DEGREE GROWTH OF BIRATIONAL MAPS OF THE PLANE

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ABSTRACT. This article studies the sequence of iterative degrees of a birational map of the plane. This sequence is known either to be bounded or to have a linear, quadratic or exponential growth.

The classification elements of infinite order with a bounded sequence of degrees is achieved, the case of elements of finite order being already known. The coefficients of the linear and quadratic growth are then described, and related to geometrical properties of the map. The dynamical number of base-points is also studied.

Applications of our results are the description of embeddings of the Baumslag-Solitar groups and $\mathrm{GL}(2,\mathbb{Q})$ into the Cremona group.

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CONTENTS

1. INTRODUCTION

A *rational map* of the complex projective plane $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ into itself is a map of the following type

$$\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad (x:y:z) \dashrightarrow (\phi_0(x,y,z):\phi_1(x,y,z):\phi_2(x,y,z)),$$

where the ϕ_i's are homogeneous polynomials of the same degree without common factor. The *degree* $\deg\phi$ of ϕ is by definition the degree of these polynomials. We will only consider *birational* maps, which are rational maps having an inverse, and denote by $\mathrm{Bir}(\mathbb{P}^2)$ the group of such maps, classically called *Cremona group*.

We are interested in the behaviour of the sequence $\left\{\deg\phi^k\right\}_{k\in\mathbb{N}}$. According to [14], the sequence is either bounded or has a linear, quadratic or exponential growth. We will say that ϕ is

(1) *elliptic* if the growth is bounded;

(2) a *Jonquières twist* if the growth is linear;

(3) an *Halphen twist* if the growth is quadratic;

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(4) *hyperbolic* if the growth is exponential.

This terminology is classical, and consistent with the natural action of $\mathrm{Bir}(\mathbb{P}^2)$ on an hyperbolic space of infinite dimension, where Jonquières and Halphen twists are parabolic ([10, Theorem 3.6]).

Recall that the first dynamical degree of $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ is $\lambda(\phi) = \lim_{k\to\infty} (\deg \phi^k)^{1/k} \in \mathbb{R}$. This is an invariant of conjugation which allows to distinguish the first three cases (where $\lambda(\phi) = 1$) from the last case (where $\lambda(\phi) > 1$). There are plenty of articles on hyperbolic elements and the possible values for the algebraic integer $\lambda(\phi)$, we are here more interested in the growth of the first three cases.

The nature of the growth is invariant under conjugation, and induces geometric properties on ϕ, that we describe now.

An element $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ is elliptic if and only if it is conjugate to an automorphism $g \in \mathrm{Aut}(\mathrm{S})$ of a smooth projective rational surface S such that g^n belongs to the connected component $\mathrm{Aut}^0(\mathrm{S})$ of $\mathrm{Aut}(\mathrm{S})$ for some $n > 0$ (*see* [14, Theorem 0.2, Lemma 4.1]). We will precise this result in Section 2, by showing that, up to conjugation, either g is finite or $\mathrm{S} = \mathbb{P}^2$. The complete classification of finite elements of $\mathrm{Bir}(\mathbb{P}^2)$ can be found in [8]; for elements of infinite order, one has (Proposition 2.3):

Theorem A. *If $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ is elliptic of infinite order, then ϕ is conjugate to an automorphism of $\mathbb{P}^2$, which restricts to one of the following automorphisms on some open subset isomorphic to $\mathbb{C}^2$:*

(1) $(x,y) \mapsto (\alpha x, \beta y)$*, where α, $\beta \in \mathbb{C}^*$, and where the kernel of the group homomorphism $\mathbb{Z}^2 \to \mathbb{C}^*$ given by $(i,j) \mapsto \alpha^i\beta^j$ is generated by $(k,0)$ for some $k \in \mathbb{Z}$;*

(2) $(x,y) \mapsto (\alpha x, y+1)$*, where $\alpha \in \mathbb{C}^*$.*

The end of Section 2 is devoted to the description of the conjugacy classes of such maps (Proposition 2.4) and their centralisers in the Cremona group (Lemmas 2.7 and 2.8).

A Jonquières twist preserves an unique pencil of rational curves [14, Theorem 0.2]. The sequence $\{\deg \phi^k\}_{k\in\mathbb{N}}$ grows as αk for some constant $\alpha \in \mathbb{R}$. The number α is not invariant under conjugation, but one can show that the minimal value is attained when the rational curves of the pencil are lines, and is an integer divided by 2. More precisely, one has (Proposition 4.4):

Theorem B. *Let $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ be a Jonquières twist.*

(1) *The set*
$$\left\{ \lim_{k\to+\infty} \frac{\deg(\psi\phi^k\psi^{-1})}{k} \;\middle|\; \psi \in \mathrm{Bir}(\mathbb{P}^2) \right\}$$
admits a minimum $\zeta(\phi) \in \frac{1}{2}\mathbb{N}$.

(2) *There exists an integer $a \in \mathbb{N}$ such that*
$$\lim_{k\to+\infty} \frac{\deg(\phi^k)}{k} = a^2\zeta(\phi).$$
Moreover, $a = 1$ if and only if ϕ preserves a pencil of lines.

The case of Halphen twists is similar. An Halphen twist preserves an unique pencil of elliptic curves (*see* [15], [14, Theorem 0.2]). Any such pencil can be sent by a birational map onto a pencil of curves of degree $3n$ with 9 points of multiplicity n, called Halphen pencil. We obtain the following (Proposition 5.1):

Theorem C. *Let $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ be an Halphen twist.*

(1) *The set*
$$\left\{ \lim_{k\to+\infty} \frac{\deg(\psi\phi^k\psi^{-1})}{k^2} \;\middle|\; \psi \in \mathrm{Bir}(\mathbb{P}^2) \right\}$$
admits a minimum $\kappa(\phi) \in 9\mathbb{N}$.

(2) *There exists an integer $a \geq 3$ such that*
$$\lim_{k\to+\infty} \frac{\deg(\phi^k)}{k^2} = \kappa(\phi) \cdot \frac{a^2}{9}.$$
Moreover, $a = 3$ if and only if ϕ preserves an Halphen pencil.

A birational map $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ has a finite number $\mathfrak{b}(\phi)$ of base-points (that may belong to $\mathbb{P}^2$ or correspond to infinitely near points). We will call the number
$$\mu(\phi) = \lim_{k\to+\infty} \frac{\mathfrak{b}(\phi^k)}{k},$$
the *dynamical number of base-points of* ϕ. In Section 3, we study the sequence $\{\mathfrak{b}(\phi^k)\}_{k\in\mathbb{N}}$ and deduce some properties on the number $\mu(\phi)$; let us state some of them. It is a non-negative integer invariant under conjugation which is equal to zero if and only if the map ϕ is conjugate to an automorphism of a smooth projective rational surface (Corollary 3.4). In the case where ϕ is a Jonquières twist, the number $\mu(\phi)$ determines the degree growth of ϕ, namely $\zeta(\phi) = \frac{\mu(\phi)}{2}$, where $\zeta(\phi)$ is defined as in Theorem B (Proposition 4.4).

An application of our results is the non-existence of embeddings of Baumslag-Solitar groups into the Cremona group and the description of the embeddings of $\mathrm{GL}(2,\mathbb{Q})$ into the Cremona group (§6):

Theorem D. *If $|m|$, $|n|$, 1 are distinct, there is no embedding of*
$$\mathrm{BS}(m,n) = \langle r, s \,|\, rs^m r^{-1} = s^n \rangle, \qquad m,n \in \mathbb{Z}, mn \neq 0$$
into the Cremona group.

Theorem E. *Let $\rho\colon \mathrm{GL}(2,\mathbb{Q}) \to \mathrm{Bir}(\mathbb{P}^2)$ be an embedding. Up to conjugation by an element of $\mathrm{Bir}(\mathbb{P}^2)$, there exists an odd integer k and an homomorphism $\chi\colon \mathbb{Q}^* \to \mathbb{C}^*$ (with respect to multiplication) such that*
$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(x \cdot \frac{\chi(ad-bc)}{(cy+d)^k}, \frac{ay+b}{cy+d}\right), \qquad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Q}).$$

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2. Elliptic maps

2.1. **Classification of elliptic maps of infinite order.** If ϕ is a birational map of $\mathbb{P}^2$ such that $\{\deg\phi^k\}_{k\in\mathbb{N}}$ is bounded, it is conjugate to an automorphism g of a smooth rational surface S such that the action of g on $\mathrm{Pic}(\mathrm{S})$ is finite [14, Lemma 4.1]. If g has finite order, the possible conjugacy classes are completely classified in [8]. Here we deal with the case of elements of infinite order, classifying the possibilities and describing its centralisers in $\mathrm{Bir}(\mathbb{P}^2)$.

Proposition 2.1. *Let g be an automorphism of a smooth rational surface S which has infinite order but has a finite action on $\mathrm{Pic}(\mathrm{S})$. Then, there exists a birational morphism $\mathrm{S} \to \mathrm{X}$ where X is equal either to $\mathbb{P}^2$ or to an Hirzebruch surface $\mathbb{F}_n$ for $n \neq 1$, which conjugates g to an automorphism of X.*

Remark 2.2. The proof of this result follows from a study of possible minimal pairs, which is similar to the one made in [7] for finite abelian subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ (*see* [7, Lemmas 3.2, 6.1, 9.7]).

