manuscripta math. 119, 225-241 (2006)

Jérémy Blanc

Conjugacy classes of affine automorphisms of $\mathbb{K}^n$ and linear automorphisms of $\mathbb{P}^n$ in the Cremona groups

Received: 30 September 2005 / Revised version: 9 November 2005
Published online: 23 January 2006

Abstract. We describe the conjugacy classes of affine automorphisms in the group $Aut(n, \mathbb{K})$ (respectively $Bir(\mathbb{K}^n)$) of automorphisms (respectively of birational maps) of $\mathbb{K}^n$. From this we deduce also the classification of conjugacy classes of automorphisms of $\mathbb{P}^n$ in the Cremona group $Bir(\mathbb{K}^n)$.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and let $\mathbb{K}^n$ and $\mathbb{P}^n$ denote respectively the affine and projective n-spaces over $\mathbb{K}$.

The *Cremona group*, which is the group of birational maps of these two spaces, $Bir(\mathbb{K}^n) = Bir(\mathbb{P}^n)$, has been studied a lot, especially in dimension 2 and 3, see for example [Hud] and [AlC]. Its subgroup of biregular morphisms (or automorphisms) of $\mathbb{K}^n$, called the *affine Cremona group* $Aut(n, \mathbb{K})$, has been also much explored. We refer to [Kra] for a list of references.

In both cases, the question of the conjugacy classes of elements is natural. For $Bir(\mathbb{P}^2)$, the classical approach can be found in [Kan] and [Wim]. A modern classification of birational morphisms of prime order was completed in [BaB], [DeF] and [BeB]. We refer to [KrS] and their references for the group $Aut(n, \mathbb{K})$.

In the literature, the affine and projective cases are often treated separately, with different methods, although the groups are very close as we can see in the following diagram:

$$\begin{array}{ccccc} & PGL(n+1,\mathbb{K}) & \subset & & Bir(\mathbb{P}^n) \\ & \cup & & & \| \\ GL(n,\mathbb{K}) \subset & Aff(n,\mathbb{K}) & \subset Aut(n,\mathbb{K}) & \subset & Bir(\mathbb{K}^n). \end{array}$$

In this paper, we restrict ourselves neither to small dimensions nor to finite elements, but to the case of maps of degree 1. We give the conjugacy classes, in the Cremona groups, of automorphisms of $\mathbb{K}^n$ and $\mathbb{P}^n$ that maps lines to lines. Explicitely these are the group $Aff(n, \mathbb{K})$ of affine automorphisms of $\mathbb{K}^n$ and the group $PGL(n + 1, \mathbb{K})$ of automorphisms of $\mathbb{P}^n$ (or linear birational maps). These two groups are very classical and studied in many domains of mathematics.

J. Blanc: Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, CP 64, 1211 Genève 4, Switzerland. e-mail: Jeremy.Blanc@math.unige.ch

DOI: 10.1007/s00229-005-0617-7

More precisely, *the goal of this paper is to find under which conditions two affine automorphisms of* $\mathbb{K}^n$ *are conjugate in* $Aff(n,\mathbb{K})$ *itself, in* $Aut(n,\mathbb{K})$*, or in* $Bir(\mathbb{K}^n)$. For this purpose, we describe the trace on $Aff(n,\mathbb{K})$ of the conjugacy classes of an affine automorphism in $Aff(n,\mathbb{K})$, $Aut(n,\mathbb{K})$ and $Bir(\mathbb{K}^n)$, respectively in Sections 2, 3, and 4.

The similar question in the projective case is to find under which conditions two linear automorphisms of $\mathbb{P}^n$ are conjugate in the group $Bir(\mathbb{P}^n)$. This will be answered in Section 5.

Let $Aut(\mathcal{T}^n) = GL(n,\mathbb{Z})$ denote the group of automorphisms of the group $\mathcal{T}^n = (\mathbb{K}^*)^n$. As $\mathbb{K}^n$ contains $\mathcal{T}^n$ as an open subset, we get a natural injection $T : GL(n,\mathbb{Z}) \to Bir(\mathbb{K}^n)$ which image normalizes $D(n,\mathbb{K})$, the subgroup of $GL(n,\mathbb{K})$ made up of *diagonal automorphisms* $(x_1,\dots,x_n) \mapsto (\alpha_1 x_1,\dots,\alpha_n x_n)$, with $\alpha_i \in \mathbb{K}^*, i = 1,\dots,n$. Identifying $GL(n,\mathbb{Z})$ with its image, the group of monomial birational maps with coefficients 1, we will see $GL(n,\mathbb{Z})$ as a subgroup of $Bir(\mathbb{K}^n)$. For a further description of the inclusion $GL(n,\mathbb{Z}) \subset Bir(\mathbb{K}^n)$, see [GoP]. *(Note that there is another natural inclusion from* $GL(n,\mathbb{Z})$ *to* $GL(n,\mathbb{K}) \subset Bir(\mathbb{K}^n)$*, but we won't use this one in this paper.)*

Among the affine transformations of $\mathbb{K}^n$, we distinguish those that we call *almost-diagonal automorphisms*: namely maps of the form $(x_1,\dots,x_n) \mapsto (x_1 + 1, \alpha_2 x_2,\dots,\alpha_n x_n)$. The $\alpha_i \in \mathbb{K}^*, i = 2,\dots,n$ will be called the *eigenvalues* of the map. It is clear that $Aut(\mathcal{T}^{n-1}) = GL(n-1,\mathbb{Z})$ normalizes the set $AD(n,\mathbb{K})$ of almost-diagonal automorphisms.

We can now extend our diagram:

$$\begin{array}{ccccccccc}
 & & PGL(n+1,\mathbb{K}) & \subset & & & Bir(\mathbb{P}^n) & & \\
 & & \cup & & & & \| & & \\
GL(n,\mathbb{K}) & \subset & Aff(n,\mathbb{K}) & \subset & Aut(n,\mathbb{K}) & \subset & Bir(\mathbb{K}^n) & \supset Aut(\mathcal{T}^n) = & GL(n,\mathbb{Z}) \\
\cup & & \cup & & & & & \cup & \cup \\
D(n,\mathbb{K}) & & AD(n,\mathbb{K}) & & & & & Aut(\mathcal{T}^{n-1}) = & GL(n-1,\mathbb{Z}).
\end{array}$$

We will prove the following results:

Theorem 1 (Conjugacy classes of affine automorphisms of $\mathbb{K}^n$)**.**

1. *In the affine Cremona group* $Aut(n,\mathbb{K})$ *(Section 3)*
   - *The conjugacy classes of affine automorphisms that fix a point are given by their Jordan normal form, as in* $GL(n,\mathbb{K})$.
   - *Any affine automorphism that fix no point is conjugate to an almost-diagonal automorphism, unique up to a permutation of its eigenvalues.*
2. *In the Cremona group* $Bir(\mathbb{K}^n)$ *(Section 4)*
   - *Any affine automorphism of* $\mathbb{K}^n$ *is conjugate, either to a diagonal, or to an almost-diagonal automorphism of* $\mathbb{K}^n$*, exclusively.*
   - *The conjugacy classes of diagonal and almost-diagonal automorphisms of* $\mathbb{K}^n$ *are respectively given by the orbits of the actions of* $GL(n,\mathbb{Z})$ *and* $GL(n-1,\mathbb{Z})$ *by conjugation. These actions correspond respectively to the natural actions of* $Aut(\mathcal{T}^n)$ *and* $Aut(\mathcal{T}^{n-1})$ *on* $\mathcal{T}^n$ *and* $\mathcal{T}^{n-1}$.
   - *In particular, if* $n > 1$*, two affine automorphisms of the same finite order are conjugate in* $Bir(\mathbb{K}^n)$.

