EXCEPTIONAL ISOMORPHISMS BETWEEN COMPLEMENTS OF AFFINE PLANE CURVES

JÉRÉMY BLANC, JEAN-PHILIPPE FURTER, AND MATTIAS HEMMIG

Abstract. This article describes the geometry of isomorphisms between complements of geometrically irreducible closed curves in the affine plane $\mathbb{A}^2$, over an arbitrary field, which do not extend to an automorphism of $\mathbb{A}^2$.

We show that these isomorphisms are quite exceptional. In particular, they occur only when both curves are isomorphic to open subsets of the affine line $\mathbb{A}^1$. Moreover, the isomorphism is uniquely determined by one of the curves, up to left composition by an automorphism of $\mathbb{A}^2$, except in the case where the curve is isomorphic to the affine line $\mathbb{A}^1$ or to the punctured line $\mathbb{A}^1 \setminus \{0\}$. We also prove that if one curve is isomorphic to a line, then both curves are in fact equivalent to lines. In addition, for any $n \in \mathbb{N}$, we construct n (even infinitely many in characteristic 0) pairwise non-equivalent closed embeddings of the punctured line with isomorphic complements. We then give a construction that provides a large family of examples of non-isomorphic geometrically irreducible curves of $\mathbb{A}^2$ that have isomorphic complements, answering negatively the Complement Problem posed by Hanspeter Kraft [Kra96]. This also gives a negative answer to the holomorphic version of this problem in any dimension $n \geq 2$.

Contents

1.	Introduction	2
2.	Geometric description of open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$	4
2.1.	Basic properties	4
2.2.	The case of lines	9
2.3.	Embeddings into Hirzebruch surfaces	10
2.4.	Extension to regular morphisms on $\mathbb{A}^2$	13
2.5.	Completion with two curves and a boundary	16
2.6.	The case of curves isomorphic to $\mathbb{A}^1$ and the proof of Theorem 1	21
3.	Families of non-equivalent embeddings	23
3.1.	A construction using elements of $\mathrm{SL}_2(\mathrm{k}[y])$	24
3.2.	Curves isomorphic to $\mathbb{A}^1 \setminus \{0\}$	25
4.	Non-isomorphic curves with isomorphic complements	29
4.1.	A geometric construction	29
4.2.	Getting explicit formulas	34
5.	Related questions	38
5.1.	Providing higher dimensional counterexamples	38
5.2.	The holomorphic case	39
	Appendix: The case of $\mathbb{P}^2$	40
	References	44

The authors gratefully acknowledge support by the Swiss National Science Foundation Grant "Birational Geometry" PP00P2_128422 /1 and by the French National Research Agency Grant "BirPol", ANR-11-JS01-004-01.

1. Introduction

In the Bourbaki Seminar *Challenging problems on affine n-space* [Kra96], Hanspeter Kraft gives a list of eight basic problems related to the affine n-spaces. The sixth one is the following:

> *Complement Problem. Given two irreducible hypersurfaces $E, F \subset \mathbb{A}^n$ and an isomorphism of their complements, does it follow that E and F are isomorphic?*

Recently, Pierre-Marie Poloni gave a negative answer to the problem for any $n \geq 3$ [Pol16]. The construction is given by explicit formulas. There are examples where both E and F are smooth, and examples where E is singular, but F is smooth. This article deals with the case of dimension $n = 2$. The situation is much more rigid than in dimension $n \geq 3$, as we discuss in Theorem 1.

We recall that two curves $C, D \subset \mathbb{A}^2$ are *equivalent* if there is an automorphism of $\mathbb{A}^2$ that sends one onto the other. Note that equivalent curves are isomorphic. A variety defined over a field k is called *geometrically irreducible* if it is irreducible over the algebraic closure of k (it will then be reduced, as all curves in the sequel). A *line* in $\mathbb{A}^2$ is a closed curve of degree 1.

Theorem 1. *Let k be any field. Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ be an isomorphism, where $D \subset \mathbb{A}^2$ is also a closed curve.*

If φ does not extend to an automorphism of $\mathbb{A}^2$, then both C and D are isomorphic to open subsets of $\mathbb{A}^1$. More precisely, there are isomorphisms

$$C \simeq \operatorname{Spec}(\mathrm{k}[t, \frac{1}{P}]) \quad \textit{and} \quad D \simeq \operatorname{Spec}(\mathrm{k}[t, \frac{1}{Q}])$$

for some square-free polynomials $P, Q \in \mathrm{k}[t]$ that have the same number of roots in k, and the same number of roots in the algebraic closure $\overline{\mathrm{k}}$.

Moreover, the following holds:

1. *If C is isomorphic to $\mathbb{A}^1$, then both C and D are equivalent to lines.*
2. *If C is not isomorphic to $\mathbb{A}^1$ or to $\mathbb{A}^1 \setminus \{0\}$, then the isomorphism φ (which does not extend to an automorphism of $\mathbb{A}^2$) is uniquely determined by C, up to left composition by an automorphism of $\mathbb{A}^2$. In particular, there are at most two equivalence classes of curves in $\mathbb{A}^2$ with complements isomorphic to $\mathbb{A}^2 \setminus C$.*

Corollary. *If the ground field k is algebraically closed, and*

$$\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$$

is an isomorphism between the complements of two irreducible plane closed curves $C, D \subset \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$, then there exist two finite subsets F, G of the affine line $\mathbb{A}^1$ of the same cardinality such that C is isomorphic to $\mathbb{A}^1 \setminus F$ and D is isomorphic to $\mathbb{A}^1 \setminus G$.

If F contains at least 2 points, then φ is uniquely determined by C, up to left composition by an automorphism of $\mathbb{A}^2$. The equivalence class of D is uniquely determined by C, except when F contains exactly one point.

Corollary. *Let $C \subset \mathbb{A}^2$ be a singular, geometrically irreducible closed curve and $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ an isomorphism, for some closed curve D. Then φ extends to an automorphism of $\mathbb{A}^2$.*

The second corollary shows in particular that the Complement Problem for $n = 2$ has a positive answer if one of the curves is singular, contrary to the case where $n \geq 3$, as pointed out before. It is also very different to the case of $\mathbb{P}^2$, where there exist non-isomorphic irreducible curves with isomorphic complements [Bla09, Theorem 1], but where all examples are necessarily singular (see Proposition A.1 below).

Theorem 1 also shows that the Complement Problem for $n = 2$ has a positive answer if the curve is not rational (which is an easy observation, see Corollary 2.8 below) but more generally when it is not isomorphic to an open subset of $\mathbb{A}^1$. For instance, the circle $x^2 + y^2 = 1$ over $\mathbb{R}$ is a smooth rational affine curve which is not isomorphic to an open subset of $\mathbb{A}^1$.

In addition, Theorem 1 gives strong restrictions on isomorphisms between complements of curves: if C is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ and there exists an isomorphism $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ that does not extend to an automorphism of $\mathbb{A}^2$, then φ is unique (up to left composition by an automorphism of $\mathbb{A}^2$), hence the class of D is unique too. This is again quite different to the case of dimension $n \geq 3$ where there are infinitely many hypersurfaces $E \subset \mathbb{A}^n$, up to equivalence, that have isomorphic complements [Pol16, Lemma 3.1]. It is also different to the case of $\mathbb{P}^2$, where we can find algebraic families of non-equivalent curves of $\mathbb{P}^2$ that have isomorphic complements (and thus infinitely many if k is infinite). This follows from a construction made in [Cos12], see Corollary A.3 below.

All tools necessary to obtain the rigidity result (Theorem 1) are developped in Section 2. The proof is then achieved at the end of the section. Our second statement is the following existence result, which shows the optimality of Theorem 1.

Theorem 2. *Let* k *be any field.*

(1) *There exists a closed curve $C \subset \mathbb{A}^2$, isomorphic to $\mathbb{A}^1 \setminus \{0\}$, whose complement $\mathbb{A}^2 \setminus C$ admits infinitely many open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ into the affine plane, up to automorphisms of $\mathbb{A}^2$. Moreover, the set of equivalence classes of curves with this property is infinite.*
(2) *For each integer $n \geq 1$ there exist pairwise non-equivalent curves $C_1, \ldots, C_n \subset \mathbb{A}^2$, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$, such that all open surfaces $\mathbb{A}^2 \setminus C_1, \ldots, \mathbb{A}^2 \setminus C_n$ are isomorphic. Moreover, if $\mathrm{char}(\mathrm{k}) = 0$, we can find an infinite sequence of pairwise non-equivalent curves $C_i \subset \mathbb{A}^2$, $i \in \mathbb{N}$, such that all open surfaces $\mathbb{A}^2 \setminus C_i$, $i \in \mathbb{N}$, are isomorphic.*
(3) *For each polynomial $f \in \mathrm{k}[t]$ of degree ≥ 1, there exist two non-equivalent closed curves $C, D \subset \mathbb{A}^2$, both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$, such that the open surfaces $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic. Moreover, the set of such pairs of closed embeddings of $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$ is infinite, up to equivalence.*

The proof of Theorem 2, mainly via explicit constructions, is made in Section 3.

We then give counterexamples to the Complement Problem in dimension 2, over any field:

Theorem 3. *For any ground field* k, *there exist two geometrically irreducible closed curves $C, D \subset \mathbb{A}^2$ which are not isomorphic but whose complements $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$*

are isomorphic. Furthermore, these two curves can be chosen of degree 7 *if the field admits more than* 2 *elements and of degree* 13 *if the field has* 2 *elements.*

The proof of this result is detailed in Section 4. We first give a geometric construction in Lemma 4.1. Then, we show in Proposition 4.2 that, for each polynomial $P \in \mathrm{k}[t]$ of degree $d \geq 1$ and each $\lambda \in \mathrm{k}$ with $P(\lambda) \neq 0$, this construction yields two closed curves $C, D \subset \mathbb{A}^2$ of degree $d^2 - d + 1$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that the following isomorphisms hold:

$$C \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{P}]\right) \text{ and } D \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{Q}]\right), \text{ where } Q(t) = P\left(\lambda + \frac{1}{t}\right) \cdot t^{\deg(P)}.$$

The proof of Theorem 3 follows by providing an appropriate pair (P, λ) for each field. The case of infinite fields is quite easy. Indeed, if k is infinite and $P \in \mathrm{k}[t]$ is a polynomial with at least 3 roots in $\overline{\mathrm{k}}$, then $\operatorname{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\operatorname{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic, for a general element $\lambda \in \mathrm{k}$ (Lemma 4.5). This shows that the isomorphism type of counterexamples to the Complement Problem is as large as possible.

We finish this introduction by giving some easy implications of Theorem 3, detailed in Section 5:

(i) The negative answer to the Complement Problem for $n = 2$ directly yields a negative answer for any $n \geq 3$ (Lemma 5.1): Our construction gives, for each $n \geq 3$, two geometrically irreducible smooth closed hypersurfaces $E, F \subset \mathbb{A}^n$ which are not isomorphic but whose complements $\mathbb{A}^n \setminus E$ and $\mathbb{A}^n \setminus F$ are isomorphic (Corollary 5.2). All the hypersurfaces provided this way are isomorphic to $\mathbb{A}^{n-2} \times C$ for some open subset $C \subset \mathbb{A}^1$. This does not allow to give singular examples like the ones of [Pol16], but gives a different type of examples.

(ii) Choosing $\mathrm{k} = \mathbb{C}$, our construction also gives families of closed complex curves $C, D \subset \mathbb{C}^2$ such that $\mathbb{C}^2 \setminus C$ and $\mathbb{C}^2 \setminus D$ are biholomorphic (because they are isomorphic as algebraic varieties) but C and D are not biholomorphic (Lemma 5.3). This directly provides, for each $n \geq 2$, the existence of algebraic hypersurfaces $E, F \subset \mathbb{C}^n$ which are complex manifolds that are not biholomorphic but have biholomorphic complements (Corollary 5.4). This is, to our knowledge, the first family of such examples, and yields the answer to a problem asked in [Pol16]. Note that in the counterexamples of [Pol16], if both hypersurfaces are smooth, then they are always biholomorphic (even if they are not isomorphic as algebraic varieties).

The authors thank Hanspeter Kraft, Lucy Moser-Jauslin and Pierre-Marie Poloni for interesting discussions during the preparation of this article.

2. Geometric description of open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$

In the sequel, we work over an arbitrary field k. When we say rational, resp. isomorphic, we mean k-rational, resp. k-isomorphic. The word geometrically rational or geometrically irreducible refers to the extension to the algebraic closure $\overline{\mathrm{k}}$, as usual.

2.1. **Basic properties.** In order to study isomorphisms between affine surfaces, it is often interesting to see the affine surfaces as open subsets of projective surfaces and to see then the isomorphisms as birational maps between the projective surfaces. Recall that a rational map $\varphi\colon X \dashrightarrow Y$ between smooth projective irreducible surfaces is defined on an open subset $U \subset X$ such that $F = X \setminus U$ is finite. If C is an irreducible curve of the surface X, its image is defined by $\varphi(C) := \overline{\varphi(C \setminus F)}$. We then say that C is *contracted by* φ if $\varphi(C)$ is a point. The aim of this section is to establish Lemma 2.7, that we use often in the sequel. Its proof relies on some easy results that we recall before: Lemmas 2.2, 2.4, 2.6 and Corollary 2.5.

Example 2.1. The morphism

$$\begin{array}{rcl} \mathbb{A}^2 & \hookrightarrow & \mathbb{P}^2 \\ (x,y) & \mapsto & [x:y:1] \end{array}$$

gives an isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$, where $L_{\mathbb{P}^2} \subset \mathbb{P}^2$ denotes the "line at infinity" given by $z = 0$. The above embedding of $\mathbb{A}^2$ into $\mathbb{P}^2$ will be often used in the sequel, and called the *standard embedding.*

With this standard embedding, every line of $\mathbb{A}^2$, given by an equation $ax + by = c$ where a, b, c are elements of k and a, b are not both zero, is the restriction of a line of $\mathbb{P}^2$, given by the equation $ax + by = cz$, and distinct from $L_{\mathbb{P}^2}$.

Lemma 2.2. *Let $\varphi\colon X \dashrightarrow Y$ be a birational map between two projective surfaces and assume that φ restricts to an isomorphism $U \xrightarrow{\simeq} V$ where $U \subset X$ and $V \subset Y$ are two open subsets.*

(1) *Any geometrically irreducible closed curve $\Gamma \subset X \setminus U$ is sent either to a point of $Y \setminus V$ or to a curve contained in $Y \setminus V$.*
(2) *Assume that $\eta\colon Z \to X$ and $\pi\colon Z \to Y$ are birational morphisms that yield a minimal resolution of φ as shown on the following diagram:*

$$\begin{tikzcd} & Z \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\varphi"] & & Y \\ U \arrow[u, hook] \arrow[rr, "\simeq"] & & V \arrow[u, hook] \end{tikzcd}$$

Then, we have $\eta^{-1}(U) = \pi^{-1}(V)$.

Proof. The morphisms η and π are obtained by blowing up the base-points of φ and φ^{-1} respectively, which are by assumption not contained in U and V respectively. This yields point (2), which in turn yields point (1). □

Definition 2.3. For each birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, one defines $J_\varphi \subset \mathbb{P}^2$ to be the reduced curve given by the union of all irreducible $\overline{\mathrm{k}}$-curves contracted by φ.

Lemma 2.4. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map.*

(1) *The curve J_φ is defined over* k*, i.e. is the zero locus of a homogeneous polynomial $f \in \mathrm{k}[x, y, z]$.*
(2) *We have an isomorphism $\mathbb{P}^2 \setminus J_\varphi \to \mathbb{P}^2 \setminus J_{\varphi^{-1}}$. Moreover, the number of $\overline{\mathrm{k}}$-irreducible components of J_φ and $J_{\varphi^{-1}}$ are equal.*

Proof. (1): If $\mathrm{char}(\mathrm{k}) = 0$ one could choose f to be the Jacobian determinant associated to φ. This does not work in positive characteristic as the Jacobian determinant can be zero. We then do as follows: we write φ as $\varphi\colon [x : y : z] \mapsto [s_0(x, y, z) : s_1(x, y, z) : s_2(x, y, z)]$, where $s_0, s_1, s_2 \in \mathrm{k}[x, y, z]$ are homogeneous polynomials of the same degree without common factor and do the same with $\varphi^{-1}\colon [x:y:z] \mapsto [q_0(x,y,z) : q_1(x,y,z) : q_2(x,y,z)]$. We then do the composition and obtain $q_0(s_0,s_1,s_2) = xA$, $q_1(s_0,s_1,s_2) = yA$, $q_2(s_0,s_1,s_2) = zA$, for some polynomial $A \in \mathrm{k}[x,y,z]$ and observe that J_φ is the zero locus of A. Indeed, the polynomial A is zero along an irreducible $\overline{\mathrm{k}}$-curve if and only if this curve is sent by φ onto a base-point of φ^{-1}.

(2) We take a minimal resolution of φ, which yields a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

where η and π are birational morphisms, the morphism η, resp. π, being the sequence of blow-ups of the $\overline{\mathrm{k}}$-base-points of φ, resp. φ^{-1}.

We can now work over $\overline{\mathrm{k}}$, forgetting the subfield k. Computing the Picard rank of X, we see that η and π contract the same number of irreducible curves of X. Let n be this number. We then denote by $E \subset X$, resp. $F \subset X$, the union of the n irreducible curves contracted by η, resp. π. The map φ restricts then to an isomorphism

$$\mathbb{P}^2 \setminus \eta(E \cup F) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \pi(E \cup F)$$

Let us observe that $\eta(E \cup F) = \eta(F)$. Since $\eta(E)$ consists of finitely many points, it suffices to show that these are contained in the curves of $\eta(F)$. Each point p of $\eta(E)$ corresponds to a connected component of E, which contains at least one (-1)-curve $\mathcal{E} \subset E$. The curve $\mathcal{E}$ is not contracted by π, by minimality, hence sent by π onto a curve $\pi(\mathcal{E}) \subset \mathbb{P}^2$, of self-intersection ≥ 1. This implies that $\mathcal{E}$ intersects F and thus $p \in \eta(F)$. One similarly gets $\pi(E \cup F) = \pi(E)$, and obtains that φ restricts to an isomorphism

$$\mathbb{P}^2 \setminus \eta(F) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \pi(E).$$

It remains to observe that $\eta(F)$ is a closed curve of $\mathbb{P}^2$ (in general not irreducible) and that each of its $\overline{\mathrm{k}}$-component is contracted by φ, so $\eta(F) = J_\varphi$. Similarly, one gets $\pi(E) = J_{\varphi^{-1}}$. Moreover, the number of $\overline{\mathrm{k}}$-irreducible components of $\eta(F)$ is equal to the number of $\overline{\mathrm{k}}$-irreducible components of $\overline{F \setminus E}$, which is equal to the number of $\overline{\mathrm{k}}$-irreducible components of $\overline{E \setminus F}$. This achieves the proof. □

Corollary 2.5. *Let $\varphi\colon \mathbb{P}^2 \setminus \Gamma \hookrightarrow \mathbb{P}^2$ be an open embedding, where Γ is a closed* k*-curve, which is a finite union of r distinct irreducible closed $\overline{\mathrm{k}}$-curves of $\mathbb{P}^2$. Then, there is a unique closed* k*-curve $\Delta \subset \mathbb{P}^2$ such that $\varphi(\mathbb{P}^2 \setminus \Gamma) = \mathbb{P}^2 \setminus \Delta$, and Δ is also a finite union of r distinct irreducible closed $\overline{\mathrm{k}}$-curves of $\mathbb{P}^2$.*

Proof. Let $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational map induced by φ. Lemma 2.4 implies that $J_{\hat{\varphi}} \subset \Gamma$, that $J_{\hat{\varphi}}$ and $J_{\hat{\varphi}^{-1}}$ are finite unions of $s \leq r$ irreducible closed distinct $\overline{\mathrm{k}}$-curves of $\mathbb{P}^2$, and that $\hat{\varphi}$ induces an isomorphism $\mathbb{P}^2 \setminus J_{\hat{\varphi}} \xrightarrow{\simeq} \mathbb{P}^2 \setminus J_{\hat{\varphi}^{-1}}$.

If $s = r$, the proof is over. Otherwise, $\Gamma' = \Gamma \setminus J_{\hat{\varphi}}$ is a closed k-curve of $\mathbb{P}^2 \setminus J_{\hat{\varphi}}$, which is the union of $r - s$ irreducible closed $\overline{\mathrm{k}}$-curves. The closed k-curve $\Delta' = \hat{\varphi}(\Gamma')$ of $\mathbb{P}^2 \setminus J_{\hat{\varphi}^{-1}}$ is again the union of $r - s$ irreducible closed $\overline{\mathrm{k}}$-curves. The result follows with $\Delta = \overline{\Delta'} \cup J_{\hat{\varphi}^{-1}}$. □

Lemma 2.6. *Let $\varphi\colon X \dashrightarrow Y$ be a birational map between two smooth projective surfaces (all defined over* k*), such that every irreducible $\overline{\mathrm{k}}$-curve contracted by φ is defined over*

k. *Then, each base-point of* φ^{-1} *is* k*-rational and each irreducible* $\overline{\mathrm{k}}$*-curve contracted by* φ *is* k*-rational.*

Proof. We argue by induction on the number of base-points of φ^{-1}. If there is no such base-point, there is nothing to show. Otherwise, let C be an irreducible $\overline{\mathrm{k}}$-curve contracted by φ to a point p of Y. Since C is defined over k, so is its image, i.e. p is k-rational (the generic point of C is defined over k and is sent onto the k-point p). Let $\pi\colon Y' \to Y$ be the blow-up at p and let $\varphi' = \pi^{-1} \circ \varphi\colon X \dashrightarrow Y'$. The base-points of $(\varphi')^{-1}$ coincide with the base-points of φ^{-1} from which the point p is removed. Moreover, the curves contracted by φ' are also contracted by φ, and if a curve is contracted by φ and not contracted by φ', then it is sent by φ' onto the exceptional divisor $\pi^{-1}(p)$ and is thus k-rational. Therefore, the result follows by induction. □

In the sequel, we will frequently use the following observation:

Lemma 2.7. *Let* $C \subset \mathbb{A}^2$ *be a geometrically irreducible closed curve and let* $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ *be an open embedding. Then, there exists a geometrically irreducible closed curve* $D \subset \mathbb{A}^2$ *such that* $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$*. Denote by* $\overline{C}$ *and* $\overline{D}$ *the closures of* C *and* D *in* $\mathbb{P}^2$*, denote as in Example 2.1 by* $L_{\mathbb{P}^2} = \mathbb{P}^2 \setminus \mathbb{A}^2$ *the line at infinity and denote by* $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *the birational map induced by* φ*. Then, one of the following three alternatives holds:*

1. *We have* $\hat{\varphi}(\overline{C}) = \overline{D}$*. Then, the map* φ *extends to an automorphism of* $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ *sending* C *onto* D*.*
2. *We have* $\hat{\varphi}(\overline{C}) = L_{\mathbb{P}^2}$*. Then, the curve* D *is a line of* $\mathbb{A}^2$*, i.e.* $\overline{D}$ *is a line of* $\mathbb{P}^2$ *and* φ *extends to an isomorphism* $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2} \xrightarrow{\simeq} \mathbb{P}^2 \setminus \overline{D}$*, that sends* C *onto* $L_{\mathbb{P}^2} \setminus \overline{D}$*. In particular,* C *is equivalent to a line via an automorphism of* $\mathbb{A}^2$*.*
3. *The map* $\hat{\varphi}$ *contracts the curve* $\overline{C}$ *to a* k*-point of* $\mathbb{P}^2$*. Then, the curve* $\overline{C}$ *(and therefore, also the curve* C*) is a rational curve (i.e. is* k*-birational to* $\mathbb{P}^1$*).*

Proof. The restriction of $\hat{\varphi}$ to $\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) = \mathbb{A}^2 \setminus C$ gives the open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. By Corollary 2.5, we obtain an isomorphism $\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \Delta$, for some k-curve $\Delta \subset \mathbb{P}^2$, which is the union of two $\overline{\mathrm{k}}$-irreducible closed curves of $\mathbb{P}^2$. Since $L_{\mathbb{P}^2}$ is included in Δ, there exists an irreducible closed $\overline{\mathrm{k}}$-curve D of $\mathbb{A}^2$ such that $\Delta = L_{\mathbb{P}^2} \cup \overline{D}$. As a conclusion, $\hat{\varphi}$ induces an isomorphism

$$\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{D}).$$

It follows that $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$. The equality $D = \mathbb{A}^2 \setminus \varphi(\mathbb{A}^2 \setminus C)$ proves us that the curve D is defined over k and is therefore geometrically irreducible. By Lemma 2.2, one of the following three alternatives holds:

1. We have $\hat{\varphi}(\overline{C}) = \overline{D}$. Hence, $\hat{\varphi}$ induces an automorphism of $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ (Lemma 2.4).
2. We have $\hat{\varphi}(\overline{C}) = L_{\mathbb{P}^2}$. Then, $\hat{\varphi}$ induces an isomorphism $\mathbb{P}^2 \setminus L_{\mathbb{P}^2} \xrightarrow{\simeq} \mathbb{P}^2 \setminus \overline{D}$ (again by Lemma 2.4). Since the Picard group of $\mathbb{P}^2 \setminus \Gamma$ is isomorphic to $\mathbb{Z}/\deg(\Gamma)\mathbb{Z}$, for each irreducible curve Γ, the curve $\overline{D}$ must be a line of $\mathbb{P}^2$.
3. The map $\hat{\varphi}$ contracts the curve $\overline{C}$ to a point of $\mathbb{P}^2$. Then, by Lemma 2.6, this point is necessary a k-point and the curve $\overline{C}$ is k-rational. □

Corollary 2.8. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. If C is not rational (i.e. not* k*-birational to $\mathbb{P}^1$), then every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. Follows from Lemma 2.7 and the fact that cases (2)-(3) only occur when C is rational. ☐

Remark 2.9. It follows from Corollary 2.8 that the group of automorphisms of $\mathbb{A}^2 \setminus C$, where C is a non-rational geometrically irreducible curve, is the subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ preserving C. By [BS15, Theorem 2], this group is finite (and in particular conjugate to a subgroup of $\mathrm{GL}_2(\mathrm{k})$ if $\mathrm{char}(\mathrm{k}) = 0$, see for instance [Kam79]).

