

Degree growth of birational maps of the plane

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Abstract. This article studies the sequence of iterative degrees of a birational map of the plane. This sequence is known either to be bounded or to have a linear, quadratic or exponential growth.

The classification elements of infinite order with a bounded sequence of degrees is achieved, the case of elements of finite order being already known. The coefficients of the linear and quadratic growth are then described, and related to geometrical properties of the map. The dynamical number of base-points is also studied.

Applications of our results are the description of embeddings of the Baumslag-Solitar groups and $GL(2, \mathbb{Q})$ into the Cremona group.

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1. Introduction

A *rational map* of the complex projective plane $\mathbb{P}^2 = \mathbb{P}_{\mathbb{C}}^2$ into itself is a map of the following type

$$\phi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x : y : z) \dashrightarrow (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)),$$

where the ϕ_i 's are homogeneous polynomials of the same degree without common factor. The *degree* $\deg \phi$ of ϕ is by definition the degree of these polynomials. We will only consider *birational maps*, which are rational maps having a rational inverse, and denote by $\text{Bir}(\mathbb{P}^2)$ the group of such maps, classically called *Cremona group*.

We are interested in the behaviour of the sequence $\{\deg \phi^k\}_{k \in \mathbb{N}}$. According to [13], the sequence is either bounded or has a linear, quadratic or exponential growth. We will say that ϕ is

- (1) *elliptic* if the growth is bounded;
- (2) a *Jonquières twist* if the growth is linear;

- (3) a *Halphen twist* if the growth is quadratic;
- (4) *hyperbolic* if the growth is exponential.

This terminology is classical, and consistent with the natural action of $\text{Bir}(\mathbb{P}^2)$ on a hyperbolic space of infinite dimension, where Jonquières and Halphen twists are parabolic ([9, Theorem 3.6]). Recall that a Jonquières twist preserves a unique pencil of rational curves [13, Theorem 0.2], and that this one can be sent by a birational map of the plane on a pencil of lines ([15, Section 43]). This explains the terminology, since an element of $\text{Bir}(\mathbb{P}^2)$ which preserves a pencil of rational curves is classically called *Jonquières transformation*. A Halphen twist preserves a unique pencil of elliptic curves ([14], [13, Theorem 0.2, Lemma 4.1]) and this one can be sent by a birational map of the plane onto a pencil of curves of degree $3n$ with 9 points of multiplicity n ([15, Section 43]). These pencils have been studied by Halphen ([16]) and are now called *Halphen pencils*; this again explains the choice of the terminology.

Recall that the first dynamical degree of $\phi \in \text{Bir}(\mathbb{P}^2)$ is $\lambda(\phi) = \lim_{k \rightarrow +\infty} (\deg \phi^k)^{1/k} \in \mathbb{R}$. This is an invariant of conjugation which allows to distinguish the first three cases (where $\lambda(\phi) = 1$) from the last case (where $\lambda(\phi) > 1$).

The nature of the growth is invariant under conjugation, and induces geometric properties on ϕ , that we describe now.

It is shown in [13, Theorem 0.2, Lemma 4.1] that an element $\phi \in \text{Bir}(\mathbb{P}^2)$ is elliptic if and only if it is conjugate to an automorphism $g \in \text{Aut}(S)$ of a smooth projective rational surface S such that g^n belongs to the connected component $\text{Aut}^0(S)$ of $\text{Aut}(S)$ for some $n > 0$. We refine this result by showing that we can choose g of finite order or $S = \mathbb{P}^2$ (and hence $n = 1$). The complete classification of elements of finite order of $\text{Bir}(\mathbb{P}^2)$ can be found in [8]; for elliptic elements of infinite order, one has (Proposition 2.3):

Theorem A. *If $\phi \in \text{Bir}(\mathbb{P}^2)$ is elliptic of infinite order, then ϕ is conjugate to an automorphism of \mathbb{P}^2 , which restricts to one of the following automorphisms on some open subset isomorphic to \mathbb{C}^2 :*

- (1) $(x, y) \mapsto (\alpha x, \beta y)$, where $\alpha, \beta \in \mathbb{C}^*$, and where the kernel of the group homomorphism $\mathbb{Z}^2 \rightarrow \mathbb{C}^*$ given by $(i, j) \mapsto \alpha^i \beta^j$ is generated by $(k, 0)$ for some $k \in \mathbb{Z}$;
- (2) $(x, y) \mapsto (\alpha x, y + 1)$, where $\alpha \in \mathbb{C}^*$.

The end of Section 2 is devoted to the description of the conjugacy classes of such maps (Proposition 2.4) and their centralisers in the Cremona group (Lemmas 2.7 and 2.8).

A birational map $\phi \in \text{Bir}(\mathbb{P}^2)$ has a finite number $\mathfrak{b}(\phi)$ of base-points (that may belong to \mathbb{P}^2 or correspond to infinitely near points). We will call the number

$$\mu(\phi) = \lim_{k \rightarrow +\infty} \frac{\mathfrak{b}(\phi^k)}{k},$$

the *dynamical number of base-points of ϕ* . In Section 3, we study the sequence $\{\mathbf{b}(\phi^k)\}_{k \in \mathbb{N}}$ and deduce some properties on the number $\mu(\phi)$. In particular, we show that $\mu(\phi)$ is a non-negative integer invariant under conjugation; it also allows us to give a characterisation of birational maps conjugate to an automorphism of a smooth projective rational surface (Proposition 3.5):

Theorem B. *Let S be a smooth projective surface; the birational map $\phi \in \text{Bir}(S)$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(\phi) = 0$.*

In the case where ϕ is a Jonquière's twist, the number $\mu(\phi)$ determines the degree growth of ϕ . The sequence $\{\text{deg } \phi^k\}_{k \in \mathbb{N}}$ grows as αk for some constant $\alpha \in \mathbb{R}$. The number α is not invariant under conjugation, but one can show that the minimal value is attained when the rational curves of the unique invariant pencil of curves are lines. More precisely, one has (Proposition 4.5):

Theorem C. *Let $\phi \in \text{Bir}(\mathbb{P}^2)$ be a Jonquière's twist.*

(1) *The set*

$$\left\{ \lim_{k \rightarrow +\infty} \frac{\text{deg}(\psi \phi^k \psi^{-1})}{k} \mid \psi \in \text{Bir}(\mathbb{P}^2) \right\}$$

admits a minimum, which is equal to $\frac{\mu(\phi)}{2} \in \frac{1}{2}\mathbb{N}$.

(2) *There exists an integer $a \in \mathbb{N}$ such that*

$$\lim_{k \rightarrow +\infty} \frac{\text{deg}(\phi^k)}{k} = \frac{\mu(\phi)}{2} \cdot a^2.$$

Moreover, $a = 1$ if and only if ϕ preserves a pencil of lines.

In the case of Halphen twists we also find a similar result (Proposition 5.1):

Theorem D. *Let $\phi \in \text{Bir}(\mathbb{P}^2)$ be a Halphen twist.*

(1) *The set*

$$\left\{ \lim_{k \rightarrow +\infty} \frac{\text{deg}(\psi \phi^k \psi^{-1})}{k^2} \mid \psi \in \text{Bir}(\mathbb{P}^2) \right\}$$

admits a minimum $\kappa(\phi) \in \mathbb{Q}_{>0}$.

(2) *There exists an integer $a \geq 3$ such that*

$$\lim_{k \rightarrow +\infty} \frac{\text{deg}(\phi^k)}{k^2} = \kappa(\phi) \cdot \frac{a^2}{9}.$$

Moreover, $a = 3$ if and only if ϕ preserves a Halphen pencil.

An application of our results is the description of birational maps whose two distinct iterates are conjugate, the non-existence of embeddings of Baumslag-Solitar groups into the Cremona group and the description of the embeddings of $\mathrm{GL}(2, \mathbb{Q})$ into the Cremona group (Section 6):

Theorem E. *Let ϕ denote a birational map of \mathbb{P}^2 of infinite order. Assume that ϕ^n and ϕ^m are conjugate and that $|m| \neq |n|$. Then, ϕ is conjugate to an automorphism of \mathbb{C}^2 of the form $(x, y) \mapsto (\alpha x, y + 1)$, where $\alpha \in \mathbb{C}^*$ such that $\alpha^{m+n} = 1$ or $\alpha^{m-n} = 1$.*

In particular, if ϕ is conjugate to ϕ^n for any positive integer n , then ϕ is conjugate to $(x, y) \mapsto (x, y + 1)$.

Theorem F. *If $|m|, |n|, 1$ are distinct, there is no embedding of*

$$\mathrm{BS}(m, n) = \langle r, s \mid r s^m r^{-1} = s^n \rangle, \quad m, n \in \mathbb{Z}, mn \neq 0$$

into the Cremona group.

Theorem G. *Let $\rho: \mathrm{GL}(2, \mathbb{Q}) \rightarrow \mathrm{Bir}(\mathbb{P}^2)$ be an embedding. Up to conjugation by an element of $\mathrm{Bir}(\mathbb{P}^2)$, there exists an odd integer k and a homomorphism $\chi: \mathbb{Q}^* \rightarrow \mathbb{C}^*$ (with respect to multiplication) such that*

$$\rho \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(x \cdot \frac{\chi(ad - bc)}{(cy + d)^k}, \frac{ay + b}{cy + d} \right), \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Q}).$$

Remark 1.1. All these statements hold replacing \mathbb{C} by an algebraically closed field of characteristic zero.

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2. Elliptic maps

2.1. Classification of elliptic maps of infinite order

If ϕ is a birational map of \mathbb{P}^2 such that $\{\deg \phi^k\}_{k \in \mathbb{N}}$ is bounded, it is conjugate to an automorphism g of a smooth rational surface S such that the action of g on $\mathrm{Pic}(S)$ is finite [13, Lemma 4.1]. If g has finite order, the possible conjugacy classes are completely classified in [8]. Here we deal with the case of elements of infinite order, classifying the possibilities and describing its centralisers in $\mathrm{Bir}(\mathbb{P}^2)$.

Proposition 2.1. *Let g be an automorphism of a smooth rational surface S which has infinite order but has a finite action on $\text{Pic}(S)$. Then, there exists a birational morphism $S \rightarrow X$ where X is equal either to \mathbb{P}^2 or to a Hirzebruch surface \mathbb{F}_n for $n \neq 1$, which conjugates g to an automorphism of X .*

Remark 2.2. The proof of this result follows from a study of possible minimal pairs, which is similar to the one made in [7] for finite abelian subgroups of $\text{Bir}(\mathbb{P}^2)$ (see [7, Lemmas 3.2, 6.1, 9.7]).