From theorem 1 we can deduce the analogue result on $\mathbb{P}^n$, which is the following Theorem (Section 5). We define *almost-diagonal automorphism of* $\mathbb{P}^n$ to be a map of the form $(x_0 : \ldots : x_n) \mapsto (x_0 : x_0 + x_1 : \alpha_2 x_2 : \ldots : \alpha_n x_n)$, with $\alpha_2, \ldots, \alpha_n \in \mathbb{K}^*$.

Theorem 2 (Conjugacy classes of automorphisms of $\mathbb{P}^n$ in the Cremona group $Bir(\mathbb{P}^n)$)**.**

- *Any automorphism of $\mathbb{P}^n$ is conjugate, either to a diagonal or to an almost-diagonal automorphism of $\mathbb{P}^n$, exclusively.*
- *The conjugacy classes of diagonal and almost-diagonal automorphisms of $\mathbb{P}^n$ are respectively given by the orbits of the actions of $GL(n, \mathbb{Z})$ and $GL(n-1, \mathbb{Z})$ by conjugation. These actions correspond respectively to the natural actions of $Aut(\mathcal{T}^n)$ and $Aut(\mathcal{T}^{n-1})$ on $\mathcal{T}^n$ and $\mathcal{T}^{n-1}$.*
- *In particular, if $n > 1$, two automorphisms of $\mathbb{P}^n$ of the same finite order are conjugate in $Bir(\mathbb{K}^n)$, (see [BeB], Proposition 2.1.).*

2. Conjugacy classes of $Aff(n, \mathbb{K})$

Denote by $\mathbb{K}[X] = \mathbb{K}[X_1, \ldots, X_n]$ the polynomial ring in the variables $X_1, \ldots, X_n$ over $\mathbb{K}$ and by $\mathbb{K}(X) = \mathbb{K}(X_1, \ldots, X_n)$ its field of fractions. Elements of $Bir(\mathbb{K}^n)$ can be written in the form $\varphi = (\varphi_1, \ldots, \varphi_n)$, where each φ_i belongs to $\mathbb{K}(X)$. Any birational map $\varphi \in Bir(\mathbb{K}^n)$ induces a map

$$\varphi^* : F \mapsto F \circ \varphi, \; (F \in \mathbb{K}(X))$$

which is a $\mathbb{K}$-automorphism of the field $\mathbb{K}(X)$. Conversely, any $\mathbb{K}$-automorphism of $\mathbb{K}(X)$ is of this form. So, $Bir(\mathbb{K}^n)$ is anti-isomorphic to the group of $\mathbb{K}$-automorphisms of $\mathbb{K}(X)$ and its subgroups $Aut(n, \mathbb{K})$, $Aff(n, \mathbb{K})$ and $GL(n, \mathbb{K})$ corresponds respectively to the groups of $\mathbb{K}$-automorphisms of $\mathbb{K}[X]$, $\mathbb{K}[X]_{\leq 1}$ and $\mathbb{K}[X]_1$. Here, $\mathbb{K}[X]_{\leq 1}$ and $\mathbb{K}[X]_1$ denote respectively the sets of polynomials of degree ≤ 1 and egal to 1.

The study of the conjugacy classes of $Aff(n, \mathbb{K})$ is elementary and well-known. Let $\alpha, \beta \in Aff(n, \mathbb{K})$; let us recall that the first dichotomy consists in separating the cases according to whether α and β fix a point or not, since an affine automorphism that fixes a point cannot be conjugate to one with no fixed point.

If both α and β fix a point, they are respectively conjugate to linear automorphisms α' and β' of $\mathbb{K}^n$ (elements of $GL(n, \mathbb{K})$), and are then conjugate if and only if these have the same Jordan normal form. We will say that these Jordan normal forms are also the *Jordan normal forms* of α and β. That doesn't depend on the choice of α' and β'.

We will extend this idea to the case of affine automorphisms with no fixed point. Suppose that α has no fixed point. We consider a basis $(1, P_1, \ldots, P_n)$ of $\mathbb{K}[X]_{\leq 1}$ such that the matrix of $\alpha^*|_{\mathbb{K}[X]_{\leq 1}}$ has the Jordan normal form

$$\begin{pmatrix} J(\lambda_1, k_1) & & \\ & \ddots & \\ & & J(\lambda_r, k_r) \end{pmatrix},$$

where $J(\mu,k)=\begin{pmatrix}\mu & 1 & & \\ & \ddots & \ddots & \\ & & \mu & 1\\ & & & \mu\end{pmatrix}\in GL(k,\mathbb{K})$ is a Jordan block of size k. Observe that $\lambda_1=1$ and $k_1>1$, as α fixes no point.

Then, $\pi:(x_1,\dots,x_n)\mapsto(P_1(x_1,\dots,x_n),\dots,P_n(x_1,\dots,x_n))$ is an affine automorphism of $\mathbb{K}^n$ such that

$$\pi\alpha\pi^{-1}:\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\mapsto {}^t\!\begin{pmatrix}J(\lambda_1,k_1-1) & & \\ & \ddots & \\ & & J(\lambda_r,k_r)\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}+\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}; \quad (1)$$

by analogy, we will also say that (1) is the *Jordan normal form of* α.

By observing that an automorphism given by formula (1) is of infinite order (since $char(\mathbb{K})=0$), we see that any affine automorphism of finite order has a fixed point, which is true in general for any automorphism of finite order (see [KrS]).

Looking at the linear action of α^* on $\mathbb{K}[X]_{\leq 1}$, we observe that the pairs (λ_i,k_i) characterize the conjugacy class of α. We have thus showed the following proposition:

Proposition 1 (Conjugacy classes in $Aff(n,\mathbb{K})$)**.** *Two affine automorphisms of* $\mathbb{K}^n$ *are conjugate if and only if they have the same Jordan normal form.* □

3. Conjugation in the group $Aut(n,\mathbb{K})$

It is clear that the conjugation in $Aut(n,\mathbb{K})$ respects the dichotomy of the existence or not of a fixed point.

When there exists a fixed point, passing from the group $Aff(n,\mathbb{K})$ to $Aut(n,\mathbb{K})$ does not bring anything new:

Proposition 2. *Two affine automorphisms of* $\mathbb{K}^n$ *that fix a point are conjugate in* $Aut(n,\mathbb{K})$ *if and only if they are already conjugate in* $Aff(n,\mathbb{K})$.

Proof. Let α, β be two affine automorphisms which have fixed points. Changing α and β within their $Aff(n,\mathbb{K})$-conjugacy classes, we can suppose that α and β belong to $GL(n,\mathbb{K})$. Let $\pi\in Aut(n,\mathbb{K})$ be such that $\pi\alpha=\beta\pi$ and let $\rho\in GL(n,\mathbb{K})$ denote the tangent map of π at the origin; then $\rho\alpha=\beta\rho$. □

On the other hand, if there are no fixed points, by means of elements of $Aut(n,\mathbb{K})$, it is possible to modify the size of the Jordan blocks as the following shows:

Example 1. The affine automorphisms

$$\alpha:\begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}$$
$$\beta:\begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}.$$

are conjugate by the automorphism $\pi : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 - \frac{x_1(x_1-1)}{2} \end{pmatrix}$.

Geometrically, the affine automorphisms α and β on $\mathbb{K}^2$ respectively leave invariant the conics C_t given by $x_2 + \frac{x_1 - x_1{}^2}{2} = t$ and the lines L_t given by $x_2 = t$ where $t \in \mathbb{K}$; the action of these automorphisms on this curves is just a translation by 1 on the axis x_1. The automorphism π sends every conic C_t on the line L_t without changing the coordinate x_1, and so sends the orbits of α on those of β.