Proof. Contracting the possible sets of disjoint (-1)-curves on S which are invariant by g, we can assume that the action of g on S is minimal. The action of g on $\mathrm{Pic}(\mathrm{S})$ being of finite order, the process corresponds to applying a G-Mori program, where G is a finite group acting on $\mathrm{Pic}(\mathrm{S})$ (we only look at parts of the Picard group which are invariant). Then one of the following occurs ([17, 16]):

(1) $\mathrm{Pic}(\mathrm{S})^g$ has rank 1 and S is a del Pezzo surface;

(2) $\mathrm{Pic}(\mathrm{S})^g$ has rank 2, and there exists a conic bundle $\pi\colon \mathrm{S}\to\mathbb{P}^1$ on S, together with an automorphism h of $\mathbb{P}^1$ such that $h\circ\pi=\pi\circ g$.

We want to show that S is $\mathbb{P}^2$ or an Hirzebruch surface $\mathbb{F}_n$ for $n\neq 1$, and exclude the other cases.

In the case where $\mathrm{Pic}(\mathrm{S})^g$ has rank 1, the fact that g has infinite order but finite action on $\mathrm{Pic}(\mathrm{S})$ implies that the kernel of the group homomorphism $\mathrm{Aut}(\mathrm{S})\to\mathrm{Aut}(\mathrm{Pic}(\mathrm{S}))$ is infinite. So S is a del Pezzo surface of degree $(K_{\mathrm{S}})^2\geq 6$. The surface cannot be $\mathbb{F}_1$ otherwise the exceptional section would be invariant. Similarly, it cannot be the unique del Pezzo surface of degree 7, which has exactly three (-1)-curves, forming a chain (one touches the two others, which are disjoint), because the curve of the middle (and also the union of the two others) would be invariant. The only possibilities are thus $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1=\mathbb{F}_0$, and the del Pezzo surface of degree 6.

If S is the del Pezzo surface of degree 6, any element $h\in\mathrm{Aut}(\mathrm{S})$ acting minimally on S has finite order [7, Lemma 9.7]. Let us recall the simple argument. The del Pezzo surface of degree 6 is isomorphic to

$$\mathrm{S}=\left\{(x:y:z),(u:v:w)\in\mathbb{P}^2\times\mathbb{P}^2\,|\,xu=yv=zw\right\}.$$

The projections π_1, π_2 on each factor is a birational morphism contracting three (-1)-curves on $p_1=(1:0:0)$, $p_2=(0:1:0)$ and $p_3=(0:0:1)$. The group $\mathrm{Pic}(\mathrm{S})$ is generated by the six (-1)-curves of S, which are $E_i=(\pi_1)^{-1}(p_i)$ and $F_i=(\pi_2)^{-1}(p_i)$ for $1\leq i\leq 3$ and form an hexagon. The action of g on S being minimal, g permutes cyclically the curves, and either g or g^{-1} acts as $(E_1\to F_2\to E_3\to F_1\to E_2\to F_3)$. This implies that g or g^{-1} is equal to

$$((x:y:z),(u:v:w))\to((\alpha v:\beta w:u),(\beta y:\alpha z:\alpha\beta x)),$$

for some $\alpha,\beta\in\mathbb{C}^*$, and has order 6.

We can now assume the existence of an invariant conic bundle $\pi\colon \mathrm{S}\to\mathbb{P}^1$. If π has no singular fibre, then S is a Hirzebruch surface $\mathbb{F}_n$ and $n\neq 1$ because of the minimality of the action. It remains to exclude the case where π has at least one singular fibre. The minimality of the action on S implies that the two components of any singular fibre F (which are two (-1)-curves) are exchanged by a power g^k of g, and in particular that the whole singular fibre is invariant by g^k. Note that k a priori depends on F.

We prove now that g^k does not act trivially on the basis of the conic bundle. If g^k acts trivially on the basis of the fibration, the automorphism g^{2k} acts trivially on $\mathrm{Pic}(\mathrm{S})$; taking a birational morphism $\mathrm{S}\to\mathbb{F}_n$ which contracts a component in each singular fibre one conjugates g^{2k} to an automorphism of $\mathbb{F}_n$, which fixes pointwise at least one section. The pull-back on S of this section intersects only one component in each singular fibre and its image by g^k gives thus another section, also fixed by g^{2k}. The action of g^k on a general fibre of π exchanges the two points of the two sections and hence has order 2: contradiction.

The action of g^k on the basis is non-trivial and fixes the point of $\mathbb{P}^1$ corresponding to the singular fibre; so the same holds for g. This implies that F is invariant by g, so its two components are exchanged by it (and thus k is odd).

In particular, g exchanges the two components of any singular fibre. This implies that the number of singular fibres of π is at most 2, so S is the blow-up of one or two points of an Hirzebruch surface.

The fact that the two components of at least one singular fibre are exchanged gives a symmetry on the sections, that will help us to determine S. Denote by $-m$ the minimal self-intersection of a section of π and let s be one section which realises this minimum. Contracting the components in the singular fibres which do not intersect s, one has a birational morphism $S \to \mathbb{F}_n$. The image of s is a section with minimal self-intersection, so $m = n$. If $n = 0$, then taking some section of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ of self-intersection 0 passing through at least one blown-up point, its strict transform on S would be a section of negative self-intersection, which contradicts the minimality of s^2, so $m = n > 0$. Let us denote by s' the section $g(s)$, which also has self-intersection $-m$ on S but self-intersection $-m+r$ on $\mathbb{F}_m$, where r is the number of singular fibres of π. Because any section of $\mathbb{F}_m$ distinct from the exceptional section has self-intersection $\geq m$, we get $-m+r \geq m$, so $2 \geq r \geq 2m$, which implies that $m = 1$ and $r = 2$.

The surface S is thus the blow-up of two points on $\mathbb{F}_1$, not lying on the exceptional section and not on the same fibre, so is a del Pezzo surface of degree 6. The fact that g acts minimally on S is impossible, as we already observed. □

Proposition 2.3. *Let ϕ be a birational map of $\mathbb{P}^2$ of infinite order, such that $\left\{\deg \phi^k\right\}_{k\in\mathbb{N}}$ is bounded.*

Then ϕ is conjugate to an automorphism of $\mathbb{P}^2$, which restricts to one of the following automorphisms on some open subset isomorphic to $\mathbb{C}^2$:

(1) $(x,y) \mapsto (\alpha x, \beta y)$*, where* α, $\beta \in \mathbb{C}^*$*, and where the kernel of the group homomorphism* $\mathbb{Z}^2 \to \mathbb{C}^*$ *given by* $(i,j) \mapsto \alpha^i\beta^j$ *is generated by* $(k,0)$ *for some* $k \in \mathbb{Z}$.

(2) $(x,y) \mapsto (\alpha x, y+1)$*, where* $\alpha \in \mathbb{C}^*$.

Proof. According to Proposition 2.1 the map ϕ is conjugate to an automorphism of a minimal surface S, equal to either $\mathbb{P}^2$ or an Hirzebruch surface.

Suppose first that $S = \mathbb{P}^2$. Looking at the Jordan normal form, any automorphism of $\mathbb{P}^2$ is conjugate to
– either $(x:y:z) \mapsto (\alpha x : \beta y : z)$,
– or $(x:y:z) \mapsto (\alpha x : y+z : z)$,
– or $(x:y:z) \mapsto (x+y : y+z : z)$.

This latter automorphism is conjugate to $(x:y:z) \mapsto (x : y+z : z)$ in $\mathrm{Bir}(\mathbb{P}^2)$ (for instance by $(x:y:z) \dashrightarrow (xz - \frac{1}{2}y(y-z) : yz : z^2)$, as already observed in [6, Example 1]). It remains to study the case of diagonal automorphisms to show the assertion on the kernel stated in the proposition. As in the proof of [6, Proposition 6], we associate to each diagonal automorphism $\psi\colon (x:y:z) \mapsto (\alpha x : \beta y : z)$ the kernel Δ_ψ of the following homomorphism of groups:

$$\delta_\psi\colon \mathbb{Z}^2 \to \mathbb{C}^*, \qquad\qquad (i,j) \mapsto \alpha^i\beta^j.$$

For any $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Z})$, we denote by $M(\psi)$ the diagonal automorphism $(x:y:z) \mapsto (\alpha^a\beta^b x : \alpha^c\beta^d sy : z)$, which is the conjugate of ψ by the birational map $(x,y) \dashrightarrow (x^Ay^B, x^Cy^D)$ (viewed in the chart $z = 1$). We can check that

$$\delta_{M(\psi)} = \delta_\psi \circ {}^tM,$$

which implies that $\Delta_{M(\psi)} = {}^tM^{-1}(\Delta_\psi)$. We can always choose M (by theorem on Smith's normal form) such that $\Delta_{M(\phi)}$ is generated by k_1e_1 and $k_1k_2e_2$, where e_1, e_2 are the canonical basis vectors of $\mathbb{Z}^2$, and k_1, k_2 are non-negative integers, and replace ϕ with $M(\phi)$, which is conjugate to it. Since ϕ and $M(\phi)$ have infinite order, we see that $k_2 = 0$, and get the assertion on the kernel stated in the proposition.

If $\mathrm{S}=\mathbb{F}_0=\mathbb{P}^1\times\mathbb{P}^1$, we can reduce to the case of $\mathbb{P}^2$ by blowing-up a fixed point and contracting the strict transform of the members of the two rulings passing through the point.