Proof. Contracting the possible sets of disjoint (-1) -curves on S which are invariant by g , we can assume that the action of g on S is minimal. The action of g on $\text{Pic}(S)$ being of finite order, the process corresponds to applying a G -Mori program, where G is a finite group acting on $\text{Pic}(S)$ (we only look at parts of the Picard group which are invariant). Denoting by $\text{Pic}(S)^g \subset \text{Pic}(S)$ the submodule $\{L \in \text{Pic}(S) \mid g(L) = L\}$, one of the following occurs ([17, 18]):

- (1) $\text{Pic}(S)^g$ has rank 1 and S is a del Pezzo surface;
- (2) $\text{Pic}(S)^g$ has rank 2, and there exist a conic bundle $\pi : S \rightarrow \mathbb{P}^1$ on S , together with an automorphism h of \mathbb{P}^1 such that $h \circ \pi = \pi \circ g$.

We want to show that S is \mathbb{P}^2 or a Hirzebruch surface \mathbb{F}_n for $n \neq 1$, and exclude the other cases.

In the case where $\text{Pic}(S)^g$ has rank 1, the fact that g has infinite order but finite action on $\text{Pic}(S)$ implies that the kernel of the group homomorphism $\text{Aut}(S) \rightarrow \text{Aut}(\text{Pic}(S))$ is infinite. So S is a del Pezzo surface of degree $(K_S)^2 \geq 6$. The surface cannot be \mathbb{F}_1 otherwise the exceptional section would be invariant. Similarly, it cannot be the unique del Pezzo surface of degree 7, which has exactly three (-1) -curves, forming a chain (one touches the two others, which are disjoint), because the curve of the middle (and also the union of the two others) would be invariant. The only possibilities are thus \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$, and the del Pezzo surface of degree 6.

If S is the del Pezzo surface of degree 6, any element $h \in \text{Aut}(S)$ acting minimally on S has finite order [7, Lemma 9.7]. Let us recall the simple argument. The del Pezzo surface of degree 6 is isomorphic to

$$S = \left\{ (x : y : z), (u : v : w) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid xu = yv = zw \right\}.$$

The projections π_1, π_2 on each factor are birational morphisms contracting three (-1) -curves on $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$ and $p_3 = (0 : 0 : 1)$. The group $\text{Pic}(S)$ is generated by the six (-1) -curves of S , which are $E_i = (\pi_1)^{-1}(p_i)$ and $F_i = (\pi_2)^{-1}(p_i)$ for $1 \leq i \leq 3$ and form an hexagon. In fact, the action on the hexagon gives rise to an isomorphism $\text{Aut}(S) \simeq (\mathbb{C}^*)^2 \rtimes (\text{Sym}_3 \times \mathbb{Z}/2\mathbb{Z}) = (\mathbb{C}^*)^2 \rtimes D_6$. The action of g on S being minimal, g permutes cyclically the curves, and either g or g^{-1} acts as $(E_1 \rightarrow F_2 \rightarrow E_3 \rightarrow F_1 \rightarrow E_2 \rightarrow F_3)$. This implies that g or g^{-1} is equal to

$$((x : y : z), (u : v : w)) \rightarrow ((\alpha v : \beta w : u), (\beta y : \alpha z : \alpha \beta x)),$$

for some $\alpha, \beta \in \mathbb{C}^*$, and has order 6.

We can now assume the existence of an invariant conic bundle $\pi: S \rightarrow \mathbb{P}^1$. If π has no singular fibre, then S is a Hirzebruch surface \mathbb{F}_n and $n \neq 1$ because of the minimality of the action. It remains to exclude the case where π has at least one singular fibre. The minimality of the action on S implies that the two components of every singular fibre F (which are two (-1) -curves) are exchanged by a power g^k of g , and in particular that the whole singular fibre is invariant by g^k . Note that k a priori depends on F .

We now prove that g^k does not act trivially on the base of the conic bundle, *i.e.* that g^k sends a general fibre onto another one. If g^k acts trivially on the base of the fibration, the automorphism g^{2k} acts trivially on $\text{Pic}(S)$; taking a birational morphism $S \rightarrow \mathbb{F}_n$ which contracts a component in each singular fibre one conjugates g^{2k} to an automorphism of \mathbb{F}_n , which fixes pointwise at least one section. The strict transform on S of this section intersects only one component in each singular fibre and its image by g^k gives thus another section, also fixed by g^{2k} . The action of g^k on a general fibre of π exchanges the two points of the two sections and hence has order 2: contradiction.

The action of g^k on the base is non-trivial and fixes the point of \mathbb{P}^1 corresponding to the singular fibre; so the same holds for g (recall that the fixed points of an element of $\text{Aut}(\mathbb{P}^1)$ and any of its non-trivial powers are the same). This implies that F is invariant by g , so its two components are exchanged by it (and thus k is odd).

In particular, g exchanges the two components of every singular fibre. This implies that the number of singular fibres of π is at most 2, so S is the blow-up of one or two points of a Hirzebruch surface.

The fact that the two components of at least one singular fibre are exchanged gives a symmetry on the sections, that will help us to determine S . Denote by $-m$ the minimal self-intersection of a section of π and let s be one section which realises this minimum. Contracting the components in the singular fibres which do not intersect s , one has a birational morphism $S \rightarrow \mathbb{F}_n$. The image of s is a section with minimal self-intersection, so $m = n$. If $n = 0$, then taking some section of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ of self-intersection 0 passing through at least one blown-up point, its strict transform on S would be a section of negative self-intersection, which contradicts the minimality of s^2 , so $m = n > 0$. Let us denote by s' the section $g(s)$, which also has self-intersection $-m$ on S but self-intersection $-m + r$ on \mathbb{F}_m , where r is the number of singular fibres of π . Because any section of \mathbb{F}_m distinct from the exceptional section has self-intersection $\geq m$, we get $-m + r \geq m$, so $2 \geq r \geq 2m$, which implies that $m = 1$ and $r = 2$.

The surface S is thus the blow-up of two points on \mathbb{F}_1 , not lying on the exceptional section and not on the same fibre, so is a del Pezzo surface of degree 6. The fact that g acts minimally on S is impossible, as we already observed. \square

Proposition 2.3. *Let ϕ be a birational map of \mathbb{P}^2 of infinite order, such that $\{\deg \phi^k\}_{k \in \mathbb{N}}$ is bounded.*

Then ϕ is conjugate to an automorphism of \mathbb{P}^2 , which restricts to one of the following automorphisms on some open subset isomorphic to \mathbb{C}^2 :

- (1) $(x, y) \mapsto (\alpha x, \beta y)$, where $\alpha, \beta \in \mathbb{C}^*$, and where the kernel of the group homomorphism $\mathbb{Z}^2 \rightarrow \mathbb{C}^*$ given by $(i, j) \mapsto \alpha^i \beta^j$ is generated by $(k, 0)$ for some $k \in \mathbb{Z}$;
- (2) $(x, y) \mapsto (\alpha x, y + 1)$, where $\alpha \in \mathbb{C}^*$.

Proof. According to Proposition 2.1 the map ϕ is conjugate to an automorphism of a minimal surface S , equal to either \mathbb{P}^2 or a Hirzebruch surface.

Suppose first that $S = \mathbb{P}^2$. Looking at the Jordan normal form, any automorphism of \mathbb{P}^2 is conjugate to

- either $(x : y : z) \mapsto (\alpha x : \beta y : z)$,
- or $(x : y : z) \mapsto (\alpha x : y + z : z)$,
- or $(x : y : z) \mapsto (x + y : y + z : z)$.

This latter automorphism is conjugate to $(x : y : z) \mapsto (x : y + z : z)$ in $\text{Bir}(\mathbb{P}^2)$ (for instance by $(x : y : z) \dashrightarrow (xz - \frac{1}{2}y(y - z) : yz : z^2)$, as already observed in [6, Example 1]). It remains to study the case of diagonal automorphisms to show the assertion on the kernel stated in the proposition. As in the proof of [6, Proposition 6], we associate to each diagonal automorphism $\psi : (x : y : z) \mapsto (\alpha x : \beta y : z)$ the kernel Δ_ψ of the following homomorphism of groups:

$$\delta_\psi : \mathbb{Z}^2 \rightarrow \mathbb{C}^*, \quad (i, j) \mapsto \alpha^i \beta^j.$$

For any $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$, we denote by $M(\psi)$ the diagonal automorphism

$$(x : y : z) \mapsto (\alpha^a \beta^b x : \alpha^c \beta^d y : z),$$

which is the conjugate of ψ by the birational map $(x, y) \dashrightarrow (x^a y^b, x^c y^d)$ (viewed in the chart $z = 1$). We can check that

$$\delta_{M(\psi)} = \delta_\psi \circ {}^t M,$$

which implies that $\Delta_{M(\psi)} = {}^t M^{-1}(\Delta_\psi)$. We can always choose M (by a result on Smith’s normal form) such that $\Delta_{M(\psi)}$ is generated by $k_1 e_1$ and $k_1 k_2 e_2$, where e_1, e_2 are the canonical basis vectors of \mathbb{Z}^2 , and k_1, k_2 are non-negative integers, and replace ϕ with $M(\phi)$, which is conjugate to it. Since ϕ and $M(\phi)$ have infinite order, we see that $k_2 = 0$, and get the assertion on the kernel stated in the proposition.

If $S = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, we can reduce to the case of \mathbb{P}^2 by blowing-up a fixed point and contracting the strict transform of the members of the two rulings passing through the point.

Suppose now that $S = \mathbb{F}_n$ for $n \geq 2$. If g fixes a point of \mathbb{F}_n which is not on the exceptional section, we can blow-up the point and contract the strict transform of the fibre to go to \mathbb{F}_{n-1} . We can thus assume that all points of \mathbb{F}_n fixed by g are on the exceptional section. The action of g on the base of the fibration is, up to conjugation, $x \mapsto \alpha x$ or $x \mapsto x + 1$ for some $\alpha \in \mathbb{C}^*$. Removing the fibre at

infinity and the exceptional section, we get \mathbb{C}^2 , where the action of g is

- either $(x, y) \mapsto (\alpha x, \beta y + Q(x))$,
- or $(x, y) \mapsto (x + 1, \beta y + Q(x))$,

where $\alpha, \beta \in \mathbb{C}^*$ and Q is a polynomial of degree $\leq n$.

The action on the fibre at infinity is obtained by conjugating by $(x, y) \dashrightarrow \left(\frac{1}{x}, \frac{y}{x^n}\right)$.

In the first case, there is no fixed point on the fibre at infinity (except the point on the exceptional section) if and only if $\beta = \alpha^n$ and $\deg Q = n$. There is no fixed point on $x = 0$ if and only if $Q(0) \neq 0$ and $\beta = 1$. This implies that α is a primitive k -th root of unity, where k is a divisor of n . Conjugating by $(x, y + \gamma x^d)$ we replace $Q(x)$ with $Q(x) + \gamma(\alpha^d - 1)x^d$, so we can assume that the coefficient of x^d is trivial if d is not a multiple of k , which means that $Q(x) = P(x^k)$ for some polynomial $P \in \mathbb{C}[x]$. In particular, g is equal to $(x, y) \mapsto (\xi x, y + P(x^k))$ and is conjugate to $(x, y) \mapsto (\xi x, y + 1)$ by $(x, y) \dashrightarrow \left(x, \frac{y}{P(x^k)}\right)$.