Example 2. The affine automorphisms

$$\alpha : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\beta : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

are conjugate by the automorphism

$$\pi : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 + P_2(x_1) \\ x_3 + P_3(x_1, x_2) \\ \vdots \\ x_n + P_n(x_1, \dots, x_{n-1}) \end{pmatrix},$$

where $P_m \in \mathbb{K}[X_1, \dots, X_{m-1}]$ is defined by the formula

$$P_m = \sum_{k=1}^{m-2} \binom{x_1 + k - 1}{k} (-1)^k x_{m-k} + (-1)^{m-1}(m-1) \binom{x_1 + m - 2}{m},$$

where we denote by $\binom{Q}{r}$ the polynomial $\frac{1}{r!} Q(Q-1)(Q-2) \cdots (Q-(r-1))$, for $r \in \mathbb{N}$, $Q \in \mathbb{K}[X_1, \dots, X_n]$.

We give a more precise idea, we explicit the polynomials $P_2, \dots, P_5$:

$$P_2 = -\tfrac{1}{2} x_1(x_1 - 1) \text{(see Example 1)}$$

$$P_3 = -x_1 x_2 + \tfrac{1}{3}(x_1 - 1)x_1(x_1 + 1)$$

$$P_4 = -x_1 x_3 + \tfrac{1}{2} x_1(x_1 + 1)x_2 - \tfrac{1}{8}(x_1 - 1)x_1(x_1 + 1)(x_1 + 2)$$

$$P_5 = -x_1 x_4 + \tfrac{1}{2} x_1(x_1 + 1)x_3 - \tfrac{1}{6} x_1(x_1 + 1)(x_1 + 2)x_2 \\ -\tfrac{1}{30}(x_1 - 1)x_1(x_1 + 1)(x_1 + 2)(x_1 + 3)$$

Geometrically, the affine automorphisms α and β on $\mathbb{K}^2$ respectively leave invariant the curves given by $x_2 + P_2(x_1) = \tau_2, x_3 + P_3(x_1, x_2) = \tau_2, \ldots, x_n + P_n(x_1, \ldots, x_{n-1}) = \tau_n$ and the lines given by $x_i = \tau_i, i = 2, \ldots, n$ where $(\tau_2, \ldots, \tau_n) \in \mathbb{K}^{n-1}$; the action of these automorphisms on this curves is just a translation by 1 on the axis x_1. The automorphism π sends the curves on the lines without changing the coordinate x_1, and so sends the orbits of α on those of β.

In fact, we can generalize this example to reduce a lot the size of the Jordan blocks:

Proposition 3. *An affine automorphism of $\mathbb{K}^n$ with no fixed point is conjugate, in $Aut(n, \mathbb{K})$, to an almost-diagonal automorphism, unique up to a permutation of its eigenvalues.*

More precisely, the automorphism

$$\alpha : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto {}^t\begin{pmatrix} J(\lambda_1, k_1 - 1) & & & \\ & J(\lambda_2, k_2) & & \\ & & \ddots & \\ & & & J(\lambda_r, k_r) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad with\ \lambda_1 = 1,$$

on Jordan normal form, is conjugate, in $Aut(n, \mathbb{K})$, to the almost-diagonal automorphism

$$\alpha_D : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 \cdot Id_{k_1-1} & & & \\ & \lambda_2 \cdot Id_{k_2} & & \\ & & \ddots & \\ & & & \lambda_r \cdot Id_{k_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $\lambda \cdot Id_k = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \in GL(k, \mathbb{K})$ *is the diagonal part of* $J(\lambda, k)$.

Proof. 1. *The conjugation*

For an automorphism $\pi : (x_1, \ldots, x_n) \mapsto (P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n))$, we have $\pi \circ \alpha = \alpha_D \circ \pi$ if and only if

$$\alpha^*(P_j) = \begin{cases} P_1 + 1 & \text{if } j = 1 \\ \mu(j) P_j & \text{if } j > 1 \end{cases} \tag{2}$$

where $\mu(j)$ denotes the j-th eigenvalue of α_D.

If j is the first indice of the i-th block, set $P_j = X_j$. If not, there exists $Q_j \in \mathbb{K}[X_1, \ldots, X_{j-1}]$ such that $\alpha^*(Q_j) = \mu(j) Q_j - X_{j-1}$ (see Lemma 1 below); then let $P_j = X_j + Q_j$ so that $\alpha^*(P_j) = \alpha^*(X_j) + \alpha^*(Q_j) = (\mu(j) X_j + X_{j-1}) + (\mu(j) Q_j - X_{j-1}) = \mu(j)(X_j + Q_j) = \mu(j) P_j$. It is clear that the map defined by these P_j is an automorphism of $\mathbb{K}^n$ that conjugates α and α_D.

2. *The unicity*
   Let us suppose that the two almost-diagonal automorphisms

$$\theta_\mu : (x_1, \dots, x_n) \mapsto (x_1, \mu_2 x_2, \mu_3 x_3, \dots, \mu_n x_n) + (1, 0, \dots, 0)$$
$$\theta_\nu : (x_1, \dots, x_n) \mapsto (x_1, \nu_2 x_2, \nu_3 x_3, \dots, \nu_n x_n) + (1, 0, \dots, 0)$$

are conjugate in $Aut(n, \mathbb{K})$: there exists $\pi \in Aut(n, \mathbb{K})$ such that $\theta_\mu \circ \pi = \pi \circ \theta_\nu$. By derivation, one sees that

$$\begin{pmatrix} 1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix} \circ M(x) = M(\theta_\mu(x)) \circ \begin{pmatrix} 1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_n \end{pmatrix} \tag{3}$$

where $M(x) = (m_{ij}(x))_{i,j=1}^n$ denotes the Jacobian matrix of π at $x = (x_1, \dots, x_n)$.
So we have $\mu_i m_{ij}(x) \nu_j^{-1} = m_{ij}(x_1 + 1, \nu_2 x_2, \dots, \nu_n x_n)$, where $\mu_1 = \nu_1 = 1$. Hence the m_{ij} are eigenvectors of θ_ν^* and belong to $\mathbb{K}[X_2, \dots, X_n]$, by Lemma 2 below. In particular, we see that $M(0, \dots, 0) = M(1, 0, \dots, 0) = M(\theta_\nu(0, \dots, 0))$. Since π is an automorphism of $\mathbb{K}^n$, we have $det(M(x)) = det(M(0, \dots, 0)) \in \mathbb{K}^*$, so equality (3) evaluated at $x = (0, \dots, 0)$ shows that the diagonal matrices

$$\begin{pmatrix} 1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix} \textit{and} \begin{pmatrix} 1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_n \end{pmatrix}$$

are conjugate in $GL(n, \mathbb{K})$, i.e. the μ_i and the ν_i are equal up to permutation.

□

Remark 1. 1. In fact, Proposition 3 shows also that α and α_D are conjugate in the famous *Jonquière subgroup* of $Aut(n, \mathbb{K})$ given by $\{\varphi = (\varphi_1, \dots, \varphi_n) \in Aut(n, \mathbb{K}) \mid \varphi_i \in \mathbb{K}[X_1, \dots, X_i] \text{ for } i = 1, \dots, n\}$.
2. The characteristic 0 is very important here, because Lemma 1 (and then Proposition 3) is false in characteristic > 0: let $\mathbb{L}$ be any field of characteristic p, and $\alpha_1, \alpha_2, \dots, \alpha_p$ be the affine automorphisms

$$\alpha_k : (x_1, x_2, \dots, x_n) \mapsto (x_1 + 1, x_2 + x_1, \dots, x_k + x_{k-1}, x_{k+1}, x_{k+2}, \dots, x_n).$$

Then $\alpha_1, \alpha_2, \dots, \alpha_{p-1}$ are all of order p and conjugate in $Bir(\mathbb{L}^n)$ but not α_p which is of order p^2.

