ABELIAN QUOTIENTS OF THE CREMONA GROUPS IN HIGHER DIMENSION

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Abstract. We study groups of birational selfmaps $\mathrm{Bir}(X)$, where X is a rationally connected variety of dimension at least 3, over an algebraically closed field of characteristic zero. We are interested in the situation where these groups are large. One prominent case is when X is the projective space $\mathbb{P}^n$, in which case $\mathrm{Bir}(X)$ is the Cremona group of rank n. We produce infinitely many distinct group homomorphisms from $\mathrm{Bir}(\mathbb{P}^n)$ to $\mathbf{Z}/2$, showing that the Cremona group is not perfect hence not simple. As a consequence we also obtain that the Cremona group of rank 3 is not generated by linear and Jonquières elements.

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1. Introduction

1.A. **Higher rank Cremona groups.** The *Cremona group of rank* n, denoted by $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ or $\mathrm{Bir}(\mathbb{P}^n)$ when $\mathbf{k}$ is implicit, is the group of birational transformations of the projective space, over a ground field $\mathbf{k}$.

The classical case is $n = 2$, where the group is already quite complicated but is now well described, at least when $\mathbf{k}$ is algebraically closed. In this case the Noether-Castelnuovo Theorem [Cas01, Alb02] asserts that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and a single quadratic transformation. This fundamental result, together with the strong factorisation of birational maps between surfaces helps to have a good understanding of the group.

The dimension $n \geqslant 3$ is more difficult, as we do not have any analogue of the Noether-Castelnuovo Theorem (see §1.C for more details) and also no strong factorisation. Here is an extract from the article "Cremona group" in the Encyclopedia of Mathematics, written by V. Iskovskikh in 1987 (who uses the notation $\mathrm{Cr}(\mathbb{P}^n_{\mathbf{k}})$ for the Cremona group):

> *One of the most difficult problems in birational geometry is that of describing the structure of the group* $\mathrm{Cr}(\mathbb{P}^3_{\mathbf{k}})$*, which is no longer generated by the quadratic transformations. Almost all literature on Cremona transformations of three-dimensional space is devoted to concrete examples of such transformations. Finally, practically nothing is known about the structure of the Cremona group for spaces of dimension higher than* 3*.* [Isk87]

In fact, 30 years later there is still almost no result on the group structure of $\mathrm{Bir}(\mathbb{P}^n)$, but only about some of its subgroups. In arbitrary dimension there are descriptions of the algebraic subgroups of rank n [Dem70], and much more recently of their lattices [CX18]. For $n = 3$, there is also a classification of the connected algebraic subgroups [Ume85, BFT17], and partial classification of finite subgroups [Pro11, Pro12, Pro14, PS16, BZ17]. There are also numerous articles devoted to the study of particular classes of examples of elements in $\mathrm{Bir}(\mathbb{P}^n)$, especially for n small (we do not attempt to start a list here, as it would always be very far from exhaustive).

Nevertheless, using modern tools such as Sarkisov links and Minimal model program, we will be able in this text to give insight on the structure of the Cremona groups $\mathrm{Bir}(\mathbb{P}^n)$ and of its quotients.

1.B. **Normal subgroups.** The question of the non-simplicity of $\mathrm{Bir}(\mathbb{P}^n)$ for each $n \geqslant 2$ was also mentioned in the article of V. Iskovskikh in the Encyclopedia

> *It is not known to date* (1987) *whether the Cremona group is simple.* [Isk87]

but was in fact asked much earlier. It is explicitly mentioned in a book by F. Enriques in 1895:

> *Tuttavia altre questioni d'indole gruppale relative al gruppo Cremona nel piano* (*ed a più forte ragione in* S_n $n > 2$) *rimangono ancora insolute; ad esempio l'importante questione se il gruppo Cremona contenga alcun sottogruppo invariante* (*questione alla quale sembra probabile si debba rispondere negativamente*). [Enr95, p. 116][1]

The feeling expressed by Enriques that the Cremona group should be simple was maybe supported by the analogy with biregular automorphism groups of projective varieties, such as $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbf{k})$. In fact in the trivial case of dimension $n = 1$, we have $\mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbf{k})$, which is indeed a simple group when the ground field $\mathbf{k}$ is algebraically closed. Pushing further the analogy with algebraic groups, it was proved by the first author that when considered as a topological group, the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ is indeed simple, in the sense that any proper Zariski closed normal subgroup must be trivial [Bla10]. This result was recently extended to arbitrary dimension by the first and third authors [BZ18].

The non-simplicity of $\mathrm{Bir}(\mathbb{P}^2)$ as an abstract group was proven, over an algebraically closed field, by S. Cantat and the second author [CL13]. The idea of proof was to apply small cancellation theory to an action of $\mathrm{Bir}(\mathbb{P}^2)$ on a hyperbolic space. A first instance of roughly the same idea was [Dan74], in the context of plane polynomial automorphisms (see also [FL10]). The modern small cancellation machinery as developed in [DGO17] allowed A. Lonjou to prove the non simplicity of $\mathrm{Bir}(\mathbb{P}^2)$ over an arbitrary field, and the fact that every countable group is a quotient of $\mathrm{Bir}(\mathbb{P}^2)$ [Lon16]. It turns out that all the normal subgroups produced by this technique have infinite index, and for instance the group $\mathrm{PGL}_2(\mathbf{k}(T))$ of Jonquières maps embeds in all associated quotients, which are in particular infinite non-abelian.

Another source of normal subgroups for $\mathrm{Bir}(\mathbb{P}^2)$, of a very different nature, was discovered by the third author, when the ground field is $\mathbf{R}$ [Zim18]. In contrast with the case of an algebraically closed field where the Cremona group of rank 2 is a perfect group, she proved that the abelianisation of $\mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2)$ is an uncountable direct sum of $\mathbf{Z}/2$. Here the main idea is to use an explicit presentation by generators and relations. In fact a presentation of $\mathrm{Bir}(\mathbb{P}^2)$ over an arbitrary perfect field is available since [IKT93], but because they insist in staying inside the group $\mathrm{Bir}(\mathbb{P}^2)$, they obtain very long lists. In contrast, if one accepts to consider birational maps between non-isomorphic varieties, the Sarkisov program provides more tractable lists of generators. Using this idea together with results of Kaloghiros [Kal13], the existence of abelian quotients for $\mathrm{Bir}(\mathbb{P}^2)$ was extended to the case of many non-closed perfect fields by the second and third authors [LZ17].

The present paper is a further extension in this direction, this time in arbitrary dimension, and over any ground field $\mathbf{k}$ which is abstractly isomorphic to a subfield of $\mathbf{C}$ (this includes any field of rational functions of any algebraic variety defined over a subfield of $\mathbf{C}$). The main result of this article is then the following:

[1] "However, other group-theoretic questions related to the Cremona group of the plane (and, even more so, of $\mathbb{P}^n$, $n > 2$) remain unsolved; for example, the important question of whether the Cremona group contains any normal subgroup (a question which seems likely to be answered negatively)."

Theorem A. *For each subfield* $\mathbf{k} \subseteq \mathbf{C}$ *and each* $n \geqslant 3$*, there is a group homomorphism*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \bigoplus_{\mathbf{k}} \mathbf{Z}/2$$

such that the restriction to the subgroup given locally by

$$\{(x_1, \ldots, x_n) \dashrightarrow (x_1\alpha(x_2, \cdots, x_n), x_2, \ldots, x_n) \mid \alpha \in \mathbf{k}(x_2, \ldots, x_n)^*\}$$

is surjective. In particular, $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ *is not perfect and thus not simple.*

Let us mention that $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbf{k})$ is contained in the kernel of the group homomorphism in Theorem A, so that in particular the normal subgroup generated by $\mathrm{Aut}(\mathbb{P}^n)$ is a proper subgroup of $\mathrm{Bir}(\mathbb{P}^n)$. We will improve on this remark in Theorem B below.