In the second case, there is no fixed point on \mathbb{C}^2 , and no point on the fibre at infinity if and only if $\beta = 1$ and $\deg Q = n$. Conjugating g by $(x, y) \mapsto (x, y + \gamma x^{n+1})$ (which corresponds to performing an elementary link $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$ at the unique fixed-point and then coming back with an elementary link at a general point of the fibre at infinity), we get

$$(x, y) \mapsto (x + 1, y - \gamma x^{n+1} + Q(x) + \gamma(x + 1)^{n+1}).$$

Choosing the right element $\gamma \in \mathbb{C}$, we can decrease the degree of $Q(x)$, and get $(x, y) \mapsto (x + 1, y)$ by induction. □

2.2. Conjugacy classes of elliptic maps of infinite order

Following [6], we will call elements of the form $(x, y) \mapsto (\alpha x, \beta y)$, respectively $(x, y) \mapsto (\alpha x, y + 1)$ *diagonal* automorphisms, respectively *almost-diagonal* automorphisms of \mathbb{C}^2 (or \mathbb{P}^2). The conjugacy classes in each family are given by the following:

Proposition 2.4 ([6], Theorem 1).

- (1) A diagonal automorphism and an almost-diagonal automorphism of \mathbb{C}^2 are never conjugate in $\text{Bir}(\mathbb{C}^2)$.
- (2) Two diagonal automorphisms $(x, y) \mapsto (\alpha x, \beta y)$ and $(x, y) \mapsto (\gamma x, \delta y)$ are conjugate in $\text{Bir}(\mathbb{C}^2)$ if and only if there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$ such that $(\alpha^a \beta^b, \alpha^c \beta^d) = (\gamma, \delta)$.
- (3) Two almost diagonal automorphisms $(x, y) \mapsto (\alpha x, y + 1)$ and $(x, y) \mapsto (\gamma x, y + 1)$ are conjugate in $\text{Bir}(\mathbb{C}^2)$ if and only if $\alpha = \gamma^{\pm 1}$.

Corollary 2.5. *Let $\phi \in \text{Bir}(\mathbb{P}^2)$ be an elliptic map which has infinite order. If ϕ^m is conjugate to ϕ^n in $\text{Bir}(\mathbb{P}^2)$ for some $m, n \in \mathbb{Z}, |m| \neq |n|$, then ϕ is conjugate to an automorphism of \mathbb{C}^2 of the form $(x, y) \mapsto (\alpha x, y + 1)$, where $\alpha \in \mathbb{C}^*$ such that $\alpha^{m+n} = 1$ or $\alpha^{m-n} = 1$.*

Proof. Note that $mn \neq 0$ since ϕ has infinite order. Then ϕ is conjugate to one of the two cases of Proposition 2.3.

First of all, assume that up to conjugation ϕ is $(x, y) \mapsto (\alpha x, \beta y)$ and that the kernel Δ_ϕ of the group homomorphism $\mathbb{Z}^2 \rightarrow \mathbb{C}^*$ given by $(i, j) \mapsto \alpha^i \beta^j$ is generated by $(k, 0)$ for some $k \in \mathbb{Z}$. Since ϕ^m and ϕ^n are conjugate there exists a matrix

$$N = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$$

such that $(\alpha^m)^a (\beta^m)^b, (\alpha^m)^c (\beta^m)^d = (\alpha^n, \beta^n)$ (Proposition 2.4). This means that

$$(\alpha^{ma-n} \beta^{mb}, \alpha^{mc} \beta^{md-n}) = (1, 1),$$

so $(ma - n, mb), (mc, md - n)$ belong to Δ_ϕ . In particular $mb = md - n = 0$, which implies that $b = 0$, so $ad = \pm 1$, which is impossible since $m \neq \pm n$.

Assume now that ϕ is conjugate to $(x, y) \mapsto (\alpha x, y + 1)$ for some α in \mathbb{C}^* . The fact that ϕ^m and ϕ^n are conjugate implies that $\alpha^{m+n} = 1$ or $\alpha^{m-n} = 1$ (Proposition 2.4). □

2.3. Centralisers of elliptic maps of infinite order

If ϕ is a birational map of \mathbb{P}^2 , we will denote by $C(\phi)$ the centraliser of ϕ in $\text{Bir}(\mathbb{P}^2)$:

$$C(\phi) = \left\{ \psi \in \text{Bir}(\mathbb{P}^2) \mid \phi\psi = \psi\phi \right\}.$$

In the sequel, we describe the centralisers of elliptic maps of infinite order of $\text{Bir}(\mathbb{P}^2)$. The results are groups which contain the centralisers of some elements of $\text{PGL}(2, \mathbb{C})$. We recall the following result, whose proof is an easy exercise. Recall that $\text{PGL}(2, \mathbb{C})$ is the group of automorphisms of \mathbb{P}^1 , or equivalently the group of Möbius transformations $x \mapsto \frac{ax+b}{cx+d}$.

Lemma 2.6. *For any $\alpha \in \mathbb{C}^*$, we have*

$$\left\{ \eta \in \text{PGL}(2, \mathbb{C}) \mid \eta(\alpha x) = \alpha \eta(x) \right\} = \begin{cases} \text{PGL}(2, \mathbb{C}) & \text{if } \alpha = 1 \\ \{x \mapsto \gamma x^{\pm 1} \mid \gamma \in \mathbb{C}^*\} & \text{if } \alpha = -1 \\ \{x \mapsto \gamma x \mid \gamma \in \mathbb{C}^*\} & \text{if } \alpha \neq \pm 1. \end{cases}$$

Lemma 2.7. *Let us consider $\phi: (x, y) \mapsto (\alpha x, \beta y)$ where α, β are in \mathbb{C}^* , and where the kernel of the group homomorphism $\mathbb{Z}^2 \rightarrow \mathbb{C}^*$ given by $(i, j) \mapsto \alpha^i \beta^j$ is generated by $(k, 0)$ for some $k \in \mathbb{Z}$. Then the centraliser of ϕ in $\text{Bir}(\mathbb{P}^2)$ is*

$$C(\phi) = \left\{ (x, y) \mapsto \left(\eta(x), yR(x^k) \right) \mid R \in \mathbb{C}(x), \eta \in \text{PGL}(2, \mathbb{C}), \eta(\alpha x) = \alpha \eta(x) \right\}.$$

Proof. Let $\psi: (x, y) \dashrightarrow (\psi_1(x, y), \psi_2(x, y))$ be an element of $C(\phi)$. The fact that ψ commutes with ϕ is equivalent to

$$(\star) \quad \psi_1(\alpha x, \beta y) = \alpha \psi_1(x, y) \quad \text{and} \quad (\diamond) \quad \psi_2(\alpha x, \beta y) = \beta \psi_2(x, y).$$

Writing $\psi_i = \frac{P_i}{Q_i}$ for $i = 1, 2$, where P_i, Q_i are polynomials without common factors, we see that P_1, P_2, Q_1, Q_2 are eigenvectors of the linear automorphism ϕ^* of the \mathbb{C} -vector space $\mathbb{C}[x, y]$ given by $\phi^*: f(x, y) \mapsto f(\alpha x, \beta y)$. This means that each of the P_i, Q_i is a product of a monomial in x, y with an element of $\mathbb{C}[x^k]$. Using (\star) and (\diamond) , we get the existence of $R_1, R_2 \in \mathbb{C}(x)$ such that

$$\psi_1(x, y) = x R_1(x^k), \quad \psi_2(x, y) = y R_2(x^k).$$

The fact that ψ is birational implies that $\psi_1(x, y)$ is an element $\eta(x) \in \text{PGL}(2, \mathbb{C})$; it satisfies $\eta(\alpha x) = \alpha \eta(x)$ because of (\star) . □

Lemma 2.8. *Let us consider $\phi: (x, y) \mapsto (\alpha x, y + \beta)$ where $\alpha, \beta \in \mathbb{C}^*$. The centraliser of ϕ in $\text{Bir}(\mathbb{P}^2)$ is equal to*

$$C(\phi) = \left\{ (x, y) \dashrightarrow (\eta(x), y + R(x)) \mid \eta \in \text{PGL}(2, \mathbb{C}), \right. \\ \left. \eta(\alpha x) = \alpha \eta(x), R \in \mathbb{C}(x), R(\alpha x) = R(x) \right\}.$$

Proof. Conjugating by $(x, y) \mapsto (x, \beta y)$, we can assume that $\beta = 1$.

Let $\psi: (x, y) \dashrightarrow (\psi_1(x, y), \psi_2(x, y))$ be a birational map of \mathbb{P}^2 which commutes with ϕ . One has

$$(\star) \quad \psi_1(\alpha x, y + 1) = \alpha \psi_1(x, y) \quad \text{and} \quad (\diamond) \quad \psi_2(\alpha x, y + 1) = \psi_2(x, y) + 1.$$

Equality (\star) implies that ψ_1 only depends on x (see [6, Lemma 2]). Therefore ψ_1 is an element of $\text{PGL}(2, \mathbb{C})$ which commutes with $x \mapsto \alpha x$.

It follows from (\diamond) that

$$\frac{\partial \psi_2}{\partial y}(\alpha x, y + 1) = \frac{\partial \psi_2}{\partial y}(x, y) \quad \text{and} \quad \frac{\partial \psi_2}{\partial x}(\alpha x, y + 1) = \alpha^{-1} \frac{\partial \psi_2}{\partial x}(x, y),$$

which again means that $\frac{\partial \psi_2}{\partial x}(x, y)$ and $\frac{\partial \psi_2}{\partial y}(x, y)$ only depend on x . The second component of ψ can thus be written $ay + B(x)$, where $a \in \mathbb{C}^*, B \in \mathbb{C}(x)$. Replacing this form in (\diamond) , we get

$$B(\alpha x) = B(x) + 1 - a,$$

which implies that $\frac{\partial B}{\partial x}(\alpha x) = \alpha^{-1} \frac{\partial B}{\partial x}(x)$, and thus that $x \frac{\partial B}{\partial x}(x)$ is invariant under $x \mapsto \alpha x$.

If α is not a root of unity, this means that $\frac{\partial B}{\partial x} = c/x$ for some $c \in \mathbb{C}$; since B is a rational function, one gets $c = 0$ and B is a constant (or equivalently an element such that $B(\alpha x) = B(x)$). It implies moreover $a = 1$ and we are done.