Lemma 1. *If j is not the first indice of a block, there exists $Q_j \in \mathbb{K}[X_1, \dots, X_{j-1}]$ such that $\alpha^*(Q_j) = \mu(j) Q_j - X_{j-1}$.*

Proof. Let us recall that :

$$\alpha^*(X_k) = \begin{cases} X_1 + 1 & \text{if } k \text{ is the first indice} \\ & \text{ of the first block } (k=1) \\ \lambda(k)X_k & \text{if } k \text{ is the first indice of another block} \\ \lambda(k)X_k + X_{k-1} & \text{if } k \text{ is not the first indice of a block} \end{cases}$$

We will prove the stronger assertion (that implies the Lemma, using $k = j - 1$, as $\mu(j-1) = \mu(j)$):

$$\begin{array}{c} \textit{For any integers } 1 \leq k \leq n \textit{ and } t \geq 0, \textit{ the monomial } X_1^t X_k \textit{ is} \\ \textit{in the image of the linear map of vector spaces} \\ (\alpha^* - \mu(k)id) : \mathbb{K}[X_1, \dots, X_k] \to \mathbb{K}[X_1, \dots, X_k]. \end{array} \tag{4}$$

- Since $(\alpha^* - id)_{|\mathbb{K}[X_1]}$ is surjective, the assertion (4) is true for $k = 1$ and $t \geq 0$.
- In the same way, if $k > 1$ is the first indice of a block, since $\alpha^*(X_k) = \mu(k)X_k$, the linear map $(\alpha^* - \mu(k)id)_{|\mathbb{K}[X_1]\cdot X_k}$ is surjective and the assertion (4) is correct for k and $t \geq 0$.
- Lastly, if k is not the first indice of a block, we have

$$\begin{aligned} \alpha^*(X_1^{t+1}X_k) &= (X_1+1)^{t+1}(\mu(k)X_k + X_{k-1}) \\ &= \mu(k)X_1^{t+1}X_k + (t+1)\mu(k)X_1^t X_k + \sum_{l,s} a_{l,s} X_1^s X_k, \end{aligned}$$

  where $a_{l,s} \in \mathbb{K}$, and all the (l, s) are strictly smaller than (k, t), for the lexicographical order.

The assertion (4) is then right by induction on the indices (k, t). □

Lemma 2. *For any almost-diagonal automorphism* $\theta_\lambda\colon (x_1, x_2, \dots, x_n) \mapsto (x_1+1, \lambda_2 x_2, \dots, \lambda_n x_n)$ *of* $\mathbb{K}^n$*, the eigenvectors of* θ_λ^* *belong to* $\mathbb{K}[X_2, \dots, X_n]$.

Proof. We first observe that the map θ_λ^* leaves invariant the decomposition

$$\mathbb{K}[X] = \bigoplus_{(a_2,\dots,a_n)\in\mathbb{N}^{n-1}} \left(\prod_{i\geq 2}^{n} {X_i}^{a_i} \cdot \mathbb{K}[X_1] \right)$$

and that the $\Pi_{i\geq 2} {X_i}^{a_i}$ are eigenvectors. Since the only eigenspace of θ_λ^* in $\mathbb{K}[X_1]$ is $\mathbb{K} \cdot 1$, any eigenvector is in $\mathbb{K}[X_2, \dots, X_n]$. □

4. Conjugation in the group $Bir(\mathbb{K}^n)$

4.1. Diagonalizable and birationaly almost-diagonal affine automorphisms

The conjugation of elements of $Aff(n, \mathbb{K})$ in $Bir(\mathbb{K}^n)$ changes the dichotomy on the existence of a fixed point, as the following example shows:

Example 3. The two affine automorphisms

$$\alpha : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\beta : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are conjugate by the birational map

$$\varphi : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 x_1^{-1} \\ x_1^{-1} \end{pmatrix}.$$

In fact, choosing $\mathbb{K}^2$ as the open subset $x_0 \neq 0$ of $\mathbb{P}^2$, the two affine automorphisms become

$$\tilde{\alpha} : (x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : x_2 + x_1)$$
$$\tilde{\beta} : (x_0 : x_1 : x_2) \mapsto (x_0 : x_1 + x_0 : x_2)$$

and the birational map corresponds only to the permutation of coordinates

$$\tilde{\varphi} : (x_0 : x_1 : x_2) \mapsto (x_1 : x_2 : x_0).$$

Let us recall that $D(n, \mathbb{K})$ denotes the subgroup of $GL(n, \mathbb{K})$ made up of diagonal automorphisms and $AD(n, \mathbb{K})$ the subset of $Aff(n, \mathbb{K})$ made up of almost-diagonal automorphisms. We will say that $\alpha \in Aff(n, \mathbb{K})$ is *diagonalizable* (respectively *birationaly almost-diagonal*) if there exists $\pi \in Aff(n, \mathbb{K})$ such that $\pi\alpha\pi^{-1} \in D(n, \mathbb{K})$ (respectively if there exists $\pi \in Bir(\mathbb{K}^n)$ such that $\pi\alpha\pi^{-1} \in AD(n, \mathbb{K})$).

These two notions provide the dichotomy for conjugation in $Bir(\mathbb{K}^n)$, as the following result shows:

Proposition 4. *An affine automorphism of $\mathbb{K}^n$ is either diagonalizable or birationaly almost-diagonal, exclusively.*

Proof.

1. *Existence of the conjugation*
   Let $\alpha \in Aff(n, \mathbb{K})$. If α fixes no point, then α is birationaly almost-diagonal, by Proposition (3). Else, by a conjugation in $Aff(n, \mathbb{K})$, we may suppose that $\alpha \in GL(n, \mathbb{K})$; if α is not diagonalizable, we verify, as in the example (3) above, that α is birationaly almost-diagonal.

   Explicitly, let α be of the form $x \mapsto \begin{pmatrix} {}^t J(\mu, m_1) & \\ \hline & \ddots \end{pmatrix} x$, with $m_1 > 1$.

   Then, the birational map

$$\pi : (x_1, \dots, x_n) \dashrightarrow \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_{m_1}}{x_1}, x_{m_1+1}, x_{m_1+2}, \dots, x_n\right)$$

conjugates α to the affine automorphism

$$x \mapsto \left(\begin{array}{cccc|c} \mu^{-1} & & & & \\ & 1 & & & \\ & \mu^{-1} & 1 & & \\ & & \ddots & \ddots & \\ & & & \mu^{-1}\ 1 & \\ \hline & & & & \ddots \end{array}\right) x + \begin{pmatrix} 0 \\ \mu^{-1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

that fixes no point and so that is birationally almost-diagonal.

2. *The diagonal and almost-diagonal automorphisms of* $\mathbb{K}^n$ *are in distinct conjugacy classes of* $Bir(\mathbb{K}^n)$
   Suppose that θ_λ is an almost-diagonal automorphism of $\mathbb{K}^n$ conjugate in $Bir(\mathbb{K}^n)$ to a diagonal automorphism ρ_μ by a birational map π. We write $\rho_\mu, \theta_\lambda, \pi$ in their explicit form:

$$\begin{array}{rl} \theta_\lambda & : (x_1, x_2, \dots, x_n) \mapsto (\ x_1 + 1\ ,\ \lambda_2 x_2\ , \dots ,\ \lambda_n x_n\ ) \\ \rho_\mu & : (x_1, x_2, \dots, x_n) \mapsto (\ \mu_1 x_1\ ,\ \mu_2 x_2\ , \dots ,\ \mu_n x_n\ ) \\ \pi & : (x_1, x_2, \dots, x_n) \mapsto (\ \frac{P_1(x_1,\dots,x_n)}{Q_1(x_1,\dots,x_n)}\ ,\ \frac{P_2(x_1,\dots,x_n)}{Q_2(x_1,\dots,x_n)}\ , \dots ,\ \frac{P_n(x_1,\dots,x_n)}{Q_n(x_1,\dots,x_n)}\ ) \end{array}$$

   with $\mu_i, \lambda_i \in \mathbb{K}^*$ and $P_i, Q_i \in \mathbb{K}[X]$ without common divisors. Since θ_λ^* gives an automorphism of $\mathbb{K}[X]$ the condition $\pi\theta_\lambda = \rho_\mu\pi$ implies that $\frac{\theta_\lambda^*(P_i)}{\theta_\lambda^*(Q_i)} = \mu_i \frac{P_i}{Q_i}$, for any $i = 1, \dots, n$. Then all the P_i and Q_i must be eigenvectors of θ_λ^*, viewed as a $\mathbb{K}$-linear map. But such eigenvectors belongs to $\mathbb{K}[X_2, \dots, X_n]$ (Lemma 2) and so the map π cannot be birational. □