Suppose now that $\mathrm{S}=\mathbb{F}_n$ for $n\geq 2$. If g fixes a point of $\mathbb{F}_n$ which is not on the exceptional section, we can blow-up the point and contract the strict transform of the fibre to go to $\mathbb{F}_{n-1}$. We can thus assume that all points of $\mathbb{F}_n$ fixed by g are on the exceptional section. The action of g on the basis of the fibration is up to conjugation $x\mapsto\alpha x$ or $x\mapsto x+1$ for some $\alpha\in\mathbb{C}^*$. Removing the fibre at infinity and the exceptional section, we get $\mathbb{C}^2$, where the action of g is

– either $(x,y)\mapsto(\alpha x,\beta y+Q(x))$,
– or $(x,y)\mapsto(x+1,\beta y+Q(x))$,

where α, $\beta\in\mathbb{C}^*$ and Q is a polynomial of degree $\leq n$. The action on the fibre at infinity is obtained by conjugating by $(x,y)\dashrightarrow(\frac{1}{x},\frac{y}{x^n})$.

In the first case, there is no fixed point on the fibre at infinity (except the point on the exceptional section) if and only if $\beta=\alpha^n$ and $\deg Q=n$. There is no fixed point on $x=0$ if and only if $Q(0)\neq 0$ and $\beta=1$. This implies that α is a primitive k-th rooth of unity, where k is a divisor of n. Conjugating by $(x,y+\gamma x^d)$ we replace $Q(x)$ with $Q(x)+\gamma(\alpha^d-1)x^d$, so we can assume that the coefficient of x^d is trivial if d is not a multiple of k, which means that $Q(x)=P(x^k)$ for some polynomial $P\in\mathbb{C}[x]$. In particular, g is equal to $(x,y)\mapsto(\xi x,y+P(x^k))$ and is conjugate to $(x,y)\mapsto(\xi x,y+1)$ by $(x,y)\dashrightarrow\left(x,\frac{y}{P(x^k)}\right)$.

In the second case, there is no fixed point on $\mathbb{C}^2$, and no point on the fibre at infinity if and only if $\beta=1$ and $\deg Q=n$. Conjugating g by $(x,y)\mapsto(x,y+\gamma x^{n+1})$ (which corresponds to performing an elementary link $\mathbb{F}_n\dashrightarrow\mathbb{F}_{n+1}$ at the unique fixed-point and then coming back with an elementary link at a general point of the fibre at infinity), we get $(x,y)\mapsto\left(x+1,y-\gamma x^{n+1}+Q(x)+\gamma(x+1)^{n+1}\right)$. Choosing the right element $\gamma\in\mathbb{C}$, we can decrease the degree of $Q(x)$, and get $(x,y)\mapsto(x+1,y)$ by induction. □

2.2. **Conjugacy classes of elliptic maps of infinite order.** Following [6], we will call elements of the form $(x,y)\mapsto(\alpha x,\beta y)$, resp. $(x,y)\mapsto(\alpha x,y+1)$ *diagonal* automorphisms, resp. *almost-diagonal* automorphisms of $\mathbb{C}^2$ (or $\mathbb{P}^2$). The conjugacy classes in each family are given by the following:

Proposition 2.4 ([6], Theorem 1)**.** (1) *A diagonal automorphism and an almost-diagonal automorphism of $\mathbb{C}^2$ are never conjugate in* $\mathrm{Bir}(\mathbb{C}^2)$.

(2) *Two diagonal automorphisms $(x,y)\mapsto(\alpha x,\beta y)$ and $(x,y)\mapsto(\gamma x,\delta y)$ are conjugate in* $\mathrm{Bir}(\mathbb{C}^2)$ *if and only if there exists* $\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\mathrm{GL}(2,\mathbb{Z})$ *such that* $(\alpha^a\beta^b,\alpha^c\beta^d)=(\gamma,\delta)$.

(3) *Two almost diagonal automorphisms $(x,y)\mapsto(\alpha x,y+1)$ and $(x,y)\mapsto(\gamma x,y+1)$ are conjugate in* $\mathrm{Bir}(\mathbb{C}^2)$ *if and only if* $\alpha=\gamma^{\pm1}$.

Corollary 2.5. *Let $\phi\in\mathrm{Bir}(\mathbb{P}^2)$ be an elliptic map which has infinite order. If ϕ^m is conjugate to ϕ^n in* $\mathrm{Bir}(\mathbb{P}^2)$ *for some m, $n\in\mathbb{Z}$, $|m|\neq|n|$, then ϕ is conjugate to an automorphism of $\mathbb{C}^2$ of the form $(x,y)\mapsto(\alpha x,y+1)$, where $\alpha\in\mathbb{C}^*$ such that $\alpha^{m+n}=1$ or $\alpha^{m-n}=1$.*

Proof. Note that $mn\neq 0$ since ϕ has infinite order. Then ϕ is conjugate to one of the two cases of Proposition 2.3.

First of all, assume that up to conjugation ϕ is $(x,y)\mapsto(\alpha x,\beta y)$ and that the kernel Δ_ϕ of the group homomorphism $\mathbb{Z}^2\to\mathbb{C}^*$ given by $(i,j)\mapsto\alpha^i\beta^j$ is generated by $(k,0)$ for some $k\in\mathbb{Z}$. Since ϕ^m and ϕ^n are conjugate there exists a matrix $N=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\mathrm{GL}(2,\mathbb{Z})$ such that $\left((\alpha^m)^a(\beta^m)^b,(\alpha^m)^c(\beta^m)^d\right)=(\alpha^n,\beta^n)$

(Proposition 2.4). This means that $\left(\alpha^{ma-n}\beta^{mb},\alpha^{mc}\beta^{md-n}\right)=(1,1)$, so $(ma-n,mb),(mc,md-n)\in\Delta_\phi$. In particular $mb=md-n=0$, which implies that $b=0$, so $ad=\pm1$, which is impossible since $m\neq\pm n$.

Assume now that ϕ is conjugate to $(x,y)\mapsto(\alpha x,y+1)$ for some α in $\mathbb{C}^*$. The fact that ϕ^m and ϕ^n are conjugate implies that $\alpha^{m+n}=1$ or $\alpha^{m-n}=1$ (Proposition 2.4). □

2.3. **Centralisers of elliptic maps of infinite order.** If ϕ is a birational map of $\mathbb{P}^2$, we will denote by $\mathrm{C}(\phi)$ the centraliser of ϕ in $\mathrm{Bir}(\mathbb{P}^2)$:

$$\mathrm{C}(\phi)=\left\{\psi\in\mathrm{Bir}(\mathbb{P}^2)\,\middle|\,\phi\psi=\psi\phi\right\}.$$

In the sequel, we describe the centralisers of elliptic maps of infinite order of $\mathrm{Bir}(\mathbb{P}^2)$. The results are groups which contain the centralisers of some elements of $\mathrm{PGL}(2,\mathbb{C})$. We recall the following result, whose proof is an easy exercise. Recall that $\mathrm{PGL}(2,\mathbb{C})$ is the group of automorphisms of $\mathbb{P}^1$, or equivalently the group of Möbius transformations $x\dashrightarrow\frac{ax+b}{cx+d}$.

Lemma 2.6. *For any $\alpha\in\mathbb{C}^*$, we have*

$$\left\{\eta\in\mathrm{PGL}(2,\mathbb{C})\,\middle|\,\eta(\alpha x)=\alpha\eta(x)\right\}=\begin{cases}\mathrm{PGL}(2,\mathbb{C}) & \textit{if}\quad \alpha=1\\ \{x\dashrightarrow\gamma x^{\pm1}\,|\,\gamma\in\mathbb{C}^*\} & \textit{if}\quad \alpha=-1\\ \{x\mapsto\gamma x\,|\,\gamma\in\mathbb{C}^*\} & \textit{if}\quad \alpha\neq\pm1\end{cases}$$

□

Lemma 2.7. *Let us consider $\phi\colon(x,y)\mapsto(\alpha x,\beta y)$ where α, β are in $\mathbb{C}^*$, and where the kernel of the group homomorphism $\mathbb{Z}^2\to\mathbb{C}^*$ given by $(i,j)\mapsto\alpha^i\beta^j$ is generated by $(k,0)$ for some $k\in\mathbb{Z}$. Then the centraliser of ϕ in $\mathrm{Bir}(\mathbb{P}^2)$ is*

$$\mathrm{C}(\phi)=\left\{(x,y)\dashrightarrow(\eta(x),yR(x^k))\,\middle|\,R\in\mathbb{C}(x),\eta\in\mathrm{PGL}(2,\mathbb{C}),\eta(\alpha x)=\alpha\eta(x)\right\}.$$

Proof. Let $\psi\colon(x,y)\dashrightarrow(\psi_1(x,y),\psi_2(x,y))$ be an element of $\mathrm{C}(\phi)$. The fact that ψ commutes with ϕ is equivalent to

$$(\star)\ \ \psi_1(\alpha x,\beta y)=\alpha\psi_1(x,y)\qquad\text{and}\qquad(\diamond)\ \ \psi_2(\alpha x,\beta y)=\beta\psi_2(x,y).$$

