition 4.2.3) and also give the dynamical degrees of other affine-triangular automorphisms. In particular, we show that in any dimension $n \geq 4$, there are affine-triangular automorphisms of $\mathbb{A}^n$ whose dynamical degrees are not those of a shift-like automorphism or, more generally, of a permutation-elementary automorphism, contrary to the case of dimension $n \leq 3$. The reason is that dynamical degrees of shift-like automorphisms are special kinds of weak Perron numbers. Indeed, they are positive real numbers that are roots of a monic integral polynomial where all coefficients (except the first one) are non-positive. These numbers are called *Handelman numbers* in [**Bas97**] (see especially [**Bas97**, Lemma 10]) and they have no other positive real Galois conjugates (Lemma 3.2.7). This implies that Handelman numbers are weak Perron numbers (see Corollary 3.2.8). Theorem 1 implies that the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ is a Handelman number (and the same holds for all automorphisms of $\mathbb{A}^1$ and $\mathbb{A}^2$), but for any $n \geq 4$, there are affine-triangular automorphisms of $\mathbb{A}^n$ whose dynamical degrees are not Handelman numbers. This follows in particular from Theorem 2, applied to any weak Perron quadratic integer with a positive conjugate, for instance to $(3+\sqrt{5})/2$. We can also apply Theorem 2 to weak Perron numbers of arbitrary large degree.

1.3. *Results in the literature on dynamical degrees of endomorphisms of $\mathbb{A}^n$.* Let us recall what is known on the dynamical degrees of elements of $\mathrm{End}(\mathbb{A}^n)$.

(1)   The case where $n = 1$ is obvious: in this case, we have $\lambda(f) = \deg(f)$, so each dynamical degree is an integer, which is moreover equal to 1 in the case of automorphisms.

(2)   When $n = 2$, the case of automorphisms follows from the Jung–van der Kulk theorem [**Jun42**, **vdK53**]: every dynamical degree is an integer, as $\deg(f^r) = \deg(f)^r$ for each r, when f is taken to be cyclically reduced (this is explained in Corollary 2.4.3 or in [**Fur99**, Proposition 3]). The set of all dynamical degrees of quadratic endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$ is equal to $\{1, \sqrt{2}, (1+\sqrt{5})/2, 2\}$ by [**Gue04**,

Theorem 2.1]. Moreover, the dynamical degree of every element of $\text{End}(\mathbb{A}^2_{\mathbb{C}})$ is a quadratic integer, by [**FJ07**, Theorem A$'$].

(3) The case of dimension $n \geq 3$ is open in general: there is, for the moment, no hope of classifying all dynamical degrees, even when studying only automorphisms.

The set of dynamical degrees of all automorphisms of $\mathbb{A}^3_{\mathbb{C}}$ of degree 2 is equal to $\{1, \sqrt{2}, (1 + \sqrt{5})/2, 2\}$ by [**Mae01a**, Theorem 3.1] (and the same holds over any field [**BvS**, Theorem 2]).

Apart from the above classification results, two natural families are also known: the *monomial endomorphisms* and the *shift-like automorphisms*.

(A) A *monomial endomorphism* of $\mathbb{A}^n$ is an endomorphism of the form $f = (f_1, \ldots, f_n)$, where each f_i is a monomial. When we write $f_i = \alpha_i x_1^{m_{i,1}} \cdots x_n^{m_{i,n}}$ with $\alpha_i \in \mathbf{k}^*$ and $m_{i,1}, \ldots, m_{i,n} \in \mathbb{N}$, and assume that f is dominant, then the dynamical degree of f is the spectral radius of the corresponding matrix $M = (m_{i,j})_{i,j=1}^n \in \text{Mat}_n(\mathbb{N})$. This classical result is proven again in Corollary 3.2.5. The numbers arising this way are the weak Perron numbers (see Theorem 3.2.4).

(B) For each $n \geq 1$, a *shift-like automorphism* of $\mathbb{A}^{n+1}$ is an automorphism of the form $(x_{n+1} + p(x_1, \ldots, x_n), x_1, \ldots, x_n)$ for some polynomial $p \in \mathbf{k}[x_1, \ldots, x_n]$. These are particular examples of affine-triangular automorphisms. The dynamics of such automorphisms have been studied in various texts (see for instance [**BP98**, **Mae00**, **Mae01b**, **Ued04**, **BV18**]). The dynamical degrees of shift-like automorphisms are known, by a result of Mattias Jonsson (see Proposition 4.2.5). For a proof of this result, together with a generalization, see §4.2.

1.4. *Description of the techniques associated to degrees.* In the rest of this introduction, we describe the main technique that we introduce to compute dynamical degrees of endomorphisms of $\mathbb{A}^n$. This is related to degree functions (or monomial valuations) and may be applied to endomorphisms of $\mathbb{A}^n$, not only affine-triangular automorphisms. We also give an outline of the whole article.

Definition 1.4.1. For each $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$, we define a *degree function* $\deg_\mu : \mathbf{k}[x_1, \ldots, x_n] \to \mathbb{R}_{\geq 0} \cup \{-\infty\}$ by $\deg_\mu(0) = -\infty$ and

$$\deg_\mu \left(\sum_{(a_1,\ldots,a_n) \in \mathbb{N}^n} \underbrace{c_{(a_1,\ldots,a_n)}}_{\in \mathbf{k}} \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \right) = \max \left\{ \sum_{i=1}^n a_i \mu_i \,\middle|\, c_{(a_1,\ldots,a_n)} \neq 0 \right\}.$$

We say that a polynomial $p \in \mathbf{k}[x_1, \ldots, x_n]$ is *μ-homogeneous of degree $\theta \in \mathbb{R}$* if p is a finite sum of monomials p_i with $\deg_\mu(p_i) = \theta$ for each i (where the zero polynomial is μ-homogeneous of degree θ for each θ).

We can then write every element $q \in \mathbf{k}[x_1, \ldots, x_n] \setminus \{0\}$ uniquely as

$$q = \sum_{\theta \in \mathbb{R}_{\geq 0}} q_\theta,$$

where each $q_\theta \in \mathbf{k}[x_1, \ldots, x_n]$ is μ-homogeneous of degree θ (and only finitely many q_θ are non-zero). We then say that q_θ is the μ-*homogeneous part of q of degree θ*. The μ-*leading part* of q is the μ-homogeneous part of q of degree $\deg_\mu(q)$.

Remark 1.4.2. Note that if $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$, then

$$\mathbf{k}(x_1, \ldots, x_n) \to \mathbb{R} \cup \{\infty\}, \quad f/g \mapsto \deg_\mu(g) - \deg_\mu(f)$$

is a valuation in the sense of [**Mat89**, p. 75], where $\mathbf{k}(x_1, \ldots, x_n)$ denotes the field of rational functions in $x_1, \ldots, x_n$ over $\mathbf{k}$. Such valuations are often called 'monomial valuations' in the literature.

Definition 1.4.3. Let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. For each $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n) \setminus \{0\}$, we denote the μ-*degree of f* by

$$\deg_\mu(f) = \inf\{\theta \in \mathbb{R}_{\geq 0} \mid \deg_\mu(f_i) \leq \theta \mu_i \text{ for each } i \in \{1, \ldots, n\}\}$$

and we say that $\deg_\mu(f) = \infty$ if the above set is empty.

   Moreover, we say that f is μ-*algebraically stable* if $\deg_\mu(f) < \infty$ and $\deg_\mu(f^r) = \deg_\mu(f)^r$ for each $r \geq 1$.

Remark 1.4.4. If $\mu = (1, \ldots, 1)$, then $\deg_\mu(f) = \deg(f)$ is the standard degree and the notion of being μ-algebraically stable is the standard notion of 'algebraically stable', studied for instance in [**GS02**, **Bis08**, **Bla16**]. The fact of being algebraically stable can be interpreted geometrically by looking at the behaviour of the endomorphism at infinity: [**Bla16**, Corollary 2.16].

   To compute the dynamical degree of an endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$, the following endomorphism associated to f will be of great importance for us.

Definition 1.4.5. Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism, let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ be such that $\deg_\mu(f) = \theta < \infty$. We define the μ-*leading part of f* to be the endomorphism $g = (g_1, \ldots, g_n) \in \mathrm{End}(\mathbb{A}^n)$, where $g_j \in \mathbf{k}[x_1, \ldots, x_n]$ is the μ-homogeneous part of f_j of degree $\theta \mu_j$ for each $j \in \{1, \ldots, n\}$.

   The degree functions are studied in §2. Basic properties are given in §2.3 and the relation with μ-homogeneous endomorphisms is given in §2.5 (we explain in particular when $\deg_\mu(f) = \infty$ in Lemma 2.5.6). In §2.6, we explain how degree functions allow us to give an estimate on the dynamical degrees, and sometimes to compute it exactly. In particular, we prove the following result (at the end of §2.6).

PROPOSITION A. *Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. For each $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{>0})^n$, the following hold:*
(1) $\theta := \deg_\mu(f) < \infty$;
(2)   *the dynamical degree of f satisfies $1 \leq \lambda(f) \leq \theta$;*
(3)   *let $g \in \mathrm{End}(\mathbb{A}^n)$ be the μ-leading part of f. If $\theta > 1$, then*

$$\lambda(f) = \theta \Leftrightarrow f \text{ is } \mu\text{-algebraically stable} \Leftrightarrow g^r \neq 0 \text{ for each } r \geq 1.$$

Remark 1.4.6. Let $\mu = (1, \ldots, 1)$. In this case, the μ-degree is the classical degree and Proposition A(2) is the classical inequality $\lambda(f) \leq \deg(f)$.

Remark 1.4.7. Proposition A is false when we apply it to $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. For instance, if $f = (x_1, x_2^2)$, $\mu = (1, 0)$, then $\deg_\mu(f) = 1$ but $1 < \lambda(f) = 2$.

To apply Proposition A to compute the dynamical degree, we need to find some eigenvectors and eigenvalues. This is done here by looking at monomial maps associated to endomorphisms in $\mathrm{End}(\mathbb{A}^n)$. These behave quite well with respect to degree functions (see Corollary 3.2.5).

Definition 1.4.8. Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be an endomorphism such that $f_i \neq 0$ for each i. We will say that a square matrix $M = (m_{i,j})_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{N})$ is *contained in* f if for each $i \in \{1, \ldots, n\}$, the coefficient of the monomial $\prod_{j=1}^n x_j^{m_{i,j}}$ in f_i is non-zero. The set of matrices that are contained in f is then finite and non-empty.

The *maximal eigenvalue of* f is defined to be

$$\theta = \max\{|\xi| \in \mathbb{R} \mid \xi \text{ is an eigenvalue of a matrix that is contained in } f\}.$$

An element $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ is a *maximal eigenvector of* f if $\deg_\mu(f_i) = \theta \mu_i$ for each $i \in \{1, \ldots, n\}$. In particular, we then get $\deg_\mu(f) = \theta < \infty$.

It often happens that we cannot apply Proposition A to compute the dynamical degree, but that we can do it by allowing μ to have some coordinates, but not all, to be equal to zero. In fact, the following generalization of Proposition A is our main tool to compute dynamical degrees.

PROPOSITION B. *Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism with maximal eigenvalue θ. Then the following hold.*

(1) *There exists a maximal eigenvector $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ of f.*

(2) *We have $1 \leq \lambda(f) \leq \theta \leq \deg(f)$.*

(3) *For each maximal eigenvector μ of f, we have $\theta = \deg_\mu(f)$, and the following hold:*
   (i) *if f is μ-algebraically stable, then $\lambda(f) = \theta$;*
   (ii) *if $\lambda(f) = \theta$, $\theta > 1$ and $\mu \in (\mathbb{R}_{>0})^n$, then f is μ-algebraically stable;*
   (iii) *let $g \in \mathrm{End}(\mathbb{A}^n)$ be the μ-leading part of f. If $\theta > 1$, then f is μ-algebraically stable if and only if for each $r \geq 1$ there is $i \in \{1, \ldots, n\}$ with $\mu_i > 0$ such that the ith component of g^r is non-zero.*

Remark 1.4.9. In Proposition B(1), there are examples with no possibility for μ to be in $(\mathbb{R}_{>0})^n$, as the examples $f = (x_1, x_2^2) \in \mathrm{End}(\mathbb{A}^2)$ or $f = (x_1, x_3, x_2 + x_3^2) \in \mathrm{Aut}(\mathbb{A}^3)$ show. Hence, Proposition A cannot be directly applied to prove Proposition B. However, if some coordinates of μ are zero, then a linear projection is preserved (this follows from Lemma 2.5.6, see also Corollary 2.6.2). To prove Proposition B, we will use Lemma 2.6.1, which is a version of Proposition A that also works for $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$.

Remark 1.4.10. The implication of Proposition B(3)(i) is not an equivalence, as we show in Example 3.4.2 below.

The proof of Proposition B is given in §3. For each dominant endomorphism $f \in$ End($\mathbb{A}^n$), Proposition B(1) gives the existence of a maximal eigenvector μ. Moreover, Proposition B(3) shows that if f is μ-algebraically stable, then $\lambda(f)$ is equal to the maximal eigenvalue θ of f. We will use this to compute the dynamical degree of many endomorphisms of $\mathbb{A}^n$.

The following result allows to compute all dynamical degrees of permutation-elementary endomorphism of $\mathbb{A}^n$, and generalizes in particular Proposition 4.2.5. Its proof is given in §4.2.

PROPOSITION C. *Let* $f \in \text{Aut}(\mathbb{A}^n)$ *be a permutation-elementary automorphism. If the maximal eigenvalue* θ *of f is bigger than* 1, *there exists a maximal eigenvector* μ *of f such that f is* μ-*algebraically stable. In particular, the dynamical degree* $\lambda(f)$ *is equal to the maximal eigenvalue* θ *of f, which is a Handelman number.*

Proposition C is false if we replace 'permutation-elementary' by 'permutation-triangular' (see Example 4.3.4 for examples in dimension 3). However, we can obtain the following result, which is proven in §4.3.

PROPOSITION D. *Every affine-triangular automorphism* $f \in \text{Aut}(\mathbb{A}^3)$ *is conjugate to a permutation-triangular automorphism* $f' \in \text{Aut}(\mathbb{A}^3)$ *such that* $\deg(f') \leq \deg(f)$ *and such that* f' *has the following property: either the maximal eigenvalue* θ *of* f' *is equal to* 1 *or* f' *is* μ-*algebraically stable for each maximal eigenvector* μ. *In particular, the dynamical degrees* $\lambda(f)$ *and* $\lambda(f')$ *are equal to the maximal eigenvalue* θ *of* f', *which is a Handelman number.*

The proof of Theorem 1 is given at the end of §4.3, directly after proving Proposition D, as it follows almost directly from this result. We use these results in §4.4 to prove Theorem 2.

2. *Inequalities associated to degree functions and the proof of Proposition A*

2.1. *Definitions of elementary, affine and triangular automorphisms.* Let us recall the following classical definitions (even if our definition of elementary is slightly more restrictive than what is used in the literature).

Definition 2.1.1. An endomorphism $f = (f_1, \ldots, f_n) \in \text{End}(\mathbb{A}^n)$ is said to be:
- *triangular* if $f_i \in \mathbf{k}[x_1, \ldots, x_i]$ for each $i \in \{1, \ldots, n\}$;
- *elementary* if $f_i = x_i$ for for each $i \in \{1, \ldots, n-1\}$;
- *an affine automorphism* if $f \in \text{Aut}(\mathbb{A}^n)$ and if $\deg(f) = 1$;
- *a permutation of the coordinates* if $\{f_1, \ldots, f_n\} = \{x_1, \ldots, x_n\}$;
- *affine-triangular* if $f = \alpha \circ \tau$, where α is an affine automorphism and τ is a triangular endomorphism;
- *affine-elementary* if $f = \alpha \circ e$, where α is an affine automorphism and e is an elementary endomorphism;
- *permutation-triangular* if $f = \alpha \circ \tau$, where α is a permutation of the coordinates and τ is a triangular endomorphism;

- *permutation-elementary* if $f = \alpha \circ e$, where α is a permutation of the coordinates and e is an elementary endomorphism.

For each $n \leq 4$, if char($\mathbf{k}$) $\neq 2$, every automorphism of $\mathbb{A}^n$ of degree 2 is conjugate, by an affine automorphism, to an affine-triangular automorphism, see [**MO91**]. This result is false in dimension $n = 5$ [**Sun14**], as for example

$$f = (x_1 + x_2 x_4, x_2 + x_1 x_5 + x_3 x_4, x_3 - x_2 x_5, x_4, x_5) \in \mathrm{Aut}(\mathbb{A}^5)$$

shows: the Jacobian of the homogeneous part of degree 2 of an affine-triangular automorphism of degree ≤ 2 contains a zero-column, but the Jacobian of the homogeneous part of degree 2 of f contains linearly independent columns (see also [**Sun14**, Theorem 3.2]).

There are quite a few automorphisms of $\mathbb{A}^3$ of degree 3 that are not conjugate, by an affine automorphism, to affine-triangular automorphisms. More precisely, when $\mathbf{k}$ is algebraically closed, then each automorphism of $\mathbb{A}^3 = \mathrm{Spec}(\mathbf{k}[x, y, z])$ of degree 3 is conjugate, by an affine automorphism, either to an affine-triangular automorphism or to an automorphism of the form

$$\alpha(x + yz + za(x, z), y + a(x, z) + r(z), z) \in \mathrm{Aut}(\mathbb{A}^3), \qquad (*)$$

where $a \in \mathbf{k}[x, z] \setminus \mathbf{k}[z]$ is homogeneous of degree 2, $r \in \mathbf{k}[z]$ is of degree ≤ 3 and α is an affine automorphism, see [**BvS**, Theorem 3]. In fact, non of the automorphisms in $(*)$ is conjugated, by an affine automorphism, to an affine-triangular automorphism, see [**BvS**, Proposition 3.9.4].

For $\mathbf{k} = \mathbb{C}$, various (dynamical) properties of the affine-elementary automorphisms $(x_0 + x_1 + x_0^q x_2^d, x_0, \alpha x_2) \in \mathrm{Aut}(\mathbb{A}^3)$ with $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1, q \geq 2, d \geq 1$ are studied in [**DL18**] and, in particular, their dynamical degree is computed, which is equal to the integer q.

2.2. *Existence of dynamical degrees.* We recall the following folklore result, which implies that the dynamical degree is well defined.

LEMMA 2.2.1. *Let* $(a_r)_{r \geq 1}$ *be a sequence of real numbers in* $\mathbb{R}_{\geq 1}$ *such that* $a_{r+s} \leq a_r \cdot a_s$ *for each* $r, s \geq 1$. *Then,* $((a_r)^{1/r})_{r \geq 1}$ *is a sequence that converges towards* $\inf_{r \geq 1}((a_r)^{1/r}) \in \mathbb{R}_{\geq 1}$.