Remark 2. The fact that a non-diagonalizable automorphism of $\mathbb{K}^n$ is conjugate, in $Bir(\mathbb{K}^n)$, to an affine automorphism with no fixed point can also be viewed as follows:

Let us consider the automorphism

$$\alpha : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \left(\begin{array}{cccc|c} \mu & & & & \\ 1 & \mu & & & \\ & \ddots & \ddots & & \\ & & 1 & \mu & \\ \hline & & & & \ddots \end{array}\right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

of $\mathbb{K}^n$, and his extension to an automorphism

$$\tilde{\alpha} : \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \left(\begin{array}{c|cccc|c} 1 & & & & & \\ \hline & \mu & & & & \\ & 1 & \mu & & & \\ & & \ddots & \ddots & & \\ & & & 1 & \mu & \\ \hline & & & & & \ddots \end{array}\right) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

of $\mathbb{P}^n(\mathbb{K})$. We observe that $\tilde{\alpha}$ leaves invariant the open set $(x_1 \neq 0)$ of $\mathbb{P}^n(\mathbb{K})$ and it induces in it the affine automorphism

$$\beta : \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \left(\begin{array}{ccccc|c} \mu^{-1} & & & & & \\ & 1 & & & & \\ & \mu^{-1} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \mu^{-1} & 1 & \\ \hline & & & & & \ddots \end{array}\right) \begin{pmatrix} x_0 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

of $\mathbb{K}^n$, that fixes no point. We go from α to β by exchange of x_0 and x_1.

4.2. Actions of $GL(n, \mathbb{Z})$ and $GL(n-1, \mathbb{Z})$

Let us summarize the situation; we have

$$Aff(n, \mathbb{K}) = Aff(n, \mathbb{K}) \cap (Bir(\mathbb{K}^n) \bullet D(n, \mathbb{K}) \uplus Bir(\mathbb{K}^n) \bullet AD(n, \mathbb{K}))$$

where $Bir(\mathbb{K}^n) \bullet B = \{\pi\alpha\pi^{-1} \mid \alpha \in B, \pi \in Bir(\mathbb{K}^n)\}$.

We continue our study by describing the trace on $D(n, \mathbb{K})$ (respectively on $AD(n, \mathbb{K})$) of the conjugacy class of an element ρ_μ (respectively θ_ν) of $D(n, \mathbb{K})$ (respectively $AD(n, \mathbb{K})$). Equivalently, we are looking for:

$$\begin{array}{l} (Bir(\mathbb{K}^n) \bullet \rho_\mu) \cap \ \ D(n, \mathbb{K}) \\ (Bir(\mathbb{K}^n) \bullet \theta_\nu) \ \cap AD(n, \mathbb{K}) \end{array}$$

Action of $GL(n, \mathbb{Z})$ on $D(n, \mathbb{K})$
For this purpose, let us recall that $\mathcal{T}^n = (\mathbb{K}^*)^n$ and consider the isomorphism

$$\begin{array}{rccc} \rho : & \mathcal{T}^n & \rightarrow & D(n, \mathbb{K}) \subset Bir(\mathbb{K}^n) \\ & (\mu_1, \ldots ., \mu_n) & \mapsto & [(x_1, \ldots, x_n) \mapsto (\mu_1 x_1, \ldots, \mu_n x_n)] \end{array}$$

that is $GL(n, \mathbb{Z})$-equivariant for actions that we discribe now:

The action on $\mathcal{T}^n$ is the natural action of of $GL(n, \mathbb{Z}) = Aut(\mathcal{T}^n)$: a matrix $A = (a_{ij})_{i,j=1}^n \in GL(n, \mathbb{Z})$ maps an element $(\mu_1, \ldots ., \mu_n) \in (\mathbb{K}^*)^n$ on

$$A \bullet (\mu_1, \ldots, \mu_n) = (\mu_1^{a_{11}} \mu_2^{a_{12}} \cdots \mu_n^{a_{1n}}, \mu_1^{a_{21}} \mu_2^{a_{22}} \cdots \mu_n^{a_{2n}}, \ldots, \mu_1^{a_{n1}} \mu_2^{a_{n2}} \cdots \mu_n^{a_{nn}}).$$

On the other side, we have the injective homomorphism

$$\begin{array}{l} T : GL(n, \mathbb{Z}) \rightarrow Bir(\mathbb{K}^n) \\ \quad A = (a_{ij})_{i,j=1}^n \mapsto [(x_1, \ldots, x_n) \dashrightarrow (x_1^{a_{11}} \cdots x_n^{a_{1n}}, \ldots, x_1^{a_{n1}} \cdots x_n^{a_{nn}})] \end{array}$$

whose image normalizes $D(n, \mathbb{K})$. This gives then an action of $GL(n, \mathbb{Z})$ on $D(n, \mathbb{K})$ by conjugation.

We get the formula:

$$\boxed{T(A) \circ \rho(\mu) \circ T(A)^{-1} = \rho(A \bullet \mu)} \quad \mu \in (\mathbb{K}^*)^n, A \in GL(n, \mathbb{Z}) \tag{5}$$

that proves that ρ is equivariant and that

$$(Bir(\mathbb{K}^n) \bullet \rho(\mu)) \cap D(n, \mathbb{K}) \supset \rho(GL(n, \mathbb{Z}) \bullet \mu).$$

We will see further that this is an equality, i.e. that two diagonal automorphisms of $\mathbb{K}^n$ that are conjugate in $Bir(\mathbb{K}^n)$ are conjugate by an element of $GL(n, \mathbb{Z})$ (see Proposition 6).

Action of $GL(n-1, \mathbb{Z})$ **on** $AD(n, \mathbb{K})$
We do the same for $AD(n, \mathbb{K})$: we have the injective homomorphism

$$\begin{aligned} S : GL(n-1, \mathbb{Z}) &\rightarrow Bir(\mathbb{K}^n) \\ A = (a_{ij})_{i,j=2}^n &\mapsto [(x_1, \ldots, x_n) \dashrightarrow (x_1, x_2^{a_{21}} \cdots x_n^{a_{2n}}, \ldots, x_2^{a_{n2}} \cdots x_n^{a_{nn}})] \end{aligned}$$

whose image normalizes $AD(n, \mathbb{K})$ and who gives then an action of $GL(n-1, \mathbb{Z})$ on $AD(n, \mathbb{K})$ by conjugation.

With the action of $GL(n-1, \mathbb{Z})$ on $(\mathbb{K}^*)^{n-1}$, the bijection

$$\begin{aligned} \theta : (\mathbb{K}^*)^{n-1} &\rightarrow AD(n, \mathbb{K}) \subset Bir(\mathbb{K}^n) \\ (\nu_2, \ldots, \nu_n) &\mapsto [(x_1, \ldots, x_n) \mapsto (x_1 + 1, \nu_2 x_2 \ldots, \nu_n x_n)] \end{aligned}$$

is $GL(n-1, \mathbb{Z})$-equivariant, by the formula

$$\boxed{S(B) \circ \theta(\nu) \circ S(B)^{-1} = \theta(B \bullet \mu)} \quad \nu \in (\mathbb{K}^*)^{n-1}, B \in GL(n-1, \mathbb{Z}) \tag{6}$$

and we see too that

$$(Bir(\mathbb{K}^n) \bullet \theta(\nu)) \cap AD(n, \mathbb{K}) \supset \theta(GL(n-1, \mathbb{Z}) \bullet \nu).$$

We will see further that this is an equality, i.e. that two almost-diagonal automorphisms of $\mathbb{K}^n$ that are conjugate in $Bir(\mathbb{K}^n)$ are conjugate by an element of $GL(n-1, \mathbb{Z})$ (see Proposition 7).