Writing $\psi_i=\frac{P_i}{Q_i}$ for $i=1,2$, where P_i, Q_i are polynomials without common factors, we see that P_1,P_2,Q_1,Q_2 are eigenvectors of the linear automorphism ϕ^* of the $\mathbb{C}$-vector space $\mathbb{C}[x,y]$ given by $\phi^*\colon f(x,y)\mapsto f(\alpha x,\beta y)$. This means that each of the P_i, Q_i is a product of a monomial in x,y with an element of $\mathbb{C}[x^k]$. Using $(\star)$ and $(\diamond)$, we get the existence of R_1, $R_2\in\mathbb{C}(x)$ such that

$$\psi_1(x,y)=xR_1(x^k),\qquad\qquad\psi_2(x,y)=yR_2(x^k).$$

The fact that ψ is birational implies that $\psi_1(x,y)$ is an element $\eta(x)\in\mathrm{PGL}(2,\mathbb{C})$; it satisfies $\eta(\alpha x)=\alpha\eta(x)$ because of $(\star)$. □

Lemma 2.8. *Let us consider $\phi\colon(x,y)\mapsto(\alpha x,y+\beta)$ where $\alpha,\beta\in\mathbb{C}^*$. The centraliser of ϕ in $\mathrm{Bir}(\mathbb{P}^2)$ is equal to*

$$\mathrm{C}(\phi)=\left\{(x,y)\dashrightarrow(\eta(x),y+R(x))\,\middle|\,\eta\in\mathrm{PGL}(2,\mathbb{C}),\eta(\alpha x)=\alpha\eta(x),R\in\mathbb{C}(x),R(\alpha x)=R(x)\right\}.$$

Proof. Conjugating by $(x,y)\mapsto(x,\beta y)$, we can assume that $\beta=1$.

Let $\psi\colon(x,y)\dashrightarrow(\psi_1(x,y),\psi_2(x,y))$ be a birational map of $\mathbb{P}^2$ which commutes with ϕ. One has

$$(\star)\ \ \psi_1(\alpha x,y+1)=\alpha\psi_1(x,y)\qquad\text{and}\qquad(\diamond)\ \ \psi_2(\alpha x,y+1)=\psi_2(x,y)+1.$$

Equality $(\star)$ implies that ψ_1 only depends on x (*see* [6, Lemma 2]). Therefore ψ_1 is an element of $\mathrm{PGL}(2,\mathbb{C})$ which commutes with $x \mapsto \alpha x$.

Equality $(\diamond)$ implies that

$$\frac{\partial\psi_2}{\partial y}(\alpha x, y+1) = \frac{\partial\psi_2}{\partial y}(x,y) \qquad \text{and} \qquad \frac{\partial\psi_2}{\partial x}(\alpha x, y+1) = \alpha^{-1}\frac{\partial\psi_2}{\partial x}(x,y),$$

which again means that $\frac{\partial\psi_2}{\partial x}(x,y)$ and $\frac{\partial\psi_2}{\partial y}(x,y)$ only depend on x. The second component of ψ can thus be written $ay+B(x)$, where $a \in \mathbb{C}^*$, $B \in \mathbb{C}(x)$. Replacing this form in $(\diamond)$, we get

$$B(\alpha x) = B(x) + 1 - a,$$

which implies that $\frac{\partial B}{\partial x}(\alpha x) = \alpha^{-1}\frac{\partial B}{\partial x}(x)$, and thus that $x\frac{\partial B}{\partial x}(x)$ is invariant under $x \mapsto \alpha x$.

If α is not a root of unity, this means that $\frac{\partial B}{\partial x} = c/x$ for some $c \in \mathbb{C}$; since B is a rational function, one gets $c = 0$ and B is a constant (or equivalently an element such that $B(\alpha x) = B(x)$). It implies moreover $a = 1$ and we are done.

If α is a primitive k-th root of unity, the fact that $\psi\colon (x,y) \dashrightarrow (\eta(x), ay+B(x))$ commutes with $\phi^k\colon (x,y) \dashrightarrow (x, y+k)$ yields $a(y+k)+B(x) = ay+B(x)+k$, so $a = 1$. We again get $B(\alpha x) = B(x)$. □

3. ON THE GROWTH OF THE NUMBER OF BASE-POINTS

If S is a projective smooth surface, any element $\phi \in \mathrm{Bir}(\mathrm{S})$ has a finite number of base-points, which can belong to S or be infinitely near. We denote by $\mathfrak{b}(\phi)$ the number of such points. We will call the number

$$\mu(\phi) = \lim_{k\to+\infty} \frac{\mathfrak{b}(\phi^k)}{k},$$

the *dynamical number of base-points of* ϕ. Since $\mathfrak{b}(\phi\psi) \leq \mathfrak{b}(\phi) + \mathfrak{b}(\psi)$ for any ϕ, $\psi \in \mathrm{Bir}(\mathrm{S})$, we see that $\mu(\phi)$ is a non-negative real number. Moreover, $\mathfrak{b}(\phi^{-1})$ and $\mathfrak{b}(\phi)$ being always equal, we get $\mu(\phi^k) = |k \cdot \mu(\phi)|$ for any $k \in \mathbb{Z}$.

In this section, we precise the properties of this number, and will in particular see that it is an integer.

If $\phi \in \mathrm{Bir}(\mathrm{S})$ is a birational map, we will say that a (possibly infinitely near) base-point p of ϕ is a *persistent base-point* if there exists an integer N such that p is a base-point of ϕ^k for any $k \geq N$ but is not a base-point of ϕ^{-k} for any $k \geq N$.

If p is a point of S or a point infinitely near, which is not a base-point of $\phi \in \mathrm{Bir}(\mathrm{S})$, we define a point $\phi^\bullet(p)$, which will also be a point of S or a point infinitely near. For this, take a minimal resolution

$$\begin{tikzcd} & \mathrm{Z} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{S} \arrow[rr, dashed, "\phi"'] & & \mathrm{S}, \end{tikzcd}$$

where π_1, π_2 are sequences of blow-ups. Because p is not a base-point of ϕ, it corresponds, via π_1, to a point of Z or infinitely near. Using π_2, we view this point on S, again maybe infinitely near, and call it $\phi^\bullet(p)$.

Remark 3.1. If p is not a base-point of $\phi \in \mathrm{Bir}(\mathrm{S})$ and $\phi(p)$ is not a base-point of $\psi \in \mathrm{Bir}(\mathrm{S})$, we have $(\psi\phi)^\bullet(p) = \psi^\bullet(\phi^\bullet(p))$. If p is a general point of S, then $\phi^\bullet(p) = \phi(p) \in S$.

Example 3.2. If $\mathrm{S} = \mathbb{P}^2$, $p = (1:0:0)$ and ϕ is the birational map $(x:y:z) \dashrightarrow (yz+x^2 : xz : z^2)$, the point $\phi^\bullet(p)$ is not equal to $p = \phi(p)$, but is infinitely near to it.

Using this definition, we put an equivalence class on the set of points that belong to S or are infinitely near, by saying that p is *equivalent* to q if there exists an integer k such that $(\phi^k)^\bullet(p) = q$ (this implies that p is not a base-point of ϕ^k and that q is not a base-point of ϕ^{-k}). The set of equivalence classes is the generalisation of the notion of set of orbits for birational maps.

Proposition 3.3. *Let ϕ be a birational map of a smooth projective surface* S*. Denote by ν the number of equivalence classes of persistent base-points of ϕ. Then, the set*

$$\left\{\mathfrak{b}(\phi^k) - \nu k \;\middle|\; k \geq 0\right\} \subset \mathbb{Z}$$

is bounded.

In particular, $\mu(\phi)$ is an integer, equal to ν.

Proof. Let us say that a base-point q is periodic if $(\phi^k)^\bullet(q) = q$ for some $k \neq 0$, or if q is a base-point of ϕ^k for any $k \in \mathbb{Z}$ (which implies that $(\phi^k)^\bullet(q)$ is never defined). Let us denote by $\mathcal{P}$ the set of periodic base-points of ϕ and by $\widehat{\mathcal{P}}$ the finite set of points equivalent to a point of $\mathcal{P}$.

The number of base-points of ϕ and ϕ^{-1} being finite, there exists an integer N such that for any non-periodic base-point p and for any j, $j' \geq N$, p is a base-point of ϕ^j (respectively of ϕ^{-j}) if and only if p is a base-point of $\phi^{j'}$ (respectively of $\phi^{-j'}$).

We decompose the set of non-periodic base-points of ϕ into four sets:

$$\begin{aligned}
\mathcal{B}_{++} = &\ \left\{p \mid p \text{ is a base-point of } \phi^j, \text{and is a base-point of } \phi^{-j} \text{ for } j \geq N\right\},\\
\mathcal{B}_{+-} = &\ \left\{p \mid p \text{ is a base-point of } \phi^j, \text{but is not a base-point of } \phi^{-j} \text{ for } j \geq N\right\},\\
\mathcal{B}_{-+} = &\ \left\{p \mid p \text{ is not a base-point of } \phi^j, \text{but is a base-point of } \phi^{-j} \text{ for } j \geq N\right\},\\
\mathcal{B}_{--} = &\ \left\{p \mid p \text{ is not a base-point of } \phi^j, \text{and is not a base-point of } \phi^{-j} \text{ for } j \geq N\right\}.
\end{aligned}$$

Note that $\mathcal{B}_{+-}$ is the set of persistent base-points of ϕ and that $\mathcal{B}_{-+}$ is the set of persistent base-point of ϕ^{-1}. This decomposes the set of base-points of ϕ into five disjoint sets. Two base-points p, p' of ϕ which are equivalent belong to the same set.