We find it interesting to observe that case (3) of Lemma 2.7 only occurs when $\overline{C}$ intersects $L_{\mathbb{P}^2}$ in at most two $\overline{\mathrm{k}}$-points, even if this will not be used in the sequel.

Corollary 2.10. *If $C \subset \mathbb{A}^2$ is a closed geometrically irreducible curve such that $\overline{C}$ intersects $L_{\mathbb{P}^2} = \mathbb{P}^2 \setminus \mathbb{A}^2$ in at least three $\overline{\mathrm{k}}$-points, then every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. We can assume that $\mathrm{k} = \overline{\mathrm{k}}$. Suppose, for contradiction, that the extension $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ does not restrict to an automorphism of $\mathbb{A}^2$. By Lemma 2.7, the curve $\overline{C}$ is contracted by $\hat{\varphi}$ (because C is not equivalent to a line, so (2) is impossible). We recall that $\hat{\varphi}$ restricts to an isomorphism $\mathbb{A}^2 \setminus C = \mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{A}^2 \setminus D = \mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{D})$ (Lemma 2.7) and that $\overline{C} \subset J_{\hat{\varphi}} \subset L_{\mathbb{P}^2} \cup \overline{C}$, $J_{\hat{\varphi}^{-1}} \subset L_{\mathbb{P}^2} \cup \overline{D}$, where $J_{\hat{\varphi}}$, $J_{\hat{\varphi}^{-1}}$ have the same number of irreducible components (Lemma 2.4). We take a minimal resolution of $\hat{\varphi}$, which yields a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\hat{\varphi}"] & & \mathbb{P}^2. \end{tikzcd}$$

We first observe that the strict transforms $\tilde{L}_{\mathbb{P}^2}, \tilde{C} \subset X$ of $L_{\mathbb{P}^2}, \overline{C}$ by η intersect in at most one point. Indeed, otherwise the curve $\tilde{L}_{\mathbb{P}^2}$ is not contracted by π, because π contracts $\tilde{C}$, and sent onto a singular curve, which has then to be $\overline{D}$. We get $J_{\hat{\varphi}} = \overline{C}$, $J_{\hat{\varphi}^{-1}} = L_{\mathbb{P}^2}$ and get an isomorphism $\mathbb{P}^2 \setminus \overline{C} \to \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$, impossible because $\overline{C}$ has degree at least 3.

Secondly, the fact that $\tilde{L}_{\mathbb{P}^2}, \tilde{C} \subset X$ intersect in at most one point implies that η blows up all points of $\overline{C} \cap L_{\mathbb{P}^2}$ except at most one. Since $J_{\hat{\varphi}^{-1}} \subset D \cup L_{\mathbb{P}^2}$, there are at most two (-1)-curves contracted by η. But $L_{\mathbb{P}^2}$ and $\overline{C}$ intersect in at least three points, so we obtain exactly two proper base-points of $\hat{\varphi}$, corresponding to exactly two (-1)-curves $E_1, E_2 \subset X$ contracted to two points $p_1, p_2 \in \overline{C} \cap L_{\mathbb{P}^2}$ by η. Moreover, $J_{\hat{\varphi}^{-1}} = D \cup L_{\mathbb{P}^2}$ so $J_{\hat{\varphi}} = C \cup L_{\mathbb{P}^2}$ (Lemma 2.4). Writing $E_i' = \overline{\eta^{-1}(p_i) \setminus E_i}$, we find that π contracts $F = E_1' \cup E_2' \cup \tilde{C} \cup \tilde{L}_{\mathbb{P}^2}$.

Let us show that $E_i \cdot F \geq 2$, for $i = 1, 2$, which will imply that $\pi(E_i)$ is a singular curve for $i = 1, 2$, and yield a contradiction since E_1, E_2 are sent by π onto $L_{\mathbb{P}^2}$ and $\overline{D}$. As $E_i \cup E_i' = \eta^{-1}(p_i)$, it is a tree of rational curves, which intersects both $\tilde{C}$ and $\tilde{L}_{\mathbb{P}^2}$ since $p_i \in \overline{C} \cap L_{\mathbb{P}^2}$. If E_i' is empty, then $E_i \cdot \tilde{C} \geq 1$ and $E_i \cdot \tilde{L}_{\mathbb{P}^2} \geq 1$, whence $E_i \cdot F \geq 2$ as we claimed. If E_i' is not empty, then $E_i \cdot E_i' \geq 1$. The only possibility to get $E_i \cdot F \leq 1$ would thus be that $E_i \cdot E_i' = 1$, $E_i \cdot \tilde{C} = E_i \cdot \tilde{L}_{\mathbb{P}^2} = 0$. The equality $E_i \cdot E_i' = 1$ implies that E_i' is connected, and $E_i \cdot \tilde{C} = E_i \cdot \tilde{L}_{\mathbb{P}^2} = 0$ yields $\tilde{C} \cdot E_i' \geq 1$ and $\tilde{L}_{\mathbb{P}^2} \cdot E_i' \geq 1$. Since

$\tilde{L}_{\mathbb{P}^2}$ and $\tilde{C}$ intersect in a point disjoint from E_i', this implies that F contains a loop and thus cannot be contracted. □

Remark 2.11. In case (3) of Lemma 2.7, it is possible that $\overline{C}$ intersects the line $L_{\mathbb{P}^2}$ in two points, as it is the case in most of our examples (see for example Lemma 3.2 or Lemma 3.9). The case of one point is of course also possible (see for instance Lemma 2.13(1)).

We will also need the following basic algebraic observation.

Lemma 2.12. *Let $f \in \mathrm{k}[x,y]$ be a polynomial, irreducible over $\overline{\mathrm{k}}$, and let $C \subset \mathbb{A}^2$ be the curve given by $f = 0$. Then, the ring of functions on $\mathbb{A}^2 \setminus C$ and its subset of invertible elements are equal to*

$$\mathcal{O}(\mathbb{A}^2 \setminus C) = \mathrm{k}[x, y, f^{-1}] \subset \mathrm{k}(x, y),\ \ \mathcal{O}(\mathbb{A}^2 \setminus C)^* = \{\lambda f^n \mid \lambda \in \mathrm{k}^*, n \in \mathbb{Z}\}.$$

In particular, every automorphism of $\mathbb{A}^2 \setminus C$ exchanges the fibres of the morphism

$$\mathbb{A}^2 \setminus C \to \mathbb{A}^1 \setminus \{0\}$$

given by f.

Proof. The field of rational functions of $\mathbb{A}^2 \setminus C$ is equal to $\mathrm{k}(x, y)$. We can write any element of this field as u/v, where $u, v \in \mathrm{k}[x, y]$ are coprime polynomials, $v \neq 0$. The rational function is regular on $\mathbb{A}^2 \setminus C$ if and only if v does not vanish on any $\overline{\mathrm{k}}$-point of $\mathbb{A}^2 \setminus C$. This means that $v = \lambda f^n$, for some $\lambda \in \mathrm{k}^*$, $n \geq 0$. This provides the description of $\mathcal{O}(\mathbb{A}^2 \setminus C)$ and $\mathcal{O}(\mathbb{A}^2 \setminus C)^*$. The last remark follows from the fact that the group $\mathcal{O}(\mathbb{A}^2 \setminus C)^*$ is generated by k^* and one single element g, if and only if this element g is equal to $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$. □

2.2. **The case of lines.** Lemma 2.7 shows that one needs to study isomorphisms $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$, which extend to birational maps of $\mathbb{P}^2$ that contract the curve C to a point. One can ask whether this point can be a point of $\mathbb{A}^2$ (and thus would be contained in D) or belongs to the boundary line $L_{\mathbb{P}^2} = \mathbb{P}^2 \setminus \mathbb{A}^2$. As we will show (Corollary 2.19), the first possibility only occurs in a very special case, namely when C is equivalent to a line by an automorphism of $\mathbb{A}^2$. The case of lines is special for this reason, and is treated separately here.

Lemma 2.13. *Let $C \subset \mathbb{A}^2$ be the line given by $x = 0$.*

(1) *The group of automorphisms of $\mathbb{A}^2 \setminus C$ is given by:*

$$\mathrm{Aut}(\mathbb{A}^2 \setminus C) = \{(x, y) \mapsto (\lambda x^{\pm 1}, \mu x^n y + s(x, x^{-1})) \mid \lambda, \mu \in \mathrm{k}^*, n \in \mathbb{Z}, s \in \mathrm{k}[x, x^{-1}]\}.$$

(2) *Every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ is equal to $\psi\alpha$, where $\alpha \in \mathrm{Aut}(\mathbb{A}^2 \setminus C)$ and $\psi \colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$. In particular, the complement of its image, i.e. the complement of $\psi\alpha(\mathbb{A}^2 \setminus C) = \psi(\mathbb{A}^2 \setminus C)$, is a curve equivalent to a line by an automorphism of $\mathbb{A}^2$.*

Proof. To prove (1), we first observe that each transformation $(x, y) \mapsto (\lambda x^{\pm 1}, \mu x^n y + s(x, x^{-1}))$ actually yields an automorphism of $\mathbb{A}^2 \setminus C$. Then, we only need to show that all automorphisms of $\mathbb{A}^2 \setminus C$ are of this form. An automorphism of $\mathbb{A}^2 \setminus C$ corresponds to an automorphism of $\mathrm{k}[x, y, x^{-1}]$, which sends x onto $\lambda x^{\pm 1}$, where $\lambda \in \mathrm{k}^*$ (Lemma 2.12). Applying the inverse of $(x, y) \mapsto (\lambda x^{\pm 1}, y)$, we can assume that x is fixed. We are left with an R-automorphism of $R[y]$, where R is the ring $\mathrm{k}[x, x^{-1}]$. Such an automorphism is of the form $y \mapsto ay+b$, where $a \in R^*$, $b \in R$. Indeed, if the maps $y \mapsto p(y)$ and $y \mapsto q(y)$ are inverses of each other, the equality $y = p(q(y))$ proves us that $\deg p = \deg q = 1$. This yields the desired form.

To prove (2), we use Lemma 2.7 and write φ as an isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ where D is a geometrically irreducible closed curve, and only need to see that D is equivalent to a line by an automorphism of $\mathbb{A}^2$. We write $\psi = \varphi^{-1}$, choose an equation $f = 0$ for D (where $f \in \mathrm{k}[x, y]$ is an irreducible polynomial over $\overline{\mathrm{k}}$) and get an isomorphism $\psi^* \colon \mathcal{O}(\mathbb{A}^2 \setminus C) = \mathrm{k}[x, y, x^{-1}] \to \mathcal{O}(\mathbb{A}^2 \setminus D) = \mathrm{k}[x, y, f^{-1}]$ that sends x to $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$ (Lemma 2.12). We can thus write ψ as $(x, y) \mapsto (\lambda f(x, y)^{\pm 1}, g(x, y) f(x, y)^n)$, where $n \in \mathbb{Z}$ and $g \in \mathrm{k}[x, y]$. Replacing ψ with its composition with the automorphism $(x, y) \mapsto ((\lambda^{-1}x)^{\pm 1}, y((\lambda^{-1}x)^{\pm 1})^{-n})$ of $\mathbb{A}^2 \setminus C$, we can assume that ψ is of the form $(x, y) \mapsto (f(x, y), g(x, y))$. If g is equal to a constant $\nu \in \mathrm{k}$ modulo f, we apply the automorphism $(x, y) \mapsto (x, (y - \nu)x^{-1})$ and decrease the degree of g. After finitely many steps we obtain an isomorphism $\mathbb{A}^2 \setminus D \to \mathbb{A}^2 \setminus C$ of the form $\psi_0 \colon (x, y) \mapsto (f(x, y), g(x, y))$ where g is not a constant modulo f. The image of D by ψ_0 is then dense in C, which implies that ψ_0 extends to an automorphism of $\mathbb{A}^2$ sending D onto C (Lemma 2.7). $\square$

2.3. **Embeddings into Hirzebruch surfaces.** We will not only need embeddings of $\mathbb{A}^2$ into $\mathbb{P}^2$ but also other embeddings of $\mathbb{A}^2$ into smooth projective surfaces, and in particular into Hirzebruch surfaces. These surfaces play a natural role in the study of automorphisms of $\mathbb{A}^2$ (and of images of curves by these automorphisms), as we can decompose every automorphism of $\mathbb{A}^2$ into small links between such surfaces and then study how the singularities at infinity of the curves behave under these small links (see for instance [BS15]).

Example 2.14. For $n \geq 1$, the n-th Hirzebruch surface $\mathbb{F}_n$ is

$$\mathbb{F}_n = \{([a : b : c], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid bv^n = cu^n\}$$

and the projection $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ yields a $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$.

Let $S_n, F_n \subset \mathbb{F}_n$ be the curves given by $[1 : 0 : 0] \times \mathbb{P}^1$ and $v = 0$, respectively. The morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{F}_n \\ (x, y) & \mapsto & ([x : y^n : 1], [y : 1]) \end{array}$$

gives an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{F}_n \setminus (S_n \cup F_n)$.

We recall the following classical easy result:

Lemma 2.15. *For each $n \geq 1$, the projection $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ is the unique $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$, up to automorphism of $\mathbb{P}^1$. The curve S_n is the unique irreducible $\overline{\mathrm{k}}$-curve of $\mathbb{F}_n$ of self-intersection $-n$, and we have $(F_n)^2 = 0$.*

Proof. Since $\mathbb{F}_n \setminus (S_n \cup F_n)$ is isomorphic to $\mathbb{A}^2$, whose Picard group is trivial, one has $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}F_n + \mathbb{Z}S_n$. Moreover, F_n is a fibre of π_n and S_n is a section, so $(F_n)^2 = 0$ and $F_n \cdot S_n = 1$. Denoting by $S'_n \subset \mathbb{F}_n$ the section given by $a = 0$, one finds that S'_n is equivalent to $S_n + nF_n$, by computing the divisor of $\frac{a}{c}$.

Since S_n and S'_n are disjoint, this yields $0 = S_n \cdot (S_n + nF_n) = (S_n)^2 + n$, so $(S_n)^2 = -n$.

To get the result, it suffices to show that an irreducible $\overline{\mathrm{k}}$-curve $C \subset \mathbb{F}_n$ not equal to S_n or to a fibre of π_n has self-intersection at least equal to n. This will show in particular that a general fibre of any morphism $\mathbb{F}_n \to \mathbb{P}^1$ is equal to a fibre of π_n, since this one has self-intersection 0. We write $C = kS_n + lF_n$ for some $k, l \in \mathbb{Z}$. Since $C \neq S_n$ we have $0 \leq C \cdot S_n = l - nk$. Since C is not a fibre, it intersects every fibre, so $0 < F_n \cdot C = k$. This yields $l \geq nk > 0$ and $C^2 = -nk^2 + 2kl = kl + k(l - nk) \geq kl \geq nk^2 \geq n$. □

Lemma 2.16. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. Then, there exists an integer $n \geq 1$ and an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ such that the closure of $\iota(C)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the following two possibilities:*

(1) *$\Gamma \cdot F_n = 1$ and $\Gamma \cap F_n \cap S_n = \emptyset$.*

(2) *$\Gamma \cdot F_n \geq 2$ and the following assertions hold:*
   - (*a*) *If $n = 1$, then $2m_p(\Gamma) \leq \Gamma \cdot F_1$ for $\{p\} = S_1 \cap F_1$, and $m_r(\Gamma) \leq \Gamma \cdot S_1$ for each $r \in F_1(\mathrm{k})$.*
   - (*b*) *If $n \geq 2$, then $2m_r(\Gamma) \leq \Gamma \cdot F_n$ for each $r \in F_n(\mathrm{k})$.*

Furthermore, in Case (1), the curve C is equivalent to a curve given by an equation of the form

$$a(y)x + b(y) = 0,$$

where $a, b \in \mathrm{k}[y]$ are coprime polynomials such that $a \neq 0$ and $\deg b < \deg a$. Moreover, the following assertions are equivalent:

(*i*) *The polynomial a is constant;*

(*ii*) *The curve C is equivalent to a line by an automorphism of $\mathbb{A}^2$;*

(*iii*) *The curve C is isomorphic to $\mathbb{A}^1$;*

(*iv*) *$\Gamma \cdot S_n = 0$.*

Proof. Let us take any fixed isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ for some $n \geq 1$, and denote by Γ the closure of $\iota(C)$.

We first assume that we have $\Gamma \cdot F_n = 1$. This is equivalent to saying that Γ is a section of π_n. We can furthermore assume that $\Gamma \cap F_n \cap S_n = \emptyset$, as otherwise we blow up the point $F_n \cap S_n$, contract the curve F_n, change the embedding to $\mathbb{F}_{n+1}$ and decrease from one unity the intersection number of Γ with S_n at the point $S_n \cap F_n$. After finitely many steps we get $\Gamma \cap F_n \cap S_n = \emptyset$, i.e. we are in Case (1).

If $\Gamma \cdot F_n = 0$, then Γ is a fibre of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$. Let ψ be the unique automorphism of $\mathbb{A}^2$ such that $\iota \circ \psi$ is the standard embedding of $\mathbb{A}^2$ into $\mathbb{F}_n$ of Example 2.14. Then, the curve C is equivalent to the curve $\psi^{-1}(C)$, which has equation $y = \lambda$, for some $\lambda \in \mathrm{k}$. This proves that C is equivalent to the line $y = \lambda$, and thus to the line $x = \lambda$, sent by the standard embedding onto a curve satisfying the conditions (1).

It remains to consider the case where $\Gamma \cdot F_n \geq 2$. If Γ satisfies (2), we are done. Otherwise, we have a k-point $p \in F_n$ satisfying one of the following two possibilities:

(*a*) $n = 1$, $m_p(\Gamma) > \Gamma \cdot S_1$, and $p \in F_1$.

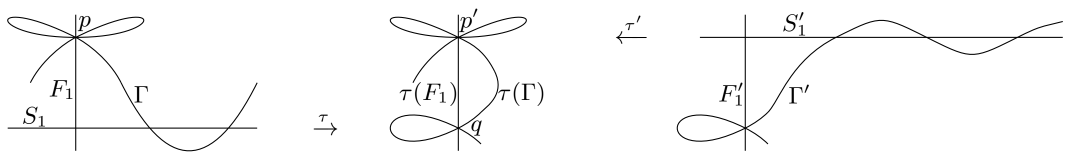
(*b*) $2m_p(\Gamma) > \Gamma \cdot F_n$ and either $n \geq 2$ or $n = 1$ and $p \in S_1 \cap F_1$.

We will replace the isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ with another one, where the singularities of the curve Γ either decrease (all multiplicities have not changed, except one multiplicity which has decreased) or stay exactly the same (as usual, the multiplicities taken into account do not only concern the proper points of $\mathbb{F}_n$ but also the infinitely near points). Moreover, the case where the multiplicities stay the same is only in (a), which cannot appear two consecutive times. We then get the result after finitely many steps.

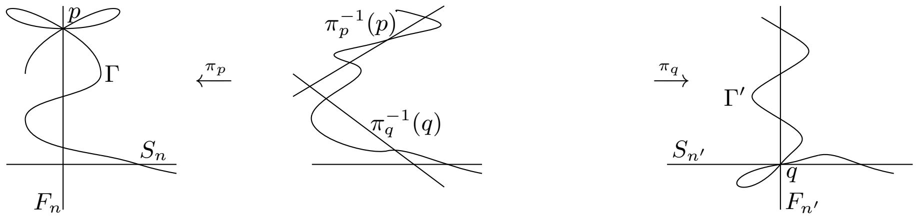
In case (a), we observe that the inequality $m_p(\Gamma) > \Gamma \cdot S_1$ joined with the inequality $\Gamma \cdot S_1 \geq (\Gamma \cdot S_1)_p \geq m_p(\Gamma) \cdot m_p(S_1)$ implies that $p \notin S_1$. We can then choose p to be a k-point of $F_1 \setminus S_1$ of maximal multiplicity and denote by $\tau\colon \mathbb{F}_1 \to \mathbb{P}^2$ the birational morphism contracting S_1 to a k-point $q \in \mathbb{P}^2$, observe that $\tau(F_1)$ is a line through q, that $\tau(\Gamma)$ is a curve of multiplicity $\Gamma \cdot S_1$ at q and of multiplicity $m_p(\Gamma) > \Gamma \cdot S_1$ at $p' = \tau(p) \in \tau(F_1)$. Moreover, p' is a k-point of $\tau(F_1)$ of maximal multiplicity on that line. Denote by $\tau'\colon \mathbb{F}_1' \to \mathbb{P}^2$ the birational morphism which is the blow-up at p'. Let S_1' be the exceptional fibre of τ', F_1' the strict transform of $\tau(F_1)$ and Γ' the strict transform of $\tau(\Gamma)$. We then replace the isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_1 \setminus (S_1 \cup F_1)$ with the analogous isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_1' \setminus (S_1' \cup F_1')$ and get

$$\forall\, r \in F_1',\ m_r(\Gamma') \leq \Gamma' \cdot S_1' = m_p(\Gamma).$$

Hence, (a) is not anymore possible. Moreover, the singularities of the new curve Γ' have either decreased or stayed the same: Indeed, the multiplicities of the singular points of $\tau(\Gamma)$ are the same as those of Γ, plus one point of multiplicity $\Gamma \cdot S_1$. Similarly, the multiplicities of the singular points of $\tau(\Gamma)$ are the same as those of Γ', plus one point of multiplicity $m_p(\Gamma)$. Of course, we do not really get a singular point if the multiplicity is 1. Therefore, the singularities of the new curve remain the same if and only if $m_p(\Gamma) = 1$ and $\Gamma \cdot S_1 = 0$. The situation is illustrated below in a simple example (which satisfies $m_p(\Gamma) = 3 > \Gamma \cdot S_1 = 2$).



In case (b), we denote by $\kappa\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n'}$ the birational map that blows up the point p and contracts the strict transform of F_n. Call q the point to which the strict transform of F_n is contracted. We have $\kappa = \pi_q \circ (\pi_p)^{-1}$, where π_p, resp. π_q, are blow-ups of the point p of $\mathbb{F}_n$, resp. the point q of $\mathbb{F}_{n'}$. The drawing below illustrates the situation in a case where $n' = n-1$. The composition of ι with κ provides a new isomorphism $\mathbb{A}^2 \to \mathbb{F}_{n'} \setminus (S_{n'} \cup F_{n'})$, where $S_{n'}$ is the image of S_n and $F_{n'}$ is the curve corresponding to the exceptional divisor of p. Note that $F_{n'}$ is a fibre of the $\mathbb{P}^1$-bundle $\pi'\colon \mathbb{F}_{n'} \to \mathbb{P}^1$ corresponding to $\pi' = \pi_n \circ \kappa^{-1}$, and that $S_{n'}$ is a section, of self-intersection $-n'$, where $n' = n+1$ if $p \in S_n$ and $n' = n-1$ if $p \notin S_n$. Hence, since $n \geq 2$ or $n = 1$ and $\{p\} = S_n \cap F_n$, we get that $(S_{n'})^2 = -n' < 0$, and obtain a new isomorphism $\iota'\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_{n'} \setminus (S_{n'} \cup F_{n'})$. The singularity of the new curve Γ' at the point q is equal to $\Gamma \cdot F_n - m_p(\Gamma)$, which is strictly smaller than $m_p(\Gamma)$ by assumption. Moreover $2m_p(\Gamma) > \Gamma \cdot F_n \geq 2$, which implies that p was indeed a singular point of Γ.