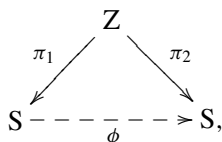
If α is a primitive k -th root of unity, the fact that $\psi: (x, y) \dashrightarrow (\eta(x), ay + B(x))$ commutes with

$$\phi^k: (x, y) \dashrightarrow (x, y + k)$$

yields $a(y+k) + B(x) = ay + B(x) + k$, so $a = 1$. We again get $B(\alpha x) = B(x)$. □

3. On the growth of the number of base-points

If S is a projective smooth surface, every $\phi \in \text{Bir}(S)$ admits a resolution



where π_1, π_2 are sequences of blow-ups. The resolution is *minimal* if and only if no (-1) -curve of Z is contracted by both π_1 and π_2 . The *base-points* of ϕ are the points blown-up by π_1 , which can be points of S or infinitely near points. We denote by $b(\phi)$ the number of such points, which is also equal to the difference of the ranks of $\text{Pic}(Z)$ and $\text{Pic}(S)$, and thus equal to $b(\phi^{-1})$.

We will call the number

$$\mu(\phi) = \lim_{k \rightarrow +\infty} \frac{b(\phi^k)}{k},$$

the *dynamical number of base-points of ϕ* . Since $b(\phi\psi) \leq b(\phi) + b(\psi)$ for any $\phi, \psi \in \text{Bir}(S)$, we see that $\mu(\phi)$ is a non-negative real number. Moreover, $b(\phi^{-1})$ and $b(\phi)$ being always equal, we get $\mu(\phi^k) = |k \cdot \mu(\phi)|$ for any $k \in \mathbb{Z}$. We can also make the following easy observation:

Lemma 3.1. *The dynamical number of base-points is an invariant of conjugation: if $\theta: S \dashrightarrow Z$ is a birational map between smooth projective surfaces and $\phi \in \text{Bir}(S)$, then*

$$\mu(\phi) = \mu(\theta\phi\theta^{-1}).$$

In particular, if ϕ is conjugate to an automorphism of a smooth projective surface, then $\mu(\phi) = 0$.

Proof. We have

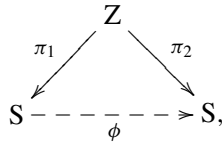
$$-2b(\theta) + b(\phi^n) \leq b(\theta\phi^n\theta^{-1}) \leq 2b(\theta) + b(\phi^n) \text{ for all } n \in \mathbb{Z}.$$

This yields $\mu(\phi) = \mu(\theta\phi\theta^{-1})$. It is thus clear that $\mu(\phi) = 0$ if ϕ is conjugate to an automorphism of a smooth projective surface. □

In this section, we precise the properties of the number μ , and will in particular see that it is an integer.

If $\phi \in \text{Bir}(S)$ is a birational map, we will say that a (possibly infinitely near) base-point p of ϕ is a *persistent base-point* if there exists an integer N such that p is a base-point of ϕ^k for any $k \geq N$ but is not a base-point of ϕ^{-k} for any $k \geq N$.

If p is a point of S or a point infinitely near, which is not a base-point of $\phi \in \text{Bir}(S)$, we define a point $\phi^\bullet(p)$, which will also be a point of S or a point infinitely near. For this, take a minimal resolution



where π_1, π_2 are sequences of blow-ups. Because p is not a base-point of ϕ , it corresponds, via π_1 , to a point of Z or infinitely near. Using π_2 , we view this point on S , again maybe infinitely near, and call it $\phi^\bullet(p)$.

Remark 3.2. If p is not a base-point of $\phi \in \text{Bir}(S)$ and $\phi(p)$ is not a base-point of $\psi \in \text{Bir}(S)$, we have $(\psi\phi)^\bullet(p) = \psi^\bullet(\phi^\bullet(p))$. If p is a general point of S , then $\phi^\bullet(p) = \phi(p) \in S$.

Example 3.3. If $S = \mathbb{P}^2$, $p = (1 : 0 : 0)$ and ϕ is the birational map $(x : y : z) \dashrightarrow (yz + x^2 : xz : z^2)$, the point $\phi^\bullet(p)$ is not equal to $p = \phi(p)$, but is infinitely near to it.

Using this definition, we put an equivalence relation on the set of points that belong to S or are infinitely near, by saying that p is *equivalent* to q if there exists an integer k such that $(\phi^k)^\bullet(p) = q$ (this implies that p is not a base-point of ϕ^k and that q is not a base-point of ϕ^{-k}). The set of equivalence classes is the generalisation of the notion of set of orbits for birational maps.

Proposition 3.4. *Let ϕ be a birational map of a smooth projective surface S . Denote by v the number of equivalence classes of persistent base-points of ϕ . Then, the set*

$$\left\{ \mathfrak{b}(\phi^k) - vk \mid k \geq 0 \right\} \subset \mathbb{Z}$$

is bounded.

In particular, $\mu(\phi)$ is an integer, equal to v .

Proof. Let us say that a base-point q is periodic if $(\phi^k)^\bullet(q) = q$ for some $k \neq 0$, or if q is a base-point of ϕ^k for any $k \in \mathbb{Z} \setminus \{0\}$ (which implies that $(\phi^k)^\bullet(q)$ is never defined for $k \neq 0$). Let us denote by \mathcal{P} the set of periodic base-points of ϕ and by $\widehat{\mathcal{P}}$ the finite set of points equivalent to a point of \mathcal{P} .

The number of base-points of ϕ and ϕ^{-1} being finite, there exists an integer N such that for any non-periodic base-point p and for any $j, j' \geq N$, p is a base-point of ϕ^j (respectively of ϕ^{-j}) if and only if p is a base-point of $\phi^{j'}$ (respectively of $\phi^{-j'}$).

We decompose the set of non-periodic base-points of ϕ into four sets:

$$\begin{aligned} \mathcal{B}_{++} &= \{p \mid p \text{ is a base-point of } \phi^j, \text{ and is a base-point of } \phi^{-j} \text{ for } j \geq N\}, \\ \mathcal{B}_{+-} &= \{p \mid p \text{ is a base-point of } \phi^j, \text{ but is not a base-point of } \phi^{-j} \text{ for } j \geq N\}, \\ \mathcal{B}_{-+} &= \{p \mid p \text{ is not a base-point of } \phi^j, \text{ but is a base-point of } \phi^{-j} \text{ for } j \geq N\}, \\ \mathcal{B}_{--} &= \{p \mid p \text{ is not a base-point of } \phi^j, \text{ and is not a base-point of } \phi^{-j} \text{ for } j \geq N\}. \end{aligned}$$

Note that \mathcal{B}_{+-} is the set of persistent base-points of ϕ and that \mathcal{B}_{-+} is the set of persistent base-point of ϕ^{-1} . This decomposes the set of base-points of ϕ into five disjoint sets. Two base-points p, p' of ϕ which are equivalent belong to the same set.

We fix an integer $k \geq 2N$ and compute the number of base-points of ϕ^k . Any such base-point being equivalent to a base-point of ϕ , we take a base-point p of ϕ , and count the number $m_{p,k}$ of base-points of ϕ^k which are equivalent to p .

If p belongs to \mathcal{P} , the number of points equivalent to p is less than $\#(\widehat{\mathcal{P}})$, so $m_{p,k} \leq \#(\widehat{\mathcal{P}})$.

If p is not in \mathcal{P} , any point equivalent to p is equal to $(\phi^i)^\bullet(p)$ for some i , and all are distinct, so we have

$$\begin{aligned} m_{p,k} &= \#I_{p,k}, \text{ where} \\ I_{p,k} &= \{i \in \mathbb{Z} \mid p \text{ is not a base-point of } \phi^i, \text{ but } p \text{ is a base-point of } \phi^{i+k}\}. \end{aligned}$$

If $p \in \mathcal{B}_{++}$, since p is not a base-point of ϕ^i one has $-N < i < N$, thus $m_{p,k} < 2N$.

A point $p \in \mathcal{B}_{--}$ is a base-point of ϕ^{i+k} hence $-N < i + k < N$ and $m_{p,k} < 2N$.

If $p \in \mathcal{B}_{-+}$, the fact that p is not a base-point of ϕ^i implies that $-N < i$ and the fact that p is a base-point of ϕ^{i+k} implies that $i + k \leq N$. With these two inequalities one has $-N < i \leq N - k$. Since $k > 2N$, we get $m_{p,k} = 0$.

If $p \in \mathcal{B}_{+-}$, the fact that p is not a base-point of ϕ^i implies that $i < N$ and the fact that p is a base-point of ϕ^{i+k} implies that $-N < i + k$. This yields $-N - k < i < N$, so $m_{p,k} \leq 2N + k$. Conversely, if $i \leq -N$ and $i + k \geq N$, p is not a base-point of ϕ^i , but p is a base-point of ϕ^{i+k} (or equivalently $i \in I_{p,k}$), so $m_{p,k} \geq \#[N - k, -N] = k - 2N + 1$. The two conditions together imply that $m_{p,k} - k \in [-2N, 2N]$.

Recall that \mathcal{B}_{+-} is the set of persistent base-points of ϕ . The above counting explains that the number of base-points of ϕ^k , for k big, behaves like νk , where ν is the number of equivalence classes of persistent base-points of ϕ . More precisely, there exist two integers c, d which do not depend on k , such that the total number of base-points of ϕ^k is between $\nu k + c$ and $\nu k + d$, for any $k \geq 2N$. Recalling that

$$\mu(\phi) = \lim_{k \rightarrow +\infty} \frac{\mathfrak{b}(\phi^k)}{k}, \text{ where } \mathfrak{b}(\phi^k) \text{ is the number of base-points of } \phi^k, \text{ we obtain } \mu(\phi) = \nu. \quad \square$$

Proposition 3.5. *Let S be a smooth projective surface, and let $\phi \in \text{Bir}(S)$. The following conditions are equivalent:*

- (1) $\mu(\phi) = 0$;
- (2) ϕ is conjugate to an automorphism of a smooth projective surface.

Proof. Lemma 3.1 yields (2) \Rightarrow (1). It suffices then to show (1) \Rightarrow (2).

By blowing-up points, we can always assume that no point of S is a base-point of ϕ^k and ϕ^{-l} for $k, l > 0$; this means that ϕ is algebraically stable (see [13]).

Using the notation of Proposition 3.4, this implies that there is no periodic base-point and that $\mathcal{B}_{++} = \emptyset$. Moreover, the condition $\mu(\phi) = 0$ corresponds to saying that $\mathcal{B}_{+-} = \mathcal{B}_{-+} = \emptyset$, so all base-points belong to \mathcal{B}_{--} .

Suppose that ϕ is not an automorphism, and write $\tau : X \rightarrow S$ the blow-up of the base-points of ϕ , and $\chi : X \rightarrow S$ the morphism $\chi = \phi\tau$, which is the blow-up of the base-points of ϕ^{-1} . Denote by $E \subset X$ a (-1) -curve (smooth rational curve of self-intersection -1) contracted by χ . The image $\chi(E)$ is a proper point of S , which is a base-point of ϕ^{-1} . Because ϕ is algebraically stable, $\chi(E)$ is not a base-point of ϕ^k , for any $k \geq 0$. This implies that $\phi^k\chi : X \dashrightarrow S$ is well-defined at any point of E , so the curve $C = \tau(E)$ is an irreducible curve of S , contracted by ϕ^{k+1} , and any base-point of ϕ^{k+1} that belongs to C , as proper or infinitely near point, is also a base-point of ϕ . This finite set of points being in \mathcal{B}_{--} , there is an integer $n > 0$ such that no base-point of ϕ^n belongs to C . Since C is contracted by ϕ^n , C is a (-1) -curve of S . Contracting C conjugates ϕ to an algebraically stable birational map having again all its base-points in \mathcal{B}_{--} .