4.3. Elements of finite order

Let us prove the equality between $(Bir(\mathbb{K}^n) \bullet \rho(\mu)) \cap D(n, \mathbb{K})$ and $\rho(GL(n, \mathbb{Z}) \bullet \mu)$, for elements $\mu \in (\mathbb{K}^*)^n$ of finite order. For $n > 1$, we do this by showing that $\rho(GL(n, \mathbb{Z}) \bullet \mu)$ contains all diagonal automorphisms of the same order as μ. This also proves that two affine automorphisms of the same finite order are conjugate in $Bir(\mathbb{K}^n)$.

If $n = 1$ the group $Bir(\mathbb{K})$ is equal to $PGL(2, \mathbb{K})$ and a simple calculation shows that two diagonal automorphisms $x \mapsto \alpha x$ and $x \mapsto \beta x$ are conjugate if and only if $\alpha = \beta^{\pm 1}$. So the equality is true in dimension one too.

Proposition 5 (Diagonal automorphisms of finite order)**.** *If $n > 1$, two diagonal automorphisms of $\mathbb{K}^n$ of the same (finite) order are conjugate by an element of $GL(n, \mathbb{Z}) \subset Bir(\mathbb{K}^n)$.*

Proof. Let m be a positive integer and ξ a primitive m-th root of the unity. A diagonal automorphism of $\mathbb{K}^n$ of order m is given by $\rho(\alpha) : (x_1, \dots, x_n) \mapsto (x_1\xi^{t_1}, \dots, x_n\xi^{t_n})$, where $\alpha = (\xi^{t_1}, \dots, \xi^{t_n})$, and the greatest common divisor of the exponents $g = gcd(t_1, \dots, t_n)$ is prime to m.

The action of $GL(n, \mathbb{Z})$ on the diagonal automorphisms corresponds here to the usual action of $GL(n, \mathbb{Z})$ on the exponents $(t_1, \dots, t_n) \in \mathbb{Z}^n$.

With elementary matrices, that add a multiple of a coordinate to another one, we can map the vector $t = (t_1, \dots, t_n)$ to the vector $(g, 0, \dots, 0)$, so the diagonal automorphism $\rho(\alpha)$ is in the same orbit as the automorphism $(x_1, \dots, x_n) \mapsto (x_1\xi^g, x_2, \dots, x_n)$. Because g is prime to m, there exist $p, q \in \mathbb{Z}$ such that $pm + gq = 1$. Since

$$\begin{pmatrix} q & -p & & & \\ m & g & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} g \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} gq \\ gm \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

we see that $\rho(\alpha)$ is conjugate, by an element of $GL(n, \mathbb{Z})$, to the diagonal automorphism $(x_1, \dots, x_n) \mapsto (x_1\xi^{gq}, x_2\xi^{gm}, x_3, \dots, x_n) = (x_1\xi, x_2, \dots, x_n)$, which concludes the proof. □

Corollary 1 (Conjugacy of affine automorphisms of finite order)**.** *Two affine automorphisms of the same finite order are conjugate in $Bir(\mathbb{K}^n)$, for $n > 1$.* □

4.4. Conjugacy classes of diagonal automorphisms

Let us now continue the work for diagonal automorphisms that are not necessary of finite order.

Proposition 6. *Two diagonal automorphisms of $\mathbb{K}^n$ that are conjugate in $Bir(\mathbb{K}^n)$ are conjugate by an element of $GL(n, \mathbb{Z})$.*

Proof. To each diagonal automorphism $\rho(\alpha) \in Aut(n, \mathbb{K}), \alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{K}^*)^n$, we associate the kernel Δ_α of the following homomorphism:

$$\begin{aligned} \delta_\alpha : \mathbb{Z}^n &\to \mathbb{K}^* \\ (t_1, t_2, \dots, t_n) &\mapsto {\alpha_1}^{t_1}{\alpha_2}^{t_2} \dots {\alpha_n}^{t_n}. \end{aligned}$$

It is easy to verify that $\delta_{M\bullet\alpha} = \delta_\alpha \circ {}^tM$, for any $M \in GL(n, \mathbb{Z})$ and so that $\Delta_{M\bullet\alpha} = {}^tM^{-1}(\Delta_\alpha)$. If δ_α is not injective, we can choose M (by theorem on Smith's normal form) such that $\Delta_{M\bullet\alpha}$ is generated by $k_1e_1, k_2e_2, \dots k_re_r$, where $e_1, e_2, \dots, e_n$ are the canonical basis vectors of $\mathbb{Z}^n$, $r \leq n$ and the k_i are positive integers such that k_i divides k_{i-1} for $i = 2, \dots, r$. Let $\alpha' = (\alpha'_1, \dots, \alpha'_n) = M \bullet \alpha$. We observe that α'_i is a primitive k_i-th root of the unity for $i = 1, \dots, r$. In particular, we have $\alpha'_2 = (\alpha'_1)^s$ for an integer s, so $(s, -1, 0, \dots 0) \in \Delta_{M\bullet\alpha}$, and then $k_2 = k_3 = \dots = k_r = 1$.

Let $\rho(\alpha)$ and $\rho(\beta)$ be two diagonal automorphisms conjugate in $Bir(\mathbb{K}^n)$. We will assume that Δ_α is generated by $ke_1, e_2, \dots, e_r$ (replacing $\rho(\alpha)$ by another element of its orbit if necessary, as we did above), so that α_1 is a primitive k-th root of the unity and $\alpha_2 = \alpha_3 = \dots = \alpha_r = 1$, if $r > 0$. By the way, if $r = n$, we see another time (Proposition 5) that any diagonal automorphism of finite order is conjugate to another one of the form $(x_1, \dots, x_n) \mapsto (\xi x_1, x_2, \dots, x_n)$.

The condition of conjugation implies that there exists a birational map

$$\varphi : (x_1, x_2, \dots, x_n) \dashrightarrow (\varphi_1(x_1, \dots, x_n), \varphi_2(x_1, \dots, x_n), \dots, \varphi_n(x_1, \dots, x_n)),$$

where $\varphi_i \in \mathbb{K}(X)$, such that $\varphi \circ \rho(\alpha) = \rho(\beta) \circ \varphi$, and then

$$\varphi_i(\alpha_1 x_1, \dots, \alpha_n x_n) = \rho(\alpha)^*(\varphi_i) = \beta_i \varphi_i \text{ for } i = 1, \dots, n.$$

So, every φ_i must be an eigenvector of $\rho(\alpha)^*$, viewed as linear map of vector spaces, and since $\rho(\alpha)^*$ gives an automorphism of $\mathbb{K}[X]$, there must exist $F_i, G_i \in \mathbb{K}[X]$ eigenvectors of $\rho(\alpha)^*_{|\mathbb{K}[X]}$ such that $\varphi_i = \frac{F_i}{G_i}$. This map can be diagonalized along the basis of eigenvectors ${X_1}^{t_1} {X_2}^{t_2} \dots {X_n}^{t_n}$, and two vectors of this basis have the same eigenvalue if and only if the difference of the exponent vectors is in the kernel group Δ_α associated to $\rho(\alpha)$.