1.C. **Generators.** As already mentioned, the Noether-Castelnuovo theorem provides simple generators of $\mathrm{Bir}(\mathbb{P}^2)$ when $\mathbf{k}$ is algebraically closed. Using Sarkisov links, there are also explicit (long) lists of generators of $\mathrm{Bir}(\mathbb{P}^2)$ for each field $\mathbf{k}$ of characteristic zero or more generally for each perfect field $\mathbf{k}$ [Isk91, Isk96]. In dimension $n \geqslant 3$, we do not have a complete list of all Sarkisov links and thus are far from having an explicit list of generators for $\mathrm{Bir}(\mathbb{P}^n)$. The lack of an analogue to the Noether-Castelnuovo Theorem for $\mathrm{Bir}(\mathbb{P}^n)$ and the question of finding good generators was already cited in the article of the Encyclopedia above, and also in the book of Enriques:

> *Questo teorema non è estendibile senz'altro allo* S_n *dove* $n > 2$*; resta quindi insoluta la questione capitale di assegnare le più semplici trasformazioni generatrici dell'intiero gruppo Cremona in* S_n *per* $n > 2$. [Enr95, p. 115][2]

A classical result, due to H. Hudson and I. Pan [Hud27, Pan99], shows that $\mathrm{Bir}(\mathbb{P}^n)$, for $n \geqslant 3$, is not generated by $\mathrm{Aut}(\mathbb{P}^n)$ and finitely many elements, or more generally by any set of elements of $\mathrm{Bir}(\mathbb{P}^n)$ of bounded degree. The reason is that one needs at least, for each irreducible variety Γ of dimension $n-2$, one birational map that contracts a hypersurface birational to $\Gamma \times \mathbb{P}^1$. These contractions can be realised in $\mathrm{Bir}(\mathbb{P}^n)$ by *Jonquières elements*, i.e. elements that preserve a family of lines through a given point, which form a subgroup

$$\mathrm{PGL}_2(\mathbf{k}(x_1, \ldots, x_n)) \rtimes \mathrm{Bir}(\mathbb{P}^{n-1}) \subseteq \mathrm{Bir}(\mathbb{P}^n).$$

Hence, it is natural to ask whether the group $\mathrm{Bir}(\mathbb{P}^n)$ is generated by $\mathrm{Aut}(\mathbb{P}^n)$ and by Jonquières elements (a question for instance asked in [PS15]).

We answer the question in dimension 3 by the negative, in the following stronger form:

Theorem B. *Let* $\mathbf{k}$ *be a subfield of* $\mathbf{C}$*. Then the normal subgroup of the Cremona group* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^3)$ *generated by* $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^3)$*, by all Jonquières elements and by any set of elements of bounded degree is a strict subgroup of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^3)$.

[2] "This theorem can not be easily extended to $\mathbb{P}^n$ where $n > 2$; therefore, the main question of finding the most simple generating transformations of the entire Cremona group of $\mathbb{P}^n$ for $n > 2$ remains open."

It is interesting to make a parallel between this statement and the classical Tame Problem in the context of the affine Cremona group $\mathrm{Aut}(\mathbb{A}^n)$, or group of polynomial automorphisms (which is one of the "challenging problems" on the affine spaces, described in the Bourbaki seminar [Kra96]). Recall that the tame subgroup $\mathrm{Tame}(\mathbb{A}^n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is defined as the subgroup generated by affine automorphisms and by the elementary subgroup of automorphisms of the form $(x_1, \ldots, x_n) \longmapsto (ax_1 + P(x_2, \ldots, x_n), x_2, \ldots, x_n)$. This elementary subgroup is an analogue of the $\mathrm{PGL}_2(\mathbf{k}(x_1, \ldots, x_n))$ factor in the Jonquières group, and of course the affine group is $\mathrm{PGL}_{n+1}(\mathbf{k}) \cap \mathrm{Aut}(\mathbb{A}^n)$. The Tame Problem asks whether the inclusion $\mathrm{Tame}(\mathbb{A}^n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is strict in dimension $n \geqslant 3$. It was solved in dimension 3 over a field characteristic zero in [SU04], and is an open problem for $n \geqslant 4$.

On the one hand, one could say that our Theorem B is much stronger, since we consider the *normal* subgroup generated by these elements, and we allow some extra generators of bounded degree. It is not known (even if not very likely) whether one can generate $\mathrm{Aut}(\mathbb{A}^3)$ with linear automorphisms, elementary automorphisms and one single automorphism, and *a fortiori* neither whether the normal subgroup generated by these is the whole group $\mathrm{Aut}(\mathbb{A}^3)$ (this last statement, even without the extra automorphism, seems more plausible).

On the other hand, even in dimension 3 we should stress that Theorem B does not recover a solution to the Tame Problem. As will become clear below, the reason is that the group $\mathrm{Aut}(\mathbb{A}^3)$, or more generally the group $\mathrm{Bir}_0(\mathbb{P}^3)$ of birational maps that contract only rational hypersurfaces (birational maps of genus 0 in the sense of Frumkin [Fru73, Lam14]), lies in the kernel of all group homomorphisms produced by Theorem A. In fact we are not aware of any elements of $\mathrm{Aut}(\mathbb{A}^3)$ which have been proved to lie outside the group generated, in $\mathrm{Bir}(\mathbb{P}^3)$, by linear and Jonquières maps (see [BH15, Proposition 6.8] for the case of the Nagata automorphism, which can be easily generalised to automorphisms given by a $\mathbb{G}_a$ action, as all algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ isomorphic to $\mathbb{G}_a$ are conjugate).

1.D. **Overwiew of the strategy.** By the Minimal model program, every rationally connected variety Z is birational to a Mori fibre space, and every birational map between two Mori fibre spaces is a composition of simple birational maps, called *Sarkisov links* (see Definition 3.7 and Theorem 4.8). We associate to such a variety Z the groupoid $\mathrm{BirMori}(Z)$ of all birational maps between Mori fibre spaces birational to Z. We then concentrate on some special Sarkisov links, called *Sarkisov links of conic bundles of type* II (see Definition 3.7 and 3.8).

To each Sarkisov links of conic bundles of type II, we associate a marked conic, which is a pair $(X/B, \Gamma)$, where X/B is a conic bundle (a terminal Mori fibre space with $\dim(B) = \dim(X) - 1$) and $\Gamma \subseteq B$ is an irreducible hypersurface (see definition 3.20 and Lemma 3.21). We then define a natural equivalence relation between marked conics (Definition 3.20), and then say that two Sarkisov links of conic bundles of type II are equivalent if they have equivalent marked conics.

We associate to each of these Sarkisov links χ an integer $\mathrm{cov.gon}(\chi)$ that measures the degree of irrationality of the base locus of χ (see § 2.G).

For each variety Z, we denote by $\mathcal{C}_Z$ the set of equivalence classes of conic bundles X/B with X birational to Z, and for each class of conic bundles $C \in \mathcal{C}_Z$ we denote by $\mathcal{M}_C$ the set of equivalence classes of marked conics $(X/B, \Gamma)$, where C is the class of X/B.

The Sarkisov program is established in every dimension [HM13] and relations among them are described in [Kal13]. Inspired by the latter, we define *rank r fibrations X/B* (see Definition 3.1); rank 1 fibrations are Mori fibres spaces and rank 2 fibrations dominate Sarkisov links (see Lemma 3.6). We prove that the relations among Sarkisov links are generated by *elementary relations* (Definition 4.6), which are relations dominated by rank 3 fibrations (see Proposition 4.8).

The BAB conjecture, proven in [Bir16a] and [Bir16b], tells us that the set of weak Fano terminal varieties of dimension n form a bounded family and the degree of their images by a (universal) multiple of the anticanonical system is bounded by a (universal) integer d (see Proposition 5.1). We show that any Sarkisov link χ of conic bundles of type II appearing in an elementary relation over a base of small dimension has $\text{cov.gon}(\chi) \leqslant d$ (see Proposition 5.2). This and the description of the elementary relations over a base of maximal dimension and including a Sarkisov link of conic bundles of type II (Proposition 5.3) allows us to prove the following statement in §5.C.

Theorem C. *Let $n \geqslant 3$. There is an integer $d \geqslant 2$ such that for every conic bundle X/B, where X is a rationally connected variety of dimension n, we have a groupoid homomorphism*

$$\text{BirMori}(X) \longrightarrow \mathop{\ast}_{C\in\mathcal{C}_X} \left(\bigoplus_{\mathcal{M}_C} \mathbf{Z}/2\right)$$

that sends each Sarkisov link of conic bundles χ of type II *with* $\text{cov.gon}(\chi) \geqslant d$ *onto the generator indexed by its marked conic, and all other Sarkisov links and all automorphisms onto zero.*

Moreover it restricts to group homomorphisms

$$\text{Bir}(X) \longrightarrow \mathop{\ast}_{C\in\mathcal{C}_X} \left(\bigoplus_{\mathcal{M}_C} \mathbf{Z}/2\right), \qquad \text{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2.$$

In order to deduce Theorem A and Theorem B, we study the image of the group homomorphism from $\text{Bir}(X)$ and $\text{Bir}(X/B)$ provided by Theorem C, for some conic bundle X/B. We give a criterion to compute the image in §6.A. We then apply this criterion to show that the image is very large if the generic fibre of X/B is $\mathbb{P}^1$ (or equivalently if X/B has a rational section, or is equivalent to $(\mathbb{P}^1 \times B)/B$). This is done in §6.$B$ and allows us to prove Theorem A. We finish by studying the more complicated case where the generic fibre X/B is not $\mathbb{P}^1$ (or equivalently if X/B has no rational section) in Section 6.C. We manage to give such examples where X is a rational threefold and where the image of the group homomorphism is big enough to deduce Theorem B.