We fix an integer $k \geq 2N$ and compute the number of base-points of ϕ^k. Any such base-point being equivalent to a base-point of ϕ, we take a base-point p of ϕ, and count the number $m_{p,k}$ of base-points of ϕ^k which are equivalent to p.

If p belongs to $\mathcal{P}$, the number of points equivalent to p is less than $\#(\widehat{\mathcal{P}})$, so $m_{p,k} \leq \#(\widehat{\mathcal{P}})$.

If p is not in $\mathcal{P}$, any point equivalent to p is equal to $(\phi^i)^\bullet(p)$ for some i, and all are distinct, so we have

$$m_{p,k} = \# I_{p,k}, \text{ where } I_{p,k} = \left\{i \in \mathbb{Z} \;\middle|\; p \text{ is not a base-point of } \phi^i, \text{ but } p \text{ is a base-point of } \phi^{i+k}\right\}.$$

If $p \in \mathcal{B}_{++}$, since p is not a base-point of ϕ^i one has $-N < i < N$, thus $m_{p,k} < 2N$.

A point $p \in \mathcal{B}_{--}$ is a base-point of ϕ^{i+k} hence $-N < i+k < N$ and $m_{p,k} < 2N$.

If $p \in \mathcal{B}_{-+}$, the fact that p is not a base-point of ϕ^i implies that $-N < i$ and the fact that p is a base-point of ϕ^{i+k} implies that $i + k \leq N$. With these two inequalities one has $-N < i \leq N - k$. Since $k > 2N$, we get $m_{p,k} = 0$.

If $p \in \mathcal{B}_{+-}$, the fact that p is not a base-point of ϕ^i implies that $i < N$ and the fact that p is a base-point of ϕ^{i+k} implies that $-N < i + k$. This yields $-N - k < i < N$, so $m_{p,k} \leq 2N + k$. Conversely, if $i \leq -N$ and $i + k \geq N$, p is not a base-point of ϕ^i, but p is a base-point of ϕ^{i+k} (or equivalently $i \in I_{p,k}$), so $m_{p,k} \geq \#[N-k, -N] = k - 2N + 1$. The two conditions together imply that $m_{p,k} - k \in [-2N, N]$.

There exist thus two integers c, d which do not depend on k, such that the total number of base-points of ϕ^k is between $\nu k + c$ and $\nu k + d$, for any $k \geq 2N$. This gives the result. $\square$

Corollary 3.4. *The dynamical number of base-points is an invariant of conjugation: if* $\theta\colon S \dashrightarrow Z$ *is a birational map between smooth projective surfaces and* $\phi \in \mathrm{Bir}(S)$*, then*

$$\mu(\phi) = \mu(\theta\phi\theta^{-1}).$$

In particular ϕ *is conjugate to an automorphism of a smooth projective surface if and only if* $\mu(\phi) = 0$.

Proof. The map θ factorises into the blow-up of a finite number of base-points followed by the contraction of a finite number of curves. The number of equivalence classes of persistent base-points of ϕ and $\theta\phi\theta^{-1}$ is thus the same, and we get the result from Proposition 3.3.

It is thus clear that $\mu(\phi) = 0$ if ϕ is conjugate to an automorphism of a smooth projective surface. The reciprocal assertion is proved in [9]. □

Remark 3.5. In [14, Theorem 0.4] one can find a characterisation of hyperbolic birational maps ϕ which are conjugate to an automorphism of a projective surface. If $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ is hyperbolic, we conjugate it to a birational map of a smooth projective surface S where the action is algebraically stable; its action on $\mathrm{H}^{1,1}(\mathbb{P}^2)$ admits the eigenvalue $\lambda(\phi) > 1$ with eigenvector θ_+. The map ϕ is birationally conjugate to an automorphism if and only if $(\theta_+)^2 = 0$.

Corollary 3.4 gives another characterisation, for all maps $\phi \in \mathrm{Bir}(\mathbb{P}^2)$, depending only on $\mu(\phi)$. This is what is done [3, 4, 5, 2, 13] to construct automorphisms with positive entropy starting from a birational map of $\mathbb{P}^2$. Let us give an example ([13]):

Example 3.6. Let $\phi = A\psi$ be the birational map where

$$A\colon (x:y:z) \dashrightarrow \big(\alpha x + 2(1-\alpha)y + (2+\alpha-\alpha^2)z : -x + (\alpha+1)z : x - 2y + (1-\alpha)z\big)$$

with $\alpha \in \mathbb{C} \setminus \{0, 1\}$ and

$$\psi\colon (x:y:z) \dashrightarrow (xz^2 + y^3 : yz^2 : z^3).$$

The map ψ (resp. ψ^{-1}) has five base-points, $p = (1:0:0)$ and four points infinitely near; we will denote $\widehat{P}_1$ (resp. $\widehat{P}_2$) the collection of these points. The automorphism A is chosen such that:

- $\widehat{P}_1, A\widehat{P}_2, A\psi A\widehat{P}_2$ have distinct supports;
- $\widehat{P}_1 = (A\psi)^2 A\widehat{P}_2$.

In particular the base-points of ϕ are non-persistent, so ϕ is conjugate to an automorphism with positive entropy on a rational surface. More precisely ϕ is conjugate to an automorphism with positive entropy on $\mathbb{P}^2$ blown up in $\widehat{P}_1, A\widehat{P}_2$ and $A\psi A\widehat{P}_2$ (*see* [13, Theorem 3.1]).

4. Growth of Jonquières twists

Lemma 4.1. *Let* ϕ *be a birational map of* $\mathbb{P}^2$ *which preserves the pencil of lines passing through some point* p_0*. The set*

$$\left\{ \deg\phi^k - k\cdot\frac{\mu(\phi)}{2} \;\middle|\; k \geq 0 \right\} \subset \mathbb{Z}$$

is bounded.

In particular, the sequence $\{\deg\phi^k\}_{k\in\mathbb{N}}$ *grows linearly if and only if* $\mu(\phi) > 0$ *and its growth is given by*

$$\frac{\mu(\phi)}{2} \in \frac{1}{2}\mathbb{N}.$$

Remark 4.2. Conjugating ϕ by a map which preserves the pencil does not change the growth $\{\deg\phi^k\}_{k\in\mathbb{N}}$, but conjugating it by a map which does not preserve the pencil can increase it (*see* Proposition 4.4).

Proof. For any k, ϕ^k preserves the pencil of lines passing through p_0. It implies that the linear system of ϕ^k (which is the pull-back of the system of lines of $\mathbb{P}^2$ by ϕ^k) has multiplicity $\deg\phi^k - 1$ at p_0 and has exactly $2(\deg\phi^k - 1)$ other base-points, all of multiplicity 1. In particular, $\deg\phi^k = \lfloor\frac{\mathfrak{b}(\phi^k)}{2}\rfloor$. The result follows then directly from Proposition 3.3. □

Example 4.3. Let us consider the family of birational maps studied in [12] and defined as

$$f_{\alpha,\beta}\colon (x:y:z) \dashrightarrow ((\alpha x+y)z : \beta y(x+z) : z(x+z)), \qquad \alpha,\beta\in\mathbb{C}^*.$$

Any of the $f_{\alpha,\beta}$ has three base-points: $(1:0:0)$, $(0:1:0)$ and $(-1:\alpha:1)$, and preserves the pencil of lines passing through $(1:0:0)$. Checking that $(-1:\alpha:1)$ is the only one persistent base-point ([12, Theorem 1.6]), the growth of $\left\{\deg f^k_{\alpha,\beta}\right\}_{k\in\mathbb{N}}$ is given by $\frac{k}{2}$ (*see* [12, Lemma 1.4]).

Proposition 4.4. *Let $\phi\in\mathrm{Bir}(\mathbb{P}^2)$ be a Jonquières twist. There exists an integer $a\in\mathbb{N}$ such that*

$$\lim_{k\to+\infty}\frac{\deg(\phi^k)}{k} = a^2\frac{\mu(\phi)}{2}.$$

Moreover, $a=1$ if and only if ϕ preserves a pencil of lines.

Proof. Since ϕ is a Jonquières twist, there exists $\psi\in\mathrm{Bir}(\mathbb{P}^2)$ such that $\tilde\phi=\psi\phi\psi^{-1}$ preserves the pencil of lines passing through some point $p\in\mathbb{P}^2$. Let $\pi\colon\mathbb{F}_1\to\mathbb{P}^2$ be the blow-up of $p\in\mathbb{P}^2$, and let $\widehat\phi\in\mathrm{Bir}(\mathbb{F}_1)$ be $\widehat\phi=\pi^{-1}\tilde\phi\pi$. Denote by $L_{\mathbb{P}^2}$ the linear system of lines of $\mathbb{P}^2$ and by Λ the linear system on $\mathbb{F}_1$ corresponding to the image by $\pi^{-1}\psi$ of the system of lines of $\mathbb{P}^2$. The degree of ϕ^k is equal to the free intersection of $L_{\mathbb{P}^2}$ with $\phi^k(L_{\mathbb{P}^2})$, which is the free intersection of Λ with $\widehat\phi(\Lambda)$.