This process decreases the rank of the Picard group of the surfaces obtained; it thus has to stop at some point. This means that ϕ is conjugate to an automorphism of a smooth projective surface. □

Remark 3.6. In [13, Theorem 0.4] one can find a characterisation of hyperbolic birational maps ϕ which are conjugate to an automorphism of a projective surface. If $\phi \in \text{Bir}(\mathbb{P}^2)$ is hyperbolic, we conjugate it to a birational map of a smooth projective surface S where the action is algebraically stable (this means that $(\phi|_{H^{1,1}(S)})^n = (\phi^n)|_{H^{1,1}(S)}$ for each n); its action on $H^{1,1}(S)$ admits the eigenvalue $\lambda(\phi) > 1$ with eigenvector θ_+ . The map ϕ is birationally conjugate to an automorphism if and only if $(\theta_+)^2 = 0$.

Proposition 3.5 gives another characterisation, for all maps $\phi \in \text{Bir}(\mathbb{P}^2)$ (not only hyperbolic maps), depending only on $\mu(\phi)$.

Example 3.7. In [2–5, 12], automorphisms with positive entropy are constructed starting from a birational map of \mathbb{P}^2 . In most cases, the authors take a birational map $\psi \in \text{Bir}(\mathbb{P}^2)$, and choose $A \in \text{Aut}(\mathbb{P}^2)$ such that $A\psi$ is conjugate to an automorphism of a surface with dynamical degree > 1 . The way to find A is exactly to ensure that $A\psi$ has no persistent base-point (i.e. $\mu(A\psi) = 0$). Let us give an example ([12]):

Let $\phi = A\psi$ be the birational map given by

$$A : (x : y : z) \dashrightarrow (\alpha x + 2(1 - \alpha)y + (2 + \alpha - \alpha^2)z : -x + (\alpha + 1)z : x - 2y + (1 - \alpha)z)$$

with $\alpha \in \mathbb{C} \setminus \{0, 1\}$ and

$$\psi : (x : y : z) \dashrightarrow (xz^2 + y^3 : yz^2 : z^3).$$

The map ψ (respectively ψ^{-1}) has five base-points, $p = (1 : 0 : 0)$ and four points infinitely near; we will denote \widehat{P}_1 (respectively \widehat{P}_2) the collection of these points. The automorphism A is chosen such that:

- $\widehat{P}_1, A\widehat{P}_2, A\psi A\widehat{P}_2$ have distinct supports;
- $\widehat{P}_1 = (A\psi)^2 A\widehat{P}_2$.

In particular the base-points of ϕ are non-persistent, so ϕ is conjugate to an automorphism of a rational surface. More precisely ϕ is conjugate to an automorphism with positive entropy on \mathbb{P}^2 blown up in $\widehat{P}_1, A\widehat{P}_2$ and $A\psi A\widehat{P}_2$ (see [12, Theorem 3.1]).

4. Growth of Jonquière's twists

As we explained at the beginning of §1, Jonquière's elements of the Cremona group are those which preserve a pencil of rational curves (which can be sent by a birational transformation of the plane onto a pencil of lines). Among them, those which have a linear degree growth are Jonquière's twists, the others are elliptic elements of the Cremona group. This section describes precisely the possible degree growth, and the relation with the pencil of rational curves. Before doing this, let us fix some convention:

Definition 4.1 (Image and preimage of a linear system, free intersection).

In this article, all linear systems considered are without fixed component (and thus of dimension ≥ 1).

Let $\phi : S \dashrightarrow S'$ be a birational map between two smooth projective surfaces.

Let Λ' be a linear system of curves on S' , without fixed component. The *preimage of Λ' by ϕ* , written $\phi^{-1}(\Lambda')$, is the linear system whose general members are the strict transforms of the general members of Λ' . By construction, $\phi^{-1}(\Lambda')$ is again a linear system of the same dimension as Λ' and has no fixed component.

If Λ is a linear system of curves on S , without fixed component, we similarly define the *image of Λ by ϕ* , written $\phi(\Lambda)$, so that $\phi^{-1}(\phi(\Lambda)) = \Lambda$ and $\phi(\phi^{-1}(\Lambda')) = \Lambda'$.

This gives in particular an action of $\text{Bir}(S)$ on the set of all linear systems of S having no fixed component.

If Λ_1, Λ_2 are two linear systems of S , the *free intersection* of Λ_1 and Λ_2 is a non-negative integer, which counts the number of "free points", *i.e.* points which are not base-points of Λ_1, Λ_2 , in the intersection between a general member of Λ_1 and a general member of Λ_2 .

By definition, the free intersection of Λ_1 and Λ_2 is the same as the free intersection of $\phi(\Lambda_1)$ and $\phi(\Lambda_2)$.

Lemma 4.2. *Let ϕ be a birational map of \mathbb{P}^2 which preserves the pencil of lines passing through some point p_0 . The set*

$$\left\{ \deg \phi^k - k \cdot \frac{\mu(\phi)}{2} \mid k \geq 0 \right\} \subset \mathbb{Z}$$

is bounded.

In particular, the sequence $\{\deg \phi^k\}_{k \in \mathbb{N}}$ grows linearly if and only if $\mu(\phi) > 0$ and its growth is given by

$$\frac{\mu(\phi)}{2} \in \frac{1}{2}\mathbb{N}.$$

Remark 4.3. Conjugating ϕ by a map which preserves the pencil does not change the growth $\{\deg \phi^k\}_{k \in \mathbb{N}}$, but conjugating it by a map which does not preserve the pencil can increase it (see Proposition 4.5).

Proof. Denote by Λ_0 the linear system of lines of \mathbb{P}^2 , and for any integer k we denote by Λ_k the linear system $\phi^k(\Lambda_0)$, which has degree $\deg \phi^k = \deg \phi^{-k}$.

For any k , ϕ^k preserves the pencil of lines passing through p_0 . Computing the free intersection of this pencil with Λ_k , we obtain 1. This implies that Λ_k has multiplicity $\deg \phi^k - 1$ at p_0 , so all other base-points of the pencil have multiplicity 1. Computing the self-intersection, we find that there are exactly $2(\deg \phi^k - 1)$ other base-points, all of multiplicity 1. In particular, $\deg \phi^k = \lfloor \frac{b(\phi^k)}{2} \rfloor$. The result follows then directly from Proposition 3.4. □

Example 4.4. Let us consider the family of birational maps studied in [11] and defined as

$$f_{\alpha,\beta}: (x : y : z) \dashrightarrow ((\alpha x + y)z : \beta y(x + z) : z(x + z)), \quad \alpha, \beta \in \mathbb{C}^*.$$

Any of the $f_{\alpha,\beta}$ has three base-points: $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(-1 : \alpha : 1)$, and preserves the pencil of lines passing through $(1 : 0 : 0)$. Checking that $(-1 : \alpha : 1)$ is the only one persistent base-point ([11, Theorem 1.6]), the growth of $\{\deg f_{\alpha,\beta}^k\}_{k \in \mathbb{N}}$ is given by $\frac{k}{2}$ (see [11, Lemma 1.4]).

Proposition 4.5. *Let $\phi \in \text{Bir}(\mathbb{P}^2)$ be a Jonquières twist. There exists an integer $a \in \mathbb{N}$ such that*

$$\lim_{k \rightarrow +\infty} \frac{\deg(\phi^k)}{k} = a^2 \frac{\mu(\phi)}{2}.$$

Moreover, a is the degree of the curves of the unique pencil of rational curves invariant by ϕ . In particular, $a = 1$ if and only if ϕ preserves a pencil of lines.

Proof. Since ϕ is a Jonquières twist, it preserves a unique pencil of rational curves. Taking a birational map $\psi \in \text{Bir}(\mathbb{P}^2)$ which sends this pencil onto a pencil of lines, the birational map $\phi' = \psi \phi \psi^{-1}$ preserves the pencil of lines passing through some point $p \in \mathbb{P}^2$. Let $\pi: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ be the blow-up of $p \in \mathbb{P}^2$, and let $\tilde{\phi} \in \text{Bir}(\mathbb{F}_1)$ be

defined by $\widehat{\phi} = \pi^{-1}\phi'\pi = \pi^{-1}\psi\phi\psi^{-1}\pi$. Denote by $L_{\mathbb{P}^2}$ the linear system of lines of \mathbb{P}^2 and write $\Lambda = \pi^{-1}\psi(L_{\mathbb{P}^2})$, which is a linear system on \mathbb{F}_1 (see Definition 4.1 for the precise definition of $\pi^{-1}\psi(L_{\mathbb{P}^2})$).

The degree of ϕ^k is equal to the free intersection of $L_{\mathbb{P}^2}$ with $\phi^k(L_{\mathbb{P}^2})$, which is the free intersection of Λ with $\widehat{\phi}^k(\Lambda)$ (see Definition 4.1).

On \mathbb{F}_1 , Λ is linearly equivalent to $aL + bf$, where $L = \pi^{-1}(L_{\mathbb{P}^2})$ and f is the divisor of a fibre of $\mathbb{F}_1 \rightarrow \mathbb{P}^1$ and where $a, b \in \mathbb{N}$. Because ϕ' preserves the pencil of lines through p , $\widehat{\phi}$ preserves the linear system given by $|f|$. To compute $\widehat{\phi}^k(\Lambda)$, we want to compute $\widehat{\phi}^k(L) = \pi^{-1}(\phi')^k\pi(L) = \pi^{-1}((\phi')^k(L_{\mathbb{P}^2}))$. Denoting by d_k the degree of $(\phi')^k(L_{\mathbb{P}^2})$, which is the degree of $(\phi')^k$, the linear system $(\phi')^k(L_{\mathbb{P}^2})$ has degree d_k and multiplicity $d_k - 1$ at p (see the proof of Lemma 4.2). Hence $\widehat{\phi}^k(L)$ is linearly equivalent to $L + (d_k - 1)f$, which implies that

$$\widehat{\phi}^k(\Lambda) \sim aL + (a(d_k - 1) + b)f,$$

so the intersection number of $\widehat{\phi}^k(\Lambda)$ with Λ is $a^2d_k + 2ab$. Because $\widehat{\phi}^k(\Lambda) \cdot f = a$, each base-point of $\widehat{\phi}^k(\Lambda)$ has at most multiplicity a . It follows from Lemma 4.2 that $\lim_{k \rightarrow +\infty} \frac{d_k}{k} = \frac{\mu(\phi)}{2}$. The number of base-points of Λ being bounded, the free intersection of $\widehat{\phi}^k(\Lambda)$ with Λ grows like $a^2 \frac{\mu(\phi)}{2} \cdot k$.