Proof. As $(\log(a_r))_{r \geq 1}$ is subadditive, $(\log(a_r)/r)_{r \geq 1}$ converges to $\inf_{r \geq 1}((\log(a_r)/r) \geq 0$ by Fekete's subadditivity lemma (see [**Fek23**, Satz II] or [**Ste97**, Lemma 1.2.1]). □

In case $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ is of the form $\mu_1 = \cdots = \mu_m = 0$ and $\mu_{m+1} = \cdots = \mu_n = 1$ for some $0 \leq m < n$, we denote for any polynomial $p \in \mathbf{k}[x_1, \ldots, x_n]$ its μ-degree $\deg_\mu(p)$ by $\deg_{x_{m+1}, \ldots, x_n}(p)$. Moreover, we denote for an endomorphism $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ by

$$\deg_{x_{m+1}, \ldots, x_n}(f) = \max_{j \in \{1, \ldots, n\}} \deg_{x_{m+1}, \ldots, x_n}(f_j).$$

If $m = 0$, then $\deg_\mu(f)$ is simply the classical degree that we denote by $\deg(f)$. If $m > 0$, then $\deg_\mu(f)$ is in general not equal to $\deg_{x_{m+1}, \ldots, x_n}(f)$. In fact, $\deg_\mu(f)$ is equal to

$\deg_{x_{m+1},\dots,x_n}(f)$ in case $\deg_{x_1,\dots,x_m}(f_i) = 0$ for all $i \in \{1, \dots, m\}$ and otherwise it is equal to ∞.

COROLLARY 2.2.2. *Let* $f \in \mathrm{End}(\mathbb{A}^n)$ *be an endomorphism. For each integer* $m \in \{0, \dots, n-1\}$, *the sequence*

$$\deg_{x_{m+1},\dots,x_n}(f^r)^{1/r}$$

converges to a real number $\mu_m \geq 1$. *This gives, in particular, the dynamical degree* $\lambda(f) = \mu_0$, *which satisfies* $\lambda(f^d) = \lambda(f)^d$ *for each* $d \geq 1$.

Proof. This follows from Lemma 2.2.1, as

$$\deg_{x_{m+1},\dots,x_n}(f^{r+s}) \leq \deg_{x_{m+1},\dots,x_n}(f^r) \cdot \deg_{x_{m+1},\dots,x_n}(f^s),$$

for all $r, s \geq 1$.   □

2.3. *Basic properties of degree functions.* Below we list several properties of degree functions (see Definition 1.4.1). Apart from the easy observations $\deg_\mu |_{\mathbf{k}^*} = 0$, $\deg_\mu(f \cdot g) = \deg_\mu(f) + \deg_\mu(g)$ and $\deg_\mu(f + g) \leq \max(\deg_\mu(f), \deg_\mu(g))$, which correspond to say that $-\deg_\mu$ is a valuation (see Remark 1.4.2), we have the following.

Remark 2.3.1. We fix $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ and get the following.
(1)   As explained in Definition 1.4.1, each polynomial $p \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$ can be written uniquely as a finite sum,

$$p = \sum_{\theta \in \mathbb{R}_{\geq 0}} p_\theta,$$

where each $p_\theta \in \mathbf{k}[x_1, \dots, x_n]$ is μ-homogeneous of degree θ. We then obtain $\deg_\mu(p) = \max\{\theta \mid p_\theta \neq 0\}$.
(2)   Let $m \in \{0, \dots, n-1\}$ and assume that $\mu_i = 0$ for $i \leq m$, but $\mu_i > 0$ for $i > m$. Then we have, for each polynomial $p \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$,

$$\mu_{\min} \cdot \deg_{x_{m+1},\dots,x_n}(p) \leq \deg_\mu(p) \leq \mu_{\max} \cdot \deg_{x_{m+1},\dots,x_n}(p),$$

where $\mu_{\min} = \min_{m+1 \leq i \leq n} \mu_i$ and $\mu_{\max} = \max_{m+1 \leq i \leq n} \mu_i$. In particular, for each dominant endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$, we have

$$\lim_{r \to \infty} \deg_{x_{m+1},\dots,x_n}(f^r)^{1/r} = \lim_{r \to \infty} \max_{i \in \{1,\dots,n\}} \deg_\mu((f^r)_i)^{1/r},$$

where $(f^r)_i$ denotes the ith coordinate function of f^r. Note that the left-hand side is the dynamical degree $\lambda(f)$ in case $m = 0$, that is, when $\mu \in (\mathbb{R}_{>0})^n$.

2.4. *Endomorphisms that preserve a linear projection.* The following is an algebraic analogue of the application of [**DN11**, Theorem 1.1] to endomorphisms of $\mathbb{A}^n$ that preserve a linear projection.

LEMMA 2.4.1. *Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. For each $r \geq 1$, we write*

$$f^r = ((f^r)_1, \ldots, (f^r)_n).$$

Let $m \in \{0, \ldots, n-1\}$ be such that $f_1, \ldots, f_m \in \mathbf{k}[x_1, \ldots, x_m]$. Then, the dynamical degree of f is given by $\lambda(f) = \max\{\lambda_1, \lambda_2\}$, where

$$\lambda_1 = \lim_{r \to \infty} \max\{\deg((f^r)_1), \ldots, \deg((f^r)_m)\}^{1/r} = \lambda((f_1, \ldots, f_m)),$$

$$\lambda_2 = \lim_{r \to \infty} \max\{\deg_{x_{m+1},\ldots,x_n}((f^r)_{m+1}), \ldots, \deg_{x_{m+1},\ldots,x_n}((f^r)_n)\}^{1/r}$$

$$= \lim_{r \to \infty} \deg_{x_{m+1},\ldots,x_n}(f^r)^{1/r}$$

are two limits which exist. (If $m = 0$, by convention we set $\lambda_1 = 1$.)

Proof. For each $r \geq 1$, we write

$$a_r = \max\{\deg((f^r)_1), \ldots, \deg((f^r)_m)\},$$

$$b_r = \max\{\deg((f^r)_{m+1}), \ldots, \deg((f^r)_n)\},$$

$$c_r = \max\{\deg_{x_{m+1},\ldots,x_n}((f^r)_{m+1}), \ldots, \deg_{x_{m+1},\ldots,x_n}((f^r)_n)\}$$

$$= \deg_{x_{m+1},\ldots,x_n}(f^r).$$

As $b_r \geq c_r$, we obtain for each $r \geq 1$,

$$\deg(f^r) = \max\{a_r, b_r\} \geq \max\{a_r, c_r\}.$$

It follows from Corollary 2.2.2 that the limits

$$\lambda_1 = \lim_{r \to \infty} a_r^{1/r}, \quad \lambda_2 = \lim_{r \to \infty} c_r^{1/r} \quad \text{and} \quad \lambda(f) = \lim_{r \to \infty} \deg(f^r)^{1/r}$$

exist (and all belong to $\mathbb{R}_{\geq 1}$). We obtain

$$\lambda(f) = \lim_{r \to \infty} \max\{a_r^{1/r}, b_r^{1/r}\} \geq \lim_{r \to \infty} \max\{a_r^{1/r}, c_r^{1/r}\} = \max\{\lambda_1, \lambda_2\}.$$

Thus we may assume that $\lambda(f) > \lambda_1$, which implies that $\lim_{r \to \infty} b_r^{1/r}$ exists and is equal to $\lambda(f)$. It remains to see that, in this case, $\lambda(f) \leq \max\{\lambda_1, \lambda_2\}$.

For all $r, s \geq 1$ and each $i \in \{m+1, \ldots, n\}$, the polynomial $(f^{r+s})_i$ is obtained by replacing $x_1, \ldots, x_n$ with $(f^r)_1, \ldots, (f^r)_n$ in $(f^s)_i$, so the degree of $(f^{r+s})_i$ is at most

$$\deg_{x_1,\ldots,x_m}((f^s)_i) \cdot \deg((f^r)_1, \ldots, (f^r)_m)$$
$$+ \deg_{x_{m+1},\ldots,x_n}((f^s)_i) \cdot \deg((f^r)_{m+1}, \ldots, (f^r)_n).$$

This gives $b_{r+s} \leq b_s \cdot a_r + c_s \cdot b_r$. Then when we choose $s = r$, we obtain

$$b_{2r} \leq b_r \cdot (a_r + c_r).$$

As $\lambda(f) = \lim_{r \to \infty} b_{2r}^{1/2r}$, we have $\lambda(f)^2 = \lim_{r \to \infty} b_{2r}^{1/r}$. The above inequality gives

$$\lambda(f)^2 = \lim_{r \to \infty} b_{2r}^{1/r}$$

$$\leq \lim_{r \to \infty} b_r^{1/r} \cdot \limsup_{r \to \infty} (a_r + c_r)^{1/r}$$

$$\leq \lambda(f) \cdot \limsup_{r \to \infty} (2 \max\{a_r, c_r\})^{1/r}$$

$$= \lambda(f) \cdot \max\{\lambda_1, \lambda_2\},$$

so $\lambda(f) \leq \max\{\lambda_1, \lambda_2\}$. $\square$

COROLLARY 2.4.2. *Let* $n \geq 2$ *and let* $f \in \mathrm{Aut}(\mathbb{A}^n)$ *be an automorphism such that* $f_1, \ldots, f_{n-2} \in \mathbf{k}[x_1, \ldots, x_{n-2}]$ *and such that the dynamical degree of* $g = (f_1, \ldots, f_{n-2}) \in \mathrm{Aut}(\mathbb{A}^{n-2})$ *is an integer. Then, the dynamical degree of* f *is an integer.*

Proof. By Lemma 2.4.1, one has $\lambda(f) = \max\{\lambda(g), \lambda_2\}$, where

$$\lambda_2 = \lim_{r \to \infty} \max\{\deg_{x_{n-1}, x_n}((f^r)_{n-1}), \deg_{x_{n-1}, x_n}((f^r)_n)\}^{1/r}.$$

It remains to see that λ_2 is an integer. As $\mathbf{k}[x_1, \ldots, x_{n-2}, f_{n-1}, f_n] = \mathbf{k}[x_1, \ldots, x_n]$, one has $K[f_{n-1}, f_n] = K[x_{n-1}, x_n]$, where $K = \mathbf{k}(x_1, \ldots, x_{n-2})$. Hence, one can see the automorphism $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, f_{n-1}, f_n)$ of $\mathbb{A}^n$ as an automorphism $F \in \mathrm{Aut}_K(\mathbb{A}^2)$ of $\mathbb{A}^2$ defined over K. For each $i \geq 0$, the automorphism $g^{-i} \circ (x_1, \ldots, x_{n-1}, f_{n-1}, f_n) \circ g^i$ of $\mathbb{A}^n$ can be seen as an element of $\mathrm{Aut}_K(\mathbb{A}^2)$ that we denote by F^{g^i}, where we identify g with the automorphism $(f_1, \ldots, f_{n-2}, x_{n-1}, x_n) \in \mathrm{Aut}(\mathbb{A}^n)$. This gives

$$\max\{\deg_{x_{n-1}, x_n}((f^r)_{n-1}), \deg_{x_{n-1}, x_n}((f^r)_n)\} = \deg(G_r),$$

where $G_r = F^{g^{r-1}} \circ \cdots \circ F^{g^2} \circ F^g \circ F \in \mathrm{Aut}_K(\mathbb{A}^2)$, because $G_r = g^{-r} \circ f^r$ when we consider G_r, g and f as automorphisms of $\mathbb{A}^n$.

According to the Jung–van der Kulk theorem [Jun42, vdK53], one can write $F = F_1 \circ \cdots \circ F_s$, where each $F_i \in \mathrm{Aut}_K(\mathbb{A}^2)$ is either triangular or affine. Moreover, one can assume that two consecutive F_i are not both affine or both triangular (as otherwise, one may reduce the description), and get then $\deg(F) = \prod_{i=1}^s \deg(F_i)$ (follows by looking at what happens at infinity or by [vdE00, Lemma 5.1.2]). We prove that λ_2 is an integer by induction on s. If $s = 1$, then F is either affine or triangular; this implies that the set $\{\deg(G_r) \mid r \geq 1\}$ is bounded, so $\lambda_2 = 1$. If $s > 1$ and F_1, F_s are both affine or both triangular, so we replace F with $(F_1)^g \circ F \circ F_1^{-1}$. This replaces $G_r = F^{g^{r-1}} \circ \cdots \circ F^g \circ F$ with $\tilde{G}_r = (F_1)^{g^r} \circ G_r \circ F_1^{-1}$. As $\deg((F_1)^{g^r}) = \deg(F_1)$ for each $r \geq 1$, one has

$$\frac{1}{\deg(F_1)^2} \deg(G_r) \leq \deg(\tilde{G}_r) \leq \deg(G_r) \cdot \deg(F_1)^2,$$

so this replacement does not change the value of λ_2. As this decreases the value of s, we may assume that F_1 and F_s are not both triangular or affine. Hence, for each $r \geq 1$, G_r is a product of rs elements that are affine or triangular, with no two consecutive in the same group. This gives $\deg(G_r) = \prod_{i=0}^{r-1} \prod_{j=1}^s \deg(F_j^{g^i}) = \prod_{i=0}^{r-1} \prod_{j=1}^s \deg(F_j) = \deg(F)^r$. Hence, $\lambda_2 = \deg(F)$ is an integer. $\square$

COROLLARY 2.4.3. *The dynamical degree of any element of* $\mathrm{Aut}(\mathbb{A}^2)$ *is an integer. Similarly, the dynamical degree of any element of* $\mathrm{Aut}(\mathbb{A}^3)$ *(respectively* $\mathrm{Aut}(\mathbb{A}^4)$*) which preserves the set of fibres of a linear projection* $\mathbb{A}^3 \to \mathbb{A}^1$ *or* $\mathbb{A}^3 \to \mathbb{A}^2$ *(respectively* $\mathbb{A}^4 \to \mathbb{A}^2$*) is an integer.*

Proof. The fact that the dynamical degree of any element of $\mathrm{Aut}(\mathbb{A}^2)$ is an integer follows from Corollary 2.4.2 applied to $n = 2$. If $f \in \mathrm{Aut}(\mathbb{A}^3)$ is an automorphism that preserves the set of fibres of a linear projection $\mathbb{A}^3 \to \mathbb{A}^1$ or $\mathbb{A}^3 \to \mathbb{A}^2$, then one may conjugate by an element of GL_3 and obtain $f = (f_1, f_2, f_3)$ with either $f_1 \in \mathbf{k}[x_1]$ or $f_1, f_2 \in \mathbf{k}[x_1, x_2]$. The fact that $\lambda(f)$ is an integer follows then from Corollary 2.4.2 and Lemma 2.4.1, respectively (in the second case, one uses the fact that the dynamical degree of $(f_1, f_2) \in \mathrm{Aut}(\mathbb{A}^2)$ is an integer). Similarly, in the case of an automorphism of $\mathbb{A}^4$ preserving a linear projection $\mathbb{A}^4 \to \mathbb{A}^2$, one restricts to the case $f = (f_1, \ldots, f_4) \in \mathrm{Aut}(\mathbb{A}^4)$ with $f_1, f_2 \in \mathbf{k}[x_1, x_2]$, and applies Corollary 2.4.2. $\square$

2.5. Homogeneous endomorphisms.

LEMMA 2.5.1. *Let* $h = (h_1, \ldots, h_n) \in \mathrm{End}(\mathbb{A}^n)$, *let* $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ *and let* $\theta \in \mathbb{R}_{\geq 0}$. *The following conditions are equivalent.*

(1)   *The polynomial* h_i *is* μ-*homogeneous of degree* $\theta \mu_i$ *for each* $i \in \{1, \ldots, n\}$.
(2)   *For each* μ-*homogeneous polynomial* p *of degree* ξ *and each integer* $r \geq 1$, *the polynomial* $p \circ h^r$ *is* μ-*homogeneous of degree* $\theta^r \xi$.

If additionally $h_i \neq 0$ *for each* $i \in \{1, \ldots, n\}$, *then conditions* (1) *and* (2) *are equivalent to the following.*

(3)   *For each matrix M contained in h, μ is an eigenvector to the eigenvalue θ.*

Proof. The implication condition (2) $\Rightarrow$ (1) is given by choosing $p = x_i$ for $i = 1, \ldots, n$, so we may assume condition (1) and prove condition (2). It suffices to prove condition (2) for $r = 1$, as the general result follows by induction.

If $p = 0$, then $h(p) = 0$ is μ-homogeneous of any degree. It then suffices to do the case where p is a monomial: we write $p = \zeta x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $\zeta \in \mathbf{k}^*, a_1, \ldots, a_n \geq 0$, which is μ-homogeneous of degree $\deg_\mu(p) = \sum_{i=1}^n a_i \mu_i$. As h_i is μ-homogeneous of degree $\theta \mu_i$, the polynomial $p \circ h = \zeta h_1^{a_1} h_2^{a_2} \cdots h_n^{a_n}$ is μ-homogeneous of degree $\sum_{i=1}^n a_i \theta \mu_i = \theta \deg_\mu(p)$.

Now, we assume additionally that $h_i \neq 0$ for each $i \in \{1, \ldots, n\}$. The equivalence between conditions (1) and (3) follows immediately from the definition of the μ-degree. $\square$

Definition 2.5.2. Let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ and let $\theta \in \mathbb{R}_{\geq 0}$. We say that $h \in \mathrm{End}(\mathbb{A}^n)$ is μ-*homogeneous of degree* θ if the conditions of Lemma 2.5.1 are satisfied.

LEMMA 2.5.3. *Let* $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. *For each* $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ *and each* $\theta \in \mathbb{R}_{\geq 0}$, *the following are equivalent:*

(1)   *we can write f as a finite sum $f = \sum_{0 \leq \xi \leq \theta} g_\xi$, where each $g_\xi \in \mathrm{End}(\mathbb{A}^n)$ is μ-homogeneous of degree ξ;*
(2) $\deg_\mu(f) \leq \theta$.

Proof. Condition (1) $\Rightarrow$ (2): For each $i \in \{1, \ldots, n\}$, the polynomial f_i is the sum of the ith components of the endomorphisms g_ξ. As each of these polynomials has degree $\xi \mu_i \leq \theta \mu_i$, the polynomial f_i is of μ-degree $\deg_\mu(f_i) \leq \theta \mu_i$.

Condition (2) $\Rightarrow$ (1): As in Remark 2.3.1(1), we write each f_i, $i \in \{1, \ldots, n\}$ as $f_i = \sum_{0 \leq \kappa \leq \theta \mu_i} p_{i,\kappa}$, where each $p_{i,\kappa}$ is μ-homogeneous of degree κ.

We define $g_0 = (p_{1,0}, \ldots, p_{n,0}) \in \text{End}(\mathbb{A}^n)$, which is μ-homogeneous of degree 0.

For each $\xi \in \mathbb{R}$ with $0 \leq \xi \leq \theta$, we define the ith component $(g_\xi)_i$ of g_ξ as follows: if $\mu_i = 0$ and $\xi > 0$, then $(g_\xi)_i = 0$ and otherwise, we choose $(g_\xi)_i = p_{i,\xi \mu_i}$. By construction, g_ξ is μ-homogeneous of degree ξ.