On $\mathbb{F}_1$, Λ is linearly equivalent to $aL+bf$, where $L=\pi^{-1}(L_{\mathbb{P}^2})$, f is the divisor of a fibre and where $a,b\in\mathbb{N}$. Note that $\phi^k(L)$ is the transform on $\mathbb{F}_1$ of the linear system of $\tilde\phi^{-k}$, and is thus equal to $L+(d_k-1)f$, where d_k is the degree of $\tilde\phi^k$ (and of $\tilde\phi^{-k}$). The system $\widehat\phi^k(\Lambda)$ is then linearly equivalent to $aL+(a(d_k-1)+b)f$, so the total intersection of $\widehat\phi^k(\Lambda)$ with Λ is a^2d_k+2ab. Because $\widehat\phi^k(\Lambda)\cdot f=a$, each base-point of $\widehat\phi^k(\Lambda)$ has at most multiplicity a. By Lemma 4.1, $\lim\limits_{k\to+\infty}\dfrac{d_k}{k}=\dfrac{\mu(\phi)}{2}$. The number of base-points of Λ being bounded, the free intersection of $\widehat\phi^k(\Lambda)$ with Λ grows like $a^2\frac{\mu(\phi)}{2}\cdot k$.

It remains to see that if $a=1$ then ϕ preserves a pencil (the other direction follows from Lemma 4.1). If $a=1$, one $f\cdot\Lambda=1$. This implies that the free intersection of $\phi^{-1}\pi(f)$ with $\phi^{-1}\pi(\Lambda)=L_{\mathbb{P}^2}$ is 1; so $\phi^{-1}\pi(f)$ is a pencil of lines, invariant by ϕ. □

Lemma 4.1 and the second assertion of Corollary 3.4 imply the following statement of [14, Theorem 0.2.]

Corollary 4.5. *Let ϕ be a Jonquières twist; then ϕ is not conjugate to an automorphism.*

We can also derive the following new results.

Corollary 4.6. *Let ϕ be a Jonquières twist. If ϕ^m and ϕ^n are conjugate in $\mathrm{Bir}(\mathbb{P}^2)$ for some $m,n\in\mathbb{Z}$, then $|m|=|n|$.*

Proof. The fact that $\left\{\deg\phi^k\right\}_{k\in\mathbb{N}}$ grows linearly implies that ϕ preserves a pencil of rational curves [14, Theorem 0.2]. In particular ϕ is conjugate to a birational map of $\mathbb{P}^2$ which preserves the pencil of lines passing through some fixed point p_0. According to Lemma 4.1, one finds $\mu(\phi)>0$.

As ϕ^m and ϕ^n are conjugate in $\mathrm{Bir}(\mathbb{P}^2)$ one has $\mu(\phi^m)=\mu(\phi^n)$ (Corollary 3.4). Since $\mu(\phi^k)=|k\cdot\mu(\phi)|$ for any k, we get $|m|=|n|$. □

Recall that $\lambda(\phi)=\lim\limits_{k\to\infty}(\deg\phi^k)^{1/k}$ is the first dynamical degree of $\phi\in\mathrm{Bir}(\mathbb{P}^2)$.

Corollary 4.7. *Let ϕ denote a birational map of $\mathbb{P}^2$. Assume that ϕ^n and ϕ^m are conjugate and assume that $|m| \neq |n|$. One has $\lambda(\phi) = 1$ and $\mu(\phi) = 0$; in particular, ϕ is either elliptic, or an Halphen twist.*

Remark 4.8. The case of Halphen twists is not possible, and excluded in Section 5 (*see* in particular Corollaries 5.2 and 5.3).

Proof. The map ϕ^m is conjugate to ϕ^n in $\mathrm{Bir}(\mathbb{P}^2)$ so one gets $\lambda(\phi)^{|m|} = \lambda(\phi)^{|n|}$ and $|m| \cdot \mu(\phi) = |n| \cdot \mu(\phi)$. This yields $\lambda(\phi) = 1$ and $\mu(\phi) = 0$.

The fact that $\lambda(\phi) = 1$ implies that ϕ is elliptic, or a Jonquières or Halphen twist. The Jonquières case is impossible since $\mu(\phi) = 0$ (Corollary 4.6). □

5. Growth of Halphen twists

In Section 4 (especially Lemma 4.1), we described the degree growth of a Jonquières twist ϕ, and showed that it is given by $\mu(\phi)$, a birational invariant given by the growth of base-points. For an Halphen twist, the dynamical number of base-points is trivial, but the growth can also be quantified by an invariant.

Proposition 5.1. *Let $\phi \in \mathrm{Bir}(\mathbb{P}^2)$ be an Halphen twist.*

(1) *The set*
$$\left\{ \lim_{k\to+\infty} \frac{\deg(\psi\phi^k\psi^{-1})}{k^2} \mid \psi \in \mathrm{Bir}(\mathbb{P}^2) \right\}$$
admits a minimum, which is a positive integer $\kappa(\phi) \in 9\mathbb{N}$.

(2) *There exists an integer $a \geq 3$ such that $\lim\limits_{k\to+\infty} \frac{\deg(\phi^k)}{k^2} = \kappa(\phi) \cdot \frac{a^2}{9}$.*

(3) *The following conditions are equivalent:*

 (a) $a = 3$;

 (b) *ϕ preserves an Halphen pencil,* i.e. *a pencil of (elliptic) curves of degree $3n$ passing through 9 points with multiplicity n;*

 (c) $\deg(\phi^k) = 1 + \kappa(\phi) \cdot k^2$ *for k big enough.*

Proof. An Halphen twist preserves an unique elliptic fibration, so there exists an element ψ in $\mathrm{Bir}(\mathbb{P}^2)$ such that $\phi' = \psi\phi\psi^{-1}$ preserves an Halphen pencil. Denoting by $\pi \colon \mathrm{S} \to \mathbb{P}^2$ the blow-up of the 9 base-points of the pencil, $\widehat{\phi} = \pi^{-1}\phi'\pi$ is an automorphism of S, which preserves the elliptic fibration $\mathrm{S} \to \mathbb{P}^1$ given by $|-mK_{\mathrm{S}}|$ for some positive integer m. Replacing $\widehat{\phi}$ by some power if needed, we can assume that $\widehat{\phi}$ is a translation on a general fibre. As explained in the proof of [15, Proposition 9, page 132], there exists thus an element $\Delta \in \mathrm{Pic}(\mathrm{S})$ with $\Delta \cdot K_{\mathrm{S}} = 0$ such that the action of $\widehat{\phi}$ on $\mathrm{Pic}(\mathrm{S})$ is given by
$$D \mapsto D - m(D \cdot K_{\mathrm{S}}) \cdot \Delta + \gamma K_{\mathrm{S}},$$
where γ is an integer depending on D which can be computed using the self-intersection:
$$\gamma = -\frac{m^2}{2}(D \cdot K_{\mathrm{S}}) \cdot \Delta^2 + m(D \cdot \Delta).$$

We denote by L the linear system of lines of $\mathbb{P}^2$ and by $\Lambda = \pi^{-1}\psi(L)$ its transform on S. The degree of ϕ^n is equal to the free intersection of L with $\phi^n(L)$, which is equal to the free intersection of Λ with $\widehat{\phi}^n(\Lambda)$.

The map $\widehat{\phi}^n$ acts on $\mathrm{Pic}(\mathrm{S})$ as
$$D \mapsto D - m(D \cdot K_{\mathrm{S}}) \cdot (n\Delta) + \left(-\frac{m^2}{2}(D \cdot K_{\mathrm{S}}) \cdot (n\Delta)^2 + m(D \cdot (n\Delta)) \right) K_{\mathrm{S}}.$$

This yields

$$\begin{aligned}\Lambda\cdot\widehat{\phi}^n(\Lambda) &= \Lambda^2 - m(\Lambda\cdot K_S)\cdot(n\Delta\cdot\Lambda) + \left(-\tfrac{m^2}{2}(\Lambda\cdot K_S)(n\Delta)^2 + m(n\Lambda\cdot\Delta)\right)(K_S\cdot\Lambda)\\ &= \Lambda^2 + \left(-\tfrac{m^2}{2}(\Lambda\cdot K_S)^2(\Delta)^2\right)\cdot n^2.\end{aligned}$$

The free intersection between Λ and $\widehat{\phi}^n(\Lambda)$ is thus equal to

$$\Lambda^2 + \left(-\frac{m^2}{2}(\Lambda\cdot K_S)^2(\Delta)^2\right)\cdot n^2 - \sum_{i=1}^{k}\mu_i(\Lambda)\cdot\mu_i(\widehat{\phi}^n(\Lambda)),$$

where $\mu_i(\Lambda)$ and $\mu_i(\widehat{\phi}^n(\Lambda))$ denote the multiplicities of respectively Λ and $\widehat{\phi}^n(\Lambda)$ at the r base-points of Λ. Since $\widehat{\phi}$ is an automorphism of S, the contribution given by the base-points is bounded, so we find that

$$\lim_{n\to+\infty}\frac{\deg(\phi^n)}{n^2} = m^2(\Lambda\cdot K_S)^2\cdot\left(\frac{-\Delta^2}{2}\right).$$

Note that Λ is the lift by π^{-1} of the homaloidal linear system $\psi(L)$, so $\Lambda\cdot(-K_S)\geq 3$, and equality holds if and only if Λ has no base-point. This shows that the minimum among all homaloidal systems is attained when Λ has no base-point; we get $\kappa(\phi) = 9m^2\cdot\left(\frac{-\Delta^2}{2}\right)$ and $a = \Lambda\cdot K_S$.