Note that $a = f \cdot \Lambda$ is also the free intersection of Λ with the linear system $|f|$, since $|f|$ has no base-point. In consequence, a is the free intersection of $\psi^{-1}\pi(\Lambda) = L_{\mathbb{P}^2}$ with $\psi^{-1}\pi(|f|)$, which is the degree of the linear system $\psi^{-1}\pi(|f|)$. This gives the result, since $\psi^{-1}\pi(|f|)$ is the unique pencil invariant by ϕ . □

Lemma 4.2 and the second assertion of Lemma 3.1 imply the following statement of [13, Theorem 0.2.]

Corollary 4.6. *Let ϕ be a Jonquières twist; then ϕ is not conjugate to an automorphism.*

We can also derive the following new results.

Corollary 4.7. *Let ϕ be a Jonquières twist. If ϕ^m and ϕ^n are conjugate in $\text{Bir}(\mathbb{P}^2)$ for some $m, n \in \mathbb{Z}$, then $|m| = |n|$.*

Proof. The fact that $\{\text{deg } \phi^k\}_{k \in \mathbb{N}}$ grows linearly implies that ϕ preserves a pencil of rational curves [13, Theorem 0.2]. In particular ϕ is conjugate to a birational map of \mathbb{P}^2 which preserves the pencil of lines passing through some fixed point p_0 . According to Lemma 4.2, one finds $\mu(\phi) > 0$.

As ϕ^m and ϕ^n are conjugate in $\text{Bir}(\mathbb{P}^2)$ it follows from Lemma 3.1 that $\mu(\phi^m) = \mu(\phi^n)$. Since $\mu(\phi^k) = |k \cdot \mu(\phi)|$ for any k , we get $|m| = |n|$. □

5. Growth of Halphen twists

In Section 4 (especially Lemma 4.2), we described the degree growth of a Jonquieres twist ϕ , and showed that it is given by $\mu(\phi)$, a birational invariant given by the growth of base-points. For a Halphen twist, the dynamical number of base-points is trivial, but the growth can also be quantified by an invariant. Recall (see §1 and references there) that a Halphen twist preserves a unique pencil of elliptic curves, which can be sent by a birational map of the plane onto a Halphen pencil. By [14, Theorem 2 and Proposition 7, page 127], a power of ϕ preserves any member of the pencil, and acts on this via a translation.

Proposition 5.1. *Let $\phi \in \text{Bir}(\mathbb{P}^2)$ be a Halphen twist.*

(1) *The set*

$$\left\{ \lim_{k \rightarrow +\infty} \frac{\deg(\psi \phi^k \psi^{-1})}{k^2} \mid \psi \in \text{Bir}(\mathbb{P}^2) \right\}$$

admits a minimum, which is a positive rational number $\kappa(\phi) \in \mathbb{Q}$. If ϕ acts via a translation on each member of its invariant pencil of elliptic curves, then $\kappa(\phi) \in 9\mathbb{N}$.

(2) *There exists an integer $a \geq 3$ such that $\lim_{k \rightarrow +\infty} \frac{\deg(\phi^k)}{k^2} = \kappa(\phi) \cdot \frac{a^2}{9}$.*

(3) *The following conditions are equivalent:*

(a) $a = 3$;

(b) ϕ *preserves a Halphen pencil, i.e. a pencil of (elliptic) curves of degree $3m$ passing through 9 points with multiplicity m .*

Remark 5.2. The proof also shows that the degree of the unique pencil of elliptic curves invariant by ϕ is equal to am , where $3m$ is the degree of the corresponding Halphen pencil.

Proof. A Halphen twist preserves a unique pencil of elliptic curves. Denoting by $\psi \in \text{Bir}(\mathbb{P}^2)$ an element which sends this pencil onto a Halphen pencil, the birational map $\phi' = \psi \phi \psi^{-1}$ preserves a Halphen pencil. Denoting by $\pi : S \rightarrow \mathbb{P}^2$ the blow-up of the 9 base-points of the pencil, $\widehat{\phi} = \pi^{-1} \phi' \pi$ is an automorphism of S , which preserves the elliptic fibration $S \rightarrow \mathbb{P}^1$ given by $| -mK_S |$ for some positive integer m .

Replacing $\widehat{\phi}$ by some power if needed, we can assume that $\widehat{\phi}$ is a translation on a general fibre. As explained in the proof of [14, Proposition 9, page 132], this yields the existence of an element $\Delta \in \text{Pic}(S)$ (depending on $\widehat{\phi}$) with $\Delta \cdot K_S = 0$ such that the action of $\widehat{\phi}$ on $\text{Pic}(S)$ is given by

$$D \mapsto D - m(D \cdot K_S) \cdot \Delta + \gamma K_S,$$

where γ is an integer depending on D which can be computed from the fact that both sides of the equation have the same self-intersection:

$$\gamma = -\frac{m^2}{2}(D \cdot K_S) \cdot \Delta^2 + m(D \cdot \Delta).$$

We denote by L the linear system of lines of \mathbb{P}^2 and by $\Lambda = \pi^{-1}\psi(L)$ the corresponding system on S (see Definition 4.1 for the precise meaning of $\pi^{-1}\psi(L)$). The degree of ϕ^n is equal to the free intersection of L with $\phi^n(L)$, which is equal to the free intersection of Λ with $\widehat{\phi}^n(\Lambda)$ (as explained in Definition 4.1).

The map $\widehat{\phi}^n$ acts on $\text{Pic}(S)$ as

$$D \mapsto D - m(D \cdot K_S) \cdot (n\Delta) + \left(-\frac{m^2}{2}(D \cdot K_S) \cdot (n\Delta)^2 + m(D \cdot (n\Delta)) \right) K_S.$$

This yields

$$\begin{aligned} \Lambda \cdot \widehat{\phi}^n(\Lambda) &= \Lambda^2 - m(\Lambda \cdot K_S) \cdot (n\Delta \cdot \Lambda) \\ &\quad + \left(-\frac{m^2}{2}(\Lambda \cdot K_S)(n\Delta)^2 + m(n\Lambda \cdot \Delta) \right) (K_S \cdot \Lambda) \\ &= \Lambda^2 + \left(-\frac{m^2}{2}(\Lambda \cdot K_S)^2(\Delta)^2 \right) \cdot n^2. \end{aligned}$$

The free intersection between Λ and $\widehat{\phi}^n(\Lambda)$ is thus equal to

$$\Lambda^2 + \left(-\frac{m^2}{2}(\Lambda \cdot K_S)^2(\Delta)^2 \right) \cdot n^2 - \sum_{i=1}^k \mu_i(\Lambda) \cdot \mu_i(\widehat{\phi}^n(\Lambda)),$$

where $\mu_i(\Lambda)$ and $\mu_i(\widehat{\phi}^n(\Lambda))$ denote the multiplicities of respectively Λ and $\widehat{\phi}^n(\Lambda)$ at the r base-points of Λ . Since $\widehat{\phi}$ is an automorphism of S , the contribution given by the base-points is bounded, so we find that

$$\lim_{n \rightarrow +\infty} \frac{\text{deg}(\phi^n)}{n^2} = m^2(\Lambda \cdot K_S)^2 \cdot \left(\frac{-\Delta^2}{2} \right).$$

Note that Λ is the lift by π^{-1} of the homaloidal linear system $\psi(L)$, so $\Lambda \cdot (-K_S) \geq 3$, and equality holds if and only if Λ has no base-point. This shows that the minimum among all homaloidal systems is attained when Λ has no base-point; we get $\kappa(\phi) = 9m^2 \cdot \left(\frac{-\Delta^2}{2} \right)$ and $a = \Lambda \cdot K_S$.

Let us prove that $-\Delta^2$ is a positive even number. To do this, we take the orthogonal basis L, E_1, \dots, E_9 of $\text{Pic}(S)$, where L is the pull-back of a line of \mathbb{P}^2 and E_1, \dots, E_9 are the exceptional divisors associated to the points blown-up. Writing

$$\Delta = dL - \sum_{i=1}^9 a_i E_i,$$

the equality $\Delta \cdot K_S = 0$ implies that $\sum a_i = 3d$. Modulo 2, we have $\sum (a_i)^2 \equiv \sum a_i \equiv 3d \equiv d^2$, so $\Delta^2 = d^2 - \sum (a_i)^2$ is even. Applying Cauchy-Schwarz to $(1, \dots, 1)$ and (a_1, \dots, a_9) , we get $9 \sum (a_i)^2 \geq (\sum a_i)^2$, and equality holds only when all a_i are equal. This latter would imply that Δ is a multiple of K_S , and thus that $\widehat{\phi}$ acts trivially on $\text{Pic}(S)$. Hence we have $9 \sum (a_i)^2 > (\sum a_i)^2 = (3d)^2$, which implies that $-\Delta^2 = \sum (a_i)^2 - d^2 > 0$.

The fact that $-\Delta^2$ is a positive even number implies that $\kappa(\phi) = 9m^2 \cdot \left(\frac{-\Delta^2}{2}\right)$ is a positive integer divisible by 9. The equality $\kappa(\phi) = \lim_{k \rightarrow +\infty} \frac{\deg(\phi^k)}{k^2}$ is equivalent to the fact that Λ has no base-point, which corresponds to say that $\psi^{-1}\pi$ is a birational morphism, or equivalently that the unique pencil of elliptic curves invariant by ϕ is a Halphen pencil. \square

Corollary 5.3. *Let $\phi \in \text{Bir}(\mathbb{P}^2)$ be a Halphen twist. The integer $\kappa(\phi)$ is a birational invariant which satisfies $\kappa(\phi^m) = m^2\kappa(\phi)$ for any $m \in \mathbb{Z}$. In particular, the maps ϕ^n and ϕ^m are not conjugate if $|m| \neq |n|$.*

Proof. Is a direct consequence of Proposition 5.1. \square

Let us give an example where $\kappa(\phi)$ is not an integer.

Example 5.4. Let Λ be the pencil of cubic curves of \mathbb{P}^2 given by $\lambda(x^2y + z^3 + y^2z) + \mu(x^2z + y^3 + yz^2) = 0$, $(\lambda : \mu) \in \mathbb{P}^1$. This pencil is invariant by the automorphism $\alpha \in \text{Aut}(\mathbb{P}^2)$ defined by $(x : y : z) \mapsto (ix : -y : z)$.