Finally, we must now prove the last statement of our lemma, which concerns Case (1). Let ψ be the unique automorphism of $\mathbb{A}^2$ such that $\iota \circ \psi$ is the standard embedding of $\mathbb{A}^2$ into $\mathbb{F}_n$ of Example 2.14. Then, replacing ι by $\iota \circ \psi$ and C by the equivalent curve $\psi^{-1}(C)$, we may assume that $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ is the standard embedding. This being done, the restriction of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$ to $\mathbb{A}^2$ is $(x,y) \to [y:1]$. The fibres of π_n, equivalent to F_n being given by $y = \text{cst}$, the degree in x of the equation of C is equal to $\Gamma \cdot F_n$ (this can be done for instance by extending the scalars to $\overline{\mathrm{k}}$ and taking a general fibre). Since $\Gamma \cdot F_n = 1$, the equation is of the form $xa(y) + b(y)$ for some polynomials $a, b \in \mathrm{k}[y]$, $a \neq 0$. Since C is geometrically irreducible, the polynomials a and b are coprime. There exist (unique) polynomials $q, \tilde{b} \in \mathrm{k}[x]$ such that $b = aq + \tilde{b}$ with $\deg \tilde{b} < \deg a$. Then, changing the coordinates by applying $(x,y) \mapsto (x + q(y), y)$, one may furthermore assume that $\deg b < \deg a$.

Let us prove that points (i)-(iv) are equivalent. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. We then prove $(iii) \Rightarrow (iv) \Rightarrow (i)$.

$(iii) \Rightarrow (iv)$: We recall that Γ is a section of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$, so that we have isomorphisms $\Gamma \simeq \mathbb{P}^1$ and $\Gamma \setminus F_n \simeq \mathbb{A}^1$. The fact that $C = \Gamma \setminus (F_n \cup S_n) \simeq \mathbb{A}^1$ implies that $C \cap (S_n \setminus F_n)$ is empty. Since $\Gamma \cap F_n \cap S_n = \emptyset$ by assumption, one gets $\Gamma \cdot S_n = 0$.

$(iv) \Rightarrow (i)$: We use the open embedding

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{F}_n \\ (u,v) & \mapsto & ([1:uv^n:u],[v:1]). \end{array}$$

The preimages of Γ and S_n by this embedding are the curves of equations $a(v)+b(v)u = 0$ and $u = 0$. Hence $\Gamma \cdot S_n = 0$ implies that a has no $\overline{\mathrm{k}}$-root and thus is a constant. $\square$

2.4. **Extension to regular morphisms on $\mathbb{A}^2$.** The following proposition, is the principal tool in the proof of Lemma 2.23, Corollary 2.24 and Proposition 2.26, which themselves give the main part of Theorem 1.

Proposition 2.17. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve, not equivalent to a line by an automorphism of $\mathbb{A}^2$, and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. Then, there exists an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the rational map $\iota \circ \varphi$ extends to a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$, and such that $\iota(\mathbb{A}^2) = \mathbb{F}_n \setminus (S_n \cup F_n)$ (where S_n and F_n are as in Example 2.14).*

Proof. By Lemma 2.7, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible curve D. If φ extends to an automorphism of $\mathbb{A}^2$ sending C onto D, the result is obvious, by taking any isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (F_n \cup S_n)$, so we can assume that φ does not extend to an automorphism of $\mathbb{A}^2$. Lemma 2.13 implies, since C is not equivalent to a line by an automorphism of $\mathbb{A}^2$, that the same holds for D. Moreover, Lemma 2.7 implies that the extension of φ^{-1} to a birational map $\mathbb{A}^2 \dashrightarrow \mathbb{P}^2$, via the standard embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$, contracts the curve D (or $\overline{D}$) to a k-point of $\mathbb{P}^2$. In particular, it does not send D birationally onto C or onto $L_{\mathbb{P}^2}$.

We choose an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$ given by Lemma 2.16, which comes from an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$, such that the closure $\iota(D)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the two possibilities (1)-(2) of Lemma 2.16.

We want to show that the open embedding $\iota \circ \varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{F}_n$ extends to a regular morphism on $\mathbb{A}^2$. Using the standard embedding of $\mathbb{A}^2$ into $\mathbb{P}^2$, one gets a birational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_n$ and needs to show that all base-points of this map are contained in $L_{\mathbb{P}^2}$. We take as usual a minimal resolution of ψ and obtain a commutative diagram

$$\begin{tikzcd}
 & & X \arrow[ld, "\eta"'] \arrow[rd, "\pi"] & & \\
\mathbb{A}^2 \arrow[r, hook, "\text{std}"] & \mathbb{P}^2 \arrow[rr, dashed, "\psi"] & & \mathbb{F}_n & \mathbb{A}^2 \arrow[l, hook', "\iota"'] \\
 & \mathbb{A}^2 \setminus C \arrow[lu, hook] \arrow[rr, "\varphi", "\simeq"'] & & \mathbb{A}^2 \setminus D. \arrow[u, hook] &
\end{tikzcd}$$

Note that ψ restricts to an isomorphism $\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{F}_n \setminus (F_n \cup S_n \cup \Gamma)$ and that by Lemma 2.2(2) we have the equality $\eta^{-1}(L_{\mathbb{P}^2} \cup \overline{C}) = \pi^{-1}(F_n \cup S_n \cup \Gamma)$. As we observed before, the map $\psi^{-1}\colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ contracts $\Gamma = \overline{\iota(D)}$ to a k-point, and thus does not send Γ birationally onto $\overline{C}$ or $L_{\mathbb{P}^2}$. The possible curves contracted by ψ are $L_{\mathbb{P}^2}, \overline{C}$ and the possible curves contracted by ψ^{-1} are Γ, F_n, S_n. Since all these are defined over k, all base-points of ψ, ψ^{-1} are defined over k (Lemma 2.6).

We suppose, for contradiction, that ψ has a base-point q in $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$, which means that one (-1)-curve $E_q \subset X$ is contracted by η to q. This curve is the exceptional divisor of a base-point infinitely near to q but not necessarily of q. The minimality of the resolution implies that π does not contract E_q, so $\pi(E_q)$ is a curve of $\mathbb{F}_n$ contracted by ψ^{-1} to q, which belongs to $\{\Gamma, F_n, S_n\}$.

We observe that ψ has also a base-point p in $L_{\mathbb{P}^2}$. Indeed, otherwise the strict transform of $L_{\mathbb{P}^2}$ would have self-intersection 1 on X: It would then not be contracted by π, and would be sent onto a curve of self-intersection ≥ 1, which belongs to $\{\Gamma, F_n, S_n\}$ by Lemma 2.2. As $(F_n)^2 = 0$ and $(S_n)^2 = -n \leq -1$, $L_{\mathbb{P}^2}$ is sent onto Γ by ψ. But Γ is not sent birationally onto $L_{\mathbb{P}^2}$ by ψ^{-1}, as we observed before. This contradiction gives us a base-point p in $L_{\mathbb{P}^2}$ and a (-1)-curve $E_p \subset X$ contracted by η to p and not contracted by π. As above, this curve is the exceptional divisor of a base-point infinitely near to p, but not necessarily of p. Again, $\pi(E_p)$ belongs to $\{\Gamma, F_n, S_n\}$.

We thus have at least two of the curves Γ, F_n, S_n that correspond to (-1)-curves of X contracted by η.

We suppose first that S_n corresponds to a (-1)-curve of X contracted by η. The fact that $(S_n)^2 = -n \leq -1$ implies that $n = 1$ and that π does not blow up any point of S_n. As there is another (-1)-curve of X contracted by η, the two curves are disjoint on X, and thus also disjoint on $\mathbb{F}_1$, since π does not blow up any point of S_1. The other curve is then Γ (since $F_1 \cdot S_1 = 1$), and $\Gamma \cdot S_1 = 0$. If moreover $\Gamma \cdot F_1 = 1$ (condition (1) of Lemma 2.16), then the contraction $\mathbb{F}_1 \to \mathbb{P}^2$ of S_1 sends Γ onto a line of $\mathbb{P}^2$, which contradicts the fact that $D \subset \mathbb{A}^2$ is not equivalent to a line. If $\Gamma \cdot F_1 \geq 2$, then condition (2) of Lemma 2.16 implies that $m_r(\Gamma) \leq \Gamma \cdot S_1 = 0$ for each $r \in F_1(\mathrm{k})$. Hence, the intersection of Γ with F_1 (which is not empty since $\Gamma \cdot F_1 \geq 2$) only consists of points not defined over k, which are therefore not blown up by π. The strict transforms $\tilde{\Gamma}$ and $\tilde{F_1}$ on X satisfy then $\tilde{\Gamma} \cdot \tilde{F_1} = \Gamma \cdot F_1 \geq 2$. As $\tilde{\Gamma}$ is contracted by η, the image $\eta(\tilde{F_1})$ is a singular curve and is then equal to $\overline{C}$. This contradicts the fact that ψ contracts $\overline{C}$ to a point.

The remaining case is when S_n does not correspond to a (-1)-curve of X contracted by η, which implies that $\{\pi(E_p), \pi(E_q)\} = \{F_n, \Gamma\}$, or equivalently that $\{E_p, E_q\} = \{\tilde{F_n}, \tilde{\Gamma}\}$, where $\tilde{F_n}$ and $\tilde{\Gamma}$ denote the strict transforms of F_n and Γ on X. Since $(F_n)^2 = 0$ and $(\tilde{F_n})^2 = -1$, there exists exactly one point $r \in F_n$ (and no infinitely near points) blown up by π, which is then a k-point (as all base-points of π are defined over k). We obtain

$$m_r(\Gamma) = \Gamma \cdot F_n \geq 1 \text{ and } \Gamma \cap F_n = \{r\},$$

since $\tilde{F_n}$ and $\tilde{\Gamma}$ are disjoint on X (and because $\Gamma \cdot F_n \geq 1$, as Γ satisfies one of the two conditions (1)-(2) of Lemma 2.16).

We now prove that $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are two disjoint connected sets of rational curves which intersect the two curves $\tilde{F_n}$ and $\tilde{\Gamma}$, i.e. the two curves E_p and E_q. To show this, it suffices to prove that $r \notin S_n$ and that $S_n \cdot \Gamma \geq 1$. Suppose first that $\Gamma \cdot F_n = 1$ (condition (1) of Lemma 2.16). Since $\Gamma \cap F_n \cap S_n = \emptyset$, we get $r \in F_n \setminus S_n$. The inequality $\Gamma \cdot S_n > 0$ is provided by the fact that D is not equivalent to a line by an automorphism of $\mathbb{A}^2$ (see again condition (1) of Lemma 2.16 and the equivalence between (*ii*) and (*iv*) given in that case). Suppose now that $\Gamma \cdot F_n \geq 2$. As $m_r(\Gamma) = \Gamma \cdot F_n \geq 2$, we have $2m_r(\Gamma) > \Gamma \cdot F_n$, which implies that $n = 1$, $r \in F_n \setminus S_n$ and $2 \leq m_r(\Gamma) \leq \Gamma \cdot S_n$ (see again possibility (2) of Lemma 2.16).

We finish by observing that, since $\eta(E_q) = q \in \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ and $\eta(E_p) = p \in L_{\mathbb{P}^2}$, any connected set of curves of $\eta^{-1}(L_{\mathbb{P}^2} \cup \overline{C})$ which touches the two curves E_q and E_p has to contain the strict transform $\tilde{C}$ of $\overline{C}$. Remembering that $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are included into $\pi^{-1}(F_n \cup S_n \cup \Gamma) = \eta^{-1}(L_{\mathbb{P}^2} \cup \overline{C})$, this contradicts the fact that $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are two disjoint connected sets of rational curves which intersect the two curves $\tilde{F_n}$ and $\tilde{\Gamma}$. □

A direct corollary of Proposition 2.17 is the following, which shows that only smooth curves $C \subset \mathbb{A}^2$ are interesting to study. This follows also from Lemma 2.23 below. Since the proof of Lemma 2.23 is more involved, we prefer to first explain the simpler argument that shows how the smoothness follows from Proposition 2.17.

Corollary 2.18. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible curve. If C is not smooth, then every open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. By Lemma 2.7, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible curve D. We apply Proposition 2.17 and obtain an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the rational map $\iota \circ \varphi$ extends to a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$. Embedding $\mathbb{A}^2$ into $\mathbb{P}^2$, we get a birational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_n$ which is regular on $\mathbb{A}^2$. In particular, the singular points of C are not blown up in the minimal resolution of ψ. So, the curve $\overline{C}$ is not contracted. Since ψ restricts to an isomorphism $\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{F}_n \setminus (F_n \cup S_n \cup \overline{D})$, Lemma 2.2 shows us that $\overline{C}$ is sent onto a singular curve of $\mathbb{F}_n$ which has to be F_n, S_n or $\overline{D}$. Since F_n and S_n are smooth, this singular curve must be $\overline{D}$. Lemma 2.7 then shows that φ extends to an automorphism of $\mathbb{A}^2$. □

Another direct consequence of Proposition 2.17 is the following result, which shows that in Case (3) of Lemma 2.7, the point where $\overline{C}$ is contracted to lies in $\mathbb{A}^2$ only in a very special situation:

Corollary 2.19. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. If the extension of φ to $\mathbb{P}^2$ contracts the curve C (or its closure) to a point of $\mathbb{A}^2$, then there exist automorphisms α, β of $\mathbb{A}^2$ and an endomorphism $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ of the form $(x, y) \mapsto (x, x^n y)$, where $n \geq 1$ is an integer, such that $\varphi = \alpha\psi\beta$. In particular, $C \subset \mathbb{A}^2$ is equivalent to a line, via β.*

Proof. By Lemma 2.7, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible curve D. Denote by $\varphi^{-1}\colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ the birational transformation which is the inverse of φ. Since C is contracted by φ to a point of $\mathbb{A}^2$, it is not possible to find an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the birational map $\iota \circ \varphi^{-1}$ actually defines a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$. By Proposition 2.17, this implies that D is equivalent to a line by an automorphism of $\mathbb{A}^2$. Hence, the same holds for C, by Lemma 2.13. Applying automorphisms of $\mathbb{A}^2$ at the source and the target, we can then assume that C and D are equal to the line $x = 0$. By Lemma 2.13(1), the map φ is of the form $(x, y) \mapsto (\lambda x, \mu x^n y + s(x))$, where $\lambda, \mu \in \mathrm{k}^*$, $n \geq 1$ and $s \in \mathrm{k}[x]$ is a polynomial. We then observe that $\varphi = \alpha\psi$, where α is the automorphism of $\mathbb{A}^2$ given by $(x, y) \mapsto (\lambda x, \mu y + s(x))$ and ψ is the endomorphism of $\mathbb{A}^2$ given by $(x, y) \mapsto (x, x^n y)$. $\square$

Corollary 2.19 also gives a simple proof of the following characterisation of birational endomorphisms of $\mathbb{A}^2$ that contract only one geometrically irreducible curve, already obtained by Daniel Daigle in [Dai91, Theorem 4.11].

Corollary 2.20. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let φ be a birational endomorphism of $\mathbb{A}^2$ which restricts to an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$. Then, the following assertions are equivalent:*

(*i*) *The endomorphism φ contracts the curve C.*
(*ii*) *The endomorphism φ is not an automorphism.*
(*iii*) *There exist automorphisms α, β of $\mathbb{A}^2$ and an endomorphism $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ of the form $(x, y) \mapsto (x, x^n y)$, where $n \geq 1$ is an integer, such that $\varphi = \alpha\psi\beta$.*

Proof. (*iii*) $\Rightarrow$ (*ii*): Follows from the fact that, for each $n \geq 1$, the map $\psi\colon (x, y) \mapsto (x, x^n y)$ is a birational endomorphism of $\mathbb{A}^2$ which is not an automorphism, as its inverse $\psi^{-1}\colon (x, y) \mapsto (x, x^{-n} y)$ is not regular.

(*ii*) $\Rightarrow$ (*i*): Denote by $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map induced by φ. Since φ is an endomorphism of $\mathbb{A}^2$ which is not an automorphism, the cases (1)-(2) of Lemma 2.7 are not possible. Hence, we are in case (3): C is contracted by $\hat{\varphi}$ to a point of $\mathbb{P}^2$, which is necessarily in $\mathbb{A}^2$ since $\varphi(\mathbb{A}^2) \subset \mathbb{A}^2$.

(*i*) $\Rightarrow$ (*iii*): Follows from Corollary 2.19. $\square$

2.5. **Completion with two curves and a boundary.** The following technical lemma (Lemma 2.23) is used to prove Corollary 2.24 and Proposition 2.26, which yield almost all statements of Theorem 1.

Definition 2.21. Let X be a smooth projective surface. A reduced closed curve $C \subset X$ is a k-*forest of* X if C is a finite union of closed curves $C_1, \ldots, C_n$, all k-isomorphic to

$\mathbb{P}^1$ and if each singular point of C is a k-point lying on exactly two components C_i, C_j intersecting transversally. We moreover ask that C does not contain any loop. If C is connected, we say that C is a k-*tree*.

Remark 2.22. If $\eta\colon X \to Y$ is a birational morphism between smooth projective surfaces such that all base-points of η^{-1} are defined over k, then the exceptional curve of η (union of curves contracted) is a k-forest $E \subset X$. Moreover, the strict transform and the preimage of any k-forest of Y is a k-forest of X. The preimage of a k-tree is a k-tree.

Lemma 2.23. *Let $C, D \subset \mathbb{A}^2$ be geometrically irreducible closed curves, not equivalent to lines by automorphisms of $\mathbb{A}^2$ and let $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ be an isomorphism which does not extend to an automorphism of $\mathbb{A}^2$. Then, there is a smooth projective surface X and two open embeddings $\rho_1, \rho_2\colon \mathbb{A}^2 \hookrightarrow X$ making the following diagram commutative*

$$\begin{tikzcd}
& X & \\
\mathbb{A}^2 \arrow[ur, hook, "\rho_1"] & & \mathbb{A}^2 \arrow[ul, hook', "\rho_2"'] \\
\mathbb{A}^2 \setminus C \arrow[u, hook] \arrow[rr, "\varphi", "\simeq"'] & & \mathbb{A}^2 \setminus D \arrow[u, hook']
\end{tikzcd}$$

and such that the following holds:

(*i*) *The curves $\Gamma = \overline{\rho_1(C)} \subset X$, $\Delta = \overline{\rho_2(D)} \subset X$ are isomorphic to $\mathbb{P}^1$.*
(*ii*) *For $i = 1, 2$, we have $\rho_i(\mathbb{A}^2) = X \setminus B_i$ for some* k*-tree B_i.*
(*iii*) *Writing $B = B_1 \cap B_2$, we have $B_1 = B \cup \Delta$ and $B_2 = B \cup \Gamma$.*
(*iv*) *There is no birational morphism $X \to Y$, where Y is a smooth projective surface, which contracts one connected component of B, and no other* $\overline{\mathrm{k}}$*-curve.*
(*v*) *The number of connected components of B is equal to the number of points of $B \cap \Gamma$ and to the number of points of $B \cap \Delta$, and is at most 2.*

Proof. By Proposition 2.17, there exist integers $m, n \geq 1$, and isomorphisms

$$\iota_1\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_m \setminus (S_m \cup F_m),\ \iota_2\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$$

such that both open embeddings $\iota_1\varphi^{-1}\colon \mathbb{A}^2 \setminus D \to \mathbb{F}_m$ and $\iota_2\varphi\colon \mathbb{A}^2 \setminus C \to \mathbb{F}_n$ extend to regular morphisms $u_1\colon \mathbb{A}^2 \to \mathbb{F}_m$ and $u_2\colon \mathbb{A}^2 \to \mathbb{F}_n$. Denoting by $\psi\colon \mathbb{F}_m \dashrightarrow \mathbb{F}_n$ the corresponding birational map, equal to $\iota_2(u_1)^{-1} = u_2(\iota_1)^{-1}$, and taking a minimal resolution of the indeterminacies of ψ, we get the following commutative diagram

$$\begin{tikzcd}
& X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\
\mathbb{F}_m \arrow[rr, dashed, "\psi"] & & \mathbb{F}_n \\
\mathbb{A}^2 \arrow[u, hook, "\iota_1"] \arrow[urr, "u_2"' near start] & & \mathbb{A}^2 \arrow[u, hook', "\iota_2"'] \arrow[ull, "u_1" near start] \\
\mathbb{A}^2 \setminus C \arrow[u, hook] \arrow[rr, "\varphi", "\simeq"'] & & \mathbb{A}^2 \setminus D \arrow[u, hook']
\end{tikzcd}$$

where η and π are birational morphisms, which are sequences of blow-ups of k-points, being the base-points of ψ and ψ^{-1} respectively: The fact that all base-points are defined over k follows from Lemma 2.6, because the irreducible $\overline{\mathrm{k}}$-curve contracted by ψ and ψ^{-1} are all defined over k, since they are equal to $S_n, F_n, S_m, F_m, \overline{u_1(D)}$ or $\overline{u_2(C)}$. Since u_1, u_2 are regular on $\mathbb{A}^2$, the base-points of ψ, resp. ψ^{-1}, belong to $F_m \cup S_m \subset \mathbb{F}_m$, resp. $F_n \cup S_n \subset \mathbb{F}_n$. In particular, we get two open embeddings

$$\rho_1 = \eta^{-1}\iota_1 \colon \mathbb{A}^2 \hookrightarrow X,\ \rho_2 = \pi^{-1}\iota_2 \colon \mathbb{A}^2 \hookrightarrow X$$

such that $\rho_2\varphi = \rho_1$ (or more precisely $\rho_2\varphi = \rho_1|_{\mathbb{A}^2\setminus C}$). We have $\rho_1(\mathbb{A}^2) = X \setminus B_1$ and $\rho_2(\mathbb{A}^2) = X \setminus B_2$, where $B_1 := \eta^{-1}(S_m \cup F_m)$ and $B_2 := \pi^{-1}(S_n \cup F_n)$ are k-trees (see Remark 2.22).

The restriction of ψ gives an isomorphism $\mathbb{F}_m \setminus (S_m \cup F_m \cup \iota_1(C)) \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n \cup \iota_2(D))$ (which corresponds to φ). By Lemma 2.2(2), the following equality holds:

$$\eta^{-1}(S_m \cup F_m \cup \iota_1(C)) = \pi^{-1}(S_n \cup F_n \cup \iota_2(D)).$$

The left-hand side is equal to $B_1 \cup \Gamma$, where $\Gamma = \overline{\rho_1(C)} \subset X$ is the strict transform of $\overline{\iota_1(C)} \subset \mathbb{F}_m$ by η and the right-hand side is equal to $B_2 \cup \Delta$, where $\Delta = \overline{\rho_2(D)} \subset X$ is the strict transform of $\overline{\iota_2(D)} \subset \mathbb{F}_n$ by π. The fact that φ does not extend to an automorphism of $\mathbb{A}^2$ implies that $B_1 \neq B_2$, whence $\Delta \neq \Gamma$. Writing $B := B_1 \cap B_2$, the equality $B_1 \cup \Gamma = B_2 \cup \Delta$ yields:

$$B_2 = B \cup \Gamma \text{ and } B_1 = B \cup \Delta \text{ (with } \Gamma = \overline{\rho_1(C)}, \Delta = \overline{\rho_2(D)} \subset X).$$

In particular, since B_1, B_2 are two k-trees, Γ and Δ are isomorphic to $\mathbb{P}^1$ (over k) and intersect transversally B in a finite number of k-points. We have then found the surface X together with the embeddings ρ_1, ρ_2, satisfying conditions (*i*)–(*ii*)–(*iii*). We will then modify X if needed, in order to also get (*iv*)–(*v*).

The number of connected components of B is equal to the number of points of $B \cap \Gamma$, and of $B \cap \Delta$: This follows from the fact that $B \cup \Gamma$ and $B \cup \Delta$ are k-trees. Let us also recall that each point of $B \cap \Gamma$, or of $B \cap \Delta$, is a k-point, as said before.

Suppose that the number of connected components of B is $r \geq 3$, and let us show that at least $r-2$ connected components of B are contractible (in the sense that there is a birational morphism $X \to Y$, where Y is a smooth projective rational surface, which contracts one component of B and no other $\overline{\mathrm{k}}$-curve). To show this, we first observe that Γ intersects r distinct curves of B. Since Γ is one of the irreducible components of $B_2 = \pi^{-1}(S_n \cup F_n)$, we can decompose π as $\pi_2 \circ \pi_1$ where $\pi_1(\Gamma)$ is an irreducible component of $(\pi_2)^{-1}(S_n \cup F_n)$ intersecting exactly two other irreducible components R_1, R_2, and such that all points blown up by π_1 are infinitely near points of $\pi_1(\Gamma) \setminus (R_1 \cup R_2)$. This proves that we can contract at least $r-2$ connected components of B.