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2. Preliminaries

From now on and until the end of Section 4, all ambient varieties are irreducible and projective over the field $\mathbf{C}$ of complex numbers. This restriction on the ground field comes from the fact that this is the setting of many references that we use, such as [BCHM10, HM13, Kal13, KKL16]. It seems to be folklore that all the results in these papers are also valid over any algebraically closed field of characteristic zero, but we let the reader take full responsibility if he is willing to deduce that our results automatically hold over such a field. However, in Section 6, we will also show how to work over fields that can be embedded in $\mathbf{C}$.

General references for this section are [KM98, Laz04, BCHM10].

2.A. **Divisors and curves.** Let X be a normal variety, and $\mathrm{Div}(X)$ the group of Cartier divisors. The *Néron-Severi* space $N^1(X) = \mathrm{Div}(X) \otimes \mathbf{R}/\equiv$ is the space of $\mathbf{R}$-divisors modulo numerical equivalence. This is a finite-dimensional vector space whose dimension $\rho(X)$ is called the *Picard rank* of X. We denote $N_1(X)$ the dual space of 1-cycles with real coefficients modulo numerical equivalence. We have a perfect pairing $N^1(X) \times N_1(X) \longrightarrow \mathbf{R}$ induced by intersection. If we need to work with coefficients in $\mathbf{Q}$ we will use notation such as $N^1(X)_{\mathbf{Q}} := \mathrm{Div}(X) \otimes \mathbf{Q}/\equiv$ or $\mathrm{Pic}_{\mathbf{Q}}(X) := \mathrm{Pic}(X) \otimes \mathbf{Q}$.

The *effective* cone $\mathrm{Eff}(X) \subseteq N^1(X)$ is the cone generated by effective divisors on X. The closure of $\mathrm{Eff}(X)$ is the cone of *pseudo-effective* classes, and its interior is the cone of *big* classes. We say that a divisor D is *nef* if for any curve C we have $D \cdot C \geqslant 0$. The cone $\mathrm{Nef}(X) \subseteq N^1(X)$ of nef classes is a closed cone, whose interior is the cone of *ample* divisors. The *movable* cone $\overline{\mathrm{Mov}}(X)$ is the closure of the cone spanned by classes of divisor whose base locus has codimension at least 2.

We denote $NE(X) \subseteq N_1(X)$ the cone of effective 1-cycles. We say that a class $C \in \overline{NE}(X)$ is *extremal* if any equality $C = C_1 + C_2$ inside $\overline{NE}(X)$ implies that C, C_1, C_2 are proportional.

We say that a Weil divisor D on X is $\mathbf{Q}$-*cartier* is mD is Cartier for some integer $m > 0$. The variety X is $\mathbf{Q}$-*factorial* if all Weil divisors on X are $\mathbf{Q}$-Cartier.

2.B. **Maps.** Let $\pi\colon X \longrightarrow Y$ be a surjective morphism between normal varieties. We shall also denote X/Y such a situation. The integer $\rho(X/Y) := \rho(X) - \rho(Y)$ is the *relative Picard rank* of π.

We say that a curve $C \subseteq X$ is *contracted* by π is $\pi(C)$ is a point. The subset $NE(X/Y) \subseteq N_1(X/Y) \subseteq N_1(X)$ are respectively the cone and the subspace generated by curves contracted by π. The relative Picard group $\mathrm{Pic}(X/Y)$ is defined as the quotient of $\mathrm{Pic}(X)$ by the orthogonal of $N_1(X/Y)$. Similarly the relative Néron-Severi group $N^1(X/Y)$ is a quotient of $N^1(X)$, and we denote by $\mathrm{Eff}(X/B)$, $\mathrm{Nef}(X/B)$, $\overline{\mathrm{Mov}}(X/B)$ the images of the corresponding cones of $N^1(X)$ in this quotient.

Let $D \in \mathrm{Div}(X)$ be a Cartier divisor. We say that D is π-*ample* if $D \cdot C > 0$ for any contracted curve, and that D is π-*big* if the restriction of D to the generic fibre is big. We have the following caracterisation for this last notion:

Lemma 2.1. *Let $\pi\colon X \longrightarrow Y$ be a surjective morphism between normal varieties. A divisor D on X is π-big if and only if we can write D has a sum*

$$D = \pi\text{-}ample + effective.$$

Proof. The restriction of the divisor D to the generic fibre is big, or equivalently it it a sum $A+E$ of an ample divisor A and an effective divisor E on the generic fibre. Then A corresponds to a π-ample divisor on X, and E to the sum of an effective divisor on X with a π-trivial divisor. We can absorb this π-trivial divisor in the π-ample factor to get the result. ∎

When the morphism $\pi\colon X \longrightarrow Y$ is birational, the *exceptional set* $\mathrm{Ex}(\pi)$ is the set covered by all contracted curves. Assume moreover that $\rho(X/Y) = 1$, and that X is $\mathbf{Q}$-factorial. Then we are in one of the following situations [KM98, Prop 2.5]: either $\mathrm{Ex}(\pi)$ is an irreducible divisor, and we say that π is a *divisorial contraction*, or $\mathrm{Ex}(\pi)$ has codimension at least 2 in X, and we say that π is a *small contraction*.

Taking three normal varieties X, Y, B together with dominant morphisms X/B, Y/B, we say that $\varphi\colon X \dashrightarrow Y$ is a *rational map over* B if we have a commutative diagram

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & Y \\ & \searrow \qquad \swarrow & \\ & B & \end{array}$$

Now let $\varphi\colon X \dashrightarrow Y$ be a birational map. Any Weil divisor D on X is sent to a well-defined cycle φ_*D on Y: we say that φ induces a map in codimension 1. If $\operatorname{codim} \varphi_*D \geqslant 2$, we say that φ contracts the divisor D. A *birational contraction* is a birational map such that the inverse does not contract any divisor, or equivalently a birational map which is surjective in codimension 1. A *pseudo-isomorphism* is a birational map which is an isomorphism in codimension 1. Birational morphisms and pseudo-isomorphisms (and compositions of those) are examples of birational contractions.

We use a dashed arrow $\dashrightarrow$ to denote a rational (or birational) map, a plain arrow $\longrightarrow$ for a morphism, and a dotted arrow $\cdots\!\!\rightarrow$, or simply a dotted line, to indicate a pseudo-isomorphism.

We denote by $\mathrm{Bir}(X)$ the group of birational selfmaps of X. Given a dominant morphism $\eta\colon X \longrightarrow B$, we denote

$$\mathrm{Bir}(X/B) := \{\varphi \in \mathrm{Bir}(X) \mid \eta \circ \varphi = \eta\} \subseteq \mathrm{Bir}(X).$$

2.C. **Minimal model program.** Let X be a normal variety, and $C \in \overline{NE}(X)$ an extremal class. We say that the *contraction* of C exists (and in that case it is unique), if there exists a dominant morphism $\pi\colon X \longrightarrow Y$ with connected fibres to a normal variety Y, with $\rho(X/Y) = 1$, and such that any curve contracted by π is numerically proportional to C. If π is a small contraction, we say that the *log-flip* of C exists (and again, in that case it is unique) is there exists $X \cdots\!\!\rightarrow X'$ a pseudo-isomorphism over Y (which is not an isomorphism), such that X' is normal and $X' \longrightarrow Y$ is a small contraction that contracts curves proportional to a class C'. For each $D \in \mathbf{N}^1(X)$, if D' is the image of D under the pseudo-isomorphism, we have a sign change between $D \cdot C$ and $D' \cdot C'$ (see [Kol91, 2.1.6]). We say that $X \cdots\!\!\rightarrow X'$ is a *D-flip*, resp. a *D-flop*, resp. a *D-antiflip* when $D \cdot C < 0$, resp. $D \cdot C = 0$, resp. $D \cdot C > 0$.