Denote by $\pi : S \rightarrow \mathbb{P}^2$ the blow-up of the base-points of Λ , which are 7 proper points of \mathbb{P}^2 and 2 infinitely near points. More precisely, the 7 proper points are

$$\begin{aligned} p_1 &= (1 : 0 : 0), \\ p_2 &= (i\sqrt{2} : 1 : 1), \\ p_3 &= \alpha(p_2) = (-\sqrt{2} : -1 : 1), \\ p_4 &= \alpha(p_3) = (-i\sqrt{2} : 1 : 1), \\ p_5 &= \alpha(p_4) = (\sqrt{2} : -1 : 1), \\ p_6 &= (0 : i : 1), \\ p_7 &= \alpha(p_6) = (0 : -i : 1). \end{aligned}$$

The last two points are the following: the point p_8 is infinitely near to p_6 , corresponding to the tangent direction of the line $y = iz$, and p_9 is infinitely near to p_7 , corresponding to the tangent direction of the line $y = -iz$.

The surface S inherits an elliptic fibration $S \rightarrow \mathbb{P}^1$, and the lift of α yields an automorphism $\widehat{\alpha} = \pi\alpha\pi^{-1}$ of S . Denote by $E_i \in \text{Pic}(S)$ the divisor of self-intersection -1 corresponding to the point p_i . If $i \neq 6, 7$, then E_i corresponds to a (-1) -curve of S ; and E_6, E_7 correspond to two reducible curves of S .

For any $\Delta \in \text{Pic}(S)$ satisfying $\Delta \cdot K_S = 0$, we denote by $\iota_\Delta \in \text{Aut}(S)$ the automorphism which restricts on a general fibre C to the translation given by the divisor $\Delta|_C$. If $\Delta^2 = -2$, the action of ι_Δ on $\text{Pic}(S)$ is given by (see the proof of Proposition 5.1)

$$D \mapsto D - (D \cdot K_S) \cdot \Delta + (D \cdot (K_S + \Delta)) \cdot K_S.$$

For any automorphism $\sigma \in \text{Aut}(S)$, one can check that $\iota_{\sigma(\Delta)} = \sigma \iota_\Delta \sigma^{-1}$. In particular, we have

$$\begin{aligned} (\widehat{\alpha} \iota_\Delta)^4 &= (\widehat{\alpha} \iota_\Delta \widehat{\alpha}^{-1})(\widehat{\alpha}^2 \iota_\Delta \widehat{\alpha}^{-2})(\widehat{\alpha}^3 \iota_\Delta \widehat{\alpha}^{-3}) \iota_\Delta \\ &= {}^t \widehat{\alpha}(\Delta) {}^t \widehat{\alpha}^2(\Delta) {}^t \widehat{\alpha}^3(\Delta) {}^t \Delta = {}^t \widehat{\alpha}(\Delta) + \widehat{\alpha}^2(\Delta) + \widehat{\alpha}^3(\Delta) + \Delta. \end{aligned}$$

Because of the action of α on the points p_i , we have

$$\begin{aligned} \widehat{\alpha}(E_6) &= E_7, & \widehat{\alpha}(E_7) &= E_6, \\ \widehat{\alpha}(E_2) &= E_3, & \widehat{\alpha}(E_3) &= E_4, \\ \widehat{\alpha}(E_4) &= E_5, & \widehat{\alpha}(E_5) &= E_2. \end{aligned}$$

We now fix $\Delta \in \text{Pic}(S)$ to be the divisor $E_2 - E_6$ (that satisfies $\Delta \cdot K_S = 0$ and $\Delta^2 = -2$), and obtain

$$\widehat{\alpha}(\Delta) + \widehat{\alpha}^2(\Delta) + \widehat{\alpha}^3(\Delta) + \Delta = E_2 + E_3 + E_4 + E_5 - 2E_6 - 2E_7,$$

which has square -8 . In particular, the number κ associated to

$$(\widehat{\alpha} \iota_\Delta)^4 = {}^t \widehat{\alpha}(\Delta) + \widehat{\alpha}^2(\Delta) + \widehat{\alpha}^3(\Delta) + \Delta$$

is

$$-9 \frac{({}^t \widehat{\alpha}(\Delta) + \widehat{\alpha}^2(\Delta) + \widehat{\alpha}^3(\Delta) + \Delta)^2}{2} = 36.$$

This shows, by Corollary 5.3, that $\kappa(\phi) = \frac{9}{4}$, where ϕ is the birational map of \mathbb{P}^2 conjugate to $\widehat{\alpha} \iota_\Delta$ by π^{-1} , namely $\alpha \pi \iota_\Delta \pi^{-1}$.

6. Applications

6.1. Birational maps having two conjugate iterates

Lemma 6.1. *Let ϕ denote a birational map of \mathbb{P}^2 . Assume that ϕ^n and ϕ^m are conjugate and assume that $|m| \neq |n|$. Then, ϕ is elliptic and satisfies $\lambda(\phi) = 1$ and $\mu(\phi) = 0$.*

Proof. The map ϕ^m is conjugate to ϕ^n in $\text{Bir}(\mathbb{P}^2)$ so one gets $\lambda(\phi)^{|m|} = \lambda(\phi)^{|n|}$ and $|m| \cdot \mu(\phi) = |n| \cdot \mu(\phi)$. This yields $\lambda(\phi) = 1$ and $\mu(\phi) = 0$.

The fact that $\lambda(\phi) = 1$ implies that ϕ is elliptic, or a Jonquières or Halphen twist. The Jonquières and Halphen cases are impossible (Corollaries 4.7 and 5.3). \square

Proposition 6.2. *Let ϕ denote a birational map of \mathbb{P}^2 of infinite order. Assume that ϕ^n and ϕ^m are conjugate and assume that $|m| \neq |n|$. Then, ϕ is conjugate to an automorphism of \mathbb{C}^2 of the form $(x, y) \mapsto (\alpha x, y + 1)$, where $\alpha \in \mathbb{C}^*$ such that $\alpha^{m+n} = 1$ or $\alpha^{m-n} = 1$.*

In particular, if ϕ is conjugate to ϕ^n for any positive integer n , then ϕ is conjugate to $(x, y) \mapsto (x, y + 1)$.

Proof. This statement follows from Corollary 2.5, Corollary 5.3 and Lemma 6.1. \square

6.2. Morphisms of Baumslag-Solitar groups in the Cremona group

For any integers m, n such that $mn \neq 0$, the Baumslag-Solitar group $\text{BS}(m, n)$ is defined by the following presentation

$$\text{BS}(m, n) = \langle r, s \mid rs^m r^{-1} = s^n \rangle.$$

Recall that if G is a group, the derived groups of G are

$$G^{(0)} = G, \quad G^{(i)} = [G^{(i-1)}, G^{(i-1)}] = \langle ghg^{-1}h^{-1} \mid g, h \in G^{(i-1)} \rangle \text{ for all } i \geq 1,$$

and that G is *solvable* if there exists an integer N such that $G^{(N)} = \{\text{id}\}$.

The group $\text{BS}(m, n)$ (respectively the subgroups of finite index of $\text{BS}(m, n)$) is solvable if and only if $|m| = 1$ or $|n| = 1$ (see [20, Proposition A.6]).

A group G is said to be *residually finite* if for any g in $G \setminus \{\text{id}\}$ there exist a finite group H and a group homomorphism $\Theta: G \rightarrow H$ such that $\Theta(g)$ belongs to $H \setminus \{\text{id}\}$. The group $\text{BS}(m, n)$ is residually finite if and only if $|m| = 1$ or $|n| = 1$ or $|m| = |n|$ (see [19]). Let V be an affine algebraic variety; according to [1] any subgroup of finite index of the automorphisms group $\text{Aut}(V)$ of V is residually finite. Therefore if $|m| \neq |n|$ and $|m|, |n| \neq 1$ there is no embedding of $\text{BS}(m, n)$ into the group of polynomial automorphisms of the plane. There is an other proof using the amalgated structure of the group of polynomial automorphisms of the plane and the fact that $\text{BS}(m, n)$ is not solvable ([10, Proposition 2.2]).

Lemma 6.3. *Let ρ be a homomorphism from $\text{BS}(m, n) = \langle r, s \mid rs^m r^{-1} = s^n \rangle$ to $\text{Bir}(\mathbb{P}^2)$. Assume that $|m|, |n|$ and 1 are distinct. If $\rho(s)$ has infinite order, the image of the subgroup of finite index $\langle r, s^m \mid rs^{m^2} r^{-1} = s^{mn} \rangle$ of $\text{BS}(m, n)$ is solvable.*

Proof. By Proposition 6.2, one can conjugate ρ so that $\rho(s) : (x, y) \dashrightarrow (\alpha x, y + 1)$ where $\alpha \in \mathbb{C}^*$ and $\alpha^{m-n} = 1$ or $\alpha^{m+n} = 1$. Denoting respectively by ψ the map $(x, \frac{n}{m}y)$ or $(x^{-1}, \frac{n}{m}y)$ one has $\psi\rho(s)^m\psi^{-1} = \rho(s)^n$. So $\rho(r) = \psi\tau$ where τ commutes with $\rho(s)^m : (x, y) \mapsto (\alpha^m x, y + m)$.

According to Lemma 2.8, one then has $\tau = (\eta(x), y + R(x))$ for some $\eta \in \text{PGL}(2, \mathbb{C})$, $\eta(\alpha^m x) = \alpha^m \eta(x)$, and some $R \in \mathbb{C}(x)$ satisfying $R(\alpha^m x) = R(x)$. And one gets $\rho(r) = (\eta(x)^{\pm 1}, \frac{n}{m}(y + R(x)))$.

The group generated by $\rho(s^m)$ and $\rho(r)$ is thus solvable. □

Corollary 6.4. *If $|m|$, $|n|$ and 1 are distinct, there is no embedding of $\text{BS}(m, n)$ into the Cremona group.*

Proof. Lemma 6.3 shows that the image of any embedding would be virtually solvable, impossible when $|m|$, $|n|$ and 1 are distinct. □

6.3. Embeddings of $\text{GL}(2, \mathbb{Q})$ into the Cremona group

To simplify the notation, we will denote in this last section by $(\phi_1(x, y), \phi_2(x, y))$ the rational map $(x, y) \dashrightarrow (\phi_1(x, y), \phi_2(x, y))$ from \mathbb{C}^2 to \mathbb{C}^2 .

Let us first give examples of embeddings of $\text{GL}(2, \mathbb{Q})$ into the Cremona group.

Examples 6.5. Let k be an odd integer and let $\chi : \mathbb{Q}^* \rightarrow \mathbb{C}^*$ be a homomorphism such that $a \mapsto \frac{\chi(a^2)}{a^k}$ is injective. The morphism ρ from $\text{GL}(2, \mathbb{Q})$ to the Cremona group given by

$$\rho \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(x \cdot \frac{\chi(ad - bc)}{(cy + d)^k}, \frac{ay + b}{cy + d} \right)$$

is an embedding. Note that $\rho(\text{GL}(2, \mathbb{Q}))$ is conjugate to a subgroup of automorphisms of the k -th Hirzebruch surface \mathbb{F}_k . Changing k gives then infinitely many non conjugate embeddings in the Cremona group.