Moreover, $f_i = \sum_{0 \leq \kappa \leq \theta \mu_i} p_{i,\kappa} = \sum_{0 \leq \xi \leq \theta} (g_\xi)_i$ for each $i \in \{1, \ldots, n\}$ with $\mu_i > 0$. If $\mu_i = 0$, then $f_i = \sum_{0 \leq \kappa \leq \theta \mu_i} p_{i,\kappa} = p_{i,0} = \sum_{0 \leq \xi \leq \theta} (g_\xi)_i$. This yields $f = \sum_{0 \leq \xi \leq \theta} g_\xi$.here $\qquad\square$

Remark 2.5.4. In the decomposition of Lemma 2.5.3(1), the ith component of each g_ξ is unique if $\mu_i > 0$, but is not unique if $\mu_i = 0$.

Example 2.5.5. We have $\deg_{(1,\ldots,1)}(f) = \deg(f)$ and $\deg_\mu(\text{id}_{\mathbb{A}^n}) = 1$ for each $\mu \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. However, $\deg_{(2,3,0)}(x_1, x_2 + x_1^2 x_3, x_3) = \frac{4}{3}$.

LEMMA 2.5.6. *Let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$. For each $f = (f_1, \ldots, f_n) \in \text{End}(\mathbb{A}^n)$, the following are equivalent:*

(1) $\deg_\mu(f) < \infty$;

(2) *for each $i \in \{1, \ldots, n\}$ such that $\mu_i = 0$, the element f_i is a polynomial in the variables $\{x_j \mid j \in \{1, \ldots, n\}, \mu_j = 0\}$.*

In particular, if $\mu \in (\mathbb{R}_{>0})^n$, then the above conditions hold.

Proof. Condition (1) $\Rightarrow$ (2): Suppose that $\theta = \deg_\mu(f) < \infty$. For each $i \in \{1, \ldots, n\}$, we get $\deg_\mu(f_i) \leq \theta \mu_i$ (Definition 1.4.3). If $\mu_i = 0$, then $\deg_\mu(f_i) = 0$, which means that f_i is a polynomial in the variables $\{x_j \mid j \in \{1, \ldots, n\}, \mu_j = 0\}$.

Condition (2) $\Rightarrow$ (1): It follows from condition (2) that $\deg_\mu(f_i) \leq 0$ for each $i \in \{1, \ldots, n\}$ such that $\mu_i = 0$. This gives $\deg_\mu(f) = \max\{\deg_\mu(f_i)/\mu_i \mid \mu_i > 0\} < \infty$. $\qquad\square$

LEMMA 2.5.7. *Let $f = (f_1, \ldots, f_n) \in \text{End}(\mathbb{A}^n)$ be a dominant endomorphism. For each maximal eigenvector μ of f, the μ-leading part $g = (g_1, \ldots, g_n) \in \text{End}(\mathbb{A}^n)$ of f has the following properties:*

(1) *the maximal eigenvalue θ of f is such that $\deg_\mu(g) = \deg_\mu(f) = \theta < \infty$;*

(2) *for each $i \in \{1, \ldots, n\}$, the polynomial g_i is non-constant.*

Proof. As $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ is a maximal eigenvector of f, we have $\deg_\mu(f_i) = \theta \mu_i$ for each $i \in \{1, \ldots, n\}$, where θ is the maximal eigenvalue of f. This gives $\deg_\mu(f) = \theta < \infty$ and therefore $\deg_\mu(g_i) = \theta \mu_i = \deg_\mu(f_i)$ for each $i \in \{1, \ldots, n\}$. Hence, we get property (1). In case $\mu_i > 0$, we have $\deg_\mu(g_i) = \theta \mu_i > 0$ and thus g_i is non-constant. In case $\mu_i = 0$, we have $\deg_\mu(f_i) = \theta \mu_i = 0$

and thus $g_i = f_i$. As f is dominant, the latter polynomial is non-constant. This shows property (2). □

2.6. *Inequalities obtained by iterations.*

LEMMA 2.6.1. *Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. Suppose that $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ and that $\theta = \deg_\mu(f) \in \mathbb{R}_{\geq 0}$. Let $g = (g_1, \ldots, g_n) \in \mathrm{End}(\mathbb{A}^n)$ be the μ-leading part of f. Then the following hold.*

(1) *We can write f as a finite sum $f = g + \sum_{0 \leq \xi < \theta} g_\xi$, where each $g_\xi \in \mathrm{End}(\mathbb{A}^n)$ is μ-homogeneous of degree ξ.*

(2) *The ith coordinate function $(g^r)_i$ of g^r is the μ-homogeneous part of degree $\theta^r \mu_i$ of $(f^r)_i$ for each $i \in \{1, \ldots, n\}$ and each $r \geq 1$.*

(3) $\deg_\mu(f^r) \leq \theta^r$ *for each $r \geq 1$.*

(4) *We have*
$$1 \leq \lim_{r \to \infty} \max_{i \in \{1,\ldots,n\}} \deg_\mu((f^r)_i)^{1/r} = \lim_{r \to \infty} (\deg_\mu(f^r))^{1/r} \leq \theta.$$

(5) *If $\theta > 1$, the following are equivalent:*
   (i) $\lim_{r \to \infty} (\deg_\mu(f^r))^{1/r} = \theta$;
   (ii) *f is μ-algebraically stable;*
   (iii) *for each $r \geq 1$, there is $i \in \{1, \ldots, n\}$ with $\mu_i > 0$ and $(g^r)_i \neq 0$.*

Proof. As $\deg_\mu(f) = \theta$, we have $\deg_\mu(f_i) \leq \theta \mu_i$ for each $i \in \{1, \ldots, n\}$. Moreover, as f is dominant and $\mu \neq 0$, there are $i, j \in \{1, \ldots, n\}$ such that $\mu_i > 0$ and $\deg_{x_i}(f_j) \geq 1$. This implies that $\deg_\mu(f_j) \geq \mu_i > 0$ and thus
$$0 < \deg_\mu(f) = \theta.$$

We now observe that $\deg_\mu(f - g) < \theta$. Indeed, for each $j \in \{1, \ldots, n\}$, the jth component g_j of g is the μ-homogeneous part of f_j of degree $\theta \mu_j \geq \deg_\mu(f_j)$. If $\mu_j = 0$, then $f_j = g_j$, and if $\mu_j > 0$, then $\deg_\mu(f_j - g_j) < \theta \mu_j$.

By Lemma 2.5.3, we can write $f - g$ as a finite sum $f - g = \sum_{0 \leq \xi < \theta} g_\xi$, where each $g_\xi \in \mathrm{End}(\mathbb{A}^n)$ is μ-homogeneous of degree ξ. This gives assertion (1).

We now prove assertions (2)–(3) by induction on $r \geq 1$. For $r = 1$, assertion (2) follows from the definition of g. Moreover, assertion (3) is given by hypothesis.

We now assume assertions (2)–(3) for some integer $r \geq 1$ and prove them for $r + 1$. For each $i \in \{1, \ldots, r\}$, we write $(f^r)_i = (g^r)_i + s_i$, where $(g^r)_i$ is μ-homogeneous of degree $\theta^r \mu_i$ and $\deg_\mu(s_i) < \theta^r \mu_i$. This gives
$$
\begin{aligned}
(f^{r+1})_i &= ((g^r)_i + s_i) \circ f \\
&\overset{(1)}{=} (g^{r+1})_i + s_i \circ g + \sum_{0 \leq \xi < \theta} ((g^r)_i + s_i) \circ g_\xi.
\end{aligned}
$$

As g is μ-homogeneous of degree θ, the polynomial $(g^{r+1})_i$ is μ-homogeneous of degree $\theta^{r+1} \mu_i$ (Lemma 2.5.1). As s_i is a sum of μ-homogeneous polynomials of degree $< \theta^r \mu_i$

and g_ξ is μ-homogeneous of degree $\xi < \theta$, we have

$$\deg_\mu \left(s_i \circ g + \sum_{0 \le \xi < \theta} ((g^r)_i + s_i) \circ g_\xi \right) < \theta^{r+1} \mu_i$$

(by using Lemma 2.5.1 again). This yields assertions (2)–(3) for $r + 1$.

We now prove assertion (4). We choose $i \in \{1, \ldots, n\}$ such that $\mu_i = \max\{\mu_1, \ldots, \mu_n\}$, and observe that for each $r \ge 1$, there is $j \in \{1, \ldots, n\}$ such that $\deg_{x_i}((f^r)_j) > 0$ (as f is dominant), so $\deg_\mu((f^r)_j) \ge \mu_i = \max\{\mu_1, \ldots, \mu_n\} > 0$. This implies that

$$1 \le \lim_{r \to \infty} \max_{i \in \{1,\ldots,n\}} \deg_\mu((f^r)_i)^{1/r}$$

(the limit exists by Remark 2.3.1(2)). Let us write $I_0 = \{i \in \{1, \ldots, n\} \mid \mu_i = 0\}$. For each $i \in I_0$, we have $\deg_\mu(f_i) \le \theta \mu_i = 0$, so f_i is a polynomial in the variables $\{x_j \mid j \in I_0\}$. This implies that the same holds for $(f^r)_i$, for each integer $r \ge 1$. Hence, $\deg_\mu((f^r)_i) = 0$ for each $i \in I_0$. Writing $I_{>0} = \{i \in \{1, \ldots, n\} \mid \mu_i > 0\}$, we get for each $r \ge 1$,

$$\deg_\mu(f^r) = \max \left\{ \frac{\deg_\mu((f^r)_i)}{\mu_i} \,\middle|\, i \in I_{>0} \right\}.$$

As $\deg_\mu(f^r) \le \theta^r$ (assertion (3)), we obtain

$$\lim_{r \to \infty} \left(\max_{i \in \{1,\ldots,n\}} \deg_\mu(f^r)_i \right)^{1/r} = \lim_{r \to \infty} (\deg_\mu(f^r))^{1/r} \le \theta.$$

It remains to prove assertion (5); for this, we assume that $\theta > 1$. For each $r \ge 1$, assertion (3) gives $\deg_\mu(f^r) \le \theta^r$, or equivalently $\deg_\mu((f^r)_i) \le \theta^r \mu_i$ for each $i \in \{1, \ldots, n\}$. The equality $\deg_\mu(f^r) = \theta^r$ holds if and only if there exists $i \in \{1, \ldots, n\}$ such that $\mu_i > 0$ and $\deg_\mu((f^r)_i) = \theta^r \mu_i$. Because $(g^r)_i$ is the μ-homogeneous part of $(f^r)_i$ of degree $\theta^r \mu_i$ (follows from assertion (2)), this gives the equivalence between assertions (ii) and (iii). It remains then to prove assertion (5) $\Leftrightarrow$ (5).

'Assertion (5) $\Rightarrow$ (5)': Suppose that for each $r \ge 1$, there is $i \in \{1, \ldots, n\}$ such that $\mu_i > 0$ and $(g^r)_i \ne 0$. There is then $j \in \{1, \ldots, n\}$ and an infinite set $I \subset \mathbb{N}$ such that $\mu_j > 0$ and $(g^r)_j \ne 0$ for each $r \in I$. Assertion (2) implies that $\deg_\mu((f^r)_j) \ge \theta^r \mu_j$, for each $r \in I$, which implies that

$$\lim_{r \to \infty} \left(\max_{i \in \{1,\ldots,n\}} \deg_\mu(f^r)_i \right)^{1/r} \ge \theta.$$

This, together with assertion (4), gives $\lim_{r \to \infty} (\deg_\mu(f^r))^{1/r} = \theta$.

'Assertion (5) $\Rightarrow$ (5)': Conversely, suppose that there exists $s \ge 1$ such that $(g^s)_i = 0$ for each $i \in \{1, \ldots, n\}$ with $\mu_i > 0$. For all such i, we obtain $\deg_\mu((f^s)_i) < \theta^s \mu_i$ (by assertions (2) and (3)). As $\theta > 1$, there exists then $\theta' \in \mathbb{R}$ with $1 < \theta' < \theta$ such that

$$\deg_\mu((f^s)_i) \le \theta'^s \mu_i$$

for each $i \in \{1, \ldots, n\}$. Applying the inequality of assertion (4) for f^s, we obtain

$$\lim_{r \to \infty} \left(\max_{i \in \{1,\ldots,n\}} (\deg_\mu(f^{sr})_i)^{1/r} \right) \le \theta'^s,$$

which gives, by taking the sth root,

$$\lim_{r \to \infty} \left(\max_{i \in \{1, \ldots, n\}} (\deg_\mu(f^r)_i)^{1/r} \right) \leq \theta' < \theta. \qquad \square$$

Now we can give a short proof of Proposition A.

Proof of Proposition A. As $\mu \in (\mathbb{R}_{>0})^n$, we have $\theta := \deg_\mu(f) < \infty$ (Lemma 2.5.6).
Using Remark 2.3.1(2), we get

$$\lambda(f) = \lim_{r \to \infty} \max_{i \in \{1, \ldots, n\}} (\deg_\mu(f^r)_i)^{1/r}.$$

By definition, g is the μ-leading part of f. Now, Lemma 2.6.1(4) implies that $1 \leq \lambda(f) \leq \theta$.
If $\theta > 1$, we moreover obtain

$$\lambda(f) = \theta \Leftrightarrow \deg_\mu(f^r) = \theta^r \text{ for each } r \geq 1 \Leftrightarrow g^r \neq 0 \text{ for each } r \geq 1$$

(by Lemma 2.6.1(4) and Lemma 2.6.1(5)). $\qquad \square$

Another consequence of Lemma 2.6.1 is the following result, that generalizes Proposition A to the case where some coordinates of μ are zero.

COROLLARY 2.6.2. *Let $f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism and let $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ be such that $\theta = \deg_\mu(f) < \infty$, and assume that $m \in \{0, \ldots, n\}$ exists, such that $\mu_i = 0$ for $i \in \{1, \ldots, m\}$ and $\mu_i > 0$ for $i \in \{m + 1, \ldots, n\}$ (which can always be obtained by conjugating with a permutation). Then, the following hold.*

(1) *For each $i \in \{1, \ldots, m\}$, we have $f_i \in \mathbf{k}[x_1, \ldots, x_m]$. Hence, the element $\hat{f} = (f_1, \ldots, f_m)$ belongs to $\mathrm{End}(\mathbb{A}^m)$.*
(2) *If $\lambda(\hat{f}) = \theta$, then $\lambda(f) = \theta$.*
(3) *If $\lambda(\hat{f}) < \theta$, then $\lambda(f) = \theta \Leftrightarrow f$ is μ-algebraically stable.*

Proof. Assertion (1) follows from the fact that $\deg_\mu(f) < \infty$ and the choice of m (Lemma 2.5.6(2)).

Lemma 2.4.1 then gives $\lambda(f) = \max\{\lambda(\hat{f}), \lim_{r \to \infty} \deg_{x_{m+1}, \ldots, x_n}(f^r)^{1/r}\}$. By using the equality $\lim_{r \to \infty} \deg_{x_{m+1}, \ldots, x_n}(f^r)^{1/r} = \lim_{r \to \infty} \deg_\mu(f^r)^{1/r}$ (see Remark 2.3.1(2) and Lemma 2.6.1(4)), we obtain

$$\lambda(f) = \max\{\lambda(\hat{f}), \lim_{r \to \infty} \deg_\mu(f^r)^{1/r}\}.$$

Moreover, Lemma 2.6.1(4) implies that $\lim_{r \to \infty} \deg_\mu(f^r)^{1/r} \leq \deg_\mu(f) = \theta$. This provides assertion (2). To show assertion (3), we assume that $\lambda(\hat{f}) < \theta$ and obtain $\lambda(f) = \theta \Leftrightarrow \lim_{r \to \infty} \deg_\mu(f^r)^{1/r} = \theta$. This is equivalent to ask that f is μ-algebraically stable, by Lemma 2.6.1(5) (note that $1 \leq \lambda(\hat{f})$, since f, and thus $\hat{f}$, is dominant). $\qquad \square$

We finish this section by the following simple observation.

LEMMA 2.6.3. *Let $f \in \mathrm{End}(\mathbb{A}^n)$ be a dominant endomorphism. For each $\mu \in (\mathbb{R}_{>0})^n$ such that $\theta = \deg_\mu(f) \in \mathbb{R}_{>1}$ and each translation $\tau = (x_1 + c_1, \ldots, x_n + c_n) \in$*

Aut($\mathbb{A}^n$), *where $c_1, \ldots, c_n \in \mathbf{k}$, the following holds:*

$$f \text{ is } \mu\text{-algebraically stable} \Leftrightarrow \tau \circ f \text{ is } \mu\text{-algebraically stable}.$$

Proof. Denote by g the μ-leading part of f. As $\mu \in (\mathbb{R}_{>0})^n$, no component of g contains any constant. Hence, g is also the μ-leading part of $\tau \circ f$. By Lemma 2.6.1(5), f (respectively $\tau \circ f$) is μ-algebraically stable if and only if for each $r \geq 1$, there is $i \in \{1, \ldots, n\}$ such that $(g^r)_i \neq 0$. □

3. *Matrices associated to endomorphisms and the proof of Proposition B*

3.1. *Spectral radii of N-uples of matrices.* In the following, we fix the usual Euclidean norm on $\mathbb{R}^n$, and on $n \times n$-matrices.

Definition 3.1.1. Let $n \geq 1$.

(1) We endow $\mathbb{R}^n$ with the usual norm:

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} \quad \text{for each } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

(2) This endows the ring $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$-real matrices with the norm

$$\|M\| = \sup \left\{ \frac{\|Mv\|}{\|v\|} \, \middle| \, v \in \mathbb{R}^n \setminus \{0\} \right\} \quad \text{for each } M \in \mathrm{Mat}_n(\mathbb{R}).$$

(3) The *spectrum* of $M \in \mathrm{Mat}_n(\mathbb{R})$ is the finite subset $\sigma(M) \subset \mathbb{C}$ of eigenvalues of M.

(4) The *spectral radius* of $M \in \mathrm{Mat}_n(\mathbb{R})$ is defined by

$$\rho(M) = \max_{\lambda \in \sigma(M)} |\lambda|$$

and satisfies

$$\rho(M) = \lim_{n \to \infty} \|M^n\|^{1/n}.$$

If $M = (m_{i,j})_{i,j=1}^n$ and $N = (n_{i,j})_{i,j=1}^n$ are matrices in $\mathrm{Mat}_n(\mathbb{R})$ such that for each (i, j), we have $0 \leq m_{i,j} \leq n_{i,j}$, then $\rho(M) \leq \rho(N)$.

(5) We have a partial order on $\mathbb{R}^n$ given by

$$x \leq y \quad \text{if only if} \quad x_i \leq y_i \quad \text{for all } i = 1, \ldots, n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Note that for $0 \leq x \leq y$, we have $\|x\| \leq \|y\|$.

(6) For $M \in \mathrm{Mat}_n(\mathbb{R})$, we denote by χ_M the characteristic polynomial of M.