If one connected component of B is contractible, there exists a morphism $X \to Y$, where Y is a smooth projective rational surface, which contracts this component of B, and no other curve. Since the component intersects Δ transversally in one point, and also Γ in one point, we can replace X with Y, ρ_1, ρ_2 with their compositions with the morphism $X \to Y$ and still have conditions (*i*)–(*ii*)–(*iii*). After finitely many steps, condition (*iv*) is satisfied. By the observation made before, the number of connected components of B, after this being done, is at most 2, giving then (*v*). □

Corollary 2.24. *Let $C, D \subset \mathbb{A}^2$ be geometrically irreducible closed curves and let $\varphi \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ be an isomorphism which does not extend to an automorphism of $\mathbb{A}^2$.*

Then, the curves C, D are isomorphic to open subsets of $\mathbb{A}^1$: there exist polynomials $P, Q \in \mathrm{k}[t]$ without square factors, such that $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$. Moreover, the numbers of $\overline{\mathrm{k}}$-roots of P and Q are the same (i.e. extending the scalars to $\overline{\mathrm{k}}$, the curves C and D become isomorphic to $\mathbb{A}^1$ minus some finite number of points, the same number for both curves). The numbers of k-roots of P and Q are also the same.

Remark 2.25. When $\mathrm{k} = \mathbb{C}$, this follows from the fact that C and D are isomorphic to open subsets of $\mathbb{A}^1$, which is given by the fact that the curves are rational (Corollary 2.8) and smooth (Corollary 2.18). Indeed, since $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic, they have the same Euler characteristic, so C and D also have the same Euler characteristic.

Proof. If C or D is equivalent to a line, so are both curves (Lemma 2.13), and the result holds. Otherwise, we apply Lemma 2.23 and get a smooth projective surface X and two open embeddings $\rho_1, \rho_2 \colon \mathbb{A}^2 \hookrightarrow X$ such that $\rho_2\varphi = \rho_1$ and satisfying the conditions (i)-(ii)-(iii)-(iv)-(v). In particular, C is isomorphic to $\Gamma \setminus B_1 = \Gamma \setminus ((\Gamma \cap B) \cup (\Gamma \cup \Delta))$. Since Γ is isomorphic to $\mathbb{P}^1$ and $\Gamma \cap B$ consists of one or two k-points, this shows that Γ is isomorphic to an open subset of $\mathbb{A}^1$. Doing the same for D, we get isomorphisms $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ where $P, Q \in \mathrm{k}[t]$ are polynomials, that we can assume without square factors.

The number of $\overline{\mathrm{k}}$-roots of P is equal to the number of $\overline{\mathrm{k}}$-points of $\Gamma \cap B_1$ minus 1. Similarly, the number of $\overline{\mathrm{k}}$-roots of Q is equal to the number of $\overline{\mathrm{k}}$-points of $\Delta \cap B_2$ minus 1. To see that these numbers are equal, we observe that $\Gamma \cap B_1 = (\Gamma \cap B) \cup (\Gamma \cap \Delta)$, that $\Delta \cap B_2 = (\Delta \cap B) \cup (\Delta \cap \Gamma)$, and that the number of points of $\Gamma \cap B$ is the same as the number of points of $\Delta \cap B$ (follows from (v)). As each point of $\Gamma \cap B$ that is also contained in $\Gamma \cap \Delta$ is also contained in $\Delta \cap B$, this shows that P and Q have the same number of $\overline{\mathrm{k}}$-roots. As each $\overline{\mathrm{k}}$-point of $\Gamma \cap B_1$ or $\Delta \cap B_2$ which is not a k-point is contained in $\Gamma \cap \Delta$, the polynomials P and Q have the same number of k-roots. $\square$

Proposition 2.26. *Let $C, D, D' \subset \mathbb{A}^2$ be geometrically irreducible closed curves, not equivalent to lines by automorphisms of $\mathbb{A}^2$, and let $\varphi \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$, $\varphi' \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D'$ be isomorphisms which do not extend to automorphisms of $\mathbb{A}^2$. Then, one of the following holds:*

(a) *The map $\varphi'(\varphi)^{-1}$ extends to an automorphism of $\mathbb{A}^2$ (sending D to D');*
(b) *The three curves C, D, D' are* k*-isomorphic to $\mathbb{A}^1$;*
(c) *The three curves C, D, D' are* k*-isomorphic to $\mathbb{A}^1 \setminus \{0\}$.*

Remark 2.27. Case (b) never happens, as we will show after. Indeed, since C is not equivalent to a line, the existence of φ, φ' is prohibited (Proposition 2.29 below).

Proof. If $C \simeq \mathbb{A}^1$ or $C \simeq \mathbb{A}^1 \setminus \{0\}$, then $D \simeq C \simeq D'$ by Corollary 2.24. We can thus assume that C is not k-isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$. We apply Lemma 2.23 with φ and φ' and get smooth projective surfaces X, X' and open embeddings $\rho_1, \rho_2, \rho_1', \rho_2' \colon \mathbb{A}^2 \hookrightarrow X$ such that $\rho_2\varphi = \rho_1$, $\rho_2'\varphi' = \rho_1'$ and satisfying the conditions (i)-(ii)-(iii)-(iv)-(v). In particular, we obtain an isomorphism $\kappa \colon X \setminus (B \cup \Gamma \cup \Delta) \xrightarrow{\simeq} X' \setminus (B' \cup \Gamma' \cup \Delta')$ (where $\Gamma = \overline{\rho_1(C)} \subset X$, $\Delta = \overline{\rho_2(D)} \subset X$, $\Gamma' = \overline{\rho_1'(C)} \subset X'$, $\Delta' = \overline{\rho_2'(D')} \subset X'$) and a commutative diagram

$$
\begin{tikzcd}
 & X \arrow[rr, dashed, "\kappa"] & & X' & \\
\mathbb{A}^2 \arrow[ur, hook, "\rho_2"] & & \mathbb{A}^2 \arrow[ul, hook', "\rho_1"'] \arrow[ur, hook, "\rho_1'"] & & \mathbb{A}^2 \arrow[ul, hook', "\rho_2'"'] \\
\mathbb{A}^2 \setminus D \arrow[u, hook] & & \mathbb{A}^2 \setminus C \arrow[u, hook] \arrow[ll, "\varphi", "\simeq"'] \arrow[rr, "\varphi'", "\simeq"'] & & \mathbb{A}^2 \setminus D' \arrow[u, hook]
\end{tikzcd}
$$

By construction, κ sends birationally $\Gamma = \overline{\rho_1(C)}$ onto $\Gamma' = \overline{\rho_1'(C)}$. If κ also sends Δ birationally onto Δ', then $\varphi'\varphi^{-1}$ extends to a birational map that sends birationally D onto D' and then extends to an automorphism of $\mathbb{A}^2$ (Lemma 2.7). It remains then to show that this holds.

Note that all base-points of κ and κ^{-1} are defined over k, since all $\overline{\mathrm{k}}$-curves contracted by κ and κ^{-1} are defined over k (Lemma 2.6). We take a minimal resolution of the indeterminacies of κ:

$$
\begin{tikzcd}
 & Z \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\
X \arrow[rr, dashed, "\kappa"] & & X'
\end{tikzcd}
$$

and observe, as before, that all points blown up by η and π are defined over k. We want to show that the strict transforms $\tilde{\Delta}$ and $\tilde{\Delta}'$ of $\Delta \subset X$, $\Delta' \subset X'$ are equal. We will do this by studying the strict transform $\tilde{\Gamma} = \tilde{\Gamma}'$ of Γ and Γ' and its intersection with $\tilde{\Delta}$ and $\tilde{\Delta}'$ and with the other components of $B_Z = \eta^{-1}(B \cup \Gamma \cup \Delta) = \pi^{-1}(B' \cup \Gamma' \cup \Delta')$.

Recall that $B_1 = B \cup \Delta$, $B_2 = B \cup \Gamma$, $B_1' = B' \cup \Delta'$, $B_2' = B' \cup \Gamma'$ are k-trees and that C is isomorphic to $\Gamma \setminus B_1$ and $\Gamma' \setminus B_1'$ (Lemma 2.23).

(i) Suppose first that $\Gamma \cap B_1$ contains some $\overline{\mathrm{k}}$-points which are not defined over k. None of these points is thus a base-point of κ and each of these points belongs to $\Gamma \cap \Delta$, so $\tilde{\Gamma} \cap \tilde{\Delta}$ contains $\overline{\mathrm{k}}$-points not defined over k. Since B_2' is a k-tree, $\pi^{-1}(B_2')$ is a k-tree, so $\tilde{\Gamma} = \tilde{\Gamma}'$ intersects all irreducible components of B_Z into k-points, except maybe $\tilde{\Delta}'$. This yields $\tilde{\Delta} = \tilde{\Delta}'$ as we wanted.

(ii) We can now assume that all $\overline{\mathrm{k}}$-points of $\Gamma \cap B_1$ are defined over k, which implies that all intersections of irreducible components of B_Z are defined over k. We will say that an irreducible component of B_Z is *separating* if the union of all other irreducible components is a k-forest (see Definition 2.21).

Since $B_1 = B \cup \Delta$ is a k-tree, its preimage on B_Z is a k-tree. The union of all components of B_Z distinct from $\tilde{\Gamma}$ being equal to the disjoint union of $\eta^{-1}(B_1)$ with some k-forest contracted to points of $\Gamma \setminus B_1$, we find that $\tilde{\Gamma}$ is separating. The same argument shows that $\tilde{\Delta}$ and $\tilde{\Delta}'$ are also separating.

It remains then to show that any irreducible component $E \subset B_Z$ which is not equal to $\tilde{\Delta}$ or $\tilde{\Gamma}$ is not separating. We use for this the fact that $C \simeq \Gamma \setminus B_1$ is not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$, so the set $\Gamma \cap B_1$ contains at least 3 points. If $\eta(E)$ is a point q, then the complement of $\eta^{-1}(q)$ in B_Z contains a loop, since Γ intersects the k-tree B_1 into at least two points distinct from q. If $\eta(E)$ is not a point, it is one of the components of B. We denote by F the union of all irreducible components of $B \cup \Gamma \cup \Delta$ not equal to $\eta(E)$, and prove that F is not a k-forest, since it contains a loop. This is true if $\Delta \cap \Gamma$ contains at least 2 points. If $\Delta \cap \Gamma$ contains one or less points, then $\Delta \cap B$ contains at least two points, so contains exactly two points, on the two connected components of B

which both intersect Γ and Δ (see Lemma 2.23(v)). We again get a loop on the union of Γ, Δ and of the connected component of B not containing $\eta(E)$. The fact that F contains a loop implies that $\eta^{-1}(F)$ contains a loop, and achieves to prove that E is not separating. □

2.6. **The case of curves isomorphic to $\mathbb{A}^1$ and the proof of Theorem 1.** To finish the proof of Theorem 1, one still needs to do the case of curves isomorphic to $\mathbb{A}^1$. The case of lines has already been treated in Lemma 2.13. In characteristic zero, this finishes the study by the Abyhankar-Moh-Suzuki theorem, but in positive characteristic, there are many closed curves of $\mathbb{A}^2$ which are isomorphic to $\mathbb{A}^1$ but are not equivalent to lines (these curves are sometimes called "bad lines" in the literature). We will show that an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ always extends to $\mathbb{A}^2$ if C is isomorphic to $\mathbb{A}^1$ but not equivalent to a line.

Lemma 2.28. *Let $n \geq 1$ and let $\Gamma \subset \mathbb{F}_n$ be a geometrically irreducible closed curve, such that $\Gamma \cdot F_n \geq 2$. If there exists a birational map $\mathbb{F}_n \dashrightarrow \mathbb{P}^2$ that contracts Γ to a point (and maybe contracts some other curves), then Γ is geometrically rational and singular. Moreover, one of the following occurs:*

(a) *There exists a point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$.*
(b) *We have $n = 1$ and there exists a point $p \in \mathbb{F}_1(\overline{\mathrm{k}}) \setminus S_1$ such that $m_p(\Gamma) > \Gamma \cdot S_1$.*

Proof. We can assume that $\mathrm{k} = \overline{\mathrm{k}}$. Denote by $\psi\colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ the birational map that contracts C to a point (and maybe some other curves). The minimal resolution of this map yields a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{F}_n \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

In $\operatorname{Pic}(\mathbb{F}_n) = \mathbb{Z}F_n \bigoplus \mathbb{Z}S_n$ we write

$$\begin{array}{rcl} \Gamma & = & aS_n + bF_n \\ -K_{\mathbb{F}_n} & = & 2S_n + (2+n)F_n \end{array}$$

for some integers a, b. Note that $a = \Gamma \cdot F_n \geq 2$ and that $b - an = \Gamma \cdot S_n \geq 0$. By hypothesis, the strict transform $\tilde{\Gamma}$ of Γ on X is a smooth curve contracted by π. In particular, Γ is rational and the divisor $2\tilde{\Gamma} + aK_X$ is not effective, since

$$(2\tilde{\Gamma} + aK_X) \cdot \pi^*(L) = aK_X \cdot \pi^*(L) = a\pi^*(K_{\mathbb{P}^2}) \cdot \pi^*(L) = aK_{\mathbb{P}^2} \cdot L = -3a < 0$$

for a general line $L \subset \mathbb{P}^2$.

Denoting by $E_1, \dots, E_r \in \operatorname{Pic}(X)$ the pull-backs of the exceptional divisors blown up by η (which satisfy $(E_i)^2 = -1$ for each i and $E_i \cdot E_j = 0$ for $i \neq j$) we have

$$\begin{array}{rcllll} \tilde{\Gamma} & = & a\eta^*(S_n) & + & b\eta^*(F_n) & - \sum_{i=1}^r m_i E_i \\ -K_X & = & 2\eta^*(S_n) & + & (2+n)\eta^*(F_n) & - \sum_{i=1}^r E_i \\ 2\tilde{\Gamma} + aK_X & = & & & (2b - a(2+n))\eta^*(F_n) & + \sum_{i=1}^r (a - 2m_i)E_i \end{array}$$

which implies, since $2\tilde{\Gamma} + aK_X$ is not effective, that either $2b < a(2+n)$ or $2m_i > a$ for some i. If $2m_i > a$ for some i, we get (a), since the m_i are the multiplicities of $\tilde{\Gamma}$ at the points blown up by η.

It remains to study the case where $2m_i \leq a$ for each i, and where $2b < a(2+n)$. Remembering that $b - an = \Gamma \cdot S_n \geq 0$, one finds $n \leq \frac{b}{a} < \frac{2+n}{2}$, whence $n = 1$ and thus $2b < 3a$. We then compute

$$3\tilde{\Gamma} + bK_X \quad = \quad (3a-2b)\eta^*(S_n) \quad + \quad \textstyle\sum_{i=1}^{r}(b-3m_i)E_i$$

which is again not effective, since $(3\tilde{\Gamma} + bK_X) \cdot \pi^*(L) = bK_X \cdot \pi^*(L) = -3b < 0$ for a general line $L \subset \mathbb{P}^2$, because $b \geq an = a \geq 2$. This implies that there exists an integer i such that $3m_i > b$. Since $2m_i \leq a$, one finds $m_i > b - a = \Gamma \cdot S_1$, which yields (b). □

Proposition 2.29. *Let $C \subset \mathbb{A}^2$ be a closed curve, isomorphic to $\mathbb{A}^1$ (i.e. isomorphic to $\mathbb{A}^1$ over* k*). The following are equivalent:*

(a) *The curve C is equivalent to a line by an automorphism of $\mathbb{A}^2$.*

(b) *There exists an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$.*

(c) *Embedding $\mathbb{A}^2$ into $\mathbb{P}^2$, via the canonical embedding, there exists a birational map of $\mathbb{P}^2$ that contracts the curve C to a point (and maybe contracts other curves).*

Proof. The implications $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ can be observed, for example by taking the map $(x, y) \mapsto (x, xy)$, which is an open embedding of $\mathbb{A}^2 \setminus \{x = 0\}$ into $\mathbb{A}^2$, which does not extend to an automorphism of $\mathbb{A}^2$, and whose extension to $\mathbb{P}^2$ contracts the line $x = 0$ to a point.

To prove $(b) \Rightarrow (c)$, we take an open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$ and look at the extension to $\mathbb{P}^2$. By Lemma 2.7, either this one contracts C, or C is equivalent to a line, in which case (c) is true as was shown before.

It remains to prove $(c) \Rightarrow (a)$. We apply Lemma 2.16, and obtain an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ such that the closure of $\iota(C)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the two cases (1)-(2) of Lemma 2.16. In case (1), the curve is equivalent to a line as it is isomorphic to $\mathbb{A}^1$ (equivalence $(ii) - (iii)$ of Lemma 2.16). It remains to study the case where Γ satisfies the conditions (2) of Lemma 2.16 (in particular $\Gamma \cdot F_n \geq 2$), and to show that these, together with (c), yield a contradiction. We prove that there is no point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$. Indeed, since $\Gamma \cdot F_n \geq 2$, the point would be a singular point of Γ, and since $\Gamma \setminus (S_n \cup F_n) = \iota(C) \simeq C$ is isomorphic to $\mathbb{A}^1$, p is a k-point and is the unique $\overline{\mathrm{k}}$-point of $\Gamma \cap (S_n \cup F_n)$. Moreover, as $\Gamma \cdot F_n \geq 2$, we find that $p \in F_n$. Since $2m_p(\Gamma) > \Gamma \cdot F_n$ and because Γ satisfies the conditions (2) of Lemma 2.16, the only possibility is that $n = 1$, $p \in F_1 \setminus S_1$ and $0 < m_p(\Gamma) \leq \Gamma \cdot S_1$. This is impossible as it contradicts the fact that $\Gamma \cap (S_1 \cup F_1)$ contains only one $\overline{\mathrm{k}}$-point.

Denote by $\psi_0\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map that contracts C (and maybe other curves) to a point. Observe that $\psi_0 \circ \iota^{-1}$ yields a birational map $\psi\colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ which contracts Γ to a point. As there is no point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$, Lemma 2.28 implies that $n = 1$ and that there exists a point $p \in \mathbb{F}_1(\overline{\mathrm{k}}) \setminus S_1$ such that $m_p(\Gamma) > \Gamma \cdot S_1$. Again, this point is a k-point, since C is k-isomorphic to $\mathbb{A}^1$. This contradicts the conditions (2) of Lemma 2.16. □

Remark 2.30. If k is algebraically closed, the equivalence between conditions (a) and (c) of Proposition 2.29 can also be proven using Kodaira dimension. Let us introduce the following conditions:

$(a)'$ The Kodaira dimension $\kappa(C, \mathbb{A}^2)$ of C is equal to $-\infty$.
$(c)'$ There exists a birational transformation of $\mathbb{P}^2$ sending C onto a line.

The equivalence between (a) and $(a)'$ follows from [Gan85, Theorem 2.4.(1)] and the equivalence between $(a)'$ and $(c)'$ is Coolidge's theorem (see e.g. [KM83, Theorem 2.6]). Let us now recall how the classical equivalence between (c) and $(c)'$ can be proven. Every simple quadratic birational transformation of $\mathbb{P}^2$ contracts three lines and no other curve. This yields $(c)' \Rightarrow (c)$. To get $(c) \Rightarrow (c)'$, we take a birational transformation φ of $\mathbb{P}^2$ that contracts C to a point and decompose φ as $\varphi = \varphi_r \circ \cdots \circ \varphi_1$, where each φ_i is a simple quadratic transformation (using Castelnuovo-Noether factorisation theorem). Choosing $i \geq 1$ as the smallest integer such that $(\varphi_i \circ \cdots \circ \varphi_1)(C)$ is a point, the curve $(\varphi_{i-1} \circ \cdots \circ \varphi_1)(C)$ is contracted by φ_i and is thus a line.

Remark 2.31. If the field k is perfect, then every curve that is geometrically isomorphic to $\mathbb{A}^1$ (i.e. over $\overline{\mathrm{k}}$) is also isomorphic to $\mathbb{A}^1$. This can be seen by embedding the curve in $\mathbb{P}^1$ and looking at the complement point, necessarily defined over k. For non-perfect fields, there exist closed curves $C \subset \mathbb{A}^2$ geometrically isomorphic to $\mathbb{A}^1$ but not isomorphic to $\mathbb{A}^1$ (see [Rus70]). Corollary 2.24 shows that every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$ for all such curves.

We can now finish the section by proving Theorem 1:

Proof of Theorem 1. We recall the hypothesis of the theorem: we have a geometrically irreducible closed curve $C \subset \mathbb{A}^2$ and an isomorphism $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ for some closed curve $C \subset \mathbb{A}^2$. Assume that φ does not extend to an automorphism of $\mathbb{A}^2$.

(1): If C is isomorphic to $\mathbb{A}^1$, then the implication $(b) \Rightarrow (a)$ of Proposition 2.29 shows that C is equivalent to a line and Lemma 2.13(2) implies that the same holds for D. In particular, the curves C and D are isomorphic.

If C is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ then so is D by Corollary 2.24.

(2): if C is not isomorphic to $\mathbb{A}^1$ or to $\mathbb{A}^1 \setminus \{0\}$, then Proposition 2.26 shows that the isomorphism $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ (not extending to an automorphism of $\mathbb{A}^2$) is uniquely determined by C, up to left composition by an automorphism of $\mathbb{A}^2$. In particular, there are at most two equivalence classes of curves of $\mathbb{A}^2$ having complements isomorphic to $\mathbb{A}^2 \setminus C$. Corollary 2.24 achieves the proof by giving the existence of isomorphisms $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ for some square-free polynomials $P, Q \in \mathrm{k}[t]$ that have the same number of roots in k, and also the same number of roots in the algebraic closure of k. $\square$

3. Families of non-equivalent embeddings

As we observed in Lemma 2.16 and its applications, the curves of $\mathbb{A}^2$ given by

$$a(y)x + b(y) = 0$$

for some coprime polynomials $a, b \in \mathrm{k}[y]$, $a \neq 0$ (where we can always assume that $\deg b < \deg a$), yield a natural family which plays an important role. We study this family here. Recall that such curves are isomorphic to a line if and only $a(y)$ is a constant (Lemma 2.16(i)-(iii)). Actually, we have the following obvious and stronger result:

Lemma 3.1. *Let $C \subset \mathbb{A}^2$ be the irreducible curve given by the equation*

$$a(y)x + b(y) = 0,$$

where $a, b \in \mathrm{k}[y]$ are coprime polynomials and a is nonzero. Then, the algebra of regular functions on C is isomorphic to $\mathrm{k}[y, 1/a(y)]$.

Proof. The algebra of regular functions on C satisfies

$$\mathrm{k}[C] = \mathrm{k}[x, y]/(a(y)x + b(y)) \simeq \mathrm{k}[y, -b(y)/a(y)] = \mathrm{k}[y, 1/a(y)],$$

where the last equality comes from the fact that there exist $c, d \in \mathrm{k}[y]$ with $ad - bc = 1$, which implies that $\frac{1}{a} = \frac{ad-bc}{a} = d - c \cdot \frac{b}{a} \in \mathrm{k}[y, \frac{b}{a}]$. □

3.1. A construction using elements of $\mathrm{SL}_2(\mathrm{k}[y])$.

Lemma 3.2. *For each matrix $\begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix} \in \mathrm{SL}_2(\mathrm{k}[y])$, we have an isomorphism*

$$\begin{array}{rccc} \varphi\colon & \mathbb{A}^2 \setminus C & \xrightarrow{\simeq} & \mathbb{A}^2 \setminus D \\ & (x, y) & \mapsto & \left(\frac{c(y)x+d(y)}{a(y)x+b(y)}, y\right) \end{array}$$

where $C, D \subset \mathbb{A}^2$ are given by $a(y)x + b(y) = 0$ and $a(y)x - c(y) = 0$ respectively.

Proof. Note first that φ is a birational transformation of $\mathbb{A}^2$, with inverse $\psi\colon (x, y) \mapsto \left(\frac{-b(y)x+d(y)}{a(y)x-c(y)}, y\right)$. It remains to prove that the isomorphism $\varphi^*\colon \mathrm{k}(x, y) \to \mathrm{k}(x, y)$, $x \mapsto \frac{cx+d}{ax+b}$, $y \mapsto y$ induces an isomorphism $\mathrm{k}[x, y, \frac{1}{ax-c}] \to \mathrm{k}[x, y, \frac{1}{ax+b}]$. This follows from the equalities:

$$\varphi^*(x) = \tfrac{cx+d}{ax+b}, \quad \varphi^*(y) = y, \quad \varphi^*\left(\tfrac{1}{ax-c}\right) = ax + b \quad \text{and}$$

$$\psi^*(x) = \tfrac{-bx+d}{ax-c}, \quad \psi^*(y) = y, \quad \psi^*\left(\tfrac{1}{ax+b}\right) = ax - c.$$ □

The curves C and D of Lemma 3.2 are always isomorphic thanks to Lemma 3.1. We now prove that they are in general not equivalent.