Remark 6.6. Taking $k = 1$ and χ the trivial map, Example 6.5 yields the embedding $\rho : \text{GL}(2, \mathbb{Q}) \rightarrow \text{Bir}(\mathbb{P}^2)$ given by

$$\rho \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(\frac{x}{cy + d}, \frac{ay + b}{cy + d} \right),$$

which is obviously conjugate (by extending the actions to \mathbb{P}^2) to the classical embedding

$$\widehat{\rho} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (ax + by, cx + dy).$$

Theorem 6.7. *Let $\rho : \text{GL}(2, \mathbb{Q}) \rightarrow \text{Bir}(\mathbb{P}^2)$ be an embedding of $\text{GL}(2, \mathbb{Q})$ into the Cremona group, then up to conjugation ρ is one of the embeddings described in Examples 6.5.*

Proof. Let us set

$$t_q = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \quad \& \quad d_{m,n} = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}, \quad q, m, n \in \mathbb{Q}.$$

Remark that t_1 is conjugate to t_n in $GL(2, \mathbb{Q})$, for any $n \in \mathbb{Z} \setminus \{0\}$; we can then assume, after conjugation, that $\rho(t_1) = (x, y + 1)$ (Proposition 6.2). As $\rho(t_{1/n})$ commutes with $\rho(t_1)$ there exist A_n in $PGL(2, \mathbb{C})$ and R_n in $\mathbb{C}(x)$ such that

$$\rho(t_{1/n}) = (A_n(x), y + R_n(x))$$

(see Lemma 2.8). Let us prove now that $A_n(x) = x$. Since $t_{1/n}^n = t_1$ the element A_n is of finite order so A_n is conjugate to some ξx where ξ is some root of unity. Hence $\rho(t_{1/n})$ is conjugate to $(\xi x, y + Q(x))$ where $Q \in \mathbb{C}(x)$ satisfies

$$Q(x) + Q(\xi x) + \dots + Q(\xi^{n-1}x) = 1.$$

The map $(\xi x, y + Q(x))$ is then conjugate to $(\xi x, y + \frac{1}{n})$ by $(x, y - \frac{\sum_{i=1}^{n-1} i Q(\xi^i x)}{n})$.

Since $t_{1/n}$ is conjugate to t_1 , Proposition 2.4 implies that $\xi = 1$, which achieves to show that $A_n(x) = x$. This implies, with equality $\rho(t_{1/n})^n = \rho(t_1)$, that $R_n(x) = 1/n$. We thus have for any q in \mathbb{Q}

$$\rho(t_q) = (x, y + q).$$

From $d_{m,n} t_1 d_{m,n}^{-1} = t_{m/n}$ one gets (using again Lemma 2.8) that

$$\rho(d_{m,n}) = \left(\eta_{m,n}(x), \frac{m}{n}y + R_{m,n}(x) \right), \quad \eta_{m,n} \in PGL(2, \mathbb{C}), R_{m,n} \in \mathbb{C}(x).$$

The map $(\mathbb{Q}^*)^2 \rightarrow PGL(2, \mathbb{C})$ given by $(m, n) \mapsto \eta_{m,n}$ is a homomorphism, which cannot be injective. There exists thus one element $d_{m,n}$ with $(m, n) \neq (1, 1)$ such that $\rho(d_{m,n}) = (x, \frac{m}{n}y + R_{m,n}(x))$. Note that $m \neq n$ since the centralisers of $\rho(d_{m,n})$ and $\rho(t_1)$ are different. Conjugating by $(x, y + \frac{R_{m,n}(x)}{m/n-1})$, we can assume that $\rho(d_{m,n}) = (x, \frac{m}{n}y)$. From $d_{m,n} d_{a,b} = d_{a,b} d_{m,n}$ one gets for any a, b in \mathbb{Q}

$$\rho(d_{a,b}) = \left(\eta_{a,b}(x), \frac{a}{b}y \right), \quad \eta_{a,b} \in PGL(2, \mathbb{C}).$$

The homomorphism $\mathbb{Q}^* \rightarrow PGL(2, \mathbb{C})$ given by $a \mapsto \eta_{a,a}$ is injective, so up to conjugation by an element of $PGL(2, \mathbb{C})$ we can assume that for any $a \in \mathbb{Q} \setminus \{0, 1\}$ there exists $\chi_{a,a} \in \mathbb{C} \setminus \{0, 1\}$ such that $\eta_{a,a}(x) = \chi_{a,a}x$. This implies the existence of $\chi_{a,b} \in \mathbb{C}^*$ for any a, b in $(\mathbb{Q}^*)^2$, such that $\eta_{a,b}(x) = \chi_{a,b}x$.

We now compute the image of $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since $\rho(M)$ commutes with $\rho(d_{2,2}) = (\chi_{2,2}x, y)$ where $\chi_{2,2} \in \mathbb{C}^*$ is of infinite order, there exist $R \in \mathbb{C}(y)$ and

$v \in \text{PGL}(2, \mathbb{C})$ such that $\rho(M) = (xR(y), v(y))$ (Lemma 2.8). For any $a \in \mathbb{Q}^*$, equality $Md_{a,1} = d_{1,a}M$ yields

$$(\chi_{a,1}x \cdot R(ay), v(ay)) = \left(\chi_{1,a}x \cdot R(y), \frac{1}{a}v(y) \right).$$

This implies that $R(y) = \alpha y^{-k}$ and $v(y) = \frac{\beta}{y}$, for some $\alpha, \beta \in \mathbb{C}^*, k \in \mathbb{Z}$, i.e. $\rho(M) = \left(\alpha \frac{x}{y^k}, \frac{\beta}{y} \right)$. We use now equality $(Mt_1)^3 = \text{id}$: the second component of $(\rho(M)\rho(t_1))^3$ being

$$\frac{\beta(y + \beta + 1)}{(\beta + 1)y + 2\beta + 1},$$

we find $\beta = -1$ and compute $(\rho(M)\rho(t_1))^3 = (x\alpha^3(-1)^k, y)$ so $\alpha^3 = (-1)^k$. Since $\rho(M)^2 = (x\alpha^2(-1)^k, y)$ has order 2, we have $-\alpha^2 = (-1)^k$, thus $\alpha = -1$ and k is odd.

Writing $\chi(a) = \chi_{a,1}$ for any $a \in \mathbb{Q}$, the map $\mathbb{Q}^* \rightarrow \mathbb{C}^*$ given by $a \rightarrow \chi(a)$ is a homomorphism, and one gets $\rho(d_{a,1}) = (\chi(a)x, ay)$. The group $\text{GL}(2, \mathbb{Q})$ is generated by the maps $d_{a,1}, t_a$ and M , so

$$\rho \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(x \cdot \frac{\chi(ad - bc)}{(cy + d)^k}, \frac{ay + b}{cy + d} \right),$$

for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Q})$. This yields an embedding if and only if the homomorphism from \mathbb{Q}^* to \mathbb{C}^* given by $a \mapsto \frac{\chi(a^2)}{a^k}$ is injective.

It remains to observe that k can be chosen to be positive. Indeed, otherwise one conjugates by $\left(\frac{1}{x}, y\right)$ and replaces χ with $\frac{1}{\chi}$ to replace k with $-k$. □

One can see that $\text{GL}(n, \mathbb{Q})$ does not embed into $\text{Bir}(\mathbb{P}^2)$ as soon as $n \geq 3$. Indeed, Theorem 6.7 implies that the diagonal matrices are sent onto diagonal elements of $\text{PGL}(3, \mathbb{C}) = \text{Aut}(\mathbb{P}^2)$, which is impossible, by considering the involutions. One can also find another less obvious corollary:

Corollary 6.8. *Let $\rho: \text{GL}(2, \mathbb{C}) \rightarrow \text{Bir}(\mathbb{P}^2)$ be an embedding of $\text{GL}(2, \mathbb{C})$ into the Cremona group. There exist a positive odd integer k , a field homomorphism $\tau: \mathbb{C} \rightarrow \mathbb{C}$ and a group homomorphism $\chi: \mathbb{C}^* \rightarrow \mathbb{C}^*$ such that*

$$\rho \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \left(x \cdot \frac{\chi(ad - bc)}{(\tau(c)y + \tau(d))^k}, \frac{\tau(a)y + \tau(b)}{\tau(c)y + \tau(d)} \right), \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{C}).$$

Remark 6.9. One sees that in the description above, ρ is an embedding if and only if the group homomorphism $\mathbb{C}^* \rightarrow \mathbb{C}^*$ given by $a \mapsto \frac{\chi(a^2)}{\tau(a)^k}$ is injective. This happens for instance by taking $\chi(a) = \tau(a)^{\frac{k+1}{2}}$, any positive odd integer k and any field homomorphism $\tau: \mathbb{C} \rightarrow \mathbb{C}$.

Proof. The map ρ induces an embedding of $\mathrm{GL}(2, \mathbb{Q})$ into $\mathrm{Bir}(\mathbb{P}^2)$. According to Theorem 6.7 one has a description of $\rho|_{\mathrm{GL}(2, \mathbb{Q})}$. Up to conjugacy, there exists an odd positive integer k and an homomorphism $\tilde{\chi}: \mathbb{Q}^* \rightarrow \mathbb{C}^*$ such that

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(x \cdot \frac{\tilde{\chi}(ad - bc)}{(cy + d)^k}, \frac{ay + b}{cy + d}\right), \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Q}).$$

Let us set

$$t_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad \& \quad d_b = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}, \quad a \in \mathbb{C}, b \in \mathbb{C}^*.$$

For any $a \in \mathbb{C}^*$, the matrix d_a commutes with all diagonal matrices with entries in \mathbb{Q} ; this implies, with the description above, that

$$\rho(d_a) = (\chi(a)x, \tau(a)y)$$

for some $\chi(a), \tau(a) \in \mathbb{C}^*$ (Lemma 2.7). This yields two group homomorphisms $\chi, \tau: \mathbb{C}^* \rightarrow \mathbb{C}^*$. Observe that χ is an extension of $\tilde{\chi}$, *i.e.* $\chi(a) = \tilde{\chi}(a)$ for any $a \in \mathbb{Q}$.

The equality $d_a t_1 d_{a^{-1}} = t_a$ implies that

$$\rho(t_a) = (x, y + \tau(a)), \quad \forall a \in \mathbb{C}^*.$$

In particular, τ extends to an (injective) field homomorphism $\mathbb{C} \rightarrow \mathbb{C}$. The group $\mathrm{GL}(2, \mathbb{C})$ being generated by $\mathrm{GL}(2, \mathbb{Q})$ and $\{d_a \mid a \in \mathbb{C}^*\}$, one has

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \left(x \cdot \frac{\chi(ad - bc)}{(\tau(c)y + \tau(d))^k}, \frac{\tau(a)y + \tau(b)}{\tau(c)y + \tau(d)}\right), \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

The map ρ is injective if and only if $\chi(a^2) \neq \tau(a^k)$ for any $a \in \mathbb{C}^* \setminus \{1\}$. □

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