3.2. *The Perron–Frobenius theorem and its applications.* The Perron–Frobenius theory was first established for matrices with positive coefficients, then generalized to irreducible matrices with non-negative coefficients and then to any matrices with non-negative coefficients. There are three equivalent definitions of reducible matrices (see [**Gan59**, Vol. 2, Ch. XIII, §1, Definitions 2,2$'$,2$''$]). Let us recall one of them.

Definition 3.2.1. [**Gan59**, Vol. 2, Ch. XIII, §1, Definition $2'$] For each $n \geq 1$, a matrix $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$ is called *reducible* if there is a permutation matrix $S \in \mathrm{GL}_n(\mathbb{Z})$ such that the matrix $SMS^{-1} \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$ is block-triangular, that is,

$$SMS^{-1} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

where A, D are square matrices, and where the zero matrix has positive dimensions.

A matrix $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$ is called *irreducible* if it is not reducible.

LEMMA 3.2.2. [**Gan59**, Vol. 2, Ch. XIII, §4] *For each reducible matrix* $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$, *there is a permutation matrix* $S \in \mathrm{GL}_n(\mathbb{Z})$ *such that* SMS^{-1} *is a lower triangular block-matrix*

$$\begin{pmatrix} A_{1,1} & 0 & \cdots & & 0 \\ A_{2,1} & A_{2,2} & \ddots & & 0 \\ \vdots & \ddots & \ddots & & 0 \\ A_{m,1} & \cdots & A_{m,m-1} & A_{m,m} \end{pmatrix},$$

where $A_{1,1}, \ldots, A_{m,m}$ *are irreducible matrices.*

THEOREM 3.2.3. (Perron–Frobenius theorem [**Gan59**, Vol. 2, Ch. XIII, §2 and §3, Theorems 2 and 3]) *For each* $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$, *there exists an eigenvector* $v \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ *to the eigenvalue* $\rho(M)$. *Moreover, if* M *is irreducible, we can choose* v *in* $(\mathbb{R}_{>0})^n$.

THEOREM 3.2.4. (Theorem of Lind on weak Perron numbers) *For each* $\lambda \in \mathbb{R}$, *the following conditions are equivalent:*

(1) λ *is a weak Perron number (see Definition 1.1.1);*
(2) λ *is the spectral radius of a non-zero square matrix with non-negative integral coefficients;*
(3) λ *is the spectral radius of an irreducible square matrix with non-negative integral coefficients;*
(4) $\lambda > 0$ *and* λ^m *is a Perron number for some* $m \geq 1$.

Proof. The equivalence between conditions (1) and (3) follows from [**Lin84**, Theorem 3, p. 291], and the equivalence between conditions (2) and (3) follows from Lemma 3.2.2. The equivalence between conditions (1) and (4) can be found, for instance, in [**Sch97**, Lemma 4] or [**Bru13**, Theorem 2]. □

As a consequence of Corollary 2.6.2 and of the Perron–Frobenius theorem, we obtain the following result (which is classical, see for instance [**FW12**, **Lin12**]).

COROLLARY 3.2.5. *For each matrix* $M = (m_{i,j})_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{N})$ *and for each* $(\alpha_1, \ldots, \alpha_n) \in (\mathbf{k}^*)^n$, *the monomial endomorphism*

$$f_M = (\alpha_1 x_1^{m_{1,1}} \cdots x_n^{m_{1,n}}, \ldots, \alpha_n x_1^{m_{n,1}} \cdots x_n^{m_{n,n}}) \in \mathrm{End}(\mathbb{A}^n)$$

is dominant if and only if $\det(M) \neq 0$. *In this case, the dynamical degree of* f_M *is equal to the spectral radius of* M:

$$\lambda(f_M) = \rho(M) \in \mathbb{R}_{\geq 1}.$$

Proof. Note that the endomorphism $f_M \in \mathrm{End}(\mathbb{A}^n)$ restricts to an endomorphism $h_M \in \mathrm{End}((\mathbb{A}^1 \setminus \{0\})^n)$.

If $\det(M) = 0$, any non-zero element of the kernel of the transpose of M gives rise to a non-constant element p in the Laurent polynomial ring $\mathbf{k}[x_1^\pm, \ldots, x_n^\pm]$ such that $p \circ h_M$ is constant, so h_M, and thus f_M, is not dominant. We then assume that $\det(M) \neq 0$. This implies that $h_M \in \mathrm{End}((\mathbb{A}^1 \setminus \{0\})^n)$ is surjective on $\overline{\mathbf{k}}$-points and thus f_M is dominant. In particular, $\lambda(f_M) \geq 1$. Thus we only have to show that $\lambda(f_M) = \rho(M)$. By the Perron–Frobenius theorem (Theorem 3.2.3), there exists an eigenvector $\mu \in (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ of M to the eigenvalue $\rho(M)$. Because the spectral radius of M and the dynamical degree of f_M do not change if we conjugate M with a permutation matrix, we may assume that there is $m < n$ such that $\mu_1 = \cdots = \mu_m = 0$ and $\mu_i > 0$ for each $i \geq m + 1$. Because $(f_M)^r = f_{M^r}$, we get for each $r \geq 1$ and each $i \in \{1, \ldots, n\}$ that $\deg_\mu(((f_M)^r)_i) = (M^r\mu)_i = \rho(M)^r\mu_i$. This implies that $\deg_\mu((f_M)^r) = \rho(M)^r$ for each $r \geq 1$. Thus f_M is μ-algebraically stable and $\deg_\mu(f_M) = \rho(M) < \infty$. By Corollary 2.6.2(1), we may write

$$M = \left(\begin{array}{c|c} \hat{M} & 0 \\ \hline * & * \end{array} \right),$$

where $\hat{M} \in \mathrm{Mat}_m(\mathbb{N})$ with $\det(\hat{M}) \neq 0$. By induction, the endomorphism $f_{\hat{M}} \in \mathrm{End}(\mathbb{A}^m)$ satisfies $\lambda(f_{\hat{M}}) = \rho(\hat{M}) \leq \rho(M)$. By Corollary 2.6.2(2), (3), we get then $\lambda(f_M) = \deg_\mu(f_M) = \rho(M)$. $\qquad\square$

COROLLARY 3.2.6. *For each endomorphism* $f \in \mathrm{End}(\mathbb{A}^n)$ *and each matrix* $M \in \mathrm{Mat}_n(\mathbb{N})$ *that is contained in* f, *we have* $\rho(M) \leq \deg(f)$.

Proof. By the Perron–Frobenius theorem (Theorem 3.2.3), there exists an eigenvector $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n$ of M to the eigenvalue $\rho(M)$. Hence, $\sum_{j=1}^n m_{i,j}\mu_j = \rho(M)\mu_i$ for each $i \in \{1, \ldots, n\}$. By choosing an integer $r \in \{1, \ldots, n\}$ such that $\mu_r = \max\{\mu_1, \ldots, \mu_n\}$, we obtain

$$\rho(M)\mu_r = \sum_{j=1}^n m_{r,j}\mu_j \leq \mu_r \sum_{j=1}^n m_{r,j}.$$

The coefficient of the monomial $\prod_{j=1}^n x_j^{m_{r,j}}$ in f_r is non-zero (as M is contained in f, see Definition 1.4.8). This monomial has degree $\sum_{j=1}^n m_{r,j}$, so $\deg(f) \geq \sum_{j=1}^n m_{r,j}$. As $\mu_r > 0$, this gives $\rho(M) \leq \deg(f)$. $\qquad\square$

In the following, we will use the next basic property of Handelman numbers. It is a straightforward application of Descarte's Rule of Signs, see e.g. [**Str86**, p. 91].

LEMMA 3.2.7. (Basic property of Handelman numbers) *Let $n \geq 1$. For each $(a_0, \ldots, a_{n-1}) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$, the polynomial $x^n - \sum_{i=0}^{n-1} a_i x^i \in \mathbb{R}[x]$ has a unique positive real root. In particular, a Handelman number has no other positive real Galois conjugate.*

COROLLARY 3.2.8. *Each Handelman number is a weak Perron number.*

Proof. Let $\lambda \in \mathbb{R}_{>0}$ be a Handelman number. There exists $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n \setminus \{0\}$ such that λ is a root of $P(x) = x^n - \sum_{i=0}^{n-1} a_i x^i \in \mathbb{Z}[x]$. By Lemma 3.2.7, all roots of P, except λ, are either non-real or real and non-positive. Because P is the characteristic polynomial of the matrix

$$A = \begin{pmatrix} a_{n-1} & \cdots & a_1 & a_0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0}),$$

it follows by the Perron-Frobenius theorem (Theorem 3.2.3) that the spectral radius of A is equal to λ. This implies that λ is a weak Perron number (Theorem 3.2.4). □

3.3. *Sequences of matrices.* To study endomorphisms of $\mathbb{A}^n$, we will need to consider finite sets of elements of $\mathrm{Mat}_n(\mathbb{R})$ that have the property that we can exchange rows. To take the norm on such sets, we will have to see them ordered, and thus see these in $\mathrm{Mat}_n(\mathbb{R})^N$ for some $N \geq 1$.

Notation 3.3.1. Let $n, N \geq 1$. We denote by $\widehat{\mathcal{M}}_{n,N} \subset \mathrm{Mat}_n(\mathbb{R})^N$ the $\mathbb{R}$-vector subspace of N-tuples $(M_1, \ldots, M_N)$ that have the following property:

> *For each $i, j \in \{1, \ldots, N\}$ and each $l \in \{1, \ldots, n\}$, the replacement of the lth row of M_i with the lth row of M_j gives a matrix which lies in $\{M_1, \ldots, M_N\}$.*

We then denote by $\mathcal{M}_{n,N} \subset \widehat{\mathcal{M}}_{n,N}$ the subset that consists of the N-tuples $(M_1, \ldots, M_N)$, where $M_1, \ldots, M_N$ are N distinct matrices with non-negative coefficients.

Remark 3.3.2. If $f \in \mathrm{End}(\mathbb{A}^n)$ is an endomorphism, then there exists some integer $N \geq 1$ and some N-tuple $(S_1, \ldots, S_N) \in \mathcal{M}_{n,N}$ such that $\{S_1, \ldots, S_N\}$ is the set of matrices that are contained in f (as in Definition 1.4.8).

The following two lemmas build the key ingredients for proving the existence of maximal eigenvectors of endomorphisms of $\mathbb{A}^n$ in the next subsection (see Proposition 3.4.1). This eventually leads then to a proof of Proposition B.

LEMMA 3.3.3. *Let $n, N \geq 1$. For each $M = (M_1, \ldots, M_N) \in \mathcal{M}_{n,N}$, there exists a sequence $(D_t)_{t \in \mathbb{N}}$ of elements $D_t = (D_{t,1}, \ldots, D_{t,N}) \in \mathcal{M}_{n,N}$ that converges towards M (with respect to the topology of $\mathrm{Mat}_n(\mathbb{R})^N$ that is given by the norm as in Definition 3.1.1) and such that for each $t \in \mathbb{N}$, there is no complex number which is an eigenvalue of two elements of $D_{t,1}, \ldots, D_{t,N}$.*

Proof. The result being trivially true for $N = 1$, we will assume $N \geq 2$. For each $i \in \{1, \ldots, n\}$, we denote by $\Gamma_i \subset \mathbb{R}^n$ the finite set of ith rows of the matrices $M_1, \ldots, M_N$:

$$\Gamma_i = \{r \in \mathbb{R}^n \mid r \text{ is the } i\text{th row of one of the matrices } M_1, \ldots, M_N\}.$$

We then write $\Gamma_i = \{r_{i,1}, \ldots, r_{i,s_i}\}$, where $s_i \geq 1$ is the cardinality of Γ_i.

As all matrices $M_1, \ldots, M_N$ are pairwise distinct and as one can 'exchange rows' (see Notation 3.3.1), we have $N = s_1 \cdots s_n$, and obtain a unique $\mathbb{R}$-linear map

$$\varphi : \prod_{i=1}^{n} (\mathbb{R}^n)^{s_i} \to \widehat{\mathcal{M}}_{n,N}$$

with the following properties.

(1) For each $k \in \{1, \ldots, N\}$, the composition of φ with the projection map $\pi_k : \operatorname{Mat}_n(\mathbb{R})^N \to \operatorname{Mat}_n(\mathbb{R})$ onto the kth factor is of the form

$$\pi_k \circ \varphi : \qquad \prod_{i=1}^{n}(\mathbb{R}^n)^{s_i} \qquad \to \qquad \operatorname{Mat}_n(\mathbb{R})$$
$$(v_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i} \quad \mapsto \quad \begin{pmatrix} v_{1,j_1} \\ \vdots \\ v_{n,j_n} \end{pmatrix},$$

where $j_i \in \{1, \ldots, s_i\}$ for each $i \in \{1, \ldots, n\}$.

(2) $(M_1, \ldots, M_N) = \varphi((r_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i})$.

Indeed, the possibilities for maps $\pi_k \circ \varphi$ as in property (1) are parametrized by the N possible choices of $j_i \in \{1, \ldots, s_i\}$ for each $i \in \{1, \ldots, n\}$, and by property (2), the image of $(r_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i}$ by the maps $\pi_1 \circ \varphi, \ldots, \pi_N \circ \varphi$ give the matrices $M_1, \ldots, M_N$; this gives the existence and the unicity of φ.

We now identify $\prod_{i=1}^{n}(\mathbb{R}^n)^{s_i}$ with the real locus $X(\mathbb{R})$ of the affine space $X = \mathbb{A}^{n \sum s_i}$.

For any two matrices $A, B \in \operatorname{Mat}_n(\mathbb{R})$, the resultant of the characteristic polynomials χ_A and χ_B is denoted by $r(A, B)$. Recall that $r(A, B) = 0$ if and only if A and B have a common eigenvalue. Hence, for any distinct $a, b \in \{1, \ldots, N\}$, the set

$$Z_{a,b} = \left\{ x \in \prod_{i=1}^{n}(\mathbb{R}^n)^{s_i} \,\middle|\, \begin{array}{l} \text{the matrices } \pi_a(\varphi(x)) \text{ and } \pi_b(\varphi(x)) \\ \text{have a common eigenvalue} \end{array} \right\}$$

corresponds to the elements of $X(\mathbb{R})$ that satisfy one polynomial equation $P_{a,b} \in \mathbb{R}[X]$.

We now prove that $P_{a,b} \neq 0$, or equivalently that $Z_{a,b} \neq X(\mathbb{R}) = \prod_{i=1}^{n}(\mathbb{R}^n)^{s_i}$, by showing that $\pi_a(\varphi(x))$ and $\pi_b(\varphi(x))$ have no common eigenvalue for at least one $x \in X(\mathbb{R})$. We consider $j_1, \ldots, j_n$ and $j'_1, \ldots, j'_n$ so that $\pi_a \circ \varphi$ and $\pi_b \circ \varphi$ are respectively given by

$$\prod_{i=1}^{n}(\mathbb{R}^n)^{s_i} \quad \to \quad \operatorname{Mat}_n(\mathbb{R}) \qquad\qquad \prod_{i=1}^{n}(\mathbb{R}^n)^{s_i} \quad \to \quad \operatorname{Mat}_n(\mathbb{R})$$
$$(v_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq s_i}} \quad \mapsto \quad \begin{pmatrix} v_{1,j_1} \\ \vdots \\ v_{n,j_n} \end{pmatrix} \quad \text{and} \quad (v_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq s_i}} \quad \mapsto \quad \begin{pmatrix} v_{1,j'_1} \\ \vdots \\ v_{n,j'_n} \end{pmatrix}.$$

Because the matrices M_a and M_b are distinct, the linear maps $\pi_a \circ \varphi$ and $\pi_b \circ \varphi$ are also distinct. Thus, there is $l \in \{1, \ldots, n\}$ such that $j_l \neq j'_l$. Suppose first that $l = 1$, that is, $j_1 \neq j'_1$. We may choose $x \in X(\mathbb{R})$ such that

$$\pi_a(\varphi(x)) = \left(\begin{array}{c|c} 0 & 1 \\ \hline I_{n-1} & 0 \end{array} \right) \quad \text{and} \quad \pi_b(\varphi(x)) = \left(\begin{array}{c|c} 0 & 0 \\ \hline I_{n-1} & 0 \end{array} \right).$$

These matrices have characteristic polynomials $t^n - 1$ and t^n, respectively. If $l > 1$, we simply consider conjugation of the above matrices by permutations. In all cases, we find an $x \in X(\mathbb{R})$ such that $\pi_a(\varphi(x))$ and $\pi_b(\varphi(x))$ are matrices without common eigenvalue in $\mathbb{C}$. This shows that $Z_{a,b} \neq X(\mathbb{R})$, that is, $P_{a,b} \neq 0$.

The product of all polynomials $P_{a,b}$ with distinct $a, b \in \{1, \ldots, n\}$ gives a non-zero polynomial $P \in \mathbb{R}[X]$. Thus we can take a real affine linear map $\ell \colon \mathbb{A}^1 \to X = \mathbb{A}^{n \sum s_i}$ such that $\ell(0) = (r_{i,j})_{1 \leq i \leq n, 1 \leq j \leq s_i}$, such that the coordinates of $\ell(\mathbb{R}_{\geq 0})$ are non-negative and such that the restriction of P to $\ell(\mathbb{R})$ is non-zero. We obtain that $P(\ell(1/n)) \neq 0$ for any sufficiently large positive integer n. It suffices then to fix a sufficiently large $c \geq 1$ and to define $D_t = \varphi(\ell(1/(t + c)))$ for each integer $t \geq 0$. $\qquad\square$

LEMMA 3.3.4. *Let $S = (S_1, \ldots, S_N) \in \mathcal{M}_{n,N}$ and let $v \geq 0$ be an eigenvector of S_1 to the eigenvalue $\lambda \geq 0$. Moreover, suppose that $\lambda > \rho(S_i)$ for each $i \in \{2, \ldots, N\}$. Then $S_i v \leq \lambda v$ for each $i \in \{1, \ldots, N\}$.*

Proof. Assume for contradiction that there is $i \in \{2, \ldots, N\}$ such that $S_i v \not\leq \lambda v$. Denote by v_j the jth component of v for each $j \in \{1, \ldots, n\}$. Because we may replace each row R_j in S_i, such that $R_j v < \lambda v_j$, with the jth row from S_1 and still get an element in $\{S_1, \ldots, S_N\}$, we may assume that $S_i v \geq \lambda v \geq 0$. As the coefficients of v and S_i are non-negative, we obtain by induction that $(S_i)^r v \geq \lambda^r v \geq 0$ for each $r \geq 1$. In particular,

$$\|(S_i)^r\| \geq \frac{\|(S_i)^r v\|}{\|v\|} \geq \lambda^r,$$

and we obtain $\rho(S_i) = \lim_{r \to \infty} \|(S_i)^r\|^{1/r} \geq \lambda$. This contradicts the assumption that $\lambda > \rho(S_i)$. $\qquad\square$

3.4. *Existence of maximal eigenvectors of endomorphisms of $\mathbb{A}^n$.*

PROPOSITION 3.4.1. *For each $n, N \geq 1$ and each $S = (S_1, \ldots, S_N) \in \mathcal{M}_{n,N}$, there exists $j \in \{1, \ldots, N\}$ and an eigenvector $v \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ of S_j to the eigenvalue $\lambda = \max\{\rho(S_1), \ldots, \rho(S_N)\}$ such that for each $i \in \{1, \ldots, N\}$, we have*

$$S_i v \leq S_j v = \lambda v.$$

Proof. Let $S = (S_1, \ldots, S_N) \in \mathcal{M}_{n,N}$. By Lemma 3.3.3, there exists a sequence $(D_t)_{t \in \mathbb{N}}$ of elements $D_t = (D_{t,1}, \ldots, D_{t,N}) \in \mathcal{M}_{n,N}$ that converges towards S and such that for each $t \in \mathbb{N}$, there is no complex number which is an eigenvalue of two elements of $D_{t,1}, \ldots, D_{t,N}$. In particular, $\rho(D_{t,i}) \neq \rho(D_{t,j})$ for distinct i, j by the Perron–Frobenius theorem (Theorem 3.2.3).