A *step* in the D-Minimal Model Program (or in the D-MMP for short) is the removal of an extremal class C with $D \cdot C < 0$, either via a divisorial contraction, or via a D-flip. If D is nef on X, we say that X is a *D-minimal model*. If there exists a contraction $X \longrightarrow Y$ with $\rho(X/Y) = 1$, $\dim Y < \dim X$ and $-D$ relatively ample, we say that X/Y is a *D-Mori fibre space* (or D-Mfs for short).

When $D = K_X$ is the canonical divisor, we usually omit the mention of the divisor in the previous notations.

2.D. **Singularities.** We recalled the basic terminology of MMP without assumption on singularities, but to get existence results, one needs such assumption. A first basic fact is that $\mathbf{Q}$-factoriality is preserved under all operations of the MMP.

Precisely, assume that X is a normal $\mathbf{Q}$-factorial variety. If $\pi\colon X \longrightarrow Y$ is a divisorial contraction or a Mori fibre space, then Y is $\mathbf{Q}$-factorial. If $\pi\colon X \longrightarrow Y$ is a small contraction and $X \dashrightarrow X'$ is the associated log-flip, then X' is $\mathbf{Q}$-factorial, but Y is not (see [KM98, 3.36 & 3.37]).

Now let X be a normal variety such that the canonical divisor K_X is $\mathbf{Q}$-Cartier, and let $\pi\colon Z \longrightarrow X$ be a resolution of singularities, with exceptional divisors E_1, $\dots$, E_r. We say that X has *terminal singularities*, or that X is terminal, if in the ramification formula

$$K_Z = \pi^* K_X + \sum a_i E_i,$$

we have $a_i > 0$ for each i. Similarly we say that X has *Kawamata log terminal* (*klt* for short) singularities, or that X is klt, if $a_i > -1$ for each i.

Lemma 2.2. *Let $\pi\colon X \longrightarrow Y$ be a divisorial contraction, with exceptional divisor $E = \mathrm{Ex}(\pi)$, and contracting the class of a curve C. If $D \in \mathrm{Div}(X)$ and $D' = \pi_* D$, then in the ramification formula*

$$D = \pi^* D' + aE,$$

the numbers a and $D \cdot C$ have opposite signs.

Proof. We have $D \cdot C = aE \cdot C$, so the claim is that $E \cdot C < 0$. For this, see for instance [Mat02, proof of 8-2-1(i)]. □

In particular, if we start with a $\mathbf{Q}$-factorial terminal variety and we run the classical MMP (that is, relatively to the canonical divisor), then each step (divisorial contraction or flip) of the MMP keeps us in the category of $\mathbf{Q}$-factorial terminal varieties. Moreover, when one reaches a Mori fibre space X/B, the base B is $\mathbf{Q}$-factorial as mentioned above, but might not be terminal. However by the following result B has at worst klt singularities.

Proposition 2.3 ([Fuj99, Corollary 4.6])**.** *Let X/B be a Mori fibre space, where X is a $\mathbf{Q}$-factorial klt variety. Then B also is a $\mathbf{Q}$-factorial klt variety.*

The following class of Mori fibre spaces will be of special importance to us.

Definition 2.4. A *conic bundle* is a $\mathbf{Q}$-factorial terminal Mori fibre space X/B with $\dim B = \dim X - 1$. The *discriminant locus* of X/B is defined as the union of irreducible hypersurfaces $\Gamma \subseteq B$ such that the preimage of a general point of Γ is not irreducible. We emphasise that the terminology of conic bundle is often used in a broader sense (for instance, for any morphism whose general fibre is isomorphic to $\mathbb{P}^1$, with no restriction on the singularities of X or on the relative Picard rank), but for our purpose we will stick to the above more restricted definition.

We say that two conic bundles X/B and X'/B' are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \\ B & \overset{\theta}{\dashrightarrow} & B' \end{array}$$

where ψ, θ are birational.

The singular locus of a terminal variety has codimension at least 3. This fact is crucial in the following result.

Lemma 2.5 ([KM92, Theorem 4.9])**.** *Let $\pi\colon X \longrightarrow Y$ be a divisorial contraction between $\mathbf{Q}$-factorial terminal varieties, with exceptional divisor E, and assume that $\Gamma = \pi(E)$ has codimension 2 in Y. Then π is the blow-up of the symbolic powers of the sheaf of ideals $\mathcal{I}$ defining the reduced scheme Γ. In particular, the fibre f over a general point of Γ (precisely, a point that is smooth for both Γ and Y) is a smooth rational curve such that $K_X \cdot f = E \cdot f = -1$.*

2.E. **Mori dream spaces.**

Definition 2.6. Let $\eta\colon X \longrightarrow B$ be a dominant morphism between normal varieties, with X $\mathbf{Q}$-factorial. We say that X/B is a (relative) *Mori dream space* if the following conditions hold:

(1) $\mathrm{Pic}_{\mathbf{Q}}(X/B) = N^1_{\mathbf{Q}}(X/B)$;
(2) $\mathrm{Nef}(X/B)$ is the affine hull of finitely many semiample models;
(3) There are finitely many birational maps $f_i\colon X \dashrightarrow X_i$ over B to projective $\mathbf{Q}$-factorial varieties X_i such that each f_i is an isomorphism in codimension 1 and each X_i satisfies (2), and $\overline{\mathrm{Mov}}(X/B) = \bigcup f_i^*(\mathrm{Nef}(X_i/B))$.

We will often ensure condition (1) in the definition via the following lemma. Recall that a variety is *rationally connected* if any two general points are contained in a rational curve (see [Kol96, IV.3]).

Lemma 2.7. *Let X be a rationally connected $\mathbf{Q}$-factorial terminal variety. Then* $\mathrm{Pic}_{\mathbf{Q}}(X) = N^1_{\mathbf{Q}}(X)$*, hence also*

$$\mathrm{Pic}_{\mathbf{Q}}(X/B) = N^1_{\mathbf{Q}}(X/B)$$

for any morphism $X \longrightarrow B$.

Proof. Recall that when we discard torsion by tensoring with $\mathbf{Q}$, then algebraic and numerical equivalences of divisors coincide. So the equality $\mathrm{Pic}_{\mathbf{Q}}(X) = N^1_{\mathbf{Q}}(X)$ is about the coincidence between linear and algebraic equivalences. Using the exponential sequence (see [Har77, Appendix B.5]), one sees that the Picard variety of divisors algebraically equivalent to zero modulo linear equivalence is isomorphic to the torus $H^1(X, \mathcal{O}_X)/H^1(X, \mathbf{Z})$. In particular it is sufficient to get $H^1(X, \mathcal{O}_X) = 0$, since then the Picard variety is trivial and we get $\mathrm{Pic}_{\mathbf{Q}}(X) = N^1_{\mathbf{Q}}(X)$.

Now consider a resolution of singularities $\pi\colon Y \to X$. Since X is $\mathbf{Q}$-factorial and terminal, it has at most rational singularities. This means, by definition, that $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$. Then [Har77, III, Ex.8.1] implies that $H^i(Y, \mathcal{O}_Y) \simeq H^i(X, \pi_*\mathcal{O}_Y) = H^i(X, \mathcal{O}_X)$ for all $i \geqslant 0$. Finally $H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0$ by [Kol96, IV.3.8] □

Let $L_1, \dots, L_\rho \in \mathrm{Pic}(X)$ be representatives of a basis of $\mathrm{Pic}_{\mathbf{Q}}(X/B)$. Then a Cox ring of X/B is defined as

$$\mathrm{Cox}(X/B) = \bigoplus_{(m_1,\dots,m_\rho)\in\mathbf{Z}^\rho} H^0(m_1L_1 + \cdots + m_\rho L_\rho).$$

The property of finite generation for such a ring does not depend on the choice of a basis, even if the Cox ring do (see [KKL16, Lemma 3.1]).

Lemma 2.8. *Let X/B be a dominant morphism between $\mathbf{Q}$-factorial normal varieties, such that* $\mathrm{Pic}_{\mathbf{Q}}(X/B) = N^1_{\mathbf{Q}}(X/B)$*. Then X/B is a relative Mori dream space if and only if its Cox ring* $\mathrm{Cox}(X/B)$ *is finitely generated.*

Proof. Similar to the proofs in the non-relative setting of [KKL16, Corollaries 4.4 and 5.7]. □

Corollary 2.9. *If X/B is a Mori dream space, then for any divisor D on X one can run a relative D-MMP from X over B: contraction morphisms exist, in the case of a small contraction the D-flip exists, and there is no infinite sequence of D-flips. Moreover $\mathbf{Q}$-factoriality is preserved under divisorial contractions and log-flips.*

Example 2.10. Standard examples of Mori dream spaces (in the non relative case) are toric varieties and Fano varieties. Both of these classes of varieties are special examples of log Fano varieties, which are Mori dream spaces by [BCHM10, Corollary 1.3.2].