Lemma 3.3. *Let $C_1, C_2 \subset \mathbb{A}^2$ be two irreducible curves given by*

$$a_1(y)x + b_1(y) = 0 \text{ and } a_2(y)x + b_2(y) = 0$$

respectively, for some polynomials $a_1, a_2, b_1, b_2 \in \mathrm{k}[y]$ such that $\deg a_1 > \deg b_1 \geq 0$ and $\deg a_2 > \deg b_2 \geq 0$. Then, the curves C_1 and C_2 are equivalent if and only if there exist constants $\alpha, \lambda, \mu \in \mathrm{k}^*$ and $\beta \in \mathrm{k}$ such that

$$a_2(y) = \lambda \cdot a_1(\alpha y + \beta), \quad b_2(y) = \mu \cdot b_1(\alpha y + \beta).$$

Proof. We first observe that if $a_2(y) = \lambda \cdot a_1(\alpha y + \beta)$ and $b_2(y) = \mu \cdot b_1(\alpha y + \beta)$ for some $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$, then the automorphism $(x, y) \mapsto (\frac{\lambda}{\mu}x, \alpha y + \beta)$ of $\mathbb{A}^2$ sends C_2 onto C_1.

Conversely, we assume the existence of $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ that sends C_2 onto C_1 and want to find $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ as above. Writing φ as $(x, y) \mapsto (f(x, y), g(x, y))$ for some polynomials $f, g \in \mathrm{k}[x, y]$, one gets

$$\text{(A)} \qquad \mu\big(a_1(g)f + b_1(g)\big) = a_2(y)x + b_2(y)$$

for some $\mu \in \mathrm{k}^*$.

(i) If $g \in \mathrm{k}[y]$, the fact that $\mathrm{k}[f,g] = \mathrm{k}[x,y]$ implies that $g = \alpha y + \beta$, $f = \gamma x + s(y)$ for some $\alpha, \gamma \in \mathrm{k}^*, \beta \in \mathrm{k}$ and $s(y) \in \mathrm{k}[y]$. This yields $a_1(g)f + b_1(g) = a_1(g)(\gamma x + s(y)) + b_1(g)$, so that Equation ($A$) gives:

$$a_2 = \mu\gamma \cdot a_1(g), \quad b_2 = \mu \cdot \big(a_1(g)s(y) + b_1(g)\big).$$

This shows in particular that $\deg a_1 = \deg a_2$, whence $\deg b_2 < \deg a_1(g)$. Since $\deg b_1(g) < \deg a_1(g)$, we find that $s = 0$, and thus that $b_2 = \mu \cdot b_1(g)$, as desired. This ends the proof, by choosing $\lambda = \mu\gamma$.

(ii) It remains to study the case where $g \notin \mathrm{k}[y]$, which corresponds to $\deg_x(g) \geq 1$. This yields $\deg_x a_1(g) = \deg a_1 \cdot \deg_x(g) > \deg b_1 \cdot \deg_x(g) = \deg_x b_1(g)$, which implies that $\deg_x \big(a_1(g)f + b_1(g)\big) = \deg(a_1) \cdot \deg_x(g) + \deg_x(f)$. Equation ($A$) shows that this degree is 1, and since $\deg a_1 \geq 1$, we find $\deg a_1 = 1$. Similarly, the automorphism sending C_1 onto C_2 satisfies the same condition, so $\deg a_2 = 1$. This implies that $b_1, b_2 \in \mathrm{k}^*$. There exist thus some $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that $a_2(y) = \lambda \cdot a_1(\alpha y + \beta)$ and $b_2(y) = \mu \cdot b_1(\alpha y + \beta)$. □

Corollary 3.4. *For each polynomial $f \in \mathrm{k}[t]$ of degree ≥ 1, there exist two closed curves $C, D \subset \mathbb{A}^2$, both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$, that are non-equivalent and have isomorphic complements. Moreover, the set of all such pairs (C, D), up to equivalence, is infinite.*

Proof. We choose an irreducible polynomial $b \in \mathrm{k}[t]$ which does not divide f. For each $n \geq 1$ such that $\deg(f^n) > 2\deg(b)$, we then choose two polynomials $c, d \in \mathrm{k}[t]$ such that $f^n d - bc = 1$ (possible since $\gcd(f^n, b) = 1$). Replacing c, d with $c + \alpha f^n, d + \alpha b$, we can moreover assume that $\deg c < \deg f^n$. The curves $C_n, D_n \subset \mathbb{A}^2$ given by $f(y)^n x + b(y) = 0$ and $f(y)^n x - c(y) = 0$ are both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f^n}]) = \mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$ by Lemma 3.1 and have isomorphic complements by Lemma 3.2. Moreover, as $\deg bc = \deg(f^n d - 1) \geq \deg(f^n) > 2\deg(b)$, we find that $\deg c > \deg b$, which implies that C_n and D_n are not equivalent by Lemma 3.3. Moreover, the curves C_n are all non-equivalent, again by Lemma 3.3. □

3.2. Curves isomorphic to $\mathbb{A}^1 \setminus \{0\}$.

We consider now families of curves of $\mathbb{A}^2$ of the form $xy^d + b(y) = 0$ for some $d \geq 1$ and some polynomial $b(y) \in \mathrm{k}[y]$ satisfying $b(0) \neq 0$. Note that all these curves are isomorphic to $\mathrm{Spec}(\mathrm{k}[y, \frac{1}{y^d}]) = \mathrm{Spec}(\mathrm{k}[y, \frac{1}{y}]) \simeq \mathbb{A}^1 \setminus \{0\}$ by Lemma 3.1.

Lemma 3.5. *Let $d \geq 1$ be an integer and $b(y) \in \mathrm{k}[y]$ be a polynomial satisfying $b(0) \neq 0$. We define $D_b \subset \mathbb{A}^2$ to be the curve given by the equation*

$$xy^d + b(y) = 0$$

and φ_b to be the birational endomorphism of $\mathbb{A}^2$ given by

$$\varphi_b(x, y) = (xy^d + b(y), y).$$

Denote by L_x, resp. L_y, the line of $\mathbb{A}^2$ given by the equation $x = 0$, resp. $y = 0$.

(1) *The transformation φ_b induces an automorphism of $\mathbb{A}^2 \setminus L_y$ and an isomorphism*

$$\mathbb{A}^2 \setminus (L_y \cup D_b) \xrightarrow{\simeq} \mathbb{A}^2 \setminus (L_y \cup L_x).$$

(2) *Assume now that b has degree $\leq d-1$ and fix an integer $m \geq 1$. Then, there exists a unique polynomial $c \in \mathrm{k}[y]$ of degree $\leq d-1$ satisfying*

$$b(y) \equiv c(yb(y)^m) \pmod{y^d}. \tag{B}$$

Furthermore, we have $c(0) \neq 0$.

(3) *Define the birational transformations τ and $\psi_{b,m}$ of $\mathbb{A}^2$ by*

$$\tau(x,y) = (x, xy) \text{ and } \psi_{b,m} = (\varphi_c)^{-1}\tau^m\varphi_b.$$

Then, $\psi_{b,m}$ induces an isomorphism $\mathbb{A}^2 \setminus D_b \xrightarrow{\simeq} \mathbb{A}^2 \setminus D_c$ whose expression is

$$\psi_{b,m} = (x,y) = \left(\frac{x+\lambda+yf(x,y)}{\left(xy^d+b(y)\right)^{md}}, y\left(xy^d+b(y)\right)^m \right),$$

for some constant $\lambda \in \mathrm{k}$ and some polynomial $f \in \mathrm{k}[x,y]$ (depending on b and m).

(4) *Fixing the polynomial b, all open embeddings $\mathbb{A}^2 \setminus D_b \hookrightarrow \mathbb{A}^2$ given by $\psi_{b,m}$, $m \geq 1$, are different, up to left composition by automorphisms of $\mathbb{A}^2$.*

Proof. (1): The automorphism $(\varphi_b)^*$ of $\mathrm{k}(x,y)$ satisfies

$$(\varphi_b)^*(x) = xy^d + b(y) \text{ and } (\varphi_b)^*(y) = y.$$

The result follows from the two following equalities:

$$\begin{array}{rcl} (\varphi_b)^*(\mathrm{k}[x,y,\frac{1}{y}]) & = & \mathrm{k}[xy^d+b(y), y, \frac{1}{y}] = \mathrm{k}[x,y,\frac{1}{y}] \quad \text{and} \\ (\varphi_b)^*(\mathrm{k}[x,y,\frac{1}{x},\frac{1}{y}]) & = & \mathrm{k}[xy^d+b(y), \frac{1}{xy^d+b(y)}, y, \frac{1}{y}] = \mathrm{k}[x,y,\frac{1}{y},\frac{1}{xy^d+b(y)}]. \end{array}$$

(2): Since $b(0) \neq 0$, the endomorphism of the algebra $\mathrm{k}[y]/(y^d)$ defined by $y \mapsto yb(y)^m$ is an automorphism. If the inverse automorphism is given by $y \mapsto u(y)$, note that (B) is equivalent to $c(y) \equiv b(u(y)) \pmod{y^d}$. This determines uniquely the polynomial c. Finally, replacing x by zero in (B), we get $c(0) = b(0) \neq 0$.

(3): Since τ induces an automorphism of $\mathbb{A}^2 \setminus (L_y \cup L_x)$, assertion (1) implies that ψ induces an isomorphism $\mathbb{A}^2 \setminus (L_y \cup D_b) \xrightarrow{\simeq} \mathbb{A}^2 \setminus (L_y \cup D_c)$ (this would be true for any choice of c). It remains to see that the choice of c that we have made implies that ψ extends to an isomorphism $\mathbb{A}^2 \setminus D_b \xrightarrow{\simeq} \mathbb{A}^2 \setminus D_c$ of the desired form.

Since $(\varphi_c)^{-1}(x,y) = \left(\frac{x-c(y)}{y^d}, y\right)$, $\tau^m(x,y) = (x, x^m y)$ and $\psi_{b,m} = (\varphi_c)^{-1}\tau^m\varphi_b$, we get:

$$\begin{array}{rcl} \psi_{b,m}(x,y) & = & (\varphi_c)^{-1}\tau^m(xy^d+b(y), y) \\ & = & \left(\frac{xy^d+b(y)-c(y\Delta)}{y^d\Delta^d}, y\Delta\right), \text{ with } \Delta = (xy^d+b(y))^m. \end{array} \tag{C}$$

To show that $\psi_{b,m}$ has the desired form, we use $b(y) \equiv c(yb(y)^m) \pmod{y^d}$ (Equation (B)), which yields $\lambda \in \mathrm{k}$ such that $b(y) \equiv c(yb(y)^m) + \lambda y^d \pmod{y^{d+1}}$. Since $y\Delta \equiv yb(y)^m \pmod{y^{d+1}}$, we get $b(y) \equiv c(y\Delta) + \lambda y^d \pmod{y^{d+1}}$. There is thus $f \in \mathrm{k}[x,y]$ such that

$$xy^d + b(y) - c(y\Delta) = y^d(x + \lambda + yf(x,y)).$$

This yields the desired form for $\psi_{b,m}$ and shows that $\psi_{b,m}$ restricts to the automorphism $x \mapsto x + \lambda$ on L_y and then restricts to an isomorphism $\mathbb{A}^2 \setminus D_b \xrightarrow{\simeq} \mathbb{A}^2 \setminus D_c$.

(4): It is enough to check that for $m > n \geq 1$ the birational transformation $\theta = \psi_{b,n} \circ (\psi_{b,m})^{-1}$ of $\mathbb{A}^2$ does not correspond to an automorphism of $\mathbb{A}^2$. Setting $l = m-n \geq 1$

and denoting by c_m and c_n the elements of $\mathrm{k}[y]$ associated to b and to the integers m and n respectively, we get

$$\theta = \left((\varphi_{c_n})^{-1}\tau^n\varphi_b\right) \circ \left((\varphi_{c_m})^{-1}\tau^m\varphi_b\right)^{-1} = (\varphi_{c_n})^{-1}\tau^{-l}\varphi_{c_m}.$$

The second component of $\theta(x,y)$ is thus equal to the second component of $\tau^{-l}\varphi_{c_m}(x,y)$ which is $\frac{y}{(xy^d+c_m(y))^l} \in \mathrm{k}(x,y) \setminus \mathrm{k}[x,y]$. This shows that θ is not an automorphism of $\mathbb{A}^2$ (and even not an endomorphism) and achieves the proof. $\square$

Remark 3.6. Note that Lemma 3.5(1) provides us an isomorphism $\mathbb{A}^2 \setminus (L_y \cup D_b) \xrightarrow{\simeq} \mathbb{A}^2 \setminus (L_y \cup L_x)$ where the reducible curves $(L_y \cup D_b)$ and $(L_y \cup L_x)$ are not isomorphic. Indeed, the reducible curve $(L_y \cup D_b)$ has two connected components (since $L_y \cap D_b = \emptyset$), while the reducible curve $(L_y \cup L_x)$ is connected (since $L_y \cap L_x \neq \emptyset$). As noted in [Kra96], this kind of easy examples explains why the complement problem in $\mathbb{A}^n$ has only been formulated for irreducible hypersurfaces.

Remark 3.7. Geometrically, the construction of Lemma 3.5(3) can be interpreted as follows: the birational morphism $\varphi_b\colon (x,y) \mapsto (xy^d + b(y), y)$ contracts the line $y = 0$ to the point $(b(0), 0)$. If $d = 1$ then φ_b just sends the line onto the exceptional divisor of $(b(0), 0)$. If $d \geq 2$, it sends the line onto the exceptional divisor of a point in the $(d-1)$-th neighbourhood of $(b(0), 0)$. The coordinates of these points are determined by the polynomial b. The fact that $\tau^m\colon (x,y) \mapsto (x, x^m y)$ contracts the line $x = 0$ implies that $\psi_{b,m}$ contracts the curve D_b given by $xy^d + b(y) = 0$. Moreover, τ^m fixes the point $(b(0), 0)$ and induces a local isomorphism around it, so acts on the set of infinitely near points. This action changes the polynomial b and replaces it with another one, which is the polynomial $c = c_{b,m}$ provided by Lemma 3.5(2).

Corollary 3.8. *For each field* k*, there exists an infinite sequence of curves $C_i \subset \mathbb{A}^2$, $i \in \mathbb{N}$, all pairwise non-equivalent under automorphisms, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$ and such that for each i there are infinitely many open embeddings $\mathbb{A}^2 \setminus C_i \hookrightarrow \mathbb{A}^2$, up to automorphisms of $\mathbb{A}^2$.*

Proof. It suffices to choose the curve C_i given by $xy^{i+2} + y + 1$, for each $i \geq 2$. These curves are all isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by Lemma 3.1 and are pairwise non-equivalent, under automorphisms of $\mathbb{A}^2$, by Lemma 3.3. The existence of infinitely many open embeddings $\mathbb{A}^2 \setminus C_i \hookrightarrow \mathbb{A}^2$, up to automorphisms of $\mathbb{A}^2$, is then ensured by Lemma 3.5(4). $\square$

One can compute the polynomial $c = c_{b,m}$ provided by Lemma 3.5(2), in terms of b and m, and find explicit formulas. We find in particular the following result:

Lemma 3.9. *For each $\mu \in \mathrm{k}$ define the curve $C_\mu \subset \mathbb{A}^2$ by*

$$xy^3 + \mu y^2 + y + 1 = 0.$$

Then, there exists an isomorphism $\mathbb{A}^2 \setminus C_\mu \xrightarrow{\simeq} \mathbb{A}^2 \setminus C_{\mu-1}$. In particular, if $\mathrm{char}(\mathrm{k}) = 0$, we obtain infinitely many closed curves of $\mathbb{A}^2$, not equivalent under automorphisms of $\mathbb{A}^2$, which have isomorphic complements.

Proof. The isomorphism between $\mathbb{A}^2 \setminus C_\mu$ and $\mathbb{A}^2 \setminus C_{\mu-1}$ follows from Lemma 3.5 applied with $d = 3$, $m = 1$, $b = \mu y^2 + y + 1$ and $c = (\mu - 1)y^2 + y + 1$.

To get the last statement, one assumes that $\mathrm{char}(\mathrm{k}) = 0$ and observes that the affine surfaces $\mathbb{A}^2 \setminus C_n$ are all isomorphic for each $n \in \mathbb{Z}$. To show that the curves C_n, $n \in \mathbb{Z}$ are pairwise non-equivalent, we apply Lemma 3.3: for $m, n \in \mathbb{Z}$, the curves C_m and C_n are equivalent only if there exist $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that

$$y^3 = \lambda \cdot (\alpha y + \beta)^3,\ my^2 + y + 1 = \mu \cdot \big(n(\alpha y + \beta)^2 + (\alpha y + \beta) + 1\big).$$

The first equality yields $\beta = 0$, which yields, together with the second equation $my^2 + y + 1 = \mu \cdot (n\alpha^2 y^2 + \alpha y + 1)$, so $\mu = 1$, $\alpha = 1$ and thus $m = n$, as we wanted. □

If $\mathrm{char}(\mathrm{k}) = p > 0$, Lemma 3.9 only gives p non-equivalent curves that have isomorphic complements. We can get more curves by applying Lemma 3.3 to polynomials of higher degree:

Lemma 3.10. *For each integer $n \geq 1$ there exist curves $C_1, \dots, C_n \subset \mathbb{A}^2$, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$, pairwise non-equivalent, such that all surfaces $\mathbb{A}^2 \setminus C_1, \dots, \mathbb{A}^2 \setminus C_n$ are isomorphic.*

Proof. The case where $\mathrm{char}(\mathrm{k}) = 0$ is provided by Lemma 3.9 so we can assume that $\mathrm{char}(\mathrm{k}) = p \geq 2$. Set $b(y) = 1 + y$ and $d = p^n + 2$. For each integer i satisfying $1 \leq i \leq n$, we apply Lemma 3.5(2) with $m = p^i$. Hence, there exists a unique polynomial $c_i \in \mathrm{k}[y]$ of degree $\leq d - 1$ satisfying

$$\text{(D)} \qquad b(y) \equiv c_i(yb(y)^{p^i}) \pmod{y^d}.$$

Define $C_i \subset \mathbb{A}^2$ to be the curve given by the equation

$$xy^d + c_i(y) = 0.$$

By Lemma 3.5(3), all surfaces $\mathbb{A}^2 \setminus C_1, \dots, \mathbb{A}^2 \setminus C_n$ are isomorphic to $\mathbb{A}^2 \setminus D$, where $D \subset \mathbb{A}^2$ is given by

$$xy^d + b(y) = 0.$$

It remains to check that $C_1, \dots, C_n$ are pairwise non-equivalent. Assume therefore that C_i and C_j are equivalent. By Lemma 3.3, there exist $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that

$$y^d = \lambda \cdot (\alpha y + \beta)^d, \quad c_j(y) = \mu \cdot c_i(\alpha y + \beta).$$

The first equality yields $\beta = 0$, so that we get:

$$\text{(E)} \qquad c_j(y) = \mu \cdot c_i(\alpha y).$$

However, by Equation (D) we have

$$1 + y \equiv c_i(y + y^{p^i+1}) \pmod{y^{p^i+2}}$$

and this equation admits the unique solution

$$c_i = 1 + y - y^{p^i+1} + (\text{terms of higher order}).$$

(The unicity follows for example again from Lemma 3.5(2)). Hence, looking at the equation (E) modulo y^2, we obtain $1 + y = \mu(1 + \alpha y)$, so that we get $\alpha = \mu = 1$. Equation (E) finally yields $c_i = c_j$, so that the above (partial) computation of c_i gives us $i = j$. □

The proof of Theorem 2 is now finished:

Proof of Theorem 2. Part (1) corresponds to Corollary 3.8. Part (2) is given by Corollary 3.9 ($\mathrm{char}(\mathrm{k}) = 0$) and Lemma 3.10 ($\mathrm{char}(\mathrm{k}) > 0$). Part (3) corresponds to Corollary 3.4. □

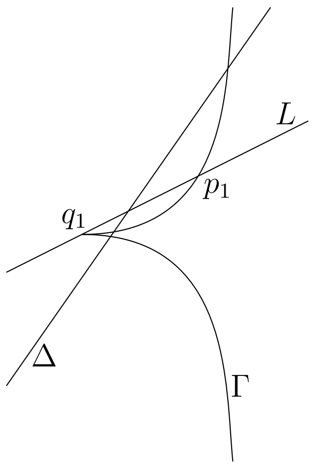
4. Non-isomorphic curves with isomorphic complements

4.1. **A geometric construction.** We begin with the following fundamental construction:

Lemma 4.1. *Let $d \geq 3$ be an integer. Let $\Gamma \subset \mathbb{P}^2$ be the cuspidal rational curve of equation*

$$y^{d-1}z = P_d(x, y),$$

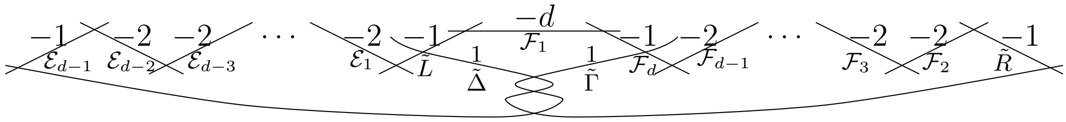
where P_d is a homogeneous polynomial of degree d which is not a multiple of y. Note that the point $q_1 = [0 : 0 : 1] \in \mathbb{P}^2(\mathrm{k})$ has multiplicity $d-1$ on Γ and that it is therefore the unique singular point of this curve (this comes for example from the genus formula of a plane curve). Let Δ, L be two lines of $\mathbb{P}^2$ such that $L \cap \Gamma = \{p_1, q_1\}$ for some point $p_1 \in \mathbb{P}^2(\mathrm{k}) \setminus \{q_1\}$, and assume that Δ does not pass through p_1 nor q_1.



Denote by $\pi \colon X \to \mathbb{P}^2$ the birational morphism given by the blow-up of p_1, q_1, followed by the blow-up of the points $p_2, \ldots, p_{d-1}$ and $q_2, \ldots, q_d$ infinitely near p_1 and q_1 respectively and all belonging to the strict transform of Γ. Denote by $\tilde{\Gamma}$, $\tilde{\Delta}$, $\tilde{L}$, $\mathcal{E}_1, \ldots, \mathcal{E}_{d-1}$, $\mathcal{F}_1, \ldots, \mathcal{F}_d \subset X$ the strict transforms of Γ, Δ, L and of the exceptional divisors above $p_1, \ldots, p_{d-1}$, $q_1, \ldots, q_d$. Consider the tree (which is even a chain)

$$B = \tilde{L} \cup \bigcup_{i=1}^{d-2} \mathcal{E}_i \cup \bigcup_{i=1}^{d} \mathcal{F}_i.$$

Then, the two surfaces $X \setminus (\tilde{\Gamma} \cup B)$ and $X \setminus (\tilde{\Delta} \cup B)$ are both isomorphic to $\mathbb{A}^2$. Moreover, the situation on X is depicted on the symmetric diagram (F),

(F)


where all curves are isomorphic to $\mathbb{P}^1$, all intersections indicated are transversal in exactly one k-point, except for $\tilde{\Gamma} \cap \tilde{\Delta}$, which can be more complicated (the picture just shows *the case where we get* 3 *points with transversal intersection), and where the numbers indicated are the self-intersections. The curve* $\tilde{R}$ *is the strict transform of the line* R *of equation* $y = 0$ *(tangent to* Γ *at* q_1*).*

In particular, this construction provides an isomorphism $\mathbb{A}^2 \setminus C \simeq \mathbb{A}^2 \setminus D$*, where* $C, D \subset \mathbb{A}^2$ *are closed curves isomorphic to* $\tilde{\Gamma} \setminus (B \cup \tilde{\Delta}) \simeq \Gamma \setminus (\Delta \cup \{q_1\})$ *and* $\tilde{\Delta} \setminus (B \cup \tilde{\Gamma}) \simeq \Delta \setminus (\Gamma \cup L)$ *respectively, both of degree* $d^2 - d + 1$*. Moreover, the closure* $\overline{C}$ *of* $C \subset \mathbb{A}^2$ *in* $\mathbb{P}^2$ *admits exactly two singular points, which lie on the line at infinity. The sequences of multiplicities of these two singular points are* $(\underbrace{d, \ldots, d}_{d-1})$ *and* $((d-1)^2, \underbrace{d-1, \ldots, d-1}_{d})$*.*

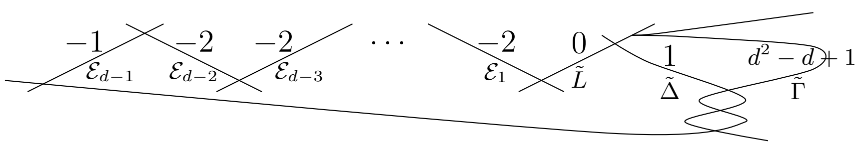
Proof. Blowing up once the singular point q_1 of Γ, the strict transform of Γ becomes a smooth rational curve having $(d-1)$-th order contact with the exceptional divisor. The unique point of intersection between the strict transform and the exceptional divisor corresponds to the direction of the tangent line R. Hence, all points $q_2, \ldots, q_d$ belong to the strict transform of the exceptional divisor of q_1. This gives the self-intersections of $\mathcal{F}_1, \ldots, \mathcal{F}_d$ and their configurations, as shown on diagram (F). As p_1 is a smooth point of Γ, the curves $\mathcal{E}_1, \ldots, \mathcal{E}_{d-1}$ form a chain of curves, as shown on diagram (F). The rest of the diagram is checked by looking at the definitions of the curves Γ, Δ, L, R.