By possibly replacing this sequence with a subsequence, we may assume that there is a $j \in \{1, \ldots, N\}$ such that $\rho(D_{t,j}) > \rho(D_{t,i})$ for all $i \in \{1, \ldots, N\} \setminus \{j\}$ and each $t \in \mathbb{N}$. After exchanging the ordering of $S_1, \ldots, S_N$, we may assume that $j = 1$. For each $i \in \{1, \ldots, N\}$, the sequence $(D_{t,i})_{t \in \mathbb{N}}$ converges towards S_i, so $(\rho(D_{t,i}))_{t \in \mathbb{N}}$ converges towards $\rho(S_i)$ [**Ost73**, Theorem in Appendix A]. In particular, $\rho(S_1) = \lambda = \max\{\rho(S_1), \ldots, \rho(S_n)\}$. By the Perron–Frobenius theorem (Theorem 3.2.3), there is for each $t \in \mathbb{N}$ an eigenvector $v_t \geq 0$ of $D_{t,1}$ to the eigenvalue $\rho(D_{t,1})$. Lemma 3.3.4 then gives for each $i \in \{1, \ldots, N\}$ and each $t \in \mathbb{N}$,

$$D_{t,i} v_t \leq \rho(D_{t,1}) v_t.$$

Now, we may assume that $\|v_t\| = 1$ for all t (after normalizing v_t). Let

$$\mathbb{S}^{n-1} = \{w \in \mathbb{R}^n \mid \|w\| = 1\}.$$

Because $\mathbb{S}^{n-1}$ is compact (with respect to the Euclidean topology), we may take a subsequence and assume that $(v_t)_{t \in \mathbb{N}}$ converges to a $v \geq 0$ in $\mathbb{S}^{n-1}$. Thus we get

$$\lambda v = \rho(S_1) v = \lim_{t \to \infty} \rho(D_{t,1}) v_t = \lim_{t \to \infty} D_{t,1} v_t = S_1 v$$

and for each $i \in \{1, \ldots, N\}$,

$$S_i v = \lim_{t \to \infty} D_{t,i} v_t \leq \lim_{t \to \infty} \rho(D_{t,1}) v_t = \rho(S_1) v = \lambda v.$$

This finishes the proof of the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition B. By Remark 3.3.2, there exists $(S_1, \ldots, S_N) \in \mathcal{M}_{n,N}$ such that $\{S_1, \ldots, S_N\}$ is the set of matrices contained in f. By Proposition 3.4.1, there exists $j \in \{1, \ldots, N\}$ and an eigenvector $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\geq 0})^n \setminus \{0\}$ of S_j to the eigenvalue $\theta = \max\{\rho(S_1), \ldots, \rho(S_N)\}$ such that $S_i \mu \leq S_j \mu = \theta \mu$ for each $i \in \{1, \ldots, N\}$. We now prove that this implies that $\deg_\mu(f_l) = \theta \mu_l$ for each $l \in \{1, \ldots, n\}$, which shows that $\mu = (\mu_1, \ldots, \mu_n)$ is a maximal eigenvector of f, and thus proves assertion (1). For each monomial $m = \chi x_1^{r_1} \cdots x_n^{r_n}$ of f_l with $\chi \in \mathbf{k}^*$, there is a matrix S_i with its lth line equal to $(r_1 \; r_2 \; \cdots \; r_n)$. The lth component of $S_i \mu$ is equal to $r_1 \mu_1 + \cdots + r_n \mu_n = \deg_\mu(m)$. The inequality $S_i \mu \leq \theta \mu$ then yields $\deg_\mu(m) \leq \theta \mu_l$. As this holds for each monomial of f_l, we obtain $\deg_\mu(f_l) \leq \theta \mu_l$. The equality follows from $S_j \mu = \theta \mu$, because the monomial m that corresponds to the lth row of S_j has μ-degree equal to $\theta \mu_l$.

We now prove assertion (2). The dominance of f implies that $1 \leq \deg(f^r)$ for each r and this in turn gives $1 \leq \lambda(f)$. The inequality $\theta \leq \deg(f)$ follows from Corollary 3.2.6, so we only need to prove $\lambda(f) \leq \theta$. This is done by induction on n. If $n = 1$, then $\mu \in (\mathbb{R}_{>0})^1$ and the statement follows from Proposition A(2). Now, let $n > 1$. We may assume (after a permutation of the coordinates) that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Now, let $m \in \{0, \ldots, n-1\}$ with $\mu_i = 0$ for $i \leq m$ and $\mu_i > 0$ for $i > m$. From Remark 2.3.1(2), we get

$$\lambda_2 := \lim_{r \to \infty} \deg_{x_{m+1}, \ldots, x_n}(f^r)^{(1/r)} = \lim_{r \to \infty} \max_{i \in \{1, \ldots, n\}} \deg_\mu((f^r)_i)^{(1/r)}.$$

From Lemma 2.5.6, we get that for each $i \in \{1, \ldots, m\}$, the element f_i is a polynomial in the variables $\{x_1, \ldots, x_m\}$. Thus we get from Lemma 2.4.1 that $\lambda(f) = \max\{\lambda_1, \lambda_2\}$,

where

$$\lambda_1 = \lambda(\hat{f}) = \lim_{r \to \infty} \deg(\hat{f}^r)^{(1/r)} \quad \text{and} \quad \hat{f} := (f_1, \ldots, f_m) \in \text{End}(\mathbb{A}^m).$$

Because $m \leq n - 1$, by induction hypothesis we have

$$\lambda_1 \leq \theta_1 := \max\{|\xi| \in \mathbb{R} \mid \xi \text{ is an eigenvalue of a matrix that is contained in } \hat{f}\}.$$

Note that each eigenvalue of a matrix that is contained in $\hat{f}$ is an eigenvalue of a matrix that is contained in f. Thus we get $\theta_1 \leq \theta$. From Lemma 2.6.1(4), it follows that $\lambda_2 \leq \theta$. In summary, we proved that $\lambda(f) = \max\{\lambda_1, \lambda_2\} \leq \theta$, that is, assertion (2) holds for n.

We now prove assertion (3). We take a maximal eigenvector μ of f. As $\deg_\mu(f_i) = \theta\mu_i$ for each $i \in \{1, \ldots, n\}$, we have $\deg_\mu(f) = \theta$. If $\theta = 1$, (i) follows from assertion (2) and (ii) is trivially true, so we may assume that $\theta > 1$. If f is μ-algebraically stable, then Lemma 2.6.1(5) gives $\lambda_2 = \theta$ and thus $\lambda(f) = \theta$, so (i) is proven. Conversely, if $\mu \in (\mathbb{R}_{>0})^n$ and $\lambda(f) = \theta > 1$, then f is μ-algebraically stable by Proposition A(3). This achieves the proof of (3). As $\theta = \deg_\mu(f) \in \mathbb{R}_{\geq 0}$ (that is, is not equal to $+\infty$), (3) is a direct consequence of Lemma 2.6.1(5). $\qquad \square$

We now give an example that shows that the implication of Proposition B(3)(i) is not an equivalence.

Example 3.4.2. We consider the automorphism

$$f = (f_1, f_2, f_3, f_4) = ((x_1)^2 + x_2, x_1, x_3 + (x_3 + x_4)^2, x_4 - (x_3 + x_4)^2) \in \text{Aut}(\mathbb{A}^4).$$

As $\deg(f) = 2$, the maximal eigenvalue θ of f (see Definition 1.4.8) satisfies $\theta \leq 2$ (Corollary 3.2.6). Moreover, $\theta = 2$, as the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is contained in f. When we choose $\mu = (0, 0, 1, 1)$, we get $\deg_\mu(f) = 2$, and we see that f is not μ-algebraically stable, as $\deg_\mu(f^2) = 2 < 4$. Moreover, $\deg_\mu(f_i) = 0$ for $i \in \{1, 2\}$ and $\deg_\mu(f_i) = 2$ for $i \in \{3, 4\}$. Thus μ is a maximal eigenvector of f (see Definition 1.4.8). However, $\lambda(f) = \theta$. Indeed, $\lambda(f) \leq \deg(f) = 2$, and $((x_1)^2 + x_2, x_1)$ is algebraically stable for the standard degree, as its homogeneous part of degree 2 is $((x_1)^2, 0)$, which satisfies $((x_1)^2, 0)^r = ((x_1)^{2^r}, 0)$ for each $r \geq 1$ (see Proposition A).

4. *Explicit calculation of dynamical degrees of affine-triangular automorphisms*
In this section, we apply Proposition B to compute the dynamical degrees of affine-triangular dominant endomorphisms of $\mathbb{A}^n$. We prove Proposition 4.2.3, which implies Propositions 4.2.5 and C.

Notation 4.0.1. We denote by $\text{TEnd}(\mathbb{A}^n)$ and $\text{TAut}(\mathbb{A}^n)$ (respectively $\text{EEnd}(\mathbb{A}^n)$ and $\text{EAut}(\mathbb{A}^n)$) the monoid and group of triangular (respectively elementary) endomorphisms

and automorphisms of $\mathbb{A}^n$. We denote by $\mathrm{Aff}(\mathbb{A}^n)$ the group of affine automorphisms of $\mathbb{A}^n$ and by $\mathrm{Sym}(\mathbb{A}^n) \subset \mathrm{Aff}(\mathbb{A}^n)$ the group of permutations of the coordinates.

4.1. *From affine-triangular to permutation-triangular endomorphisms.* We can restrict ourselves to permutation-triangular endomorphisms, as the next simple result shows.

PROPOSITION 4.1.1. *Each affine-triangular endomorphism of $\mathbb{A}^n$ is conjugate by an element of $\mathrm{Aff}(\mathbb{A}^n)$ to a permutation-triangular endomorphism.*

Proof. We take $\alpha \in \mathrm{Aff}(\mathbb{A}^n)$ and $\tau \in \mathrm{TEnd}(\mathbb{A}^n)$ and show that we can conjugate $f = \alpha \circ \tau$ to a permutation-triangular endomorphism by an element of $\mathrm{Aff}(\mathbb{A}^n)$.

Let $p = (p_1, \ldots, p_n) \in \mathbb{A}^n$ be the point such that $\alpha(p) = 0$ and consider the translation $\tau_p = (x_1 + p_1, \ldots, x_n + p_n) \in \mathrm{Aff}(\mathbb{A}^n) \cap \mathrm{TAut}(\mathbb{A}^n)$. Then $\alpha' = \alpha \circ \tau_p \in \mathrm{Aff}(\mathbb{A}^n)$ fixes the origin $(0, \ldots, 0) \in \mathbb{A}^n$. We then replace α with α' and τ with $\tau_p^{-1} \circ \tau$, and may assume that α belongs to the subgroup $\mathrm{GL}_n = \mathrm{GL}_n(\mathbf{k}) \subset \mathrm{Aff}(\mathbb{A}^n)$ of elements that fix the origin.

The group $B = \mathrm{TAut}(\mathbb{A}^n) \cap \mathrm{GL}_n$ is a Borel subgroup of GL_n. It consists of all lower triangular matrices. The so-called Bruhat decomposition of GL_n:

$$\mathrm{GL}_n = B\,\mathrm{Sym}(\mathbb{A}^n)B$$

yields $\beta, \gamma \in B$ and $\sigma \in \mathrm{Sym}(\mathbb{A}^n)$ such that $\alpha = \beta \circ \sigma \circ \gamma$. This gives

$$\beta^{-1} \circ f \circ \beta = \beta^{-1} \circ \alpha \circ \tau \circ \beta = \sigma \circ \gamma \circ \tau \circ \beta,$$

where $\gamma \circ \tau \circ \beta \in \mathrm{TEnd}(\mathbb{A}^n)$. This achieves the proof. $\qquad \square$

4.2. *Permutation-elementary automorphisms.* Up to conjugation, each permutation-elementary automorphism has a particular form. This shows the following easy observation.

LEMMA 4.2.1. *Let $n \geq 1$ and let $h \in \mathrm{End}(\mathbb{A}^{n+1})$ be a permutation-elementary automorphism. There is a permutation of the coordinates $\alpha \in \mathrm{Sym}(\mathbb{A}^{n+1})$ such that*

$$f = \alpha \circ h \circ \alpha^{-1} = (f_1, \ldots, f_m, \xi x_{n+1} + p(x_1, \ldots, x_n), x_{m+1}, \ldots, x_n),$$

where $0 \leq m \leq n$, $\{x_1, \ldots, x_m\} = \{f_1, \ldots, f_m\}$, $\xi \in \mathbf{k}^*$ and $p \in \mathbf{k}[x_1, \ldots, x_n]$.

Proof. We write $h = \sigma \circ \tau$, where $\sigma \in \mathrm{Sym}(\mathbb{A}^{n+1})$ and $\tau \in \mathrm{EAut}(\mathbb{A}^{n+1})$. We may choose $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathrm{Sym}(\mathbb{A}^{n+1})$ such that $\alpha_{n+1} = x_{n+1}$ and $\alpha \circ \sigma \circ \alpha^{-1}$ induces the following cyclic permutation on the last coordinates

$$(\alpha \circ \sigma \circ \alpha^{-1})_{m+1} = x_{n+1}, \ (\alpha \circ \sigma \circ \alpha^{-1})_{m+2} = x_{m+1}, \ \ldots, \ (\alpha \circ \sigma \circ \alpha^{-1})_{n+1} = x_n,$$

for some integer m with $0 \leq m \leq n$. This gives

$$\alpha \circ \sigma \circ \alpha^{-1} = (f_1, \ldots, f_m, x_{n+1}, x_{m+1}, \ldots, x_n),$$

where $\{x_1, \ldots, x_m\} = \{f_1, \ldots, f_m\}$. As $\alpha_{n+1} = x_{n+1}$, we obtain

$$\alpha \circ \tau \circ \alpha^{-1} = (x_1, \ldots, x_n, \xi x_{n+1} + p(x_1, \ldots, x_n))$$

for some $\xi \in \mathbf{k}^*$ and $p \in \mathbf{k}[x_1, \dots, x_n]$. This implies that $\alpha \circ h \circ \alpha^{-1}$ is equal to

$$(\alpha \circ \sigma \circ \alpha^{-1}) \circ (\alpha \circ \tau \circ \alpha^{-1})$$
$$= (f_1, \dots, f_m, \xi x_{n+1} + p(x_1, \dots, x_n), x_{m+1}, \dots, x_n). \qquad \square$$

We will need the following result to obtain Proposition 4.2.3. Proposition 4.2.3 will be the key ingredient to show Proposition 4.2.5 and Proposition C.

LEMMA 4.2.2. *Let* $0 \le m \le n$, *let* $\hat{f} = (f_1, \dots, f_m) \in \mathrm{Aut}(\mathbb{A}^m)$ *and let* $q \in \mathbf{k}[x_1, \dots, x_{n+1}] \setminus \{0\}$. *For each* $r \ge 1$, *every component of* g^r *is non-zero where*

$$g = (f_1, \dots, f_m, q, x_{m+1}, \dots, x_n) \in \mathrm{End}(\mathbb{A}^{n+1}).$$

Proof. For each $r \ge 1$, we write $g^r = ((g^r)_1, \dots, (g^r)_{n+1})$. The result is true by assumption when $r = 1$. For each $r \ge 1$ and $1 \le i \le m$, we have $(g^r)_i = (\hat{f}^r)_i \ne 0$.

As $(f_1, \dots, f_m) \in \mathrm{Aut}(\mathbb{A}^m)$, we also have $(f_1, \dots, f_m, x_{m+1}, \dots, x_n) \in \mathrm{Aut}(\mathbb{A}^n)$. In particular, g is dominant if $q \notin \mathbf{k}[x_1, \dots, x_n]$, that is, if $\deg_{x_{n+1}}(q) \ge 1$. Thus we assume that $q \in \mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$.

Suppose first that $m = n$, in which case $g = (f_1, \dots, f_m, q)$. For each $r \ge 2$, we get $g^r = ((g^r)_1, \dots, (g^r)_m, q((g^{r-1})_1, \dots, (g^{r-1})_m))$. As $\hat{f}$ is dominant and q is not the zero polynomial, every component of g^r is not zero.

We then assume that $n > m$ and prove the result by induction on $n - m$. As $(f_1, \dots, f_m, x_{m+1}, \dots, x_n) \in \mathrm{Aut}(\mathbb{A}^n)$, there is a polynomial $h \in \mathbf{k}[x_1, \dots, x_n]$ such that $h(f_1, \dots, f_m, x_{m+1}, \dots, x_n) = q$, because $q \in \mathbf{k}[x_1, \dots, x_n]$. We denote by $\phi \colon \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$ the closed embedding that is given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_m, h(x_1, \dots, x_n), x_{m+1}, \dots, x_n)$$

and we write $\tau = (f_1, \dots, f_m, h, x_{m+1}, \dots, x_{n-1}) \in \mathrm{End}(\mathbb{A}^n)$. We now prove that $g \circ \phi = \phi \circ \tau$:

$$g \circ \phi(x_1, \dots, x_n)$$
$$= (f_1, \dots, f_m, q(x_1, \dots, x_m, h, x_{m+1}, \dots, x_{n-1}), h, x_{m+1}, \dots, x_{n-1})$$
$$= (f_1, \dots, f_m, h(f_1, \dots, f_m, h, x_{m+1}, \dots, x_{n-1}), h, x_{m+1}, \dots, x_{n-1})$$
$$= \phi \circ \tau(x_1, \dots, x_n).$$

Hence, $g^r \circ \phi = \phi \circ \tau^r$ for each $r \ge 1$. By induction, every component of τ^r is non-zero, so every component of g^r is non-zero, except maybe the $(m+1)$th component. However, if the $(m+1)$th component of g^r were zero, then the $(m+2)$th of g^{r+1} would be zero, which is impossible as the $(m+2)$th component of $g^{r+1} \circ \phi = \phi \circ \tau^{r+1}$ is not equal to zero. $\qquad \square$

PROPOSITION 4.2.3. *Let* $0 \le m < n$, *let* $\hat{f} = (f_1, \dots, f_m) \in \mathrm{Aut}(\mathbb{A}^m)$, $\xi \in \mathbf{k}^*$ *and* $p \in \mathbf{k}[x_1, \dots, x_n]$. *Denote by* $I \subset \mathbb{N}^n$ *the finite subset of indices of the monomials of p, and*

define

$$\theta = \max \left\{ \lambda \in \mathbb{R} \mid \lambda^{n-m} = \sum_{j=m+1}^{n} i_j \lambda^{n-j} \text{ for some } (i_1, \ldots, i_n) \in I \right\}.$$

Then,

$$f = (f_1, \ldots, f_m, \xi x_{n+1} + p(x_1, \ldots, x_n), x_{m+1}, \ldots, x_n) \in \mathrm{Aut}(\mathbb{A}^{n+1})$$

has the following properties.