One can verify that $-\Delta^2$ is a positive even number, so $\kappa(\phi)$ is a positive integer divisible by 9. The equality $\kappa(\phi) = \lim\limits_{k\to+\infty}\frac{\deg(\phi^k)}{k^2}$ is equivalent to the fact that Λ has no base-point, which corresponds to say that $\psi^{-1}\pi$ is a birational morphism, or equivalently that the unique pencil of elliptic curves invariant by ϕ is an Halphen pencil. □

Direct consequences of Proposition 5.1 are the following statements.

Corollary 5.2. *Let $\phi\in\mathrm{Bir}(\mathbb{P}^2)$ be an Halphen twist. The integer $\kappa(\phi)$ is a birational invariant which satisfies $\kappa(\phi^m) = m^2\kappa(\phi)$ for any $m\in\mathbb{Z}$. In particular, the maps ϕ^n and ϕ^m are not conjugate if $|m|\neq|n|$.*

Corollary 5.3. *Let ϕ denote a birational map of $\mathbb{P}^2$ of infinite order. Assume that ϕ^n and ϕ^m are conjugate and assume that $|m|\neq|n|$. Then, ϕ is conjugate to an automorphism of $\mathbb{C}^2$ of the form $(x,y)\mapsto(\alpha x, y+1)$, where $\alpha\in\mathbb{C}^*$ such that $\alpha^{m+n}=1$ or $\alpha^{m-n}=1$.*

In particular, if ϕ is conjugate to ϕ^n for any positive integer n, then ϕ is conjugate to $(x,y)\mapsto(x,y+1)$.

Proof. Follows from Corollaries 2.5, 4.7 and 5.2. □

6. Applications

6.1. **Morphisms of Baumslag-Solitar groups in the Cremona group.** For any integers m, n such that $mn\neq 0$, the Baumslag-Solitar group $\mathrm{BS}(m,n)$ is defined by the following presentation

$$\mathrm{BS}(m,n) = \langle r, s\,|\, rs^m r^{-1} = s^n\rangle.$$

Recall that if G is a group, the derived groups of G are

$$\mathrm{G}^{(0)} = \mathrm{G}, \qquad \mathrm{G}^{(i)} = \left[\mathrm{G}^{(i-1)},\mathrm{G}^{(i-1)}\right] = \langle ghg^{-1}h^{-1}\,|\, g,h\in\mathrm{G}^{(i-1)}\rangle \text{ for all } i\geq 1,$$

and that G is *solvable* if there exists an integer N such that $\mathrm{G}^{(N)} = \{\mathrm{id}\}$.

The groups $\mathrm{BS}(m,n)$ (resp. the subgroups of finite index of $\mathrm{BS}(m,n)$) are solvable if and only if $|m|=1$ or $|n|=1$ (*see* [19, Proposition A.6]).

A group G is said to be *residually finite* if for any g in $\mathrm{G}\setminus\{\mathrm{id}\}$ there exists a finite group H and a group homomorphism $\Theta\colon \mathrm{G}\to\mathrm{H}$ such that $\Theta(g)$ belongs to $\mathrm{H}\setminus\{\mathrm{id}\}$. The group $\mathrm{BS}(m,n)$ is residually finite if and only if $|m|=1$ or $|n|=\ 1$ or $|m|=|n|$ (*see* [18]). Let V be an affine algebraic variety; according to [1] any subgroup of finite index of the automorphisms group of V is residually finite. Therefore if $|m|\neq|n|$ and $|m|,|n|\neq 1$ there is no embedding of $\mathrm{BS}(m,n)$ into the group of polynomial automorphisms of the plane. There is an other proof using the amalgated structure of the group of polynomial automorphisms of the plane and the fact that $\mathrm{BS}(m,n)$ is not solvable ([11, Proposition 2.2]).

Lemma 6.1. *Let ρ be a homomorphism from $\mathrm{BS}(m,n)=\langle r,s\,|\,rs^mr^{-1}=s^n\rangle$ to $\mathrm{Bir}(\mathbb{P}^2)$. Assume that $|m|,|n|$ and 1 are distinct. If $\rho(s)$ has infinite order, the image of the subgroup of finite index $\langle r,s^m\,|\,rs^{m^2}r^{-1}=s^{mn}\rangle$ of $\mathrm{BS}(m,n)$ is solvable.*

Proof. By Corollary 5.3, one can conjugate ρ so that $\rho(s)\colon (x,y)\dashrightarrow(\alpha x,y+1)$ where $\alpha\in\mathbb{C}^*$ and $\alpha^{m-n}=1$ or $\alpha^{m+n}=\ 1$. Denoting respectively by ψ the map $\left(x,\frac{n}{m}y\right)$ or $\left(x^{-1},\frac{n}{m}y\right)$ one has $\psi\rho(s)^m\psi^{-1}=\rho(s)^n$. So $\rho(r)=\psi\tau$ where τ commutes with $\rho(s)^m\colon (x,y)\mapsto(\alpha^m x,y+m)$.

According to Lemma 2.8, one then has $\tau=(\eta(x),y+R(x))$ for some $\eta\in\mathrm{PGL}(2,\mathbb{C}),\eta(\alpha^m x)=\alpha^m\eta(x)$, and some $R\in\mathbb{C}(x)$ satisfying $R(\alpha^m x)=R(x)$. And one gets $\rho(r)=\left(\eta(x)^{\pm1},\frac{n}{m}(y+R(x))\right)$.

The group generated by $\rho(s^m)$ and $\rho(r)$ is thus solvable. $\square$

Corollary 6.2. *If $|m|$, $|n|$ and 1 are distinct, there is no embedding of $\mathrm{BS}(m,n)$ into the Cremona group.*

Proof. Lemma 6.1 that the image of any embedding would be solvable, impossible when $|m|$, $|n|$ and 1 are distinct. $\square$

6.2. **Embeddings of $\mathrm{GL}(2,\mathbb{Q})$ into the Cremona group.** To simplify the notation, we will denote in this last section by $(\phi_1(x,y),\phi_2(x,y))$ the rational map $(x,y)\dashrightarrow(\phi_1(x,y),\phi_2(x,y))$ from $\mathbb{C}^2$ to $\mathbb{C}^2$.

Let us first give examples of embeddings of $\mathrm{GL}(2,\mathbb{Q})$ into the Cremona group.

Example 6.3. Let k be an odd integer and let $\chi\colon \mathbb{Q}^*\to\mathbb{C}^*$ be a homomorphism such that $a\mapsto\frac{\chi(a^2)}{a^k}$ is injective. The morphism ρ from $\mathrm{GL}(2,\mathbb{Q})$ to the Cremona group given by
$$\rho\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)=\left(x\cdot\frac{\chi(ad-bc)}{(cy+d)^k},\frac{ay+b}{cy+d}\right)$$
is an embedding. Note that $\rho(\mathrm{GL}(2,\mathbb{Q}))$ is conjugate to a subgroup of automorphisms of the k-th Hirzebruch surface $\mathbb{F}_k$. Changing k gives then infinitely many non conjugate embeddings int he Cremona group.

Theorem 6.4. *Let $\rho\colon \mathrm{GL}(2,\mathbb{Q})\to\mathrm{Bir}(\mathbb{P}^2)$ be an embedding of $\mathrm{GL}(2,\mathbb{Q})$ into the Cremona group, then up to conjugation ρ is one of the embeddings described in Example 6.3.*

Proof. Let us set
$$\mathrm{t}_q=\begin{bmatrix} 1 & q\\ 0 & 1\end{bmatrix}\qquad\&\qquad \mathrm{d}_{m,n}=\begin{bmatrix} m & 0\\ 0 & n\end{bmatrix},\qquad q,m,n\in\mathbb{Q}.$$
Remark that t_1 is conjugate to t_n in $\mathrm{GL}(2,\mathbb{Q})$, for any $n\in\mathbb{Z}\setminus\{0\}$; we can then assume, after conjugation, that $\rho(\mathrm{t}_1)=(x,y+1)$ (Corollary 5.3). As $\rho(\mathrm{t}_{1/n})$ commutes with $\rho(\mathrm{t}_1)$ there exist A_n in $\mathrm{PGL}(2,\mathbb{C})$ and R_n in $\mathbb{C}(x)$ such that
$$\rho(\mathrm{t}_{1/n})=(A_n(x),y+R_n(x))$$
(*see* Lemma 2.8). Let us prove now that $A_n(x)=x$. Since $\mathrm{t}_{1/n}^n=\mathrm{t}_1$ the element A_n is of finite order so A_n is conjugate to some ξx where ξ is some root of unity. Hence $\rho(\mathrm{t}_{1/n})$ is conjugate to $(\xi x,y+Q(x))$ where $Q\in\mathbb{C}(x)$ satisfies $Q(x)+Q(\xi x)+\cdots+Q(\xi^{n-1}x)=1$. The map $(\xi x,y+Q(x))$ is then conjugate to