2.F. **Two-rays game.** Let $Y \longrightarrow X \longrightarrow B$ be two dominant morphisms between normal varieties, such that Y/B is a Mori dream space and $\rho(Y/X) = 2$. Then $NE(Y/X)$ is a closed 2-dimensional cone, generated by two extremal classes C_1, C_2. Pick some classes $D_1, D_2 \subseteq N^1(Y)$ such that $D_i \cdot C_i < 0$, $i = 1, 2$. Moreover substracting a large multiple of an ample divisor we can assume that both D_i are not pseudoeffective, so that a D_i-minimal model does not exist. Then by running a D_i-MMP from Y over X, which starts by the contraction of the class C_i, one produces a commutative diagram that we call the *two-rays game* associated to Y/X (and which does not depend on the choice of D_i):

$$\begin{array}{ccccc} Y_1 & \dashleftarrow & Y & \dashrightarrow & Y_2 \\ \downarrow & & & & \downarrow \\ X_1 & & & & X_2 \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

Here $Y \dashrightarrow Y_i$ is a (possibly empty) sequence of D_i-flips, and $Y_i \longrightarrow X_i$ is either a divisorial contraction or a D_i-Mori fibre space.

Now we give a few direct consequences of the two-rays game construction.

Lemma 2.11. *Let $X_1 \dashrightarrow X_2$ be a sequence of relative log-flips over B, and $Y_1 \longrightarrow X_1$ a divisorial contraction. Assume that Y_1/B is a Mori dream space. Then there exists a sequence of log-flips $Y_1 \dashrightarrow Y_2$ over B such that the induced map $Y_2 \longrightarrow X_2$ is a divisorial contraction.*

Proof. By induction, it is sufficient to consider the case where $X_1 \dashrightarrow X_2$ is a single log-flip over a non $\mathbf{Q}$-factorial variety X dominating B, given by a diagram

$$\begin{array}{ccccc} X_1 & & \dashrightarrow & & X_2 \\ & \searrow & & \swarrow & \\ & & X & & \\ & & \downarrow & & \\ & & B & & \end{array}$$

In this situation, the two-rays game Y_1/X gives a diagram

$$\begin{array}{ccccc} Y_1 & & \cdots\cdots & & Y_2 \\ \downarrow & & & & \downarrow \\ X_1 & & & & X' \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

where $Y_1 \dashrightarrow Y_2$ is a sequence of log-flips and $Y_2 \longrightarrow X'$ is a divisorial contraction. By unicity of the log-flip associated to the small contraction $X_1 \longrightarrow X$, we conclude that $X' = X_2$. □

We recall the following criterion for a pseudo-isomorphism to be an isomorphism, which follows from the Negativity Lemma, and two corollaries (adapted from [Cor95, Proposition 3.5]).

Lemma 2.12 ([Cor95, Proposition 2.7])**.** *Let $\varphi\colon Y \dashrightarrow Y'$ be a pseudo-isomorphism between $\mathbf{Q}$-factorial varieties. If there is an ample divisor H on Y such that $\varphi_* H$ is ample on Y', then φ is an isomorphism.*

Corollary 2.13. *Let X/B and X'/B be Mori fibre spaces over the same base B, and $\varphi\colon X \dashrightarrow X'$ a pseudo-isomorphism over B, that is, the following diagram commutes:*

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X' \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\eta'} & \\ & B & \end{array}$$

Then φ is an isomorphism.

Proof. Let A_B, A_X be ample divisors respectively on B and X, and $0 < \varepsilon \ll 1$. If C is a general curve contracted by X/B, then φ is an isomorphism in a neighborhood of C, hence $\varphi_* A_X$ is relatively ample on X'/B. Then $\eta^* A_B + \varepsilon A_X$ and $\eta'^* A_B + \varphi_* \varepsilon A_X$ are both ample, and we conclude by Lemma 2.12. □

Corollary 2.14. *Consider a commutative diagram*

$$\begin{array}{ccc} Y & \overset{\varphi}{\dashrightarrow} & Y' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

where X, Y, Y' are $\mathbf{Q}$-factorial varieties, π and π' are divisorial contractions with respective exceptional divisor E, E', and φ is a pseudo-isomorphism. If $\varphi_* E = E'$, then φ is an isomorphism.

Proof. Pick A a general ample divisor on X and $0 < \varepsilon \ll 1$, and consider $H = \pi^* A - \varepsilon E$, $H' = \pi'^* A - \varepsilon E'$. Both H and H' are ample, and we have $H' = \varphi_* H$, so by Lemma 2.12 we conclude that $Y \dashrightarrow Y'$ is an isomorphism. □

Lemma 2.15. *Let $T \longrightarrow Y$ and $Y \longrightarrow X$ be two divisorial contractions between $\mathbf{Q}$-factorial varieties, with respective exceptional divisors E and F. Assume that there exists a morphism $X \longrightarrow B$ such that T/B is a Mori dream space. Then there exist two others $\mathbf{Q}$-factorial varieties T' and Y', with a pseudo-isomorphism $T \dashrightarrow T'$ and birational contractions $T' \longrightarrow Y' \longrightarrow X$, with respective exceptional divisors the strict transforms of F and E, such that the following diagram commutes:*

$$\begin{array}{ccc} T & \cdots\cdots\cdots & T' \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle F} \\ Y & & Y' \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle E} & \\ & X & \end{array}$$

Proof. The diagram comes from the two-rays game associated to T/X. The only thing to prove is that the divisors are not contracted in the same order on the two sides of the two-rays game. Assume that both $\pi\colon Y \longrightarrow X$ and $\pi'\colon Y' \longrightarrow X$ contract the strict transforms of the same divisor F. Then by Corollary 2.14 the pseudo-isomorphism $Y \dashrightarrow Y'$ is an isomorphism. Then applying again Corollary 2.14 to the two divisorial contractions from T, T' to $Y \simeq Y'$, with same exceptional divisor E, we obtain that $T \dashrightarrow T'$ also is an isomorphism, contradicting the assumption that the diagram was produced by a two-rays game. □

2.G. **Gonality.** The following definitions are taken from [BDPE+17].

The *gonality* $\mathrm{gon}(C)$ of a curve C is defined to be the least degree of the field extension associated to a dominant rational map $C \dashrightarrow \mathbb{P}^1$.

If C is a curve, then $\mathrm{gon}(C) = 1$ if and only if C is rational. For each smooth curve $C \subseteq \mathbb{P}^2$ of degree > 1 we moreover have $\mathrm{gon}(C) = \deg(C) - 1$. The inequality $\mathrm{gon}(C) \leqslant \deg(C) - 1$ is given by the projection from a general point of C and the other inequality is given a result of Noether (see for instance [BDPE+17]).

Definition 2.16. For each variety X we define the *covering gonality* of X to be

$$\mathrm{cov.gon}(X) = \min \left\{ c > 0 \;\middle|\; \begin{array}{c} \text{Given a general point } x \in X, \exists \text{ an} \\ \text{irreducible curve } C \subseteq X \text{ through } x \text{ with} \\ \mathrm{gon}(C) = c. \end{array} \right\}.$$

The condition $\mathrm{cov.gon}(X) = 1$ corresponds to ask that the union of all rational curves of X is dense in X. This is sometimes called *uniruled* and is always satisfied if X is rationally connected. The converse does not hold, as $\mathrm{cov.gon}(X \times \mathbb{P}^n) = 1$ for each variety X and each $n \geqslant 1$.

The covering gonality is an integer which is invariant under birational maps. Moreover, if $X \subseteq \mathbb{P}^n$ is an irreducible closed subvariety, then $\mathrm{cov.gon}(X) \leqslant \deg(X)$, by taking a general projection from X to a linear subspace $\mathbb{P}^{\dim(X)} \subseteq \mathbb{P}^n$ and taking the preimage of lines of $\mathbb{P}^n$, which are curves $C \subseteq X$ with $\mathrm{gon}(C) \leqslant \deg(C)$.