(1) *If* $\deg_{x_{m+1},\ldots,x_n}(p) \le 1$, *then* $\lambda(f) = \lambda(\hat{f})$.
(2) *If* $\deg_{x_{m+1},\ldots,x_n}(p) \ge 2$, *define*

$$\mu = (\mu_1, \ldots, \mu_{n+1}) = (0, \ldots, 0, \theta^{n-m}, \theta^{n-m-1}, \ldots, \theta, 1),$$

that is, $\mu_j = 0$ *for* $j \le m$ *and* $\mu_j = \theta^{n+1-j}$ *for* $j \ge m+1$. *Then we have* $\theta > 1$, $\deg_\mu(f_j) = \theta \mu_j$ *for each* j *(in particular* $\deg_\mu(f) = \theta$*) and* f *is* μ-*algebraically stable. Moreover, if* $\lambda(\hat{f}) \le \theta$ *(in particular, if* $m = 0$*), then* $\lambda(f) = \theta$.
(3) *Assume* $\{f_1, \ldots, f_m\} = \{x_1, \ldots, x_m\}$. *If* $\deg_{x_{m+1},\ldots,x_n}(p) \le 1$, *then the maximal eigenvalue of* f *is equal to* 1 *and otherwise it is equal to* θ.

Remark 4.2.4. The case $m = n$, not treated in Proposition 4.2.3, is rather trivial. We have $f = (f_1, \ldots, f_n, \xi x_{n+1} + p(x_1, \ldots, x_n))$, where $\{f_1, \ldots, f_n\} = \{x_1, \ldots, x_n\}$. Every matrix contained in f is then a block-matrix with a $(n \times n)$-permutation-matrix and a (1×1)-matrix with a 0 or a 1 on the diagonal, so every eigenvalue is either 0 or a root of unity. This implies that $\theta = 1$ is the only possible maximal eigenvalue.

Proof of Proposition 4.2.3. (1) Because $\deg_{x_{m+1},\ldots,x_n}(p) \le 1$, Lemma 2.4.1 implies that

$$\lambda(f) = \max\{\lambda(\hat{f}), \lim_{r \to \infty} \deg_{x_{m+1},\ldots,x_{n+1}}(f^r)^{1/r}\} = \lambda(\hat{f}),$$

where, by convention, $\lambda(\hat{f}) = 1$ in case $m = 0$.

(2) For each $i = (i_1, \ldots, i_n) \in I$, we set

$$p_i = \sum_{j=m+1}^{n} i_j x^{n-j} \in \mathbb{Z}[x]$$

and $q_i = x^{n-m} - p_i \in \mathbb{Z}[x]$. Then θ is the biggest real root of one of the polynomials in $\{q_i \mid i \in I\}$. Note that q_i is monic and of degree $n - m > 0$. As $\deg_{x_{m+1},\ldots,x_n}(p) \ge 2$, there is $i = (i_1, \ldots, i_n) \in I$ such that $p_i(1) \ge 2$. This implies that $q_i(1) = 1 - p_i(1) < 0$, so q_i has a real root that is bigger than 1. This proves that $\theta > 1$. Moreover, for each $i \in I$, we have $q_i(\theta) \ge 0$, because q_i has no real root bigger than θ. This gives $\theta^{n-m} \ge p_i(\theta)$, with equality for at least one $i \in I$.

We now prove that $\deg_\mu(f_j) = \theta \mu_j$ for each $j \in \{1, \ldots, n+1\}$, where f_j denotes the jth component of f. For each $j \in \{1, \ldots, m\}$, we have $\deg_\mu(f_j) = 0 = \theta \mu_j$ and for each $j \in \{m+2, \ldots, n+1\}$, we have $\deg_\mu(f_j) = \deg_\mu(x_{j-1}) = \mu_{j-1} = \theta \mu_j$. Moreover, we

have

$$\deg_\mu(f_{m+1}) = \max\left(\{\deg_\mu(x_{n+1})\} \cup \left\{\sum_{j=m+1}^{n} i_j \mu_j \mid (i_1, \ldots, i_n) \in \mathbb{N}^n\right\}\right)$$

$$= \max(\{1\} \cup \{\theta \cdot p_i(\theta) \mid i = (i_1, \ldots, i_n) \in \mathbb{N}^n\}) = \theta^{n-m+1} = \theta\mu_{m+1}.$$

This gives in particular $\theta = \deg_\mu(f)$.

It remains to prove that f is μ-algebraically stable, that is, that $\deg_\mu(f^r) = \theta^r$ for each $r \geq 1$; this will then give the result by Corollary 2.6.2.

By Lemma 2.6.1(5), this corresponds to ask that for each $r \geq 1$, there exists $j \in \{m+1, \ldots, n\}$ such that $(g^r)_j \neq 0$, where $g = (g_1, \ldots, g_{n+1}) \in \mathrm{End}(\mathbb{A}^{n+1})$ is the μ-leading part of f and $(g^r)_j$ denotes the jth component of g^r. We observe that

$$g = (f_1, \ldots, f_m, g_{m+1}, x_{m+1}, \ldots, x_n),$$

where $g_{m+1} \in \mathbf{k}[x_1, \ldots, x_{n+1}] \setminus \{0\}$. The result then follows from Lemma 4.2.2

(3) The maximal eigenvalue of f is the biggest real number that is an eigenvalue of one of the matrices contained in f. Each such matrix is either contained in $(f_1, \ldots, f_m, \xi x_{n+1}, x_{m+1}, \ldots, x_n)$, but then has spectral radius equal to 1, or is contained in $(f_1, \ldots, f_m, \prod_{j=1}^{n} x_j^{i_j}, x_{m+1}, \ldots, x_n)$ for some $(i_1, \ldots, i_n) \in I$. In this latter case, the spectral radius is the one of the matrix

$$\begin{pmatrix} i_{m+1} & \cdots & i_n & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

and thus equal to the biggest real root of the polynomial $x^{n-m} - \sum_{j=m+1}^{n} i_j x^{n-j}$. If $\deg_{x_{m+1}, \ldots, x_n}(p) \leq 1$, the maximal eigenvalue is again equal to 1, and if $\deg_{x_{m+1}, \ldots, x_n}(p) \geq 2$, we get that θ is the maximal eigenvalue of f. $\square$

As mentioned in the §1, the following result is attributed to Mattias Jonsson (unpublished).

PROPOSITION 4.2.5. *For each $n \geq 1$ and each polynomial $p \in \mathbf{k}[x_1, \ldots, x_n]$ of degree ≥ 2, let $e_p \in \mathrm{Aut}(\mathbb{A}^{n+1})$ be the automorphism*

$$e_p = (x_{n+1} + p(x_1, \ldots, x_n), x_1, \ldots, x_n) \in \mathrm{Aut}(\mathbb{A}^{n+1}).$$

Let $I \subset \mathbb{N}^n$ be the finite subset of indices of the monomials of p. We get

$$\lambda(e_p) = \max\left\{\lambda \in \mathbb{R} \mid \lambda^n = \sum_{j=1}^{n} i_j \lambda^{n-j} \text{ for some } (i_1, \ldots, i_n) \in I\right\}.$$

Proof. Apply Proposition 4.2.3(2) with $m = 0$ and $\xi = 1$. $\square$

Proof of Proposition C. Let $h \in \mathrm{End}(\mathbb{A}^{n+1})$ be a permutation-elementary automorphism. By Lemma 4.2.1, there is a permutation of the coordinates $\alpha \in \mathrm{Sym}(\mathbb{A}^{n+1})$ such that

$$f = \alpha \circ h \circ \alpha^{-1} = (f_1, \ldots, f_m, \xi x_{n+1} + p(x_1, \ldots, x_n), x_{m+1}, \ldots, x_n),$$

where $0 \le m \le n$, $\{x_1, \ldots, x_m\} = \{f_1, \ldots, f_m\}$, $\xi \in \mathbf{k}^*$ and $p \in \mathbf{k}[x_1, \ldots, x_n]$. In particular, $\lambda(\hat{f}) = 1$, where $\hat{f} = (f_1, \ldots, f_m) \in \mathrm{Sym}(\mathbb{A}^m)$.

As the maximal eigenvalue θ of f is bigger than 1, we have $m < n$ (see Remark 4.2.4). Moreover, Proposition 4.2.3(3) yields that $\deg_{x_{m+1}, \ldots, x_n}(p) \ge 2$. Then, Proposition 4.2.3(2), (3) give the existence of a maximal eigenvector μ such that f is μ-algebraically stable and prove that the dynamical degree $\lambda(f)$ is equal to the maximal eigenvalue θ of f (this latter fact also follows from Proposition B). Because $\alpha \in \mathrm{Sym}(\mathbb{A}^{n+1})$, we get that $\alpha^{-1}(\mu)$ is a maximal eigenvector of $h = \alpha^{-1} \circ f \circ \alpha$, h is $\alpha^{-1}(\mu)$-algebraically stable and θ is the maximal eigenvalue of h. Moreover, $\lambda(h) = \lambda(f)$. Proposition 4.2.3(2) shows that θ is the root of a monic integral polynomial where all coefficients (except the first one) are non-positive, so it is a Handelman number by definition. $\qquad\square$

4.3. *Affine-triangular automorphisms of $\mathbb{A}^3$.*

In this section, we apply Proposition B to affine-triangular automorphisms $f \in \mathrm{Aut}(\mathbb{A}^3)$ and prove Proposition D and Theorem 1. By Proposition 4.1.1, we can reduce to the case of permutation-triangular automorphisms. If the maximal eigenvalue θ of f is equal to 1, then Proposition B gives $\lambda(f) = \theta$. If $\theta > 1$, there is a maximal eigenvector $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{R}_{\ge 0})^n \setminus \{0\}$ of f, and if f is μ-algebraically stable, we obtain $\lambda(f) = \theta$ (Proposition B(3)). We will then study the cases where f is not μ-algebraically stable. This implies that the μ-leading part g of f is such that one component of g^r is equal to zero for some $r \ge 1$. The possibilities for such endomorphisms g are studied in Lemma 4.3.2 below. The following result is a simple observation, whose proof is left as an exercise.

LEMMA 4.3.1. *Let $n \ge 1$ and let $f = (f_1, \ldots, f_n) \in \mathrm{TEnd}(\mathbb{A}^n)$ be a triangular endomorphism. Then:*

(1) *f is dominant if and only if $\deg_{x_i}(f_i) \ge 1$ for each $i \in \{1, \ldots, n\}$; and*
(2) *f is an automorphism if and only if $\deg_{x_i}(f_i) = 1$ for each $i \in \{1, \ldots, n\}$.*

LEMMA 4.3.2. *Let $g = (g_1, g_2, g_3) = \sigma \circ \tau \in \mathrm{End}(\mathbb{A}^3)$, where $\tau \in \mathrm{TEnd}(\mathbb{A}^3)$ is a triangular endomorphism and $\sigma \in \mathrm{Sym}(\mathbb{A}^3)$ is a permutation of the coordinates, where all g_i are non-constant and such that one of the components of g^r is a constant for some $r \ge 2$. Then, one of the following holds:*

(1) *$g_2, g_3 \in \mathbf{k}[x_1]$, $g_1 \in \mathbf{k}[x_1, x_2, x_3] \setminus (\mathbf{k}[x_1, x_2] \cup \mathbf{k}[x_1, x_3])$ and there exists $\zeta \in \mathbf{k}$ such that $g_1(t, g_2, g_3) = \zeta$ for each $t \in \mathbf{k}$;*
(2) *$g_1, g_3 \in \mathbf{k}[x_1]$, $g_2 \in \mathbf{k}[x_1, x_3] \setminus \mathbf{k}[x_1]$;*
(3) *$g_1, g_2 \in \mathbf{k}[x_1]$, $g_3 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$; and*
(4) *$g_1, g_2 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$, $g_3 \in \mathbf{k}[x_1]$ and $g_1(g_1, g_2) = \zeta_1$, $g_2(g_1, g_2) = \zeta_2$ for some $\zeta_1, \zeta_2 \in \mathbf{k}$.*

Proof. We distinguish some cases, depending on which of the polynomials g_1, g_2, g_3 belong to $\mathbf{k}[x_1]$.

We first observe that $g_1, g_2, g_3 \in \mathbf{k}[x_1]$ is impossible, as each component of g^r, for each $r \geq 1$, would then be obtained by composing dominant endomorphisms of $\mathbb{A}^1$ and thus would not be constant.

- Suppose that $g_1, g_3 \in \mathbf{k}[x_1]$. By induction, we obtain $(g^r)_1, (g^r)_3 \in \mathbf{k}[x_1] \setminus \mathbf{k}$ for each $r \geq 1$, so $(g^r)_2 \in \mathbf{k}$ for some $r \geq 2$. If $g_2 \in \mathbf{k}[x_1, x_3]$, we obtain possibility (2). Otherwise, $\deg_{x_2}(g_2) = d \geq 1$ and proceeding by induction, we obtain $\deg_{x_2}((g^r)_2) = d^r \geq 1$ for each $r \geq 1$, impossible.

- If $g_1, g_2 \in \mathbf{k}[x_1]$, we do the same argument as before (by exchanging the roles of x_2 and x_3) and obtain possibility (3).

- Suppose now that $g_2, g_3 \in \mathbf{k}[x_1]$. As $g_1 \in \mathbf{k}[x_1, x_2, x_3] \setminus \mathbf{k}[x_1]$, the closure of the image of $g \in \mathrm{End}(\mathbb{A}^3)$ is then equal to $\mathbb{A}^1 \times \Gamma$, where $\Gamma \subset \mathbb{A}^2$ is the irreducible curve that is the closure of the image of $\mathbb{A}^1 \to \mathbb{A}^2$, $x_1 \mapsto (g_2(x_1), g_3(x_1))$. The restriction of g gives an endomorphism $h = g|_{\mathbb{A}^1 \times \Gamma} \in \mathrm{End}(\mathbb{A}^1 \times \Gamma)$.

  We now prove that h is not dominant. For each $r \geq 1$ and each $i \in \{1, 2, 3\}$, the restriction of $(g^r)_i$ to $\mathbb{A}^1 \times \Gamma$ is equal to $\pi_i \circ h^r$, where $\pi_i \colon \mathbb{A}^1 \times \Gamma \to \mathbb{A}^1$ is the ith projection. Choosing i and r such that $(g^r)_i$ is constant, we find that $\pi_i \circ h^r$ is constant, so h^r is not dominant, as π_i is dominant. This proves that h is not dominant.

  Denote by $\Gamma' \subset \mathbb{A}^1 \times \Gamma$ the closure of $h(\mathbb{A}^1 \times \Gamma)$, which is an irreducible curve that contains $\{(g_1(x, g_2(y), g_3(y)), g_2(x), g_3(x)) \mid (x, y) \in \mathbb{A}^2\}$. This implies that the polynomial $s = g_1(x, g_2(y), g_3(y)) \in \mathbf{k}[x, y]$ is contained in $\mathbf{k}[x]$. Moreover, we observe that s is a constant. Indeed, otherwise the restriction of h to Γ' would be a dominant map $\Gamma' \to \Gamma'$ and because $\pi_i|_{\Gamma'} \colon \Gamma' \to \mathbb{A}^1$ is non-constant for each $i \in \{1, 2, 3\}$, the restriction of $(g^r)_i$ to Γ' would be non-constant for each $r \geq 1$ and each $i \in \{1, 2, 3\}$, which is a contradiction. Hence, $\pi_1 \circ h = g_1|_{\mathbb{A}^1 \times \Gamma} \colon \mathbb{A}^1 \times \Gamma \to \mathbb{A}^1$ is equal to a constant $\zeta \in \mathbf{k}$. This yields $g_1(t, g_2, g_3) = \zeta$ for each $t \in \mathbf{k}$ and implies that $g_1 \notin \mathbf{k}[x_1, x_2] \cup \mathbf{k}[x_1, x_3]$, because g_1, g_2, g_3 are non-constant, whence possibility (1).

- It remains to assume that, at most, one of the g_i belongs to $\mathbf{k}[x_1]$. We write $\tau = (\tau_1, \tau_2, \tau_3)$ and observe that $\{g_1, g_2, g_3\} = \{\tau_1, \tau_2, \tau_3\}$. As $\tau_1 \in \mathbf{k}[x_1] \setminus \mathbf{k}$, we get that exactly one of the g_i belongs to $\mathbf{k}[x_1]$ and that $\tau_2 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$. As g is not dominant, neither is τ; Lemma 4.3.1 then implies that $\tau_3 \in \mathbf{k}[x_1, x_2]$. So $g_1, g_2, g_3 \in \mathbf{k}[x_1, x_2]$ and exactly one of the three belongs to $\mathbf{k}[x_1]$. Note that the endomorphism $h = (g_1, g_2) \in \mathrm{End}(\mathbb{A}^2)$ is not dominant. Indeed, otherwise no component of g^r is constant for each $r \geq 1$, as $g_3 \in \mathbf{k}[x_1, x_2]$ is non-constant. Thus it is impossible that $g_1 \in \mathbf{k}[x_1]$ or $g_2 \in \mathbf{k}[x_1]$, as (g_1, g_2) (respectively (g_2, g_1)) would be a dominant triangular endomorphism of $\mathbb{A}^2$ (Lemma 4.3.1). Hence, $g_3 \in \mathbf{k}[x_1] \setminus \mathbf{k}$ and $g_1, g_2 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$. As h is not dominant, the closure of $h(\mathbb{A}^2)$ is an irreducible curve $\Gamma \subset \mathbb{A}^2$.