By a choice of representants in the classes mod Δ_α, we can put the transformation φ on the following form:

$$\begin{gathered} \varphi : (x_1, \dots, x_n) \dashrightarrow \\ ({x_1}^{a_{11}} \cdots {x_n}^{a_{1n}} \psi_1({x_1}^k, x_2, \dots, x_r), \dots, {x_1}^{a_{n1}} \cdots {x_n}^{a_{nn}} \psi_n({x_1}^k, x_2, \dots, x_r)) \end{gathered}$$

where each ψ_i belongs to $\mathbb{K}(X_1, \dots, X_r)$.

Hence, the map φ defines a matrix $A = (a_{ij})_{i,j=1}^n$ in $Mat_{n,n}(\mathbb{Z})$, which is not unique because of the choice of the representants. More precisely, φ defines an unique element of the quotient of $Mat_{n.n}(\mathbb{Z})$ by the relation

$$(b_{ij})_{i,j=1}^n \sim (c_{ij})_{i,j=1}^n \text{ if } b_{i1} - c_{i1} \in k\mathbb{Z} \text{ and } b_{ij} = c_{ij} \text{ for } i = 1, \dots, n \text{ and } j > r.$$

We observe that $\beta = (\beta_1, \dots, \beta_n) = ({\alpha_1}^{a_{11}} \dots {\alpha_n}^{a_{1n}}, \dots, {\alpha_1}^{a_{n1}} \dots {\alpha_n}^{a_{nn}})$. The birationaliy of φ implies, (see Lemma 3 below), that there exists a matrix $B = (b_{ij})_{i,j=1}^n$ in the equivalence class associated to φ which is invertible. So, we get from this that $T(B)\rho(\alpha)T(B)^{-1} = \rho(\beta)$, by the fact that

$$\begin{aligned} B \bullet \alpha &= ({\alpha_1}^{b_{11}} \cdots {\alpha_n}^{b_{1n}}, \dots, {\alpha_1}^{b_{n1}} \cdots {\alpha_n}^{b_{nn}}) \\ &= ({\alpha_1}^{a_{11}} \cdots {\alpha_n}^{a_{1n}}, \dots, {\alpha_1}^{a_{n1}} \cdots {\alpha_n}^{a_{nn}}) \\ &= (\beta_1, \dots, \beta_n) = \beta. \end{aligned}$$

□

Lemma 3. *If the rational map*

$$\begin{aligned} \varphi(x_1, \dots, x_n) = (&x_1^{a_{11}} \cdots x_n^{a_{1n}} \psi_1({x_1}^k, x_2, \dots, x_r), \dots, \\ &x_1^{a_{n1}} \cdots x_n^{a_{nn}} \psi_n({x_1}^k, x_2, \dots, x_r)) \end{aligned}$$

is birational, then there exists an invertible matrix $B = (b_{ij})_{i,j=1}^n \in GL(n, \mathbb{Z})$ *such that* $b_{i1} - a_{i1} \in k\mathbb{Z}$ *and* $b_{ij} = a_{ij}$ *for* $i = 1, \dots, n$ *and* $j > r$.

Proof. First of all, if $n = 1$ and $r = 0$, the lemma is deduced from the fact that the map $x \dashrightarrow x^a$ is birational if and only if $a = \pm 1$.

Suppose that $r = n \geq 1$ and let ξ a primitive k-th root of the unity. The conjugation of the linear automorphism of finite order $\nu_1 : (x_1, \ldots, x_n) \mapsto (x_1\xi, x_2, \ldots, x_n)$ by φ gives the linear automorphism (of the same order) $\nu_2 : (x_1, \ldots, x_n) \mapsto (x_1\xi_1^{a_{11}}, \ldots, x_n\xi^{a_{n1}})$. If $n = 1$, the lemma follows from the fact that $x \mapsto \xi x$ and $x \mapsto \xi^{a_{11}}x$ are conjugated in $PGL(2, \mathbb{K})$ if and only if $\xi^{a_{11}} = \xi^{\pm 1}$, so $a_{11} = \pm 1 \; mod \; k$. If $n > 1$, by Proposition 5, there exists an invertible matrix $B = (b_{ij})_{i,j=1}^n \in GL(n, \mathbb{Z})$ such that $T(B)$ conjugates ν_1 to ν_2. Explicitly, it gives that $\xi^{b_{i1}} = \xi^{a_{i1}}$ for $i = 1, \ldots, n$, so $b_{i1} - a_{i1} \in k\mathbb{Z}$ for $i = 1, \ldots, n$. The lemma is then proved in this case.

It remains to prove the lemma when r is strictly smaller than n and $n > 1$. In this case, the matrix $(a_{ij})_{1 \leq i \leq n, r+1 \leq j \leq n}$ has the maximal rank $n - r$, since $x_{r+1}, \ldots, x_n$ appear only in the monomial part. There exists then an invertible matrix $M \in GL_n(\mathbb{Z})$ such that the coefficients of $A' = MA$ are the same as the ones of the identity, for columns $r + 1, \ldots, n$. The composition of $T(M)$ and φ gives a new birational map $T(M) \circ \varphi$:

$$\begin{aligned} T(M) \circ \varphi(x_1, \ldots, x_n) = (& x_1^{a'_{11}} \cdots x_r^{a'_{1,r}} \psi'_1(x_1^k, x_2, \ldots, x_r), \ldots, \\ & x_1^{a'_{r,1}} \cdots x_r^{a'_{r,r}} \psi'_r(x_1{}^k, x_2, \ldots, x_r), \\ & x_1{}^{a'_{r+1,1}} \cdots x_r^{a'_{r+1,r}} x_{r+1} \psi'_{r+1}(x_1{}^k, x_2, \ldots, x_r), \ldots, \\ & x_1{}^{a'_{n,1}} \cdots x_r^{a'_{n,r}} x_n \psi'_n(x_1{}^k, x_2, \ldots, x_r)), \end{aligned}$$

which has the same structure as φ, with matrix MA.

This map exchanges affine spaces of codimension r of type $x_1 = \tau_1, x_2 = \tau_2, \ldots, x_r = \tau_r$ and so must be its inverse. So, the map of $\mathbb{K}^r$ given by the r first coordinates of $T(M) \circ \varphi$ must be birational.

By induction, there exists an invertible matrix $C' = (c'_{ij})_{i,j=1}^{n-1} \in GL_r(\mathbb{Z})$ such that $c'_{i1} - a'_{i1} \in k\mathbb{Z}$ for $i = 1, \ldots, r$. The matrix $B' = \left(\begin{array}{c|c} C' & 0 \\ \hline 0 & Id \end{array}\right)$ (where $Id \in GL_{n-r}(\mathbb{Z})$ denotes the identity matrix) satisfy the condition that $b'_{i1} - a'_{i1} \in k\mathbb{Z}$ and $b'_{ij} = a'_{ij}$ for $i = 1, \ldots, n$ and $j > r$. The same occurs for $B = M^{-1}B'$ and $A = M^{-1}A'$. □

4.5. *Conjugacy classes of almost-diagonal automorphisms*

The case of almost-diagonal automorphism is deduced from the situation of diagonal automorphisms:

Proposition 7. *Two almost-diagonal automorphisms of $\mathbb{K}^n$ are conjugate in the group $Bir(\mathbb{K}^n)$ if and only if they are conjugate by an element of $GL(n - 1, \mathbb{Z})$.*

Proof. Let $\theta(\alpha)$ and $\theta(\beta)$ be two almost-diagonal automorphisms of $\mathbb{K}^n$:

$$\theta(\alpha) : (x_1, x_2, \ldots, x_n) \mapsto (x_1 + 1, \alpha_2 x_2, \alpha_3 x_3, \ldots, \alpha_n x_n)$$
$$\theta(\beta) : (x_1, x_2, \ldots, x_n) \mapsto (x_1 + 1, \beta_2 x_2, \beta_3 x_3, \ldots, \beta_n x_n).$$