We want to show the existence of isomorphisms

$$\psi_1 \colon X \setminus (B \cup \tilde{\Delta}) \xrightarrow{\simeq} \mathbb{A}^2 \text{ and } \psi_2 \colon X \setminus (B \cup \tilde{\Gamma}) \xrightarrow{\simeq} \mathbb{A}^2$$

such that $C = \psi_1(\tilde{\Gamma} \setminus (B \cup \tilde{\Delta}))$ and $D = \psi_2(\tilde{\Delta} \setminus (B \cup \tilde{\Gamma}))$ are of degree $d^2 - d + 1$. Indeed, the existence of ψ_1, ψ_2 implies that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic, and that C and D are isomorphic to $\tilde{\Gamma} \setminus (B \cup \tilde{\Delta})$ and $\tilde{\Delta} \setminus (B \cup \tilde{\Gamma})$ respectively. The morphism π gives then isomorphisms of these curves with $\Gamma \setminus (\Delta \cup \{q\})$ and $\Delta \setminus (\Gamma \cup L)$ respectively.

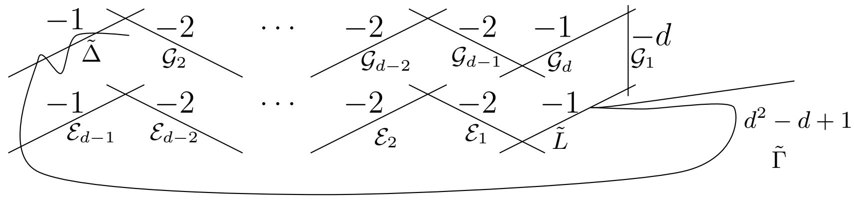
We first show that ψ_1 exists. We observe that since π is the blow-up of $2d - 1$ points which are defined over k, the Picard group of X is of rank $2d$, over k and over its algebraic closure $\overline{\mathrm{k}}$. We contract the curves $\mathcal{F}_d, \ldots, \mathcal{F}_1$ and obtain a smooth projective surface Y of Picard rank d (again over k and $\overline{\mathrm{k}}$). The configuration of the image of the curves $\mathcal{E}_1, \ldots, \mathcal{E}_{d-1}, \tilde{L}, \tilde{\Gamma}$ is then depicted on diagram (G) (we omit to write the curve $\tilde{R}$ as we will not need it):



(G)

In fact, Y is just the blow-up of the points $p_1, \ldots, p_{d-1}$ starting from $\mathbb{P}^2$.

In order to show that $X \setminus (B \cup \tilde{\Delta}) \simeq Y \setminus (\tilde{\Delta} \cup \tilde{L} \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{d-2})$ is isomorphic to $\mathbb{A}^2$, we will construct a birational map $\hat{\psi}_1 \colon Y \dashrightarrow \mathbb{P}^2$ which restricts to an isomorphism $Y \setminus (\tilde{\Delta} \cup \tilde{L} \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{d-2}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \mathcal{L}$ for some line $\mathcal{L}$. Let us now describe this map. Denote by r_1 the unique point of Y such that $\{r_1\} = \tilde{\Delta} \cap \tilde{L}$ in Y. We blow up r_1 and then the point r_2 lying on the intersection of the exceptional curve of r_1 and of the strict transform of $\tilde{\Delta}$. For $i = 3, \ldots, d$, denoting by r_i the point lying on the intersection of the exceptional curve of r_{i-1} and on the strict transform of the exceptional curve of r_1, we successively blow up r_i. We obtain thus a birational morphism $\theta \colon Z \to Y$. The configuration of curves on Z is depicted on diagram (H) (we again use the same name for a curve on Y and its strict transform on Z, and denote by $\mathcal{G}_i \subset Z$ the strict transform of the exceptional divisor of r_i):



(H)

We can then contract the curves $\tilde{\Delta}, \mathcal{G}_2, \dots, \mathcal{G}_{d-1}, \tilde{L}, \mathcal{E}_1, \dots, \mathcal{E}_{d-2}, \mathcal{G}_1$ and obtain a birational morphism $\rho\colon Z \to \mathbb{P}^2$: The image of the target is $\mathbb{P}^2$ because it has Picard rank 1; note also that the image $\mathcal{L}$ of $\mathcal{G}_d$ is actually a line of $\mathbb{P}^2$ since it has self-intersection 1. The birational map $\hat{\psi}_1\colon Y \dashrightarrow \mathbb{P}^2$ given by $\hat{\psi}_1 = \rho\theta^{-1}$ is the desired birational map. The closure $\overline{C}$ of $C \subset \mathbb{A}^2$ in $\mathbb{P}^2$ is then equal to the image of $\tilde{\Gamma}$ by ρ.

For each contracted curve above, the multiplicity (on $\overline{C}$) at the point where it is contracted, is equal to d for $\tilde{\Delta}, \mathcal{G}_2, \dots, \mathcal{G}_{d-1}$, it is equal to $d-1$ for $\tilde{L}, \mathcal{E}_1, \dots, \mathcal{E}_{d-2}$ and it is equal to $(d-1)^2$ for $\mathcal{G}_1$. Adding the singular point of multiplicity $d-1$ of $\tilde{\Gamma}$, we obtain the two sequences of multiplicities $(\underbrace{d, \dots, d}_{d-1})$ and $((d-1)^2, \underbrace{d-1, \dots, d-1}_{d})$.

The self-intersection of $\overline{C}$ is then

$$(d^2-d+1) + (d-1)\cdot d^2 + (d-1)\cdot(d-1)^2 + ((d-1)^2)^2 = (d^2-d+1)^2,$$

which implies that the curve has degree d^2-d+1.

The case of ψ_2 is similar, as the diagram (F) is symmetric. $\square$

Proposition 4.2. *For each polynomial $P \in \mathrm{k}[t]$ of degree $d \geq 3$ and each $\lambda \in \mathrm{k}$ with $P(\lambda) \neq 0$, there exist two closed curves $C, D \subset \mathbb{A}^2$ of degree d^2-d+1 such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that the following isomorphisms hold:*

$$C \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{P}]\right) \text{ and } D \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{Q}]\right), \text{ where } Q(t) = P\Big(\lambda + \frac{1}{t}\Big)\cdot t^d.$$

Proof. Write $P_d(x,y) = P(\frac{x}{y})y^d \in \mathrm{k}[x,y]$, which is a homogeneous polynomial of degree d, such that $P_d(x,1) = P(x)$. Let then $\Gamma, \Delta, L \subset \mathbb{P}^2$ be the curves given by the equations

$$\Gamma : y^{d-1}z = P_d(x,y), \quad \Delta : z = 0, \quad L : x = \lambda y.$$

By construction, P_d is not divisible by y. Moreover, the two lines L and Δ satisfy $L \cap \Gamma = \{p_1, q_1\}$ where $p_1 = [\lambda : 1 : P(\lambda)]$, $q_1 = [0:0:1]$ and Δ does not pass through p_1 nor q_1.

We apply Lemma 4.1, and obtain an isomorphism $\mathbb{A}^2 \setminus C \simeq \mathbb{A}^2 \setminus D$, where $C, D \subset \mathbb{A}^2$ are closed curves isomorphic to $\Gamma \setminus (\Delta \cup \{q_1\})$ and $\Delta \setminus (\Gamma \cup L)$ respectively. Furthermore, both curves have degree d^2-d+1.

Since $\Gamma \setminus \{q_1\}$ is isomorphic to $\mathbb{A}^1$ via $t \mapsto [t : 1 : P_d(t,1)] = [t : 1 : P(t)]$, we obtain that $C \simeq \Gamma \setminus (\Delta \cup \{q_1\})$ is isomorphic to $\operatorname{Spec}(\mathrm{k}[t, \frac{1}{P}])$.

We then take the isomorphism $\mathbb{A}^1 \to \Delta \setminus L = \Delta \setminus \{[\lambda : 1 : 0]\}$ given by $t \mapsto [\lambda t + 1 : t : 0]$. The pull-back of $\Delta \cap \Gamma$ corresponds to the zeroes of $P_d(\lambda t + 1, t) = t^d P_d(\lambda + \frac{1}{t}, 1) = Q(t)$. Hence, D is isomorphic to $\operatorname{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ as desired. $\square$

Corollary 4.3. *For each $d \geq 0$ and each distinct points $a_1, \dots, a_d, b_1, b_2 \in \mathbb{P}^1(\mathrm{k})$, there are two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathbb{P}^1 \setminus \{a_1, \dots, a_d, b_1\}$ and $D \simeq \mathbb{P}^1 \setminus \{a_1, \dots, a_d, b_2\}$.*

Proof. The case where $d \leq 2$ is obvious: Since $\mathrm{PGL}_2(\mathrm{k})$ acts 3-transitively on $\mathbb{P}^1(\mathrm{k})$, one can take $C = D$ given by the equation $x = 0$, resp. $xy = 1$, resp. $x(x-1)y = 1$, if $d = 0$, resp. $d = 1$, resp. $d = 2$. Let us now assume that $d \geq 3$. Since $\mathrm{PGL}_2(\mathrm{k})$ acts transitively on $\mathbb{P}^1(\mathrm{k})$, we may assume without restriction that b_1 is the point at infinity $[1:0]$. Therefore, there exist distinct constants $\mu_1, \dots, \mu_d, \lambda \in \mathrm{k}$ such that $a_1 = [\mu_1 : 1], \dots, a_d = [\mu_d : 1]$ and $b_2 = [\lambda : 1]$. We now apply Proposition 4.2 with $P = \prod_{i=1}^d (t - \mu_i)$. We get two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}]) \simeq \mathbb{A}^1 \setminus \{\mu_1, \dots, \mu_d\} \simeq \mathbb{P}^1 \setminus \{a_1, \dots, a_d, b_1\}$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}]) \simeq \mathbb{A}^1 \setminus \{\frac{1}{\mu_1 - \lambda}, \dots, \frac{1}{\mu_d - \lambda}\}$, where $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^d$. It remains to see that D is isomorphic to $\mathbb{P}^1 \setminus \{[\mu_1 : 1], \dots, [\mu_d : 1], [\lambda : 1]\}$ via $t \mapsto [\lambda t + 1 : t]$. $\square$

Corollary 4.4. *If k is infinite and $P \in \mathrm{k}[t]$ is a polynomial with at least 3 roots in $\overline{\mathrm{k}}$, we can find two curves $C, D \subset \mathbb{A}^2$ that have isomorphic complements, such that C is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ but not D.*

Proof. By Lemma 4.5 below, there exists a constant λ in k such that $P(\lambda) \neq 0$ and such that the curves $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic. The result now follows from Proposition 4.2. $\square$

Lemma 4.5. *If k is infinite and $P \in \mathrm{k}[t]$ is a polynomial with at least 3 roots in $\overline{\mathrm{k}}$, then for a general $\lambda \in \mathrm{k}$, the polynomial $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^{\deg(P)}$ is such that the curves $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic.*

Proof. Let $\lambda_1, \dots, \lambda_d \in \overline{\mathrm{k}}$ be the single roots of P. It is enough to check that for a general λ, there exists no automorphism of $\mathbb{P}^1$ sending $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\frac{1}{\lambda_1 - \lambda}, \dots, \frac{1}{\lambda_d - \lambda}, \infty\}$, or equivalently that there exists no automorphism sending $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\lambda_1, \dots, \lambda_d, \lambda\}$. But, if an automorphism sends $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\lambda_1, \dots, \lambda_d, \lambda\}$, it necessarily belongs to the set $\mathcal{A}$ of automorphisms φ such that $\varphi^{-1}(\{\lambda_1, \lambda_2, \lambda_3\}) \subset \{\lambda_1, \dots, \lambda_d, \infty\}$. An automorphism of $\mathbb{P}^1$ being determined by the image of 3 points, the set $\mathcal{A}$ has at most $6\binom{d+1}{3} = (d+1)d(d-1)$ elements. As a conclusion, if λ is not of the form $\varphi(\mu)$ for some $\varphi \in \mathcal{A}$ and some $\mu \in \{\lambda_1, \dots, \lambda_d, \infty\}$, then no automorphism of $\mathbb{P}^1$ sends $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\lambda_1, \dots, \lambda_d, \lambda\}$. $\square$

Remark 4.6. If k is a finite field (with at least 3 elements), then the conclusion of Corollary 4.4 is false for the polynomial $P = \prod_{\alpha \in \mathrm{k}} (x - \alpha)$. Indeed, if $C, D \subset \mathbb{A}^2$ are two curves such that C is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathbb{A}^2 \setminus C$ is isomorphic to $\mathbb{A}^2 \setminus D$, then D is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ for some polynomial Q without square factors having the same number of roots in k and in $\overline{\mathrm{k}}$ as P (Theorem 1(2)). This implies that Q is equal to μP for some $\mu \in \mathrm{k}^*$ and thus that C and D are isomorphic.

A similar argument holds for $P = \prod_{\alpha \in \mathrm{k}^*} (x - \alpha)$ and $P = \prod_{\alpha \in \mathrm{k} \setminus \{0,1\}} (x - \alpha)$ (when the field has at least 4, respectively 5 elements) since $\mathrm{PGL}_2(\mathrm{k})$ acts 3-transitively on $\mathbb{P}^1(\mathrm{k})$.

Corollary 4.7. *For each ground field* k *having more than* 27 *elements, one gets two geometrically irreducible curves* $C, D \subset \mathbb{A}^2$ *of degree* 7 *which are not isomorphic but such that* $\mathbb{A}^2 \setminus C$ *and* $\mathbb{A}^2 \setminus D$ *are isomorphic.*

Proof. We fix some element $\zeta \in \mathrm{k} \setminus \{0,1\}$. For each $\lambda \in \mathrm{k} \setminus \{0,1,\zeta\}$, one can apply Corollary 4.3 with $d = 3$, $a_1 = [0 : 1]$, $a_2 = [1 : 1]$, $a_3 = [\zeta : 1]$, $b_1 = [1 : 0]$, $b_2 = [\lambda : 1]$ and get two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathbb{A}^1 \setminus \{0,1,\zeta\} = \mathbb{P}^1 \setminus \{[0 : 1],[1 : 1],[\zeta : 1],[1 : 0]\}$ and $D \simeq \mathbb{P}^1 \setminus \{[0 : 1],[1 : 1],[\zeta : 1],[\lambda : 1]\}$. It remains to see that one can find at least one λ such that C and D are not isomorphic. Note that C and D are isomorphic if and only if there is an element of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathrm{k})$ that sends $\{[0 : 1],[1 : 1],[\zeta : 1],[\lambda : 1]\}$ onto $\{[0 : 1],[1 : 1],[\zeta : 1],[1 : 0]\}$. The image of this element is determined by the image of $[0 : 1],[1 : 1],[\zeta : 1]$, so one gets at most 24 automorphisms to avoid, hence at most 24 elements of $\mathrm{k} \setminus \{0,1,\zeta\}$ to avoid. The field k having at least 28 elements, we find at least one λ having the right property. □

We can now give the proof of Theorem 3.

Proof of Theorem 3. If the field is infinite (or simply has more than 27 elements), it follows from Corollary 4.7. Let us therefore assume that k is a finite field. We again apply Proposition 4.2 (with $\lambda = 0$). Therefore, if $|\mathrm{k}| > 2$ (resp. $|\mathrm{k}| = 2$), it is enough to give a polynomial $P \in \mathrm{k}[t]$ of degree 3 (resp. 4) such that $P(0) \neq 0$ and such that if we set $Q := P(\frac{1}{t})t^{\deg P}$, then the k-algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are not isomorphic.

We begin with the case where the characteristic of k is odd. Then, the kernel of the morphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^2$ is equal to $\{-1,1\}$, so that this map is not surjective. Let us pick an element $\alpha \in \mathrm{k}^* \setminus (\mathrm{k}^*)^2$. Let us check that we can take $P = (t-1)((t-1)^2 - \alpha)$. Indeed, up to a multiplicative constant, we have $Q = (t-1)((t-1)^2 - \alpha t^2)$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, these algebras would still be isomorphic if we replace P and Q by

$$\tilde{P} = P(t+1) = t\big(t^2 - \alpha\big) \text{ and } \tilde{Q} = Q(t+1) = t\big(t^2 - \alpha(t+1)^2\big).$$

This would yield an automorphism of $\mathbb{P}^1$, via the embedding $t \mapsto [t : 1]$, which sends the polynomial $uv(u^2 - \alpha v^2)$ onto a multiple of $uv(u^2 - \alpha(u+v)^2)$. This automorphism preserving the set of k-roots: $\{[0 : 1],[1 : 0]\}$, it is either of the form $[u : v] \mapsto [\mu u : v]$ or $[u : v] \mapsto [\mu v : u]$ where $\mu \in \mathrm{k}^*$. The polynomial $u^2 - \alpha v^2$ has to be sent to a multiple of $u^2 - \alpha(u+v)^2$, which is not possible because of the term uv.

We now handle the case where k has characteristic 2. We divide it into three cases, depending whether the cube homomorphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^3$ is injective or not (which corresponds to ask if 4 divides $|\mathrm{k}|$ or not), and putting the field with two elements apart.

If the cube homomorphism is not surjective, we can pick an element $\alpha \in \mathrm{k}^* \setminus (\mathrm{k}^*)^3$. Let us check that we can take the irreducible polynomial $P = t^3 - \alpha \in \mathrm{k}[t]$. Indeed, up to a multiplicative constant, we have $Q = t^3 - \alpha^{-1}$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, there should exist constants $\lambda, \mu, c \in \mathrm{k}$ with $\lambda c \neq 0$ such that

$$c(t^3 - \alpha^{-1}) = (\lambda t + \mu)^3 - \alpha.$$

This gives us $\mu = 0$ and $\lambda^3 = c = \alpha^2$. The square homomorphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^2$ being bijective, there is a unique square root for each element of k^*. Taking the square root of the equality $\alpha^2 = \lambda^3$, we obtain $\alpha = (\nu)^3$, where ν is the square root of λ. This is impossible since α was chosen not to be a cube.

If the cube homomorphism is surjective, then 1 is the only root of $t^3-1 = (t-1)(t^2+t+1)$, so $t^2+t+1 \in \mathrm{k}[t]$ is irrreducible. If moreover k has more than 2 elements, we can choose $\alpha \in \mathrm{k}\setminus\{0,1\}$ and take $P = (t-\alpha)(t^2+t+1)$. Up to a multiplicative constant, we have $Q = (t-\alpha^{-1})(t^2+t+1)$. Let us assume by contradiction that the algebras $\mathrm{k}[t,\frac{1}{P}]$ and $\mathrm{k}[t,\frac{1}{Q}]$ are isomorphic. Then, these algebras would still be isomorphic if we replace P and Q by

$$\tilde{P} = P(t+\alpha) = t(t^2+t+\alpha^2+\alpha+1) \text{ and } \tilde{Q} = Q(t+\alpha^{-1}) = t(t^2+t+\alpha^{-2}+\alpha^{-1}+1).$$

This would yield an automorphism of $\mathbb{P}^1$, via the embedding $t \mapsto [t:1]$, which sends the polynomial $uv(u^2+uv+(\alpha^2+\alpha+1)v^2)$ onto a multiple of $uv(u^2+uv+(\alpha^{-2}+\alpha^{-1}+1)v^2)$. The same argument as before yields $\alpha^2+\alpha+1 = \alpha^{-2}+\alpha^{-1}+1$, i.e. $\alpha^2+\alpha+1 = \alpha^{-2}(\alpha^2+\alpha+1)$. This is impossible since $\alpha^2+\alpha+1 \neq 0$ and $\alpha^2 \neq 1$.

The last case is when $\mathrm{k} = \{0,1\}$ is the field with two elements. Here the construction does not work with polynomials of degree 3: the only ones which are not symmetric and do not vanish in 0 are t^3+t^2+1 and t^3+t+1, which are equivalent via $t \mapsto t+1$. We then choose for P the irreducible polynomial $P = t^4+t+1$ (it has not root and is not equal to $(t^2+t+1)^2 = t^4+t^2+1$). This gives $Q = t^4+t^3+1$. Let us assume by contradiction that the algebras $\mathrm{k}[t,\frac{1}{P}]$ and $\mathrm{k}[t,\frac{1}{Q}]$ are isomorphic. Then, there should exist constants $\lambda,\mu,c \in \mathrm{k}$ with $\lambda c \neq 0$ such that

$$c(t^4+t^3+1) = (\lambda t+\mu)^4 + (\lambda t+\mu) + 1.$$

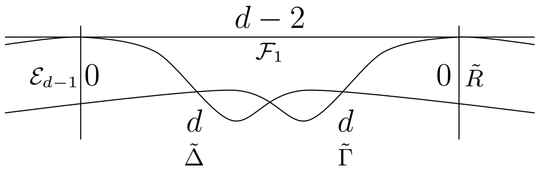
This is impossible since $(\lambda t+\mu)^4 + (\lambda t+\mu)+1 = \lambda^4 t^4 + \lambda t + (\mu^4+\mu+1)$. ☐

4.2. **Getting explicit formulas.** To obtain the equations of the curves C, D and the isomorphism $\mathbb{A}^2\backslash C \xrightarrow{\simeq} \mathbb{A}^2\backslash D$ given by Lemma 4.1, one could follow the construction and explicit the birational maps described: The lemma yields the existence of isomorphisms

$$\psi_1\colon X\setminus(B\cup\tilde{\Delta}) \xrightarrow{\simeq} \mathbb{A}^2 \text{ and } \psi_2\colon X\setminus(B\cup\tilde{\Gamma}) \xrightarrow{\simeq} \mathbb{A}^2$$

such that $C = \psi_1(\tilde{\Gamma}\setminus(B\cup\tilde{\Delta}))$ and $D = \psi_2(\tilde{\Delta}\setminus(B\cup\tilde{\Gamma}))$ are of degree d^2-d+1, where $B = \tilde{L}\cup\bigcup_{i=1}^{d-2}\mathcal{E}_i\cup\bigcup_{i=1}^{d}\mathcal{F}_i$, and ψ_1,ψ_2 are given by blow-ups and blow-downs, so it is possible to explicit $\psi_i\pi^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with formulas (looking at the linear systems), and then get the isomorphism $\psi_2\pi^{-1}\circ(\psi_1\pi^{-1})^{-1}\colon \mathbb{A}^2\setminus C \to \mathbb{A}^2\setminus D$. The formulas for $\psi_1\pi^{-1},\psi_2\pi^{-1}$ are however quite complicated.

Another possibility can be done as follows: we choose a birational morphism $X \to W$ that contracts $\tilde{L},\mathcal{E}_1,\ldots,\mathcal{E}_{d-2}$ and $\mathcal{F}_d,\ldots,\mathcal{F}_2$ to two smooth points of W, passing through the image of $\mathcal{F}_1$ (possible, see Diagram (F)). The situation of the image of the curves $\tilde{R},\mathcal{E}_{d-1},\mathcal{F}_1,\tilde{\Gamma},\tilde{\Delta}$ (that we again denote by the same name) in W is as follows:



Computing the dimension of the Picard group, one finds that W is a Hirzebruch surface. Hence, the curves $\mathcal{E}_{d-1}, \tilde{R}$ are fibres of a $\mathbb{P}^1$-bundle $W \to \mathbb{P}^1$ and $\mathcal{F}_1, \tilde{\Delta}, \tilde{\Gamma}$ are sections of self-intersection $d-2, d, d$. One can then find plenty of examples in $\mathbb{F}_1$ and $\mathbb{F}_0$ (depending on the parity of d) but also in $\mathbb{F}_m$ for $m \geq 2$ if the polynomial chosen at the beginning is special enough.