  If $g_j(\Gamma)$ is not a point for $j = 1$ or $j = 2$, then the restriction $h|_\Gamma \colon \Gamma \to \Gamma$ would be dominant. As g_3 is not constant on Γ (because $g_3(g_1(x_1, x_2))$ is not constant), we get that $(g^r)_i$ is non-constant for each $r \geq 1$ and each $i \in \{1, 2, 3\}$, which is a contradiction. Thus $g_i(\Gamma) = \{\zeta_i\}$ for $i = 1, 2$, where $\zeta_i \in \mathbf{k}$. This gives possibility (4). $\square$

LEMMA 4.3.3. *Let* $f = \sigma \circ \nu \in \mathrm{Aut}(\mathbb{A}^3)$ *be a permutation-triangular automorphism, where* $\sigma \in \mathrm{Sym}(\mathbb{A}^3)$ *and* $\nu \in \mathrm{TAut}(\mathbb{A}^3)$. *Suppose that the maximal eigenvalue* θ *of* f *is bigger than 1 and let* μ *be a maximal eigenvector of* f *such that* f *is not* μ-*algebraically stable. Then, one of the following cases holds.*

(i) $f = (\xi_3 x_3 + p_3(x_1, x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1))$, *where* $\xi_2, \xi_3 \in \mathbf{k}^*$, $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$, $\deg(p_1) = 1$, *and* $\deg(p_2) = \theta^2 > 1$. *Moreover, there exists* $s \in \mathbf{k}[x_2]$ *such that the conjugation of* f *by* $(x_1, x_2, x_3 + s(x_2))$ *does not increase the degree of* p_3 *and (strictly) decreases the degree of* p_2.

(ii) $f = (\xi_2 x_2 + p_2(x_1), \xi_3 x_3 + p_3(x_1, x_2), p_1(x_1))$, *where* $\xi_2, \xi_3 \in \mathbf{k}^*$, $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$, $\deg(p_1) = 1$ *and* $\deg(p_2) = \theta > 1$. *Moreover, there exists* $s \in \mathbf{k}[x_1]$ *such that the conjugation of* f *by* $(x_1, x_2 + s(x_1), x_3)$ *(strictly) decreases the degrees of* p_2 *and* p_3.

Proof. Denote by $g = (g_1, g_2, g_3)$ the μ-leading part of f. As $\mu = (\mu_1, \mu_2, \mu_3) \in (\mathbb{R}_{\geq 0})^3 \setminus \{0\}$ is a maximal eigenvector of f, $g_i \notin \mathbf{k}$ for each $i \in \{1, 2, 3\}$ (Lemma 2.5.7). Moreover, as f is not μ-algebraically stable, there is some $r \geq 1$ such that $(g^r)_i = 0$ for all $i \in \{1, 2, 3\}$ with $\mu_i > 0$ (Lemma 2.6.1(5)). We write $g = \sigma \circ \tau$, where $\tau = (\tau_1, \tau_2, \tau_3) \in \mathrm{TEnd}(\mathbb{A}^3)$; Lemma 4.3.2 gives then four possibilities (1)–(4) for g, which we consider separately. We will show that cases (i) and (ii) occur in possibilities (1) and (4), respectively and that possibilities (2)–(3) do not occur.

Possibilities (2)–(3): Let us first observe that possibility (2) (respectively (3)) of Lemma 4.3.2 does not occur. Indeed, otherwise the first and the last (respectively the first two) components of g^r belong to $\mathbf{k}[x_1] \setminus \mathbf{k}$ for each $r \geq 1$, so $\mu = (0, \mu_2, 0)$ (respectively $\mu = (0, 0, \mu_3)$), because $(g^r)_i = 0$ for each $i \in \{1, 2, 3\}$ with $\mu_i > 0$. This gives $\deg_\mu(g_i) = 0$ for $i = 1, 2, 3$, as g_1, g_2, g_3 belong to $\mathbf{k}[x_1, x_3]$ (respectively $\mathbf{k}[x_1, x_2]$), which is impossible as $\deg_\mu(g) = \deg_\mu(f) = \theta > 1$ (Lemma 2.5.7).

Possibility (1): Suppose now that possibility (1) of Lemma 4.3.2 occurs. As $g_1 \in \mathbf{k}[x_1, x_2, x_3] \setminus (\mathbf{k}[x_1, x_2] \cup \mathbf{k}[x_1, x_3])$ and because the monomials of g_1 are some of those of f_1, that is, one of the coordinates of the triangular automorphism $\nu \in \mathrm{TAut}(\mathbb{A}^3)$, the polynomial f_1 is equal to the third coordinate of ν and g_1 is of the form $g_1 = \xi_3 x_3 + q(x_1, x_2)$ for some $\xi_3 \in \mathbf{k}^*$ and $q \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$. Because $g_1(t, g_2, g_3) = \zeta \in \mathbf{k}$ for each $t \in \mathbf{k}$, we obtain $\xi_3 g_3 + q(t, g_2) = \zeta$ for each $t \in \mathbf{k}$, so $q \in \mathbf{k}[x_2] \setminus \mathbf{k}$ and

$$g = (\xi_3 x_3 + q(x_2), g_2, (\zeta - q(g_2))/\xi_3),$$

where $g_2 \in \mathbf{k}[x_1]$. By definition (Definition 1.4.5), g_i is the μ-homogeneous part of f_i of degree $\theta \mu_i$, for each $i \in \{1, 2, 3\}$ so each monomial of g_i is of μ-degree $\theta \mu_i$. The explicit form of g_1, g_2, g_3 directly gives

$$\theta \mu_1 = \mu_3 = \deg(q)\mu_2, \ \theta \mu_2 = \deg(g_2)\mu_1 \ \text{and} \ \theta \mu_3 = \deg(g_3)\mu_1 = \deg(q)\deg(g_2)\mu_1.$$

In particular, $\mu_1, \mu_2, \mu_3 \in \mathbb{R}_{>0}$ and $\deg(g_3) = \deg(q)\deg(g_2) = \theta^2 > 1$. Because two monomials in the same variables have distinct μ-degrees, we find that q, g_2 and g_3 are monomials, so $\zeta = 0$.

One component of f (and of τ) belongs to $\mathbf{k}[x_1]$ and is of degree 1. As $g_1 \notin \mathbf{k}[x_1]$ and $\deg(g_3) > 1$, we find that $f_2 \in \mathbf{k}[x_1]$ is of degree 1. This yields $\sigma = (x_3, x_1, x_2)$ and

$\deg(f_2) = \deg(g_2) = 1$, whence $\deg(q) = \deg(g_3) = \theta^2 > 1$. We obtain the form given in case (i): the automorphism f is equal to

$$f = (\xi_3 x_3 + p_3(x_1, x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1)),$$

where $\xi_2, \xi_3 \in \mathbf{k}^*$, $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$, $\deg(p_1) = 1$. Moreover, $g_3 = -q(g_2)/\xi_3 \in \mathbf{k}[x_1]$ is the μ-leading part of $f_3 = \xi_2 x_2 + p_2(x_1)$, so g_3 is only one monomial, of degree $\theta^2 = \deg(g_3) = \deg(p_2)$.

To prove that we are indeed in case (i), it remains to show that the conjugation by $h = (x_1, x_2, x_3 + \xi_3^{-1} q(x_2))$ does not increase the degree p_3 and strictly decreases the degree of p_2. We calculate

$$h \circ f \circ h^{-1} = (\xi_3 x_3 + p_3(x_1, x_2) - q(x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1) + q(p_1(x_1))/\xi_3).$$

As every monomial of $g_1 = \xi_3 x_3 + q(x_2)$ is contained in $f_1 = \xi_3 x_3 + p_3(x_1, x_2)$, the degree of $p_3(x_1, x_2) - q(x_2)$ is at most that of $p_3(x_1, x_2)$. It remains to see that $\deg(p_2 + q(p_1)/\xi_3) < \deg(p_2)$, which follows from the fact that $g_3 = -q(g_2)/\xi_3 \in \mathbf{k}[x_1]$ is the μ-leading part of $f_3 = \xi_2 x_2 + p_2(x_1)$, and that g_2 is the leading monomial of p_1 (of degree 1).

Possibility (4): It remains to consider possibility (4) of Lemma 4.3.2. As $g_1, g_2 \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$, the only component of f which belongs to $\mathbf{k}[x_1]$ (and is of degree 1) is f_3, so $\sigma = (x_3, x_2, x_1)$ or $\sigma = (x_2, x_3, x_1)$. Let $j \in \{1, 2\}$ be such that $f_j = \nu_2$, where $\nu = (\nu_1, \nu_2, \nu_3)$. We then have $f_j = \xi_2 x_2 + p_2(x_1)$ for some $\xi_2 \in \mathbf{k}^*$ and some $p_2 \in \mathbf{k}[x_1]$. As $g_j \in \mathbf{k}[x_1, x_2] \setminus \mathbf{k}[x_1]$, we get $g_j = \xi_2 x_2 + q(x_1)$ for some $q \in \mathbf{k}[x_1]$, which consists of some monomials of p_2. Because $\zeta_j = g_j(g_1, g_2)$, we obtain $\zeta_j = \xi_2 g_2 + q(g_1)$.

We now show that $j = 2$ leads to a contradiction. It gives

$$g_2 = \xi_2 x_2 + q(x_1) = \xi_2^{-1}(\zeta_2 - q(g_1)).$$

Because $\xi_2 x_2 + q(x_1)$ is irreducible, the polynomial $\zeta_2 - q(g_1)$ is irreducible, and thus $\deg(q) = 1$, which in turn implies that g_2 and thus g_1 is of degree 1. Hence, g_1, g_2, g_3 are of degree 1, which is impossible, as $\theta > 1$ is the eigenvalue of a matrix that is contained in g (Lemma 2.5.1).

This contradiction proves that $j = 1$, so $\sigma = (x_2, x_3, x_1)$. This yields

$$f = (\xi_2 x_2 + p_2(x_1), \xi_3 x_3 + p_3(x_1, x_2), p_1(x_1)),$$

where $\xi_2, \xi_3 \in \mathbf{k}^*$, $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$ and $\deg(p_1) = 1$, as in case (ii).

We also have $g_1 = \xi_2 x_2 + q(x_1)$ and $\zeta_1 = \xi_2 g_2 + q(g_1)$, which yields $g_2 = (\zeta_1 - q(g_1))/\xi_2 = (\zeta_1 - q(\xi_2 x_2 + q(x_1)))/\xi_2$. As g is the μ-leading part of f, the polynomial g_2 is not constant (Lemma 2.5.7), so $\deg(q) \geq 1$. Recall that g_i is the μ-homogeneous part of f_i of degree $\theta\mu_i$, for each $i \in \{1, 2, 3\}$ (Definition 1.4.5) so each monomial of g_i is of μ-degree $\theta\mu_i$. We thus obtain

$$\theta\mu_1 = \mu_2 = \deg(q)\mu_1 \quad \text{and} \quad \theta\mu_3 = \mu_1.$$

This proves that $\mu_1, \mu_2, \mu_3 \in \mathbb{R}_{>0}$, that $\deg(q) = \theta > 1$ and that $\mu = (\theta\mu_3, \theta^2\mu_3, \mu_3)$.

Because two monomials in the same variables have distinct μ-degrees, we moreover find that q is a monomial, the leading monomial of p_2, so $\deg(p_2) = \deg(q) = \theta > 1$, as stated in case (ii).

To prove that we are indeed in case (ii), it remains to show that the conjugation by $h = (x_1, x_2 + q(x_1)/\xi_2, x_3)$ strictly decreases the degree of p_2 and p_3. We calculate

$$h \circ f \circ h^{-1} = (\xi_2 x_2 + p_2'(x_1), \xi_3 x_3 + p_3'(x_1, x_2), p_1(x_1)),$$

where

$$p_2'(x_1) = p_2(x_1) - q(x_1),$$
$$p_3'(x_1, x_2) = p_3(x_1, x_2 - q(x_1)/\xi_2) + q(\xi_2 x_2 + p_2'(x_1))/\xi_2.$$

As q is the leading monomial of p_2, this conjugation decreases the degree of p_2, that is, $\deg(p_2') < \deg(p_2) = \theta$. It remains to see that $\deg(p_3') < \deg(p_3)$. To simplify the calculations, we replace μ by a multiple of itself (this is still a maximal eigenvector) and may assume that $\mu = (1, \theta, \theta^{-1})$. As $g_2 = (\zeta_1 - q(\xi_2 x_2 + q(x_1)))/\xi_2$ is the μ-homogeneous part of $f_2 = \xi_3 x_3 + p_3(x_1, x_2)$ of μ-degree $\theta \mu_2 = \theta^2$, the polynomial $\Delta = p_3 - g_2 \in \mathbf{k}[x_1, x_2]$ is equal to

$$\Delta = \sum_{i=0}^{\theta-1} x_2^i \Delta_i,$$

where each $\Delta_i \in \mathbf{k}[x_1]$ is such that $\deg(\Delta_i) + i\theta < \theta^2$. As $\theta > 1$, this implies that $\deg(x_2^i \Delta_i) = i + \deg(\Delta_i) < \theta^2$ for each i, so $\deg(\Delta) < \theta^2$, which implies that the degree of $p_3 = \Delta + g_2$ is equal to θ^2, because $\deg(g_2) = \theta^2$. We then need to show that $\deg(p_3') < \theta^2$. Because $\deg(p_2') < \deg(q) = \theta$, we have $\deg(q(\xi_2 x_2 + p_2'(x_1))/\xi_2) < \theta^2$, so we only need to show that $\deg(p_3(x_1, x_2 - q(x_1)/\xi_2)) < \theta^2$. This is given by

$$p_3(x_1, x_2 - q(x_1)/\xi_2) = \Delta(x_1, x_2 - q(x_1)/\xi_2) + g_2(x_1, x_2 - q(x_1)/\xi_2)$$
$$= \sum_{i=0}^{\theta-1} (x_2 - q(x_1)/\xi_2)^i \Delta_i + (\zeta_1 - q(\xi_2 x_2))/\xi_2$$

and by the fact that $\deg(\Delta_i) + i\theta < \theta^2$ for each i. $\square$

Example 4.3.4. We now give two distinct examples to show that cases (1) and (2) of Lemma 4.3.3 indeed occur.

(1)   Let $n \geq 2$, and let $f = (x_3 - x_2^n, x_1, x_2 + x_1^n) \in \mathrm{Aut}(\mathbb{A}^3)$. Because of the matrix contained in (x_3, x_1, x_1^n), the maximal eigenvalue satisfies $\theta \geq \sqrt{n} > 1$ and as $f^2 = (x_2, x_3 - x_2^n, x_1 + (x_3 - x_2^n)^n)$ and $f^3 = (x_1, x_2, x_3)$, the map f is not μ-algebraically stable for any maximal eigenvector μ of f. It has then to satisfy case (i) of Lemma 4.3.3, so $\theta = \sqrt{n}$.

(2)   Let $n \geq 2$ and let $f = (x_2 - x_1^n, x_3 + (x_2 - x_1^n)^n, x_1) \in \mathrm{Aut}(\mathbb{A}^3)$. Because of the matrix contained in $(-x_1^n, x_3, x_1)$, the maximal eigenvalue satisfies $\theta \geq n > 1$ and as $f^2 = (x_3, x_1 + x_3^n, x_2 - x_1^n)$ and $f^3 = (x_1, x_2, x_3)$, the element f is not μ-algebraically stable for each maximal eigenvector μ of f. It has then to satisfy case (ii) of Lemma 4.3.3, so $\theta = n$.

We now give examples of permutation-triangular automorphisms of $\mathbb{A}^3$, which are μ-algebraically stable. These will be useful in the proof of Theorem 1.

LEMMA 4.3.5. *For all $a, b, c \in \mathbb{N}$ such that $\lambda = (a + \sqrt{a^2 + 4bc})/2 \neq 0$, the maximal eigenvalue and the dynamical degree of the automorphisms*

$$f = (x_1^a x_2^b + x_3, x_2 + x_1^c, x_1) \quad and \quad f' = (x_3 + x_1^a x_2^{bc}, x_1, x_2)$$

are equal to λ. Both automorphisms are μ-algebraic stable for each maximal eigenvector μ.

Proof. The matrices that are contained in f are

$$\begin{pmatrix} a & b & 0 \\ c & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ c & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

whose characteristic polynomials are $x(x^2 - ax - bc)$, $x(x - a)(x - 1)$, $x(x^2 - 1)$ and $(x + 1)(x^2 - 1)$, respectively. The corresponding spectral radii are respectively λ, a, 1 and 1. Hence, the maximal eigenvalue of f is λ.

Similarly, the matrices contained in f' are

$$\begin{pmatrix} a & bc & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

whose characteristic polynomials are $x(x^2 - ax - bc)$ and $x^3 - 1$. The maximal eigenvalue of f' is then also λ.

As neither f nor f' satisfies any of the two cases (i) and (ii) of Lemma 4.3.3, both f and f' are μ-algebraically stable for each maximal eigenvector μ (of f and f', respectively). This gives then $\lambda(f) = \lambda(f') = \lambda$ (Proposition B) and achieves the proof. □

LEMMA 4.3.6. *The maximal eigenvalue θ of a permutation-triangular automorphism $f \in$ Aut$(\mathbb{A}^3)$ of degree $d \geq 1$ is a non-zero number equal to $(a + \sqrt{a^2 + 4bc})/2$ for some $(a, b, c) \in \mathbb{N}^3$, where $a + b \leq d$ and $c \leq d$. It is thus a positive integer or a quadratic integer and a Handelman number.*

Proof. Each real number $\theta = (a + \sqrt{a^2 + 4bc})/2 \neq 0$, where $(a, b, c) \in \mathbb{N}^3$ is a root of the polynomial $P(x) = x^2 - ax - bc$, with $a, b, c \in \mathbb{N}^2 \setminus \{0\}$ so it is a Handelman number. If P is irreducible, then θ is a quadratic integer, and otherwise it is a positive integer. It remains to see that the maximal eigenvalue of every f is of the desired form.

We write $f = \sigma \circ \tau$, where $\sigma \in$ Sym$(\mathbb{A}^3)$ and $\tau \in$ TAut$(\mathbb{A}^3)$ is a triangular automorphism, which we write as $\tau = (\nu_1 x_1 + \epsilon, \nu_2 x_2 + p(x_1), \nu_3 x_3 + q(x_1, x_2))$, where $\nu_1, \nu_2, \nu_3 \in \mathbf{k}^*$, $\epsilon \in \mathbf{k}$, $p \in \mathbf{k}[x_1]$ and $q \in \mathbf{k}[x_1, x_2]$. The matrices contained in τ are all of the form

$$\begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} m & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ i & j & 0 \end{pmatrix}, \begin{pmatrix} m & 0 & 0 \\ k & 0 & 0 \\ i & j & 0 \end{pmatrix},$$

where m, k, i, j are non-negative integers and $0 \leq m \leq 1$, $k \leq \deg(p) \leq d$ and $i + j \leq \deg(q) \leq d$. Because the spectral radius is order-preserving on real square matrices with non-negative coefficients (see Definition 3.1.1(4)) and because $\nu_1 \neq 0$, the maximal eigenvalue is the spectral radius of a matrix where $m = 1$. The matrices contained in f are obtained from one of the above four types by permuting the rows. Permuting the rows of the identity matrix only gives a spectral radius equal to 1. In the second case, we conjugate by the permutation of the last two. In any case, we obtain that θ is either equal to 1 or is the spectral radius of a matrix $\sigma' M$, where σ' is a permutation matrix and M is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & j & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ k & 0 & 0 \\ i & j & 0 \end{pmatrix},$$

where $k \leq d$ and $i + j \leq d$. We obtain

$$\sigma' M = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & 0 \end{pmatrix}$$

for some $m_{ij} \in \mathbb{N}$, so θ is the spectral radius of the matrix

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

This last matrix is one of the following:

$$\begin{pmatrix} r & 0 \\ s & 0 \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} r & 0 \\ i & j \end{pmatrix}, \begin{pmatrix} i & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & j \\ r & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ i & j \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix},$$

where $r, s \in \{1, i, j, k\}$. In the first four cases, θ is an integer in $\{1, \ldots, d\}$, so has the desired form, with $a = \theta$, and $b = c = 0$. In the fifth case, the characteristic polynomial is $x^2 - ix - jr$. Choosing $a = i$, $b = j$ and $c = r$, we get $\theta = (a + \sqrt{a^2 + 4bc})/2$. In the sixth case, the characteristic polynomial is $x^2 - jx - i$. When we choose $a = j, b = i$ and $c = 1$, we get again $\theta = (a + \sqrt{a^2 + 4bc})/2$. In the last case, the characteristic polynomial is $x^2 - rs$. We then choose $a = 0, b = r$ and $c = s$. □

We can now give the proof of Proposition D.