We suppose that $\theta(\alpha)$ and $\theta(\beta)$ are conjugate in $Bir(\mathbb{K}^n)$, so that there exists a birational map

$$\varphi : (x_1, x_2, \ldots, x_n) \dashrightarrow \left(\frac{P_1(x_1, \ldots, x_n)}{Q_1(x_1, \ldots, x_n)}, \frac{P_2(x_1, \ldots, x_n)}{Q_2(x_1, \ldots, x_n)}, \ldots, \frac{P_n(x_1, \ldots, x_n)}{Q_n(x_1, \ldots, x_n)}\right),$$

with $P_i, Q_i \in \mathbb{K}[X]$ without common divisor, such that $\varphi \circ \theta(\alpha) = \theta(\beta) \circ \varphi$. This implies that

$$\frac{P_i(x_1 + 1, \alpha_2 x_2, \ldots, \alpha_n x_n)}{Q_i(x_1 + 1, \alpha_2 x_2, \ldots, \alpha_n x_n)} = \beta_i \frac{P_i(x_1, \ldots, x_n)}{Q_i(x_1, \ldots, x_n)} \text{ for } i = 2, \ldots, n.$$

Since $\theta(\alpha)$ induces an automorphism $\theta(\alpha)^*$ of $\mathbb{K}[X]$, all the P_i and Q_i, for $i \geq 2$, must be eigenvectors of $\theta(\alpha)^*$, viewed as a $\mathbb{K}$-linear map. Such eigenvectors belongs to $\mathbb{K}[X_2, \ldots, X_n]$ (Lemma 2), then the map φ exchange lines of type $x_2 = \tau_2, x_3 = \tau_3, \ldots, x_n = \tau_n$ and so must be its inverse. This implies that the map $\varphi' : (x_2, x_3, \ldots, x_n) \dashrightarrow (\frac{P_2(x_2,\ldots,x_n)}{Q_2(x_2,\ldots,x_n)}, \ldots, \frac{P_n(x_2,\ldots,x_n)}{Q_n(x_2,\ldots,x_n)})$, given by the $n-1$ last coordinates of φ is birational.

Since the birational map φ' of $\mathbb{K}^{n-1}$ conjugates the diagonal automorphisms

$$\rho(\alpha) : (x_2, \ldots, x_n) \mapsto (\alpha_2 x_2, \alpha_3 x_3, \ldots, \alpha_n x_n)$$
$$\rho(\beta) : (x_2, \ldots, x_n) \mapsto (\beta_2 x_2, \beta_3 x_3, \ldots, \beta_n x_n),$$

Proposition 6 shows that $\rho(\alpha)$ and $\rho(\beta)$ are conjugate by an element of $GL(n-1, \mathbb{Z}) \subset Bir(\mathbb{K}^{n-1})$. Then, $\theta(\alpha)$ and $\theta(\beta)$ are conjugated by the same element of $GL(n-1, \mathbb{Z}) \subset Bir(\mathbb{K}^n)$. □

5. Conjugacy classes of automorphisms of $\mathbb{P}^n$ in the Cremona group

Let us work in $\mathbb{P}^n$, the projective n-space over $\mathbb{K}$. Choosing a coordinate x_i, the open subset $U_i = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n \mid x_i \neq 0\}$ is isomorphic to $\mathbb{K}^n$ via the map $(x_0 : \ldots : x_n) \overset{\nu_i}{\mapsto} (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$. The restriction map $\varphi \mapsto \nu_i \varphi {\nu_i}^{-1}$ gives an isomorphism of the group $Bir(\mathbb{P}^n)$ of birational maps of $\mathbb{P}^n$ to $Bir(\mathbb{K}^n)$, that we call both the *Cremona group*.

Let us recall that a birational map of $\mathbb{P}^n$ is given by a map $(x_0 : \ldots : x_n) \dashrightarrow (P_0(x_0, \ldots, x_n) : \ldots : P_n(x_0, \ldots, x_n))$, where $P_0, \ldots, P_n \in \mathbb{K}[X_0, \ldots, X_n]$ are homogeneous polynomials of the same degree (that will be called the degree of the map). As a birational map is biregular if and only if its degree is one, the group of automorphisms (biregular rational maps) of $\mathbb{P}^n$ is $PGL(n+1, \mathbb{K})$.

We will denote *diagonal* (respectively *almost-diagonal*) automorphisms of $\mathbb{P}^n$ maps of the form $(x_0 : \ldots : x_n) \mapsto (x_0 : \alpha_1 x_1 : \ldots : \alpha_n x_n)$, with $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{K}^*)^n$ (respectively $(x_0 : \ldots : x_n) \mapsto (x_0 : x_0 + x_1 : \alpha_2 x_2 : \ldots : \alpha_n x_n)$, with $\alpha = (\alpha_2, \ldots, \alpha_n) \in (\mathbb{K}^*)^{n-1}$).

It is clear that the restriction map $\varphi \mapsto \nu_0 \varphi {\nu_0}^{-1}$ gives an isomorphism of $D(n, \mathbb{K})$ (respectively $AD(n, \mathbb{K})$) on the group of diagonal (respectively the set of almost-diagonal) automorphisms of $\mathbb{P}^n$.

We can then explicit the conjugacy classes of automorphisms of $\mathbb{P}^n$ in the Cremona group using the work made in affine space:

Any automorphism of $\mathbb{P}^n$ is conjugate, in $Bir(\mathbb{P}^n)$, either to a diagonal automorphism or to an almost-diagonal automorphism. The conjugacy classes of diagonal and almost diagonal automorphisms of $\mathbb{P}^n$ in the Cremona group are given by the action of $GL(n, \mathbb{Z})$ on the diagonal automorphisms and of $GL(n-1, \mathbb{Z})$ on the almost-diagonal automorphisms (see Section 4).

Furthermore, if $n > 1$, two linear automorphisms of $\mathbb{P}^n$ of the same order are conjugate in $Bir(\mathbb{P}^n)$. This result was already proved in dimension 2 in ([BeB], Proposition 2.1.) with another method that works also in higher dimension.

Acknowledgements. The author would like to thank the referee for his helpful criticisms and wishes to express his sincere gratitude to P. De La Harpe, I. Pan, F. Ronga and T.Vust for many interesting discussions during the preparation of this paper. Moreover, the author acknowledges support from the *Swiss National Science Foundation*.

References

[AlC] Alberich-Carramiñana, M.: Geometry of the plane Cremona maps, Lecture Notes in Math., 1769, Springer, Berlin, 2002

[BaB] Bayle, L., Beauville, A.: Birational involutions of P^2. Asian J. Math. **4** (1), 11–17 (2000)

[BeB] Beauville, A., Blanc, J.: On Cremona transformations of prime order. C.R. Acad. Sci. Paris, Ser. I **339**, 257–259 (2004)

[DeF] De Fernex, T.: On planar Cremona maps of prime order. Nagoya Math. J. **174** (2004)

[GoP] Gonzalez-Sprinberg, G., Pan, I.: On the monomial birational maps of the projective space. An. Acad. Brasil. Ciênc. **75** (2), 129–134 (2003)

[Hud] Hudson, H.: Cremona transformations in the plane and the space. Cambridge University Press, 1927

[Kan] Kantor, S.: Theorie der endligen Gruppen von eindeutigen Transformationen in der Ebene. Mayer & Müller, Berlin, 1895

[Kra] Kraft, H.: Algebraic automorphisms of affine space. In: Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, pp. 251–274

[KrS] Kraft, H., Schwarz, G.: Finite automorphisms of affine n-space. In: Automorphisms of affine spaces (Curaçao, 1994), Kluwer Acad. Publ., Dordrecht, 1995, pp. 55–66

[Wim] Wiman, A.: Zur Theorie der endlichen Gruppen von birationalen Transformationen in der Ebene. Math. Ann., vol. xlviii, 1896, pp. 497–498