The case where $d = 3$ corresponds to curves of degree 7 in $\mathbb{A}^2$ (Lemma 4.1) which is the first interesting case, as it gives non-isomorphic curves for almost each field (Theorem 3). When $d = 3$, one finds that $\mathcal{F}_1$ is a section of self-intersection 1 in $W = \mathbb{F}_1$, so $\mathbb{F}_1 \setminus \mathcal{F}_1$ is isomorphic to the blow-up of $\mathbb{A}^2$ at one point, and $\tilde{\Gamma}, \tilde{\Delta}$ are sections of self-intersection 3 and are thus strict transforms of parabolas passing through the point blown up. This explains how the following result is derived from Lemma 4.1. However, the result can also be read independently of Lemma 4.1, since the proof that we give does not use the rest of the article:

Proposition 4.8. *Let us fix some constants $a_0, a_1, a_2, a_3 \in \mathrm{k}$ with $a_0 a_3 \neq 0$ and consider the two irreducible polynomials $P, Q \in \mathrm{k}[x,y]$ of degree 2 given by*

$$P = x^2 - a_2 x - a_3 y \quad \text{and} \quad Q = y^2 + a_0 x + a_1 y.$$

(1) *Denoting by $\eta\colon \hat{\mathbb{A}}^2 \to \mathbb{A}^2$ the blow-up of the origin and by $\tilde{\Gamma}, \tilde{\Delta} \subset \hat{\mathbb{A}}^2$ the strict transforms of the curves $\Gamma, \Delta \subset \mathbb{A}^2$ given by $P = 0$ and $Q = 0$ respectively, the rational maps*

$$\begin{array}{rccc} \varphi_P\colon & \mathbb{A}^2 & \dashrightarrow & \mathbb{A}^2 \\ & (x,y) & \mapsto & \left(-\dfrac{x}{P(x,y)}, P(x,y)\right) \end{array} \quad \textit{and} \quad \begin{array}{rccc} \varphi_Q\colon & \mathbb{A}^2 & \dashrightarrow & \mathbb{A}^2 \\ & (x,y) & \mapsto & \left(\dfrac{y}{Q(x,y)}, Q(x,y)\right) \end{array}$$

are birational maps that induce isomorphisms

$$\psi_P = (\varphi_P \eta)|_{\hat{\mathbb{A}}^2 \setminus \tilde{\Gamma}}\colon \ \hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2 \quad \textit{and} \quad \psi_Q = (\varphi_Q \eta)|_{\hat{\mathbb{A}}^2 \setminus \tilde{\Delta}}\colon \ \hat{\mathbb{A}}^2 \setminus \tilde{\Delta} \xrightarrow{\simeq} \mathbb{A}^2.$$

(2) *Define the curves $C, D \subset \mathbb{A}^2$ by $C = \psi_Q(\tilde{\Gamma} \setminus \tilde{\Delta})$, $D = \psi_P(\tilde{\Delta} \setminus \tilde{\Gamma})$ and denote by $\psi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ the isomorphism induced by the birational transformation $\psi_P(\psi_Q)^{-1}\colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$. Then, the curves $C, D \subset \mathbb{A}^2$ are given by $f = 0$ and $g = 0$ respectively, where the polynomials $f, g \in \mathrm{k}[x,y]$ are defined by:*

$$\begin{aligned} f &= \Big(1 - x(xy + a_1)\Big)\Big(y\Big(1 - x(xy + a_1)\Big) - a_0 a_2\Big) - x(a_0)^2 a_3, \\ g &= \Big(1 - x(xy + a_2)\Big)\Big(y\Big(1 - x(xy + a_2)\Big) - a_1 a_3\Big) - x a_0 (a_3)^2. \end{aligned}$$

The following isomorphisms hold

$$C \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} a_i t^i}]\right) \quad \textit{and} \quad D \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} a_{3-i} t^i}]\right).$$

Moreover, ψ and ψ^{-1} are given by

$$\begin{array}{cccc} \psi\colon & (x,y) & \mapsto & \left(\dfrac{a_0\Big(x(xy+a_1)-1\Big)}{f(x,y)}, \dfrac{y\,f(x,y)}{(a_0)^2}\right) \\ & \left(\dfrac{a_3\Big(x(xy+a_2)-1\Big)}{g(x,y)}, \dfrac{y\,g(x,y)}{(a_3)^2}\right) & \leftarrow\!\shortmid & (x,y). \end{array}$$

Proof. (1): Let us first prove that φ_P is birational and that $\varphi_P\eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2$. We observe that $\kappa\colon (x,y) \mapsto (x, x^2 - a_2x - a_3y)$ is an automorphism of $\mathbb{A}^2$ that sends Γ onto the line $L_y \subset \mathbb{A}^2$ of equation $y = 0$. Moreover $\tilde{\varphi}_P = \varphi_P\kappa^{-1}\colon (x,y) \mapsto (-\frac{x}{y}, y)$ is birational, so φ_P is birational. Since κ fixes the origin, $\eta^{-1}\kappa\eta$ is an automorphism of $\hat{\mathbb{A}}^2$, that sends $\tilde{\Gamma}$ onto the strict transform $\tilde{L}_y \subset \hat{\mathbb{A}}^2$ of L_y. The fact that $\tilde{\varphi}_P\eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{L}_y \xrightarrow{\simeq} \mathbb{A}^2$ is straightforward using the classical description of the blow-up $\hat{\mathbb{A}}^2$ in which

$$\hat{\mathbb{A}}^2 = \{((x,y),[u:v]) \mid xv = yu\} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

and $\eta\colon \hat{\mathbb{A}}^2 \to \mathbb{A}^2$ is the first projection. Actually, with this description $\tilde{L}_y = L_y \times [1:0]$ is given by the equation $v = 0$ and the following morphisms are inverses of each other:

$$\begin{array}{ll} \hat{\mathbb{A}}^2 \setminus \tilde{L}_y \to \mathbb{A}^2, & ((x,y),[u:v]) \mapsto (-\frac{u}{v}, y) \\ \mathbb{A}^2 \to \hat{\mathbb{A}}^2 \setminus \tilde{L}_y, & (x,y) \mapsto ((-xy,y),[-x:1]). \end{array}$$

It follows that $(\tilde{\varphi}_P\eta)(\eta^{-1}\kappa\eta) = \varphi_P\eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2$. The case of φ_Q and $\varphi_Q\eta$ would be done similarly, using the automorphism of $\mathbb{A}^2$ given by $(x,y) \mapsto (y^2 + a_0x + a_1y, y)$. This yields (1).

(2): Now that (1) is proven, one gets two isomorphisms

$$\psi_P|_U\colon U \xrightarrow{\simeq} \mathbb{A}^2 \setminus D, \quad \psi_Q|_U\colon U \xrightarrow{\simeq} \mathbb{A}^2 \setminus C,$$

where $U = \hat{\mathbb{A}}^2 \setminus (\tilde{\Gamma} \cup \tilde{\Gamma})$. Remembering that $\Gamma \subset \mathbb{A}^2$ is given by $x(x - a_2) = a_3y$, we have an isomorphism

$$\begin{array}{cccc} \rho\colon & \mathbb{A}^1 & \xrightarrow{\simeq} & \Gamma \\ & t & \mapsto & (ta_3 + a_2, t(ta_3 + a_2)) \\ & \frac{1}{a_3}(x - a_2) & \leftarrow\!\shortmid & (x,y). \end{array}$$

Replacing $\rho(t)$ in the polynomial $Q(x,y) = xa_0 + ya_1 + y^2$ used for defining Δ, one finds

$$Q(ta_3 + a_2, t(ta_3 + a_2)) = (ta_3 + a_2)(t^3a_3 + t^2a_2 + ta_1 + a_0).$$

The root of $ta_3 + a_2$ is sent by ρ to the origin, which is itself blown up by η. Hence, the map $\eta^{-1}\rho$ induces an isomorphism from $V = \operatorname{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} t^i a_i}]) \subset \mathbb{A}^1$ to $\tilde{\Gamma} \setminus \tilde{\Delta}$. Applying $\psi_Q = (\varphi_Q\eta)|_{\hat{\mathbb{A}}^2 \setminus \tilde{\Delta}}$, one gets an isomorphism $\theta = (\varphi_Q\rho)|_V\colon V \xrightarrow{\simeq} C$. Since $(\varphi_Q)^{-1}$ is given by

$$(\varphi_Q)^{-1}\colon (x,y) \mapsto \left(\frac{y\Big(1 - x(xy + a_1)\Big)}{a_0}, xy\right),$$

one can give explicitly θ and its inverse:

$$\begin{array}{cccc}
\theta\colon & \mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} t^i a_i}]) & \stackrel{\simeq}{\longrightarrow} & C \\
 & t & \mapsto & \left(\dfrac{t}{\sum_{i=0}^{3} t^i a_i}, (ta_3+a_2)(\sum_{i=0}^{3} t^i a_i)\right) \\
 & \dfrac{1}{a_3}\left(\dfrac{y\left(1-x(xy-a_1)\right)}{a_0} - a_2\right) & \leftarrow\!\shortmid & (x,y).
\end{array}$$

Computing the extension of θ to a morphism $\mathbb{P}^1 \to \mathbb{P}^2$, one sees that the curve $C \subset \mathbb{A}^2$ has degree 7. To get its equation, we can compute $((\varphi_Q)^{-1})^*(P)$: since $(a_0)^2 P(x,y) = (a_0 x)(a_0 x - a_0 a_2) - (a_0)^2 a_3 y$, one gets

$$\begin{array}{rcl}
(a_0)^2((\varphi_Q)^{-1})^*(P) & = & (a_0)^2 P\left(\frac{y(1-x(xy+a_1))}{a_0}, xy\right) \\
 & = & y\,(1-x(xy+a_1))\,(y\,(1-x(xy+a_1)) - a_0a_2) - xy(a_0)^2 a_3 \\
 & = & yf(x,y),
\end{array}$$

where

$$f = (1-x(xy+a_1))\,(y\,(1-x(xy+a_1)) - a_0a_2) - x(a_0)^2 a_3 \in \mathrm{k}[x,y]$$

yields the equation of C (note that the polynomial $y = 0$ appears here because it corresponds to the line contracted by $(\psi_Q)^{-1}$, corresponding to the exceptional divisor of $\hat{\mathbb{A}}^2 \to \mathbb{A}^2$ via the isomorphism $\mathbb{A}^2 \to \hat{\mathbb{A}}^2 \setminus \hat{\Delta}$). The linear involution of $\mathbb{A}^2$ given by $(x,y) \mapsto (-y,-x)$ exchanges the polynomials P and Q and the maps φ_P and φ_Q, by replacing a_0, a_1, a_2, a_3 with a_3, a_2, a_1, a_0 respectively. This shows that $D \subset \mathbb{A}^2$ has equation $g = 0$ where g is obtained from f by replacing a_0, a_1, a_2, a_3 with a_3, a_2, a_1, a_0, i.e.

$$g = (1-x(xy+a_2))\,(y\,(1-x(xy+a_2)) - a_1a_3) - xa_0(a_3)^2 \in \mathrm{k}[x,y].$$

Therefore, D is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} a_{3-i} t^i}])$. It remains to compute the isomorphism $\psi\colon \mathbb{A}^2 \setminus C \to \mathbb{A}^2 \setminus D$, which is by construction equal to the birational maps $\psi_P(\psi_Q)^{-1} = \varphi_P(\varphi_Q)^{-1}$. Using the equation $(a_0)^2 P\left(\frac{y(1-x(xy+a_1))}{a_0}, xy\right) = yf(x,y)$, one gets:

$$\begin{array}{rcl}
\psi(x,y) & = & \varphi_P\left(\dfrac{y\,(1-x(xy+a_1))}{a_0}, xy\right) \\
 & = & \left(-\dfrac{y\,(1-x(xy+a_1))}{a_0 P\left(\frac{y(1-x(xy+a_1))}{a_0}, xy\right)}, P\left(\dfrac{y\,(1-x(xy+a_1))}{a_0}, xy\right)\right) \\
 & = & \left(\dfrac{a_0\,(x(xy+a_1)-1)}{f(x,y)}, \dfrac{y\,f(x,y)}{(a_0)^2}\right).
\end{array}$$

By symmetry, the expression of ψ^{-1} is obtained from the one of ψ by replacing a_0, a_1, a_2, a_3 with a_3, a_2, a_1, a_0, i.e. it is given by $\psi^{-1}(x,y) = \left(\dfrac{a_3\,(x(xy+a_2)-1)}{g(x,y)}, \dfrac{y\,g(x,y)}{(a_3)^2}\right)$. $\square$

Remark 4.9. Proposition 4.8 yields an isomorphism $\psi^*\colon \mathrm{k}[x,y,\frac{1}{g}] \xrightarrow{\simeq} \mathrm{k}[x,y,\frac{1}{f}]$ which sends the invertible elements onto the invertible elements and thus sends g onto $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$ (see Lemma 2.12). This corresponds to say that ψ induces an isomorphism between the two fibrations

$$\mathbb{A}^2 \setminus C \xrightarrow{f} \mathbb{A}^1 \setminus \{0\} \quad \text{and} \quad \mathbb{A}^2 \setminus D \xrightarrow{g} \mathbb{A}^1 \setminus \{0\},$$

possibly exchanging the fibres. To study these fibrations, one uses the equalities

$$(\mathrm{I}) \qquad (\varphi_Q)^*(f) = \frac{(a_0)^2 P}{Q}, \quad (\varphi_P)^*(g) = \frac{(a_3)^2 Q}{P},$$

which can either be checked directly, or observed as follows: the first equality follows from $((\varphi_Q)^{-1})^*(P) = \frac{yf(x,y)}{(a_0)^2}$, applying $(\varphi_Q)^*$, and the second is obtained by symmetry.

Note that Equation (I) yields $\psi^*(g) = \frac{(a_0 a_3)^2}{f}$, since $\psi = \varphi_P(\varphi_Q)^{-1}$.

For each $\mu \in \mathrm{k}$, the fibre $C_\mu \subset \mathbb{A}^2$ given by $f(x,y) = \mu$ is an algebraic curve isomorphic to its preimage by the isomorphism $\psi_Q = (\varphi_Q \eta)|_{\hat{\mathbb{A}}^2 \setminus \tilde{\Delta}}\colon \hat{\mathbb{A}}^2 \setminus \tilde{\Delta} \xrightarrow{\simeq} \mathbb{A}^2$ of Proposition 4.8(1). By construction, $(\psi_Q)^{-1}(C_\mu)$ is equal to $\tilde{\Gamma}_\mu \setminus \tilde{\Delta}$, where $\tilde{\Gamma}_\mu \subset \hat{\mathbb{A}}^2$ is the strict transform of the curve $\Gamma_\mu \subset \mathbb{A}^2$ given by $(a_0)^2 P - \mu Q = 0$ (follows from Equation (I)). The closure of Γ_μ in $\mathbb{P}^2$ is the conic given by

$$(a_0)^2 x^2 - \mu y^2 - z\left(a_0(\mu + a_0 a_2)x - (\mu a_1 + (a_0)^2 a_3)y\right) = 0$$

which passes through $[0:0:1]$ and is irreducible for a general μ. Projecting from the point $[0:0:1]$ one obtains an isomorphism with $\mathbb{P}^1$ (still for a general μ). The curve $\tilde{\Gamma}_\mu \setminus \tilde{\Delta}$ is then isomorphic to $\mathbb{P}^1$ minus three $\overline{\mathrm{k}}$-points of $\tilde{\Delta}$, which are fixed and do not depend on μ, and minus the two points at infinity, which correspond to $(a_0)^2 x^2 - \mu y^2 = 0$.

When the field is algebraically closed, one thus gets that the general fibres of f are isomorphic to $\mathbb{P}^1$ minus 5 points, whereas the zero fibre is isomorphic to $\mathbb{P}^1$ minus 4 points (if $\sum_{i=0}^3 a_i t^i$ is chosen to have three distinct roots). Moreover, the two points of intersection with the line at infinity say that this curve is a *horizontal curve of degree* 2, or a *horizontal curve which is not a section* (in the usual notation of polynomials and components on boundary, see [NN02, AC96, CD16]), so the polynomials f and g are rational but not of simple type (see [NN02, CD16]). When $\mathrm{k} = \mathbb{C}$, this implies that the polynomial has non-trivial monodromy [ACD98, Corollary 2, page 320].

5. Related questions

5.1. **Providing higher dimensional counterexamples.** The negative answer to the Complement Problem for $n = 2$ also yields a negative answer for any $n \geq 3$. This follows from the following easy observation:

Lemma 5.1. *Let $C, D \subset \mathbb{A}^2$ be two closed geometrically irreducible curves having isomorphic complements. Then for each $m \geq 1$, the varieties $H_C = C \times \mathbb{A}^m$ and $H_D = D \times \mathbb{A}^m$ are closed hypersurfaces of $\mathbb{A}^2 \times \mathbb{A}^m = \mathbb{A}^{m+2}$ that have isomorphic complements. Moreover, C and D are isomorphic if and only if $C \times \mathbb{A}^m$ and $D \times \mathbb{A}^m$ are isomorphic.*

Proof. Denoting by $f, g \in \mathrm{k}[x, y]$ the geometrically irreducible polynomials that define the curves C, D, the varieties $H_C, H_D \subset \mathbb{A}^2 \times \mathbb{A}^m = \mathbb{A}^{m+2}$ are given by the same polynomials and are thus again closed geometrically irreducible hypersurfaces. The isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ extends then naturally to an isomorphism $\mathbb{A}^{m+2} \setminus H_C \xrightarrow{\simeq} \mathbb{A}^{m+2} \setminus H_D$.

If C and D are isomorphic, then H_C and H_D are isomorphic. The converse also holds, and is proven in [AHE72, Corollary (3.4)]. □

Corollary 5.2. *For each ground field* k *and each integer* $n \geq 3$*, there exist two geometrically irreducible smooth closed hypersurfaces* $E, F \subset \mathbb{A}^n$ *which are not isomorphic but whose complements* $\mathbb{A}^n \setminus E$ *and* $\mathbb{A}^n \setminus F$ *are isomorphic. Furthermore, the hypersurfaces can be given by polynomials* $f, g \in \mathrm{k}[x_1, x_2] \subset \mathrm{k}[x_1, \ldots, x_n]$ *of degree* 7 *if the field admits more than* 2 *elements and of degree* 13 *if the field has* 2 *elements. The hypersurfaces* E, F *are isomorphic to* $C \times \mathbb{A}^{n-2}$ *and* $D \times \mathbb{A}^{n-2}$ *for some smooth closed curves* $C, D \subset \mathbb{A}^2$ *of the same degree.*

Proof. It suffices to choose for f, g the equations of the curves $C, D \subset \mathbb{A}^2$ given by Theorem 3. The result then follows from Lemma 5.1. □

5.2. **The holomorphic case.**

Lemma 5.3. *For each* $d + 1$ *distinct points* $a_1, \ldots, a_d, a_{d+1} \in \mathbb{C}$*, with* $d \geq 3$*, there exist two closed algebraic curves* $C, D \subset \mathbb{C}^2$ *of degree* $d^2 - d + 1$ *such that* C *and* D *are algebraically isomorphic to* $\mathbb{C} \setminus \{a_1, \ldots, a_{d-1}, a_d\}$ *and* $\mathbb{C} \setminus \{a_1, \ldots, a_{d-1}, a_{d+1}\}$ *respectively, and such that* $\mathbb{C}^2 \setminus C$ *and* $\mathbb{C}^2 \setminus D$ *are algebraically isomorphic.*

In particular, choosing the points general enough, the curves C *and* D *are not biholomorphic, but their complements are biholomorphic.*

Proof. The existence of C, D directly follows from Proposition 4.2. It remains to observe that C and D are not biholomorphic if the points are general enough. If $f : C \to D$ is a biholomorphism, then f extends to a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$, as it cannot have essential singularities. The same holds for f^{-1}, so f is just an element of $\mathrm{PGL}_2(\mathbb{C})$, hence an algebraic automorphism of the projective complex line. Removing at least 4 points of $\mathbb{CP}^1$ (this is the case since $d \geq 3$) and moving one of them gives then infinitely many curves up to biholomorphism. □

Corollary 5.4. *For each* $n \geq 2$*, there are algebraic hypersurfaces* $E, F \subset \mathbb{C}^n$ *which are complex manifolds, not biholomorphic but having biholomorphic complements.*

Proof. It suffices to take polynomials $f, g \in \mathbb{C}[x_1, x_2]$ provided by Lemma 5.3, whose zero sets are smooth algebraic curves $C, D \subset \mathbb{C}^2$ not biholomorphic but having holomorphic complements. We then use the same polynomials to define $E, F \subset \mathbb{C}^n$, which are smooth complex manifolds that have biholomorphic complements and are biholomorphic to $C \times \mathbb{C}^{n-2}$ and $D \times \mathbb{C}^{n-2}$ respectively. It remains to observe that $C \times \mathbb{C}^{n-2}$ and $D \times \mathbb{C}^{n-2}$ are not biholomorphic. Denote by $p_C : C \times \mathbb{C}^{n-2} \to C$ and $p_D : D \times \mathbb{C}^{n-2} \to D$ the projections on the first factor. If $\psi : \mathbb{C}^{n-2} \times C \to \mathbb{C}^{n-2} \times D$ is a biholomorphism, then $p_D \circ \psi : \mathbb{C}^{n-2} \times C \to D$ induces, for each $c \in C$, a holomorphic map $\mathbb{C}^{n-2} \to D$ which has to be constant by Picard's theorem (since it avoids at least two values of $\mathbb{C}$). Therefore, the map $p_D \circ \psi$ factors through a holomorphic map $\chi : C \to D$: we have $p_D \circ \varphi = \chi \circ p_C$.

We get analogously a holomorphic map $\theta\colon D \to C$, which is by construction the inverse of χ, so C and D are biholomorphic, a contradiction. $\square$

Appendix: The case of $\mathbb{P}^2$

In this appendix, we describe some results on the question of complements of curves in $\mathbb{P}^2$ explained in the introduction. These are not directly related to the rest of the text and only serve as comparison with the affine case.

We recall the following simple argument, known to specialists, for lack of reference:

Proposition A.1. *Let $C, D \subset \mathbb{P}^2$ be two geometrically irreducible closed curves such that $\mathbb{P}^2 \setminus C$ and $\mathbb{P}^2 \setminus D$ are isomorphic. If C and D are not equivalent, up to automorphism of $\mathbb{P}^2$, then C and D are singular rational curves.*

Proof. Denote by $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map which restricts to an isomorphism $\mathbb{P}^2 \setminus C \xrightarrow{\simeq} \mathbb{P}^2 \setminus D$. If φ is an automorphism of $\mathbb{P}^2$, then C and D are equivalent. Otherwise, the same argument as in Lemma 2.7 shows that both C and D are rational (this also follows from [Bla09, Lemma 2.2]). If C and D are singular, we are finished, so we can assume that one of them is smooth, and then has degree 1 or 2. The Picard group of $\mathbb{P}^2 \setminus C$ being $\mathbb{Z}/\deg(C)\mathbb{Z}$, one finds that C and D have the same degree. This implies that C and D are equivalent under automorphisms of $\mathbb{P}^2$. The case of lines is obvious. For conics, it is enough to check that a rational conic over any field is necessarily equivalent to the conic of equation $xy+z^2=0$. Actually, one may always assume that the rational conic contains the point $[1:0:0]$, since it contains a rational point. One may furthermore assume that the tangent at this point has equation $y=0$. This means that the equation of the conic is of the form $xy+u(y,z)$, where u is a homogenous polynomial of degree 2. Using a change of variables of the form $(x,y,z) \mapsto (x+ay+bz, y, z)$, where $a, b \in \mathrm{k}$, one may assume that the equation is of the form $xy+cz^2=0$, where $c \in \mathrm{k}^*$. Then, using the change of variables $(x,y,z) \mapsto (cx, y, z)$, one finally gets, as announced, the equation $xy+z^2=0$. $\square$

In order to get families of (singular) curves of $\mathbb{P}^2$ having the same complement, we explicit here the construction of Paolo Costa [Cos12]. We obtain thus unicuspidal curves of $\mathbb{P}^2$ which have isomorphic complements but which are non-equivalent under the action of $\mathrm{Aut}(\mathbb{P}^2)$. We provide equations and give the details of the proof for self-containedness, and also because the results below are not explicitly stated in [Cos12].