Theorem 2.17 ([BDPE+17, Theorem A])**.** *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant n+2$. Then, $d \geqslant \mathrm{cov.gon}(X) \geqslant d - n$.*

Lemma 2.18. *Let X, Y be irreducible algebraic varieties, and let $\varphi \colon X \longrightarrow Y$ be a dominant rational morphism.*

(1) *If $\dim(X) = \dim(Y)$, we have*

$$\mathrm{cov.gon}(X) \leqslant \mathrm{cov.gon}(Y) \cdot \deg(\varphi),$$

where $\deg(\varphi)$ corresponds here to the degree of the associated field extension $\mathbf{C}(Y) \subseteq \mathbf{C}(X)$.

(2) *If $\dim(X) > \dim(Y)$, a general fibre F of φ is a union of algebraic varieties F_i satisfying $\mathrm{cov.gon}(F_i) \geqslant \mathrm{cov.gon}(X)$.*

Proof. (1): By definition of $\mathrm{cov.gon}(Y)$, the union of irreducible curves C of Y with $\mathrm{gon}(C) = \mathrm{cov.gon}(Y)$ is dense on Y. Taking the preimages of general such curves, we obtain a covering of a dense subset of X by irreducible curves D of X with $\mathrm{gon}(D) \leqslant \mathrm{cov.gon}(Y) \cdot \deg(\varphi)$.

(2): Using the Stein factorisation, we can decompose φ into $X \longrightarrow Z \longrightarrow Y$ such that X/Z has connected fibres and Z/Y is a finite morphism. A general fibre G of X/Z is then smooth and irreducible by Bertini's theorem, and satisfies $\mathrm{cov.gon}(G) \geqslant \mathrm{cov.gon}(X)$, as the union of all such fibres is dense in X. As a general fibre of φ is a finite union of general fibres of X/Z, the result follows. □

3. Sarkisov links

3.A. **Rank r fibrations.** The notion of rank r fibration is a key concept in this paper. Essentially these are relative Mori dream spaces with strong constrainsts on singularities. The cases of $r = 1, 2, 3$ will allow us to recover respectively the notion of terminal Mori fibre spaces, of Sarkisov links, and of elementary relations between those. The precise definition is as follows.

Definition 3.1. A variety X with a dominant morphism $\eta\colon X \longrightarrow B$ is a *rank r fibration* if the following conditions hold:

(1) X/B is a Mori dream space (see Definition 2.6);

(2) η has connected fibres, B is normal, $\dim X > \dim B \geqslant 0$ and the relative Picard rank $\rho(X/B)$ is equal to $r \geqslant 1$;

(3) X is $\mathbf{Q}$-factorial with at most terminal singularities, and for any divisor D on X, any D-minimal model or D-Mori fibre space obtained by running a relative D-MMP from X over B is still $\mathbf{Q}$-factorial and terminal;

(4) The anticanonical divisor $-K_X$ is η-big, meaning that the restriction of $-K_X$ to the generic fibre is big.

We say that a rank r fibration X/B *factorises through* a rank r' fibration X'/B', or that X'/B' *is dominated by* X/B, if the fibrations X/B and X'/B' fit in a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & & \longrightarrow & B \\ & \searrow & & & \nearrow \\ & & X' \longrightarrow B' & & \end{array}$$

where $X \dashrightarrow X'$ is a birational contraction, and $B' \longrightarrow B$ is a morphism. This implies $r \geqslant r'$.

Example 3.2. (1) For $r = 1$, condition (1) is empty and one recovers the usual definition of a terminal Mori fibre space.

(2) Let p_1, p_2 be two distinct points on a fibre f of $\mathbb{P}^1 \times \mathbb{P}^1/\mathbb{P}^1$, and consider $S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the blow-up of p_1 and p_2. Then S is a weak del Pezzo toric surface of Picard rank 4, hence in particular S/pt is a Mori dream space. However S/pt is *not* a rank 4 fibration, because when contracting the strict transform of f one gets a (non terminal) singular point, which is forbidden by condition (3) of Definition 3.1.

Lemma 3.3. *Let X/B be a rank r fibration. If Y is obtained from X by performing a log-flip* (*resp. a divisorial contraction*) *over B, then Y/B is a rank r fibration* (*resp. a rank* $(r-1)$*-fibration*)*.*

Proof. Condition (2) holds by construction, (3) follows from Corollary 2.9 and (4) is preserved under birational maps. So we are left with condition (1). If $\pi\colon X \dashrightarrow Y$ is a divisorial contraction or a log-flip, and $L_1, \dots, L_\rho \in \mathrm{Pic}_{\mathbf{Q}}(X)$ are representatives of a basis of $\mathrm{Pic}_{\mathbf{Q}}(X/B)$, we have

$$H^0(X, m_1L_1 + \cdots m_\rho L_\rho) \simeq H^0(Y, m_1\pi_*(L_1) + \cdots + m_\rho\pi_*(L_\rho))$$

for all $m_1, \dots, m_\rho \in \mathbf{Z}$. Therefore, if X/B is a Mori dream space, so is Y/B by Lemma 2.8. □

Lemma 3.4. *Any rank r fibration X/B is pseudo-isomorphic, via a sequence of log-flips over B, to another rank r fibration Y/B such that $-K_Y$ is relatively nef and big over B.*

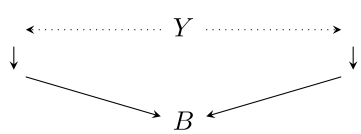
Proof. We run a $(-K)$-MMP from X over B (recall that by Corollary 2.9, one can run a D-MMP for an arbitrary divisor D). It is not possible to have a divisorial contraction, because by Lemma 2.2 the resulting singularity would not be terminal, in contradiction with assumption (3) in the definition of rank r fibration. If there exists an extremal class that gives a small contraction, we anti-flip it. After finitely many such steps, either $-K$ is nef, or there exists a fibration such that K is relatively ample. But this last situation contradicts the assumption (4) that the anti-canonical divisor is big over B. So finally $-K_X$ is also nef over B, as expected. □

Corollary 3.5. *Let $\eta\colon Y \longrightarrow B$ be a rank r fibration such that $-K_Y$ is relatively nef and big over B. Then for a general point $p \in B$, the fibre $Y_p := \eta^{-1}(p)$ is a weak Fano terminal variety, and the curves in Y_p that are trivial against the canonical divisor cover a subset of codimension at least 2.*

Proof. The fact that Y_p is terminal follows from [Kol97, 7.7] by taking successive hyperplane sections on B locally defining p. Let $\Gamma \subseteq Y$ be the subset covered by curves contracted by Y/B that are trivial against the canonical divisor. Then consider the rational map $\varphi := |-mK_Y| \times \eta\colon X \dashrightarrow \mathbb{P}^N \times B$. By [Kol93], φ is a morphism, and it is a birational contraction onto its image. If Γ contains a divisor E, then E is contracted by φ, and by Lemma 2.2 this would produce a non-terminal singularity, in contradiction with the definition of rank r fibration. So Γ has codimension at least 2 in Y, hence $\Gamma_p = \Gamma \cap Y_p$ has codimension at least 2 in Y_p for a general p. Since by adjunction $K_{Y_p} = K_Y|_{Y_p}$, Γ_p is exactly the locus of contracted curves in Y_p with trivial intersection against K_{Y_p}. The fact that $-K_{Y_p}$ is big over B follows from Lemma 2.1, by restricting to Y_p a decomposition $-K_Y = \eta$-ample + effective. □

The notion of rank 2 fibration recovers the notion of Sarkisov link:

Lemma 3.6. *Let Y/B be a rank 2 fibration. Then Y/B factorises through exactly two rank 1 fibrations $X_1/B_1, X_2/B_2$, which both fit into a diagram*



where the top dotted arrows are sequences of log-flips, and the other four arrows are morphisms of relative Picard rank 1.

Proof. The diagram comes from the two-rays game associated to Y/B, as explained in §2.F. Morever, since $\dim Y > \dim B$, on each side of the diagram exactly one of the two descending arrows corresponds to a Mori fibre space of the same dimension as Y. □

Definition 3.7. In the situation of Lemma 3.6, we say that the birational map $\chi\colon X_1 \dashrightarrow X_2$ is a *Sarkisov link*. The diagram is called a *Sarkisov diagram*. Observe that a rank 2 fibration uniquely defines a Sarkisov diagram, but the Sarkisov link is only defined up to taking inverse.

If a rank r fibration factorises through Y/B, we equivalently say that it *factorises through* the Sarkisov link associated to Y/B.