Proof of Proposition D. We take an affine-triangular automorphism $f \in \text{Aut}(\mathbb{A}^3)$. By Proposition 4.1.1, there exists $\alpha \in \text{Aff}(\mathbb{A}^3)$ such that $f' = \alpha f \alpha^{-1}$ is a permutation-triangular automorphism. We then have $\deg(f') = \deg(f)$. Moreover, Proposition B shows that there exists a maximal eigenvector of f. We denote by θ the maximal eigenvalue of f'. If $\theta = 1$ or if f' is μ-algebraically stable for each maximal eigenvector μ, the dynamical degrees $\lambda(f)$ and $\lambda(f')$ are equal to the maximal eigenvalue θ of f' (Proposition B), which is a Handelman number (Lemma 4.3.6) so the result holds.

Suppose now that $\theta > 1$ and that f' is not μ-algebraically stable for some maximal eigenvector μ. Lemma 4.3.3 gives two possibilities for f':

$$f' = (\xi_3 x_3 + p_3(x_1, x_2), p_1(x_1), \xi_2 x_2 + p_2(x_1)) \quad \text{or}$$
$$f' = (\xi_2 x_2 + p_2(x_1), \xi_3 x_3 + p_3(x_1, x_2), p_1(x_1)),$$

where $p_1, p_2 \in \mathbf{k}[x_1]$, $p_3 \in \mathbf{k}[x_1, x_2]$, $\xi_2, \xi_3 \in \mathbf{k}^*$, $\deg(p_1) = 1$ and $\deg(p_2) > 1$. In both cases, Lemma 4.3.3 shows that one can replace f' by a conjugate, decrease the degree of p_2 and not increase the degree of f'. After finitely many steps, we obtain the desired case where $\theta = 1$ or f' is μ-algebraically stable for each maximal eigenvector μ. Moreover, we still have $\deg(f') \leq \deg(f)$. $\quad\square$

Proof of Theorem 1. Let $f \in \mathrm{Aut}(\mathbb{A}^3)$ be an affine-triangular automorphism of $\mathbb{A}^3$ of degree d. Proposition D gives the existence of a permutation-triangular automorphism $f' \in \mathrm{Aut}(\mathbb{A}^3)$ such that $\deg(f') \leq \deg(f)$ and such that either the maximal eigenvalue θ of f' is equal to 1, or $\theta > 1$ and f' is μ-algebraically stable for each maximal eigenvector μ. In the first case, the dynamical degree $\lambda(f)$ is equal to $\lambda(f') = 1$, by Proposition B(2). In the second case, we obtain $\lambda(f) = \lambda(f') = \theta$, by Proposition B(3). Moreover, Lemma 4.3.6 proves that $\theta = (a + \sqrt{a^2 + 4bc})/2$ for some $a, b, c \in \mathbb{N}$ with $a + b \leq d$, $c \leq d$ (and that $\theta \neq 0$).

Conversely, for all $a, b, c \in \mathbb{N}$ such that $\theta = (a + \sqrt{a^2 + 4bc})/2 \neq 0$, the element θ is the dynamical degree of $(x_1^a x_2^b + x_3, x_2 + x_1^c, x_1)$ and $(x_3 + x_1^a x_2^{bc}, x_1, x_2)$ (Lemma 4.3.5), and thus of a permutation-triangular automorphism of $\mathbb{A}^3$. This achieves the proof. $\quad\square$

COROLLARY 4.3.7. *For each $d \geq 3$, the set of all dynamical degrees of shift-like automorphisms of $\mathbb{A}^3$ of degree d is strictly contained in the set of all dynamical degrees of affine-triangular automorphisms of degree d.*

Proof. As each shift-like automorphism is also an affine-triangular automorphism, we have an inclusion that we need to prove to be strict. From Proposition 4.2.5, it follows that the set of dynamical degrees of all shift-like automorphisms of $\mathbb{A}^3$ of degree d is equal to

$$\{ (a + \sqrt{a^2 + 4d - 4a})/2 \mid 0 \leq a \leq d \}.$$

From Theorem 1, it follows that $\lambda_d = (1 + \sqrt{1 + 4d})/2$ is the dynamical degree of the affine-triangular automorphism $(x_3 + x_1 x_2, x_2 + x_1^d, x_1)$. To show that λ_d is not the dynamical degree of any shift-like automorphism of $\mathbb{A}^3$ of degree d, for each $d \geq 3$, we only have to show that there exists no $d \geq 3$ and no $a \in \{0, \ldots, d\}$ such that

$$\sqrt{1 + 4d} = (a - 1) + \sqrt{a^2 + 4d - 4a}.$$

Indeed, if this would be the case, then $1 + 4d = (a - 1)^2 + 2(a - 1)\sqrt{a^2 + 4d - 4a} + a^2 + 4d - 4a$, which yields

$$a(3 - a) = (a - 1)\sqrt{a^2 + 4d - 4a}.$$

This implies that $a \leq 3$ and $a \notin \{0, 1\}$, that is, $a = 2$. However, in this case, $d = 2$. $\quad\square$

4.4. *Automorphisms of affine spaces associated to weak Perron numbers.* In this section, we construct some affine-triangular automorphisms associated to weak Perron numbers and prove Theorem 2.

LEMMA 4.4.1. *Let $n \geq 1$ and let $A = (a_{i,j})_{i,j=1}^{n} \in \mathrm{Mat}_n(\mathbb{N})$ be an irreducible matrix with spectral radius $\rho(A) > 1$. The automorphism $f \in \mathrm{Aut}(\mathbb{A}^{2n})$ given by*

$$\left(x_{n+1} + \prod_{i=1}^{n} x_i^{a_{1,i}}, \, x_{n+2} + \prod_{i=1}^{n} x_i^{a_{2,i}}, \ldots, x_{2n} + \prod_{i=1}^{n} x_i^{a_{n,i}}, \, x_1, \ldots, x_n \right) \qquad (*)$$

has dynamical degree $\lambda(f) = \rho(A)$.

Proof. Let us write $\theta = \rho(A)$ and choose an eigenvector $v = (v_1, \ldots, v_n) \in (\mathbb{R}_{>0})^n$ of A to the eigenvalue θ (which exists by Theorem 3.2.3). We then choose $\mu = (\theta v_1, \ldots, \theta v_n, v_1, \ldots, v_n) \in (\mathbb{R}_{>0})^{2n}$. The matrix

$$M = \begin{pmatrix} A & 0 \\ I_n & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{N})$$

is contained in f, its spectral radius is θ and μ is an eigenvector of M to the eigenvalue θ. Writing $f = (f_1, \ldots, f_{2n})$, we now prove that $\deg_\mu(f_j) = \theta \mu_j$ for each $j \in \{1, \ldots, 2n\}$, and compute the μ-homogeneous part g_j of f_j of degree $\theta \mu_j$.

(1)	For each $j \in \{1, \ldots, n\}$, we have $\deg_\mu(x_{n+j}) = v_j$ and $\deg_\mu(\prod_{i=1}^{n} x_i^{a_{j,i}}) = \sum_{i=1}^{n} \theta a_{j,i} v_i = \theta^2 v_j$, so $\deg_\mu(f_j) = \theta^2 v_j = \theta \mu_j$ and $g_j = \prod_{i=1}^{n} x_i^{a_{j,i}}$.

(2)	For each $j \in \{n+1, \ldots, 2n\}$, we have $\deg_\mu(f_j) = \deg_\mu(x_{j-n}) = \theta v_{j-n} = \theta \mu_j$ and $g_j = f_j$.

This implies that $\deg_\mu(f) = \theta$. As the endomorphism $g = (g_1, \ldots, g_{2n}) \in \mathrm{End}(\mathbb{A}^{2n})$ is monomial, it satisfies $g^r \neq 0$ for each $r \geq 1$ (and, moreover, each component of g^r is not zero). This implies that f is μ-algebraically stable and that $\lambda(f) = \theta$ (see Proposition A). $\qquad \square$

PROPOSITION 4.4.2. *Let $\lambda \in \mathbb{R}$ be a weak Perron number that is a quadratic integer and let $x^2 - ax - b$ be its minimal polynomial, with $a, b \in \mathbb{Z}$. We then have $a \geq 0$ and the following hold.*

(1)	*If $b \geq 0$, then λ is the dynamical degree of the shift-like automorphism*

$$(x_3 + x_1^a x_2^b, \, x_1, x_2) \in \mathrm{Aut}(\mathbb{A}^3).$$

(2)	*If $b < 0$, then λ is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$, but is the dynamical degree of a permutation-triangular automorphism of $\mathbb{A}^4$ of the form $(*)$ in Lemma 4.4.1.*

Proof. Let us write $x^2 - ax - b = (x - \lambda)(x - \mu)$ for some $\mu \in \mathbb{R}$. Note that $\mu \neq \lambda$, as otherwise $\lambda^2 \in \mathbb{Z}$ and $2\lambda \in \mathbb{Z}$ would imply that $\lambda \in \mathbb{Z}$, which is impossible as λ is a quadratic integer. Because λ is a weak Perron number, we have $\lambda \geq 1$ and $-\lambda \leq \mu < \lambda$. In particular, $a = \lambda + \mu \geq 0$. As $x^2 - ax - b$ is irreducible and has a real root by assumption, the discriminant is $a^2 + 4b \geq 1$.

If $b \geq 0$, assertion (1) follows from Lemma 4.3.5 (and also from Proposition 4.2.3).

Suppose now that $b < 0$. As $\lambda\mu = -b$, this implies that $\mu > 0$, so λ is not a Handelman number (Lemma 3.2.7) and thus is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ (Proposition D). It is now enough to show that

$$f = (x_3 + x_1^\alpha x_2, x_4 + x_1^{\alpha(a-\alpha)+b} x_2^{a-\alpha}, x_1, x_2) \in \mathrm{Aut}(\mathbb{A}^4)$$

is a permutation-triangular automorphism with dynamical degree $\lambda(f) = \lambda$.

First, we prove that f is a permutation-triangular automorphism of $\mathbb{A}^4$ by showing that the exponents are non-negative. As $a \geq 0$, the numbers $\alpha = \lfloor a/2 \rfloor$ and $a - \alpha$ are non-negative integers, so we only need to see that $\alpha(a - \alpha) + b \geq 0$. Because $a^2 + 4b \geq 1$, we get in case a is even, that $\alpha(a - \alpha) + b = \alpha^2 + b = (a^2 + 4b)/4 > 0$ and in case a is odd, that $\alpha = (a - 1)/2$, so $\alpha(a - \alpha) + b = ((a - 1)/2) \cdot ((a + 1)/2) + b = (a^2 + 4b - 1)/4 \geq 0$.

Second, the matrix

$$A = \begin{pmatrix} \alpha & 1 \\ \alpha(a - \alpha) + b & a - \alpha \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{N})$$

has characteristic polynomial $x^2 - ax - b$ and thus spectral radius $\rho(A) = \lambda$. As $x^2 - ax - b$ is irreducible by assumption, it follows that A is an irreducible matrix. Moreover, as $b \leq -1$ and as $x^2 - ax - b$ has a real root, we get $a \neq 0$ and hence $a \geq 1$. Because $a^2 + 4b \geq 1$, we get $\lambda = (a + \sqrt{a^2 + 4b})/2 \geq 1$. Now, if $\lambda = 1$, then $1 \leq a \leq 2$ and thus $a^2 + 4b \leq 0$ (as $b \leq -1$), which is a contradiction. Thus $\lambda > 1$ and we can apply Lemma 4.4.1 and get that the dynamical degree of f is $\lambda(f) = \rho(A) = \lambda$. $\square$

Proof of Theorem 2. Let $\lambda \geq 1$ be a weak Perron number. By Theorem 3.2.4, λ is the spectral radius of an irreducible square matrix with non-negative integral coefficients. Lemma 4.4.1 then shows that λ is the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^n$ for some integer n. We denote by n_0 the least possible such n.

If $\lambda = 1$, then $n_0 = 1$, by taking the identity.

If $\lambda > 1$ is an integer, then $n_0 \geq 2$, as every automorphism of $\mathbb{A}^1$ is affine and thus has dynamical degree 1. Moreover, $n_0 = 2$ as $f = (x_1^\lambda + x_2, x_1)$ has dynamical degree equal to λ (f is μ-algebraic stable for $\mu = (1, 0)$ and $\deg_\mu(f) = \lambda$).

If λ is not an integer, then $n_0 \geq 3$, as the dynamical degree of every automorphism of $\mathbb{A}^2$ is an integer (Corollary 2.4.3). If λ is a quadratic integer, the minimal polynomial of λ is equal to $x^2 - ax - b$ with $a \geq 0$ and $b \in \mathbb{Z}$ (Proposition 4.4.2). If the conjugate of λ is negative, we have $b > 0$, so $n_0 = 3$ by Proposition 4.4.2(1). If the conjugate of λ is positive, we have $b < 0$, so $n_0 = 4$ by Proposition 4.4.2(2). $\square$

To complement Theorem 2, we now give a family of examples of quadratic integers that do not arise as dynamical degrees of affine-triangular automorphisms of $\mathrm{Aut}(\mathbb{A}^3)$ but which arise as dynamical degrees of some other automorphisms of $\mathbb{A}^3$.

LEMMA 4.4.3. *For all integers $r, s, t \geq 1$, the dynamical degree of the automorphism*

$$f = (y + x^r z^t, z, x + z^s(y + x^r z^t)) \in \mathrm{Aut}(\mathbb{A}^3)$$

TABLE 2. Small dynamical degrees provided by Lemma 4.4.3.

r	s	t	f	$\lambda(f)$
1	1	1	$(y + xz, z, x + z(y + xz))$	$(3+\sqrt{5})/2$
1	1	2	$(y + xz^2, z, x + z(y + xz^2))$	$2+\sqrt{3}$
1	1	3	$(y + xz^3, z, x + z(y + xz^3))$	$(5+\sqrt{21})/2$
1	2	1	$(y + xz, z, x + z^2(y + xz))$	$2+\sqrt{2}$
1	2	2	$(y + xz^2, z, x + z^2(y + xz^2))$	$(5+\sqrt{17})/2$
1	3	1	$(y + xz, z, x + z^3(y + xz))$	$(5+\sqrt{13})/2$
2	3	1	$(y + x^2z, z, x + z^3(y + x^2z))$	$3+\sqrt{3}$

is the biggest root of $x^2 - ax + b \in \mathbb{R}[x]$, with $a = r + s + t$, $b = rs$ and satisfies $\lambda(f) > s + 1$. In particular, if $\lambda(f)$ is not an integer, it is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$, so f is not conjugate to an affine-triangular automorphism of $\mathbb{A}^3$.

Proof. Let θ be the biggest root of $P(x) = x^2 - ax + b = (x - r)(x - s) - tx \in \mathbb{R}[x]$. As $P(s + 1) = (s + 1 - r) - t(s + 1) = (s + 1)(1 - t) - r < 0$, we find that $\theta > s + 1$. In particular, $\mu = (\theta - s, 1, \theta) \in \mathbb{R}_{\geq 1}$.

We compute $\deg_\mu(x^r z^{s+t}) = r(\theta - s) + (s + t)\theta = (r + s + t)\theta - rs = \theta^2$ and $\deg_\mu(x^r z^t) = \theta^2 - s\theta = \theta(\theta - s)$. This gives $\deg_\mu(f) = \theta$, with μ-leading part $g = (x^r z^t, z, x^r z^{s+t})$. Hence, $\lambda(f) = \theta$ by Proposition A.

If θ is not an integer, the other root of $P(x)$ is positive, so θ is not the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ (Theorem 2). This implies that f is not conjugate to an affine-triangular automorphism of $\mathbb{A}^3$. $\square$

Example 4.4.4. We now apply Lemma 4.4.3 to small values of r, s, t, and find some examples of automorphisms $f = (y + x^r z^t, z, x + z^s(y + x^r z^t)) \in \mathrm{Aut}(\mathbb{A}^3)$ whose dynamical degree $\lambda(f)$ is not that of an affine-triangular automorphism of $\mathbb{A}^3$. We give in Table 2 all examples of $\lambda(f) \leq 5$ given by Lemma 4.4.3. Exchanging r and s does not change the value of $\lambda(f)$, so we will assume that $r \leq s \leq 3$.

Remark 4.4.5. Let λ be a weak Perron number that is a quadratic integer.

By Theorem 2, λ is the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^4$ but is the dynamical degree of an affine-triangular automorphism of $\mathbb{A}^3$ if and only if its conjugate λ' is negative. If $\lambda' > 0$, then one can ask if λ is the dynamical degree of an automorphism of $\mathbb{A}^3$ (which would then necessarily be not conjugate to an affine-triangular automorphism). Writing $x^2 - ax + b$ the minimal polynomial of λ, with a, b positive integers, Lemma 4.4.3 shows that this is indeed true if one can write $b = rs$ with $r, s \geq 1$ and $a > r + s$. In particular, this holds if $b \leq 4$, as $a^2 - 4b > 0$, so $a > 2\sqrt{b}$. If $b = 5$, then $a \geq 5$ (as $a > 2\sqrt{b}$), and Lemma 4.4.3 applies as soon as $a \geq 6$. The case where $a = b = 5$ corresponds to $\lambda = (5 + \sqrt{5})/2$, which is then the 'simplest' weak Perron quadratic integer that is not covered by Theorem 2 or Lemma 4.4.3.

According to Remark 4.4.5, it seems natural to ask if every quadratic weak Perron number is the dynamical degree of an automorphism of $\mathbb{A}^3$. A first intriguing case concerns the following question, which was in fact already asked to us by Jean-Philippe Furter and Pierre-Marie Poloni.

Question 4.4.6. Is $(5 + \sqrt{5})/2$ the dynamical degree of an automorphism of $\mathbb{A}^3$?

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