$\left(\xi x, y+\frac{1}{n}\right)$ by $\left(x, y-\frac{\sum_{i=1}^{n-1} iQ(\xi^i x)}{n}\right)$. Since $\mathrm{t}_{1/n}$ is conjugate to t_1, Proposition 2.4 implies that $\xi=1$, which achieves to show that $A_n(x)=x$. This implies, with equality $\rho(\mathrm{t}_{1/n})^n=\rho(\mathrm{t}_1)$, that $R_n(x)=1/n$. We thus have for any q in $\mathbb{Q}$

$$\rho(\mathrm{t}_q)=(x, y+q).$$

From $\mathrm{d}_{m,n}\mathrm{t}_1\mathrm{d}_{m,n}^{-1}=\mathrm{t}_{m/n}$ one gets (using again Lemma 2.8) that

$$\rho(\mathrm{d}_{m,n})=\left(\eta_{m,n}(x), \frac{m}{n}y+R_{m,n}(x)\right), \qquad \eta_{m,n}\in \mathrm{PGL}(2,\mathbb{C}),\, R_{m,n}\in\mathbb{C}(x).$$

The map $(\mathbb{Q}^*)^2\to \mathrm{PGL}(2,\mathbb{C})$ given by $(m,n)\mapsto \eta_{m,n}$ is a homomorphism, which cannot be injective. There exists thus one element $\mathrm{d}_{m,n}$ with $(m,n)\neq(1,1)$ such that $\rho(\mathrm{d}_{m,n})=\left(x, \frac{m}{n}y+R_{m,n}(x)\right)$. Note that $m\neq n$ since the centraliser of $\rho(\mathrm{d}_{m,n})$ and $\rho(\mathrm{t}_1)$ are different. Conjugating by $\left(x, y+\frac{R_{m,n}(x)}{m/n-1}\right)$, we can assume that $\rho(\mathrm{d}_{m,n})=\left(x,\frac{m}{n}y\right)$. From $\mathrm{d}_{m,n}\mathrm{d}_{a,b}=\mathrm{d}_{a,b}\mathrm{d}_{m,n}$ one gets for any a, b in $\mathbb{Q}$

$$\rho(\mathrm{d}_{a,b})=\left(\eta_{a,b}(x),\frac{a}{b}y\right), \qquad \eta_{a,b}\in\mathrm{PGL}(2,\mathbb{C}).$$

The homomorphism $\mathbb{Q}^*\to\mathrm{PGL}(2,\mathbb{C})$ given by $a\mapsto\eta_{a,a}$ is injective, so up to conjugation by an element of $\mathrm{PGL}(2,\mathbb{C})$ we can assume that for any $a\in\mathbb{Q}\backslash\{0,1\}$ there exists $\chi_{a,a}\in\mathbb{C}\backslash\{0,1\}$ such that $\eta_{a,a}(x)=\chi_{a,a}x$. This implies the existence of $\chi_{a,b}\in\mathbb{C}^*$ for any a, b in $(\mathbb{Q}^*)^2$, such that $\eta_{a,b}(x)=\chi_{a,b}x$.

We now compute the image of $M=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$. Since $\rho(M)$ commutes with $\rho(\mathrm{d}_{2,2})=(\chi_{2,2}x,y)$ where $\chi_{2,2}\in\mathbb{C}^*$ is of infinite order, there exist $R\in\mathbb{C}(y)$ and $\nu\in\mathrm{PGL}(2,\mathbb{C})$ such that $\rho(M)=(xR(y),\nu(y))$ (Lemma 2.8). For any $a\in\mathbb{Q}^*$, equality $M\mathrm{d}_{a,1}=\mathrm{d}_{1,a}M$ yields

$$\left(\chi_{a,1}x\cdot R(ay),\nu(ay)\right)=\left(\chi_{1,a}x\cdot R(y),\frac{1}{a}\nu(y)\right).$$

This implies that $R(y)=\alpha y^{-k}$ and $\nu(y)=\frac{\beta}{y}$, for some α, $\beta\in\mathbb{C}^*$, $k\in\mathbb{Z}$, *i.e.* $\rho(M)=\left(\alpha\frac{x}{y^k},\frac{\beta}{y}\right)$. We use now equality $(M\mathrm{t}_1)^3=\mathrm{id}$: the second component of $(\rho(M)\rho(\mathrm{t}_1))^3$ being

$$\frac{\beta(y+\beta+1)}{(\beta+1)y+2\beta+1},$$

we find $\beta=-1$ and compute $(\rho(M)\rho(\mathrm{t}_1))^3=(x\alpha^3(-1)^k,y)$ so $\alpha^3=(-1)^k$. Since $\rho(M)^2=(x\alpha^2(-1)^k,y)$ has order 2, we have $-\alpha^2=(-1)^k$, thus $\alpha=-1$ and k is odd.

Writing $\chi(a)=\chi_{a,1}$ for any $a\in\mathbb{Q}$, the map $\mathbb{Q}^*\to\mathbb{C}^*$ given by $a\to\chi(a)$ is an homomorphism, and one gets $\rho(\mathrm{d}_{a,1})=(\chi(a)x,ay)$. The group $\mathrm{GL}(2,\mathbb{Q})$ is generated by the maps $\mathrm{d}_{a,1},\mathrm{t}_a$ and M, so

$$\rho\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)=\left(x\cdot\frac{\chi(ad-bc)}{(cy+d)^k},\frac{ay+b}{cy+d}\right),$$

for any $\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathrm{GL}(2,\mathbb{Q})$. This yields an embedding if and only if the homomoprhism from $\mathbb{Q}^*$ to $\mathbb{C}^*$ given by $a\mapsto\frac{\chi(a^2)}{a^k}$ is injective.

It remains to observe that k can be chosen to be positive. Indeed, otherwise one conjugates by $\left(\frac{1}{x},y\right)$ and replaces χ with $\frac{1}{\chi}$ to replace k with $-k$. □

One can see that $\mathrm{GL}(n,\mathbb{Q})$ does not embedd into $\mathrm{Bir}(\mathbb{P}^2)$ as soon as $n \geq 3$. Indeed, Theorem 6.4 implies that the diagonal matrices are sent onto diagonal elements of $\mathrm{PGL}(3,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2)$, which is impossible, by considering the involutions. One can also find another less obvious corollary:

Corollary 6.5. *Let* $\rho\colon \mathrm{GL}(2,\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^2)$ *be an embedding of* $\mathrm{GL}(2,\mathbb{C})$ *into the Cremona group. There exist a positive odd integer k, a field homomorphism* $\tau\colon \mathbb{C} \to \mathbb{C}$ *and a group homomorphism* $\chi\colon \mathbb{C}^* \to \mathbb{C}^*$ *such that*

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(x \cdot \frac{\chi(ad-bc)}{(\tau(c)y+\tau(d))^k}, \frac{\tau(a)y+\tau(b)}{\tau(c)y+\tau(d)}\right), \qquad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{C}).$$

Remark 6.6. One sees that in the description above, ρ is an embedding if and only if the group homomorphism $\mathbb{C}^* \to \mathbb{C}^*$ given by $a \to \frac{\chi(a^2)}{\tau(a)^k}$ is injective. This happens for instance by taking $\chi(a) = \tau(a)^{\frac{k\pm1}{2}}$ any positive odd integer k and any field homomorphism $\tau\colon \mathbb{C} \to \mathbb{C}$.

Proof. The map ρ induces an embedding of $\mathrm{GL}(2,\mathbb{Q})$ into the Cremona group. According to Theorem 6.4 one has a description of $\rho_{|\mathrm{GL}(2,\mathbb{Q})}$. Up to conjugation, there exists an odd positive integer k and an homomorphism $\tilde{\chi}\colon \mathbb{Q}^* \to \mathbb{C}^*$ such that

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(x \cdot \frac{\tilde{\chi}(ad-bc)}{(cy+d)^k}, \frac{ay+b}{cy+d}\right), \qquad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Q}).$$

Let us set

$$\mathrm{t}_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad \& \qquad \mathrm{d}_b = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}, \qquad a \in \mathbb{C}, b \in \mathbb{C}^*.$$

For any $a \in \mathbb{C}^*$, the matrix d_a commutes with all diagonal matrices with entries in $\mathbb{Q}$; this implies, with the description above, that

$$\rho(\mathrm{d}_a) = (\chi(a)x, \tau(a)y)$$

for some $\chi(a)$, $\tau(a)$ in $\mathbb{C}^*$ (Lemma 2.7). This yields two group homomorphisms χ, $\tau\colon \mathbb{C}^* \to \mathbb{C}^*$. Observe that χ is an extension of $\tilde{\chi}$, *i.e.* $\chi(a) = \tilde{\chi}(a)$ for any $a \in \mathbb{Q}$.

The equality $\mathrm{d}_a \mathrm{t}_1 \mathrm{d}_{a^{-1}} = \mathrm{t}_a$ implies that

$$\rho(\mathrm{t}_a) = (x, y+\tau(a)), \qquad \forall a \in \mathbb{C}^*.$$

In particular, τ extends to an (injective) field homomorphism $\mathbb{C} \to \mathbb{C}$. The group $\mathrm{GL}(2,\mathbb{C})$ being generated by $\mathrm{GL}(2,\mathbb{Q})$ and $\{\mathrm{d}_a \,|\, a \in \mathbb{C}^*\}$, one has

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(x \cdot \frac{\chi(ad-bc)}{(\tau(c)y+\tau(d))^k}, \frac{\tau(a)y+\tau(b)}{\tau(c)y+\tau(d)}\right), \qquad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{C})$$

The map ρ is injective if and only if $\chi(a^2) \neq \tau(a^k)$ for any $a \in \mathbb{C}^*\setminus\{1\}$. □

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