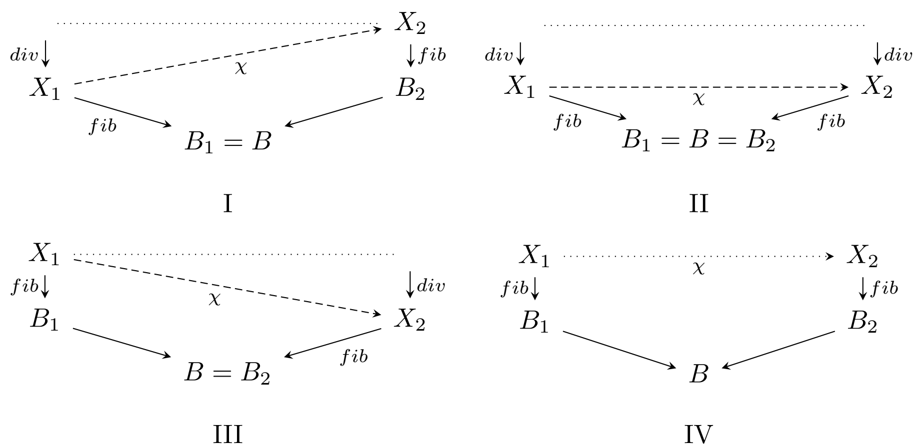
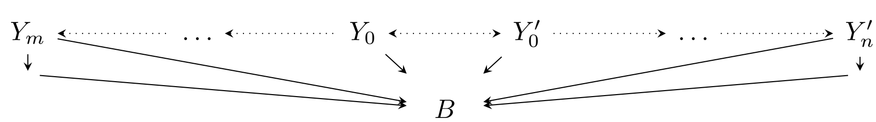


Figure 1. The four types of Sarkisov links.

We say that the Sarkisov link associated with a rank 2 fibration Y/B is a *Sarkisov link of conic bundles* if $\dim B = \dim X - 1$. Observe that in this situation both X_1/B_1 and X_2/B_2 are indeed conic bundles in the sense of Definition 2.4.

Definition 3.8. In the diagram of Lemma 3.6, there are two possibilities for the sequence of two morphisms on each side of the diagram: either the first arrow is already a Mori fibre space, or it is divisorial and in this case the second arrow is a Mori fibre space. This gives 4 possibilities, which correspond to the usual definition of *Sarkisov links of type* I*,* II*,* III *and* IV, as illustrated on Figure 1.

Remark 3.9. The definitions of a Sarkisov link in the literature are usually not very precise about the pseudo-isomorphism involved in the top row of the diagram. An exception is [CPR00, Definition 3.1.4(b)], but even there they do not make clear that there is at most one flop, and that all varieties admit morphisms to a common B. In fact, there are strong constraints about the sequence of anti-flips, flops and flips (that is, about the sign of the intersection of the exceptional curves against the canonical). Precisely, the top row of a Sarkisov diagram has the following form:



where $Y_0 \dashrightarrow Y_0'$ is a flop over B (or an isomorphism), $m, n \geqslant 0$, and each $Y_i \dashrightarrow Y_{i+1}$, $Y_i' \dashrightarrow Y_{i+1}'$ is a flip over B. This follows from the fact that for $Y = Y_i$ or Y_i', a general contracted curve C of the fibration Y/B satisfies $K_Y \cdot C < 0$, hence at least one of the two extremal rays of the cone $NE(Y/B)$ is strictly negative against K_Y.

Observe also that both Y_0/B and Y_0'/B are relatively weak Fano (or Fano if the flop is an isomorphism) over B, as predicted by Lemma 3.4. All other Y_i/B and Y_i'/B are not weak Fano over B, but still each is a rank 2 fibration and uniquely defines the Sarkisov diagram.

In the case of threefolds, we also emphasise that the unique flop $Y_0 \dashrightarrow Y_0'$ might involve several components, which by definition are all numerically proportional. For instance in [AZ17, §5.2], each arrow labeled "n flops" really corresponds to a single flop with n components.

Lemma 3.10. *Consider a Sarkisov link of type* II*:*

$$\begin{array}{ccccc} Y_1 & & \overset{\varphi}{\dashrightarrow} & & Y_2 \\ {\scriptstyle \pi_1}\downarrow & & & & \downarrow{\scriptstyle \pi_2} \\ X_1 & & \overset{\chi}{\dashrightarrow} & & X_2 \\ & \searrow & & \swarrow & \\ & & B & & \end{array}$$

and denote E_1, E_2 the respective exceptional divisors of π_1, π_2. Then $\varphi_* E_1 \neq E_2$.

Proof. Assume that $\varphi_* E_1 = E_2$. Then $\chi\colon X_1 \dashrightarrow X_2$ is a pseudo-isomorphism, hence an isomorphism by Corollary 2.13. Then Corollary 2.14 implies that the pseudo-isomorphism $\varphi\colon Y_1 \dashrightarrow Y_2$ also is an isomorphism, contradicting the assumption that the diagram was the result of the two-rays game from Y_1/B. □

Lemma 3.11. *Let X/B be a rank 2 fibration that factorises through a rank 1 fibration $\sigma\colon X \longrightarrow B'$, with $\dim X - 1 = \dim B' > \dim B$. Then $\eta\colon B' \longrightarrow B$ is a klt Mori fibre space, and in particular for any $p \in B$ we have $\operatorname{cov.gon}(\eta^{-1}(p)) = 1$.*

Proof. First B' is klt by Proposition 2.3. We need to show that $-K_{B'}$ is η-ample, and then $\operatorname{cov.gon}(\eta^{-1}(p)) = 1$ by [Kaw91, Theorem 1].

By assumption $\rho(B'/B) = 1$, so we only need to show that there exists a contracted curve $C \subset B'$ such that $-K_{B'} \cdot C > 0$. Since $\dim B' > \dim B$, the contracted curves cover B', so we can choose C not contained in the discriminant locus $\Delta \subset B'$ of $\sigma\colon X \longrightarrow B'$, and such that the surface $\sigma^{-1}(C)$ does not contain any of the curves $C' \subset X$ with $-K_X \cdot C' \leqslant 0$ (such curves cover at most a codimension 2 subset of X). As in [MM86, Corollary 4.6], we have $-4K_{B'} \equiv \sigma_*(-K_X)^2 + \Delta$. Intersecting with C, we obtain

$$\begin{aligned} -4K_{B'} \cdot C &= \sigma_*(-K_X)^2 \cdot C + \Delta \cdot C \\ &\geqslant (-K_X)^2 \cdot \sigma^* C \\ &> 0 \text{ by our choice of } C. \end{aligned}$$
□

3.B. **Rank r fibrations with general fibre a curve.** Let $\eta\colon T \longrightarrow B$ be a rank r fibration, with $\dim B = \dim T - 1$. If $\Gamma \subseteq B$ is an irreducible divisor, we define $\eta^\sharp(\Gamma) \subseteq T$ to be the Zariski closure of all fibres of dimension 1 over Γ. The reason for introducing this notion is twofold: first B might not be $\mathbf{Q}$-factorial, so we cannot consider the pull-back of Γ as a $\mathbf{Q}$-Cartier divisor, and second the preimage $\eta^{-1}(\Gamma)$ might contain superfluous components (see Example 3.12).

Now we distinguish two classes of special divisors in T, and we shall show in Proposition 3.14 below that they account for the relative rank of T/B. Let $D \subseteq T$ be an irreducible divisor. If $\eta(D)$ has codimension at least 2 in B, we say that D is a *divisor of type* I. If $\eta(D)$ is a divisor in B, and the inclusion $D \subsetneq \eta^\sharp(\eta(D))$ is strict, we say that D is a *divisor of type* II.

Example 3.12. We give an example illustrating the definitions above, which also shows that the inclusion $\eta^\sharp(\Gamma) \subseteq \eta^{-1}(\Gamma)$ might be strict. For B an arbitrary base variety, consider $Y = \mathbb{P}^1 \times B$ with Y/B the first projection. Let $\Gamma \subseteq B$ be any irreducible smooth divisor, $D \simeq \mathbb{P}^1 \times \Gamma$ the pull-back of Γ in Y, $\Gamma' = \{t\} \times \Gamma \subseteq D$ a section and $p \in D \smallsetminus \Gamma'$ a point. Let $T \longrightarrow Y$ be the blow-up of Γ' and p, with respective exceptional divisors D' and E, and denote again D the strict transform of $\mathbb{P}^1 \times \Gamma$ in T. Then the induced morphism $\eta\colon T \longrightarrow B$ is a rank 3 fibration , E is a divisor of type I, $D \cup D'$ is a pair of divisors of type II, and

$$\eta^\sharp(\Gamma) = D \cup D' \subsetneq D \cup D' \cup E = \eta^{-1}(\Gamma).$$

Remark 3.13. The similarity between the terminology for Sarkisov links and for special divisors of type I or II is intentional. See Lemma 3.17(2) below.