Lemma A.2. *Let k be a field. Let $d \geq 1$ be an integer and $P \in \mathrm{k}[x,y]$ be a homogenous polynomial of degree d which is not a multiple of y. We define the homogeneous polynomial $f_P \in \mathrm{k}[x,y,z]$ of degree $4d+1$ by the following formula, where $w := xz - y^2$:*

$$f_P = zw^{2d} + 2yw^d P(x^2, w) + xP^2(x^2, w).$$

Denote by $C_P, \mathcal{L}, \mathcal{Q} \subset \mathbb{P}^2$ the curves of equations $f_P = 0$, resp. $z = 0$, resp. $w = 0$, and by $V_P, V_{\mathcal{L}}, V_{\mathcal{Q}} \subset \mathbb{A}^3$ their corresponding cones (given by the same equations). Then:

(1) *The polynomial f_P is geometrically irreducible (i.e. irreducible in $\overline{\mathrm{k}}[x,y,z]$).*
(2) *The rational map $\psi_P\colon \mathbb{A}^3 \dashrightarrow \mathbb{A}^3$ sending (x,y,z) to*

$$\Big(x, y + xP\left(x^2w^{-1}, 1\right), z + 2yP\left(x^2w^{-1}, 1\right) + xP^2\left(x^2w^{-1}, 1\right)\Big)$$

is a birational map of $\mathbb{A}^3$ *that restricts to isomorphisms*

$$\mathbb{A}^3 \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\mathcal{Q}},\ V_P \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} V_{\mathcal{L}} \setminus V_{\mathcal{Q}} \text{ and } \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\mathcal{L}}).$$

Since ψ_P *is homogeneous, the same formula induces a birational map of* $\mathbb{P}^2$ *that restricts to isomorphisms*

$$\mathbb{P}^2 \setminus \mathcal{Q} \xrightarrow{\simeq} \mathbb{P}^2 \setminus \mathcal{Q},\ C_P \setminus \mathcal{Q} \xrightarrow{\simeq} \mathcal{L} \setminus \mathcal{Q} \text{ and } \mathbb{P}^2 \setminus (\mathcal{Q} \cup C_P) \xrightarrow{\simeq} \mathbb{P}^2 \setminus (\mathcal{Q} \cup \mathcal{L}).$$

Since $[0:0:1]$ *is the unique point of intersection between* C_P *and* $\mathcal{Q}$*, this proves in particular that this point is the unique singular point of* C_P*.*

(3) *Let* λ *be a nonzero element of* k*. Then, the rational map*

$$\varphi_\lambda \colon (x,y,z) \mapsto \big(x + (\lambda-1)wz^{-1}, y, z\big) = (\lambda x - (\lambda-1)y^2z^{-1}, y, z)$$

is a birational map of $\mathbb{A}^3$ *that restricts to automorphisms of* $\mathbb{A}^3 \setminus V_{\mathcal{L}}$*,* $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ *and* $\mathbb{A}^3 \setminus (V_{\mathcal{L}} \cup V_{\mathcal{Q}})$*. The same formula yields then automorphisms of* $\mathbb{P}^2 \setminus \mathcal{L}$*,* $\mathcal{Q} \setminus \mathcal{L}$ *and* $\mathbb{P}^2 \setminus (\mathcal{L} \cup \mathcal{Q})$*.*

(4) *Set* $\tilde{P}(x,y) = P(\lambda x, y)$ *and* $\kappa = (\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P$*. Then, the rational map* κ *restricts to an isomorphism* $\mathbb{A}^3 \setminus V_P \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\tilde{P}}$*. In particular,* κ *also induces an isomorphism* $\mathbb{P}^2 \setminus C_P \xrightarrow{\simeq} \mathbb{P}^2 \setminus C_{\tilde{P}}$*.*

(5) *For each homogeneous polynomial* $\tilde{P} \in \mathrm{k}[x,y]$ *of degree* d *which is not divisible by* y*, the curves* C_P *and* $C_{\tilde{P}}$ *are equivalent up to automorphisms of* $\mathbb{P}^2$*, if and only if there exist some constants* $\rho \in \mathrm{k}^*, \mu \in \mathrm{k}$ *such that*

$$\tilde{P}(x,y) = \rho P(\rho^2 x, y) + \mu y^d.$$

Proof. (1)-(2): As each rational map $\mathbb{A}^3 \dashrightarrow \mathbb{A}^3$, the rational map ψ_P yields a morphism of k-algebras $(\psi_P)^* \colon \mathrm{k}[x,y,z] \to \mathrm{k}(x,y,z)$. This sends x, y, z onto $x, y + xP(x^2w^{-1}, 1), z + 2yP(x^2w^{-1},1) + xP^2(x^2w^{-1},1)$. One also observes that $(\psi_P)^*$ fixes x and w. This implies that $(\psi_P)^*$ extends to an endomorphism of $\mathrm{k}[x,y,z,w^{-1}]$, which is moreover an automorphism since $(\psi_P)^* \circ (\psi_{-P})^* = \mathrm{id}$. Extending to the quotient field $\mathrm{k}(x,y,z)$, one gets an automorphism of $\mathrm{k}(x,y,z)$, that we again denote by $(\psi_P)^*$, so ψ_P is a birational map of $\mathbb{A}^3$ and induces moreover an isomorphism of $\mathbb{A}^3 \setminus V_{\mathcal{Q}}$, because $(\psi_P)^*(\mathrm{k}[x,y,z,w^{-1}]) = \mathrm{k}[x,y,z,w^{-1}]$. One then observes that $(\psi_P)^*(z) = f_P w^{-2d}$ where f_P and $w = xz - y^2$ are coprime since $f_P(1,0,0) = P^2(1,0) \neq 0$. Let us also observe that $V_P \cap V_{\mathcal{Q}} = \{(x,y,z) \in \mathbb{A}^3 \mid x = y = 0\}$ and that $V_{\mathcal{L}} \cap V_{\mathcal{Q}} = \{(x,y,z) \in \mathbb{A}^3 \mid y = z = 0\}$. Hence ψ_P restricts to an isomorphism of surfaces $V_P \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} V_{\mathcal{L}} \setminus V_{\mathcal{Q}}$. This implies that V_P and C_P are rational, and that f_P is geometrically irreducible, proving (1). This also implies that ψ_P restricts to an isomorphism $\mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\mathcal{L}})$. As ψ_P is homogeneous, one gets the analogous results by replacing $\mathbb{A}^3$, V_P, $V_{\mathcal{L}}$, $V_{\mathcal{Q}}$ by $\mathbb{P}^2$, C_P, $\mathcal{L}$, $\mathcal{Q}$ respectively.

(3): One checks that $\varphi_\lambda \circ \varphi_{\lambda^{-1}} = \mathrm{id}$, so φ_λ is a birational map of $\mathbb{A}^3$, which restricts to an automorphism of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, since the denominators only involve z. Moreover, $(\varphi_\lambda)^*(w) = \lambda w$ (where $(\varphi_\lambda)^*$ is the automorphism of $\mathrm{k}(x,y,z)$ corresponding to φ_λ), so the surface $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ is preserved, hence φ_λ restricts to automorphisms of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ and $\mathbb{A}^3 \setminus (V_{\mathcal{L}} \cup V_{\mathcal{Q}})$. Since φ_λ is homogeneous, the same formula yields then automorphisms of $\mathbb{P}^2 \setminus \mathcal{L}$, $\mathcal{Q} \setminus \mathcal{L}$ and $\mathbb{P}^2 \setminus (\mathcal{L} \cup \mathcal{Q})$.

(4): By (2)-(3), the transformation $\kappa = (\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P$ restricts to an isomorphism $\mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\tilde{P}})$. Let us prove that with the special choice of $\tilde{P}$ that we have made, then κ restricts to an isomorphism $\mathbb{A}^3 \setminus V_P \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\tilde{P}}$. For this, we prove that the restriction of κ yields the identity automorphism on $V_{\mathcal{Q}} \setminus V_P = V_{\mathcal{Q}} \setminus V_{\tilde{P}} = V_{\mathcal{Q}} \setminus \{(x,y,z) \in \mathbb{A}^3 \mid x = y = 0\}$. We compute

$$\varphi_\lambda \psi_P(x,y,z) = \left(x + (\lambda-1)w^{2d+1}f_P^{-1}, y + xP(x^2,w)w^{-d}, f_P w^{-2d}\right)$$

which satisfies $(\varphi_\lambda\psi_P)^*(w) = (\varphi_\lambda)^*(w) = \lambda w$. To simplify the notation, we write $\delta = (\lambda-1)w^{2d+1}f_P^{-1}$ and get that $\kappa(x,y,z) = (\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P(x,y,z)$ is equal to

$$\left(x+\delta, y + xP(x^2,w)w^{-d} - (x+\delta)\tilde{P}\left(\lambda^{-1}(x+\delta)^2 w^{-1}, 1\right), z + \zeta\right)$$

for some $\zeta \in \mathrm{k}(x,y,z)$. Since $\tilde{P}(x,y) = P(\lambda x, y)$, the second component is

$$\kappa^*(y) = y + \frac{xP(x^2,w) - P((x+\delta)^2,w)(x+\delta)}{w^d}.$$

As w^{d+1} divides the numerator of δ, one can write $\kappa^*(y)$ as $y + w(f_P)^{-n}R$, for some $R \in \mathrm{k}[x,y,z]$ and $n \geq 0$. Similarly, $\kappa^*(x) = x + wf_P^{-1}S$, where $S \in \mathrm{k}[x,y,z]$. Since $\kappa^*(w) = \lambda w$, we get

$$\lambda w = (x + wf_P^{-1}S)(z+\zeta) - (y + wf_P^{-n}R)^2$$

which shows that $\zeta(x + wf_P^{-1}S) = wf_P^{-\tilde{m}}\tilde{T}$ for some $\tilde{T} \in \mathrm{k}[x,y,z]$, $\tilde{m} \geq 0$, hence one can write $\kappa^*(z) = z + \zeta = z + wf_P^{-m}T$ for some $T \in \mathrm{k}[x,y,z]$ and $m \geq 0$. This shows that κ is well defined on $V_{\mathcal{Q}} \setminus V_P = V_{\mathcal{Q}} \setminus V_{\tilde{P}} = V_{\mathcal{Q}} \setminus \{(x,y,z) \in \mathbb{A}^3 \mid x = y = 0\}$ and restricts to the identity on this surface.

Since κ is homogeneous, the isomorphism $\mathbb{A}^3 \setminus V_P \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\tilde{P}}$ also induces an isomorphism $\mathbb{P}^2 \setminus C_P \xrightarrow{\simeq} \mathbb{P}^2 \setminus C_{\tilde{P}}$, which fixes pointwise the curve $\mathcal{Q} \setminus C_P = \mathcal{Q} \setminus C_{\tilde{P}}$.

(5): Suppose first that $\tilde{P}(x,y) = \rho P(\rho^2 x, y) + \mu y^d$ for some $\rho \in \mathrm{k}^*, \mu \in \mathrm{k}$. Define the transformation $\alpha \in \mathrm{GL}_3(\mathrm{k})$ by

$$\alpha(x,y,z) = (x, \rho y - \mu x, \rho^2 z - 2\rho\mu y + \mu^2 x)$$

and the birational transformation $s \in \mathrm{Bir}(\mathbb{A}^3)$ by $s = \psi_{\tilde{P}}\alpha(\psi_P)^{-1}$. Let us note that $s^* = (\psi_P^*)^{-1}\alpha^*\psi_{\tilde{P}}^*$. One checks that $\alpha^*(w) = \rho^2 w$, from which we get $s^*(w) = \rho^2 w$. The equality

$$\begin{aligned} \alpha^*(\psi_{\tilde{P}}^*(y)) &= \alpha^*(y + x\tilde{P}(x^2w^{-1},1)) = \rho y - \mu x + x\tilde{P}(\rho^{-2}x^2w^{-1},1) \\ &= \rho y + \rho x P(x^2w^{-1},1) = \rho\, \psi_P^*(y) \end{aligned}$$

gives us $s^*(y) = \rho y$. The relation $z = x^{-1}(w - y^2)$ joined to the equality $s^*(x) = x$ now proves us that $s^*(z) = \rho^2 z$. But we have $(\psi_P)^*(z) = f_P w^{-2d}$ and $(\psi_{\tilde{P}})^*(z) = f_{\tilde{P}} w^{-2d}$, so that we get $\alpha^*(f_{\tilde{P}} w^{-2d}) = \rho^2 f_P w^{-2d}$. In turn, this latter equality yields us

$$\alpha^*(f_{\tilde{P}}) = \rho^{4d+2} f_P.$$

This shows that α induces an automorphism of $\mathbb{P}^2$ sending C_P onto $C_{\tilde{P}}$.

Conversely, suppose that there exists $\tau \in \mathrm{Aut}(\mathbb{P}^2)$ sending C_P onto $C_{\tilde{P}}$.

We begin by proving that τ preserves the conic $\mathcal{Q}$. Since $C_P \setminus \mathcal{Q} \simeq C_{\tilde{P}} \setminus \mathcal{Q} \simeq L \setminus \mathcal{Q} \simeq \mathbb{A}^1$, the irreducible conic $\mathcal{Q} \subset \mathbb{P}^2$ intersects C_P (respectively $C_{\tilde{P}}$) in exactly one $\overline{\mathrm{k}}$-point, the unique singular point $[0:0:1]$ of C_P (resp. $C_{\tilde{P}}$). The irreducible conic $\tau(\mathcal{Q})$ intersects thus also $C_{\tilde{P}}$ into one $\overline{\mathrm{k}}$-point, namely $[0:0:1]$. Let us observe that this implies that $\tau(\mathcal{Q}) = \mathcal{Q}$. One first notices that $C_{\tilde{P}} \setminus \{[0:0:1]\} \simeq \mathbb{A}^1$, so there is one k-point at each step of the resolution of $C_{\tilde{P}}$. We can then write $q_1 = [0:0:1]$ and define a sequence of points $(q_i)_{i\geq 1}$ such that q_i is the point infinitely near q_{i-1} belonging to the strict transform of $C_{\tilde{P}}$, for each $i \geq 2$. Denote by r the biggest integer such that q_r belongs to the strict transform of $\mathcal{Q}$ and by r' the biggest integer such that $q_{r'}$ belongs to the strict transform of $\tau(\mathcal{Q})$. Bézout Theorem yields (since $\mathcal{Q}$ and $\tau(\mathcal{Q})$ are smooth)

$$\sum_{i=1}^{r} m_{q_i}(C_{\tilde{P}}) = \deg(\mathcal{Q})\deg(C_{\tilde{P}}) = \deg(\tau(\mathcal{Q}))\deg(C_{\tilde{P}}) = \sum_{i=1}^{r'} m_{q_i}(C_{\tilde{P}}),$$

which yields $r = r'$. On the blow-up $X \to \mathbb{P}^2$ of $q_1, \ldots, q_r$, the strict transform of the curve $C_{\tilde{P}}$ is then disjoint from those of $\mathcal{Q}$ and $\tau(\mathcal{Q})$, which are linearly equivalent. Assume by contradiction that we have $\tau(\mathcal{Q}) \neq \mathcal{Q}$. Then, we claim that the strict transform of any irreducible conic $\mathcal{Q}'$ in the pencil generated by $\mathcal{Q}$ and $\tau(\mathcal{Q})$ is also disjoint from the strict transform of $C_{\tilde{P}}$. Indeed, we first note that $C_{\tilde{P}}$ and $\mathcal{Q}'$ have no common irreducible component since $C_{\tilde{P}}$ is an irreducible curve whose degree satisfies

$$\deg C_{\tilde{P}} \geq 5 > 2 = \deg \mathcal{Q}'.$$

Finally, since the (infinitely near) points $q_1, \ldots, q_r$ belong to both $\mathcal{Q}'$ and $C_{\tilde{P}}$ and since we have the equality $\sum_{i=1}^{r} m_{q_i}(C_{\tilde{P}}) = \deg(\mathcal{Q}')\deg(C_{\tilde{P}})$, the curves $\mathcal{Q}'$ and $C_{\tilde{P}}$ do not have any other common (infinitely near) point.

Choose now a general point q of $\mathbb{P}^2$ which belong to $C_{\tilde{P}} \setminus \{q_1\} \simeq \mathbb{A}^1$ and choose the conic $\mathcal{Q}'$ in the pencil generated by $\mathcal{Q}$ and $\tau(\mathcal{Q})$ which passes through q. Then, the strict transforms of $\mathcal{Q}'$ and $C_{\tilde{P}}$ intersect in X (at the point q). This contradiction shows that $\mathcal{Q}$ is preserved by τ.

Since $\tau \in \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathrm{k})$ fixes the point $[0:0:1]$ (which is the unique singular point of both C_P and $C_{\tilde{P}}$) and preserves the line $x = 0$ (which is the tangent line of both C_P and $C_{\tilde{P}}$ at the point $[0:0:1]$), it admits a (unique) lift $\alpha \in \mathrm{GL}_3(\mathrm{k})$ which is triangular and which satisfies $\alpha^*(x) = x$. This means that α is of the form:

$$\alpha\colon (x, y, z) \mapsto (x, \rho y - \mu x, \gamma z + \delta y + \varepsilon x),$$

for some constants $\rho, \mu, \gamma, \delta, \varepsilon \in \mathrm{k}$ (satisfying $\rho\gamma \neq 0$). Since $\alpha^*(w)$ is proportional to w, we get $\gamma = \rho^2$, $\delta = -2\rho\mu$ and $\varepsilon = \mu^2$, i.e. α is of the form

$$\alpha\colon (x, y, z) \mapsto (x, \rho y - \mu x, \rho^2 z - 2\rho\mu y + \mu^2 x).$$

Set $s := \psi_{\tilde{P}}\alpha(\psi_P)^{-1} \in \mathrm{Bir}(\mathbb{A}^3)$. Since $\alpha^*(w) = \rho^2 w$, we also get $s^*(w) = \rho^2 w$. Since $(\psi_P)^*(z) = f_P w^{-2d}$, $(\psi_{\tilde{P}})^*(z) = f_{\tilde{P}} w^{-2d}$ and since $\alpha^*(f_{\tilde{P}})$ and f_P are proportional, the fractions $s^*(z)$ and z are also proportional. Therefore, there exists a nonzero constant $\xi \in \mathrm{k}$ such that

$$\text{(J)} \qquad s^*(x) = x, \quad s^*(w) = \rho^2 w, \quad s^*(z) = \xi z.$$

Moreover, s induces a birational map $\hat{s}$ of $\mathbb{P}^2$ which is an automorphism of $\mathbb{P}^2 \setminus \mathcal{Q}$, because the same holds for α, ψ_P and $\psi_{\tilde{P}}$. Let us observe that $\hat{s}$ is in fact an automorphism of $\mathbb{P}^2$. Indeed, otherwise $\hat{s}$ would contract $\mathcal{Q}$ to one point. This is impossible: Since $\hat{s}$ preserves the two pencils of conics given by $[x:y:z] \mapsto [w:x^2]$ and $[x:y:z] \mapsto [w:z^2]$, which have distinct base-points $[0:0:1]$ and $[1:0:0]$, these base-points are fixed by $\hat{s}$. Hence, there exist some constants $\zeta, \eta, \theta \in \mathrm{k}$ such that

$s^*(y) = \zeta x + \eta y + \theta z$. Hence ($J$) gives us $\zeta = \theta = 0$, i.e. $s^*(y) = \eta y$. But the equality $s = \psi_{\tilde{P}}\alpha(\psi_P)^{-1}$ yields us $\psi_{\tilde{P}}\alpha = s\psi_P$ and by taking the second coordinate we get

$$(\rho y - \mu x) + x\tilde{P}\left(\rho^{-2}x^2w^{-1}, 1\right) = (\psi_{\tilde{P}}\alpha)^*(y) = (s\psi_P)^*(y) = \eta\left(y + xP\left(x^2w^{-1}, 1\right)\right)$$

which yields $\rho = \eta$ and $\tilde{P}(\rho^{-2}x^2w^{-1}, 1) = \rho P(x^2w^{-1}, 1) + \mu$. By substituting $\rho^{-2}y + x^{-1}y^2$ for z and by noting that $w(x, y, \rho^{-2}y + x^{-1}y^2) = \rho^{-2}xy$, we obtain $\tilde{P}(xy^{-1}, 1) = \rho P(\rho^2 xy^{-1}, 1) + \mu$, which is equivalent to $\tilde{P}(x, y) = \rho P(\rho^2 x, y) + \mu y^d$, as we wanted. □

The construction of Lemma A.2 yields, for each $d \geq 1$, families of curves of degree $4d + 1$ having isomorphic complements. These are equivalent for $d = 1$, at least when k is algebraically closed (Lemma A.2(5)), but not for $d \geq 2$. One can now easily provide explicit examples:

Corollary A.3. *Let $d \geq 2$ be an integer. Set $P = x^d + x^{d-1}y$ and $w = xz - y^2 \in \mathrm{k}[x, y]$. All curves of $\mathbb{P}^2$ given by*

$$zw^{2d} + 2yw^dP(\lambda x^2, w) + xP^2(\lambda x^2, w) = 0$$

for $\lambda \in \mathrm{k}^*$, have isomorphic complements and are pairwise not equivalent up to automorphisms of $\mathbb{P}^2$.

Proof. The curves correspond to the curves $C_{P(\lambda x,y)}$ of Lemma A.2 and have thus isomorphic complements by Lemma A.2(4). It remains to show that if $C_{P(\lambda x,y)}$ is equivalent to $C_{P(\tilde{\lambda}x,y)}$, then $\lambda = \tilde{\lambda}$. Lemma A.2(4) yields the existence of $\rho \in \mathrm{k}^*, \mu \in \mathrm{k}$ such that $P(\tilde{\lambda}x, y) = \rho P(\rho^2\lambda x, y) + \mu y^d$. Since $d \geq 2$, both $P(\tilde{\lambda}x, y)$ and $\rho P(\rho^2\lambda x, y)$ do not have component with y^d, so $\mu = 0$. We then compare the coefficients of x^d and $x^{d-1}y$ and get

$$\tilde{\lambda}^d = \rho(\rho^2\lambda)^d, \quad \tilde{\lambda}^{d-1} = \rho(\rho^2\lambda)^{d-1},$$

which yields $\tilde{\lambda} = \rho^2\lambda$, whence $\rho = 1$ and $\tilde{\lambda} = \lambda$ as desired. □

References

[AHE72] Abhyankar Shreeram Shankar, Heinzer William, Eakin Paul, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra 23 (1972), 310–342. 5.1

[AC96] Artal-Bartolo Enrique, Cassou-Nogués Pierrette, *One remark on polynomials in two variables*, Pacific J. Math. **176** (1996), no. 2, 297–309. 4.9

[ACD98] Artal-Bartolo Enrique, Cassou-Nogués Pierrette and Dimca Alexandru, *Sur la topologie des polynômes complexes*, Progress in Mathematics, **162** (1998), 317–343. 4.9

[Bla09] Blanc Jérémy, *The correspondence between a plane curve and its complement*, J. Reine Angew. Math. 633 (2009), 1–10. 1, 5.2

[BS15] Blanc Jérémy, Stampfli Immanuel, *Automorphisms of the plane preserving a curve*, Algebr. Geom. 2 (2015), no. 2, 193–213. 2.9, 2.3

[CD16] Cassou-Noguès Pierrette, Daigle Daniel, *Rational polynomials of simple type: a combinatorial proof*, to appear in Proceedings of Kyoto Workshop 2014, Advanced Studies in Pure Mathematics (ASPM). 4.9

[Cos12] Costa Paolo, *New distinct curves having the same complement in the projective plane*, Math. Z. 271 (2012), no. 3-4, 1185–1191. 1, 5.2

[Dai91] Daigle Daniel, *Birational endomorphisms of the affine plane*, J. Math. Kyoto Univ. **31** (1991), no. 2, 329–358. 2.4

[Gan85] Ganong Richard, *Kodaira dimension of embeddings of the line in the plane*, J. Math. Kyoto Univ. **25** (1985), no. 4, 649–657. 2.30

[Kam79] Kambayashi Tatsuji, *Automorphism group of a polynomial ring and algebraic group action on an affine space*, J. Algebra 60 (1979), no. 2, 439–451. 2.9

[KM83] Kumar Mohan, Murthy Pavaman, *Curves with negative self intersection on rational surfaces*, J. Math. Kyoto Univ. **22** (1983), no. 4, 767–777. 2.30

[Kra96] Kraft Hanspeter, *Challenging problems on affine n-space*, Séminaire Bourbaki, Vol. 1994/95. Astérisque No. 237 (1996), Exp. No. 802, 5, 29–317. (document), 1, 3.6

[NN02] Neumann Walter D., Norbury Paul, *Rational polynomials of simple type*, Pacific J. Math. **204** (2002), no. 1, 177–207. 4.9

[Pol16] Poloni Pierre-Marie, *Counterexamples to the Complement Problem*, https://arxiv.org/abs/1605.05169 1, 1, 1

[Rus70] Russell Peter, *Forms of the affine line and its additive group*, Pacific J. Math. **32** 1970 527–539. 2.31

Jérémy Blanc, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland

E-mail address: `jeremy.blanc@unibas.ch`

Jean-Philippe Furter, Dpt. of Math., Univ. of La Rochelle, av. Crépeau, 17000 La Rochelle, France

E-mail address: `jpfurter@univ-lr.fr`

Mattias Hemmig, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland

E-mail address: `mattias.hemmig@gmail.com`