Proposition 3.14. *Let $\eta\colon T \longrightarrow B$ be a rank r fibration, with* $\dim B = \dim T - 1$.

(1) *For any rank r' fibration T'/B' such that T/B factorises through T'/B', any divisor contracted by the birational contraction $T \dashrightarrow T'$ is a divisor of type* I *or* II *for T/B.*

(2) *Divisors of type* II *always come in pair: for each divisor D_1 of type* II*, there exists another divisor D_2 of type* II *such that*

$$D_1 \cup D_2 = \eta^\sharp(\eta(D_1)) = \eta^\sharp(\eta(D_2)).$$

(3) *If $D_1 \cup D_2$ is a pair of divisors of type* II*, and p a general point of $\eta(D_1) = \eta(D_2)$, then $\eta^{-1}(p) = f_1 \cup f_2$ with $f_i \subseteq D_i$, $i = 1, 2$, some smooth rational curves satisfying*

$$K_T \cdot f_i = -1, \qquad D_i \cdot f_i = -1, \qquad D_1 \cdot f_2 = D_2 \cdot f_1 = 1.$$

(4) *Let $D \subseteq T$ be a divisor of type* I *or* II*. Then there exists a birational contraction over B*

$$T \dashrightarrow X \longrightarrow B$$

that contracts D and such that $\rho(X) = \rho(T) - 1$.

(5) *Assume B is* **Q***-factorial. Let d_1 (resp. d_2) be the number of divisors of type* I *(resp. the number of pairs of divisors of type* II*). Then*

$$r = 1 + d_1 + d_2.$$

Proof. (1): Let D be a prime divisor contracted by $T \dashrightarrow T'$ and suppose that it is neither of type I nor of type II for T/B. By running a D-MMP over B we produce a sequence of log-flips (which do not change the type of special divisors) and then a divisorial contraction. Replacing T by the result of the sequence of log-flips, and T' by the image of the divisorial contraction, we can assume $T \longrightarrow T'$ is a divisorial contraction. By Lemma 2.5, a general fibre f in the exceptional divisor D is a smooth irreducible rational curve. Now $\eta(D) \subseteq B$ is a divisor because D is not of type I, and $D = \eta^\sharp(\eta(D))$ because D is a prime divisor and is not of type II. But then $f = \eta^{-1}(p)$ for some $p \in \eta(D)$, so f is proportional to the fibre of η, in contradiction with the fact that the extremal contraction of f is divisorial.

(2) and (3): Let D_1 be a divisor of type II, and let $D_2, \dots, D_s$ be the other divisors of type II such that

$$\eta^\sharp(\eta(D)) = D_1 \cup \dots \cup D_s.$$

By definition $s \geqslant 2$, we want to prove $s = 2$. Let $p \in \eta(D)$ be a general point, and write $\eta^{-1}(p) = f_1 \cup \dots \cup f_s$ with f_i a curve in D_i. Let f be a general fibre of η. We have $D_i \cdot f = 0$ for each i, $D_i \cdot f_j > 0$ for at least one j (because $\eta^{-1}(p)$ is connected) and $f \equiv f_1 + \dots + f_s$, which gives

$$D_i \cdot f_i < 0.$$

Then by running a D_i-MMP from T over B, one obtains a sequence of log-flips that does not affect the general fibre $\eta^{-1}(p)$, and then a divisorial contraction between **Q**-factorial and terminal varieties, with exceptional divisor D_i and centre of codimension 2. By Lemma 2.5, this implies that f_i is smooth with $K_T \cdot f_i = D_i \cdot f_i = -1$. But $K_T \cdot f = -2$, so we conclude that $s = 2$ as expected. The equality $D_1 \cdot f_2 = 1$ (and similarly $D_2 \cdot f_1 = 1$) follows from $D_1 \cdot f = 0$, $f \equiv f_1 + f_2$ and $D_1 \cdot f_1 = -1$.

To prove (4), it suffices to show that the divisor D is covered by curves ℓ such that $D \cdot \ell < 0$, since then we can get the expected birational contraction by running a D-MMP. When D has type II we already showed that D is covered by such curves. Now let D be a divisor of type I, p a general point in $\eta(D)$, and let $d \geqslant 2$ be the dimension of $\eta^{-1}(p) \cap D$. Now consider a surface $S \subseteq T$ obtained as

$$S = \bigcap_{i=1}^{d-1} H_i \cap \bigcap_{j=1}^{n-d-1} \eta^* H'_j$$

where the H_i are general hyperplane sections of T, and the H'_j general hyperplane sections of B through p. By construction, the curve $\ell = S \cap D$ is contracted to p by η, and $\eta(S)$ is a surface. We obtain $D \cdot \ell = (\ell \cdot \ell)_S < 0$ as expected.

To prove (5) we assume $d_1 = d_2 = 0$, and we want to show $r = 1$, or equivalently, that T/B is a Mori fibre space. Then we run a MMP from T over B. A flip does not change d_1 nor d_2, so we can assume that we have a divisorial contraction or a Mori fibre space. A divisorial contraction would produce a divisor of type I or II (depending on the codimension of the centre), in contradiction with our assumption $d_1 = d_2 = 0$. On the other hand, if $T \longrightarrow B'$ is a Mori fibre space, and $B' \longrightarrow B$ is not an isomorphism, then we can continue the MMP from B' over B (both are $(n-1)$ dimensional **Q**-factorial klt varieties). After a sequence of flips this has to produce a divisorial contraction, hence a divisor of type I in T by pulling-back, and again a contradiction. In conclusion, we have a Mori fibre space T/B, as expected. □

Lemma 3.15. *Let $\eta\colon T \longrightarrow B$ be a rank r fibration with* $\dim B = \dim T - 1$*. Assume that D is a divisor of type* II *for T/B, with* $\operatorname{cov.gon}(\eta(D)) > 1$*. Then for any rank r' fibration T'/B' that factorises through T/B, the strict transform of D is also of type* II *for T'/B'.*

Proof. If D is a divisor of type I for T'/B', then the divisor $\eta(D) \subseteq B$ is one of the divisors contracted by the birational morphism $B \longrightarrow B'$ between klt varieties. By [Kaw91], this implies that $\eta(D)$ is covered by rational curves, in contradiction with our assumption $\operatorname{cov.gon}(\eta(D)) > 1$. □

Lemma 3.16. *Let T/B be a rank r fibration with* $\dim B = \dim T - 1$ *and B* **Q**-*factorial. Assume that for each divisor D of type* II *for T/B, we have* $\operatorname{cov.gon}(\eta(D)) > 1$*. Then T/B factorises through a rank 1 fibration T'/B' such that $T' \dashrightarrow T$ is a pseudo-isomorphism if and only if T/B does not admit any divisor of type* II*. In particular in this situation we have* $\dim B' = \dim T - 1$*, $B' \longrightarrow B$ is a birational morphism and $\rho(B'/B) = r - 1$.*

Proof. First observe that the existence of a pair of divisors $D \cup D'$ of type II is an obstruction to the existence of a Mori fibre space on T, since by Proposition 3.14(3) the general fibre of T/B is equivalent to $f + f'$ with f, f' non proportional.

To prove the converse, we assume that T/B does not admit any divisor of type II, and we proceed by induction on the number d_1 of divisor of type I. If $d_1 = 0$ then by Proposition 3.14(5), T/B is already a rank 1 fibration, so we just take $T'/B' = T/B$. Now if $d_1 > 0$, by Proposition 3.14(4) there exists a birational contraction over B, $T \dashrightarrow X_1 \longrightarrow B$, which contracts one of the divisor D of type I. Since the contraction is obtained by running a D-MMP, in fact it factorises as $T \dashrightarrow T_1 \longrightarrow X_1$, where $T \dashrightarrow T_1$ is a sequence of D-flips and $T_1 \longrightarrow X_1$ is a divisorial contraction. Then by induction hypothesis X_1/B factorises through a rank 1 fibration X_2/B_2 with $X_1 \dashrightarrow X_2$ a pseudo-isomorphism. By Lemma 2.11, there exist a pseudo-isomorphism $T_1 \dashrightarrow T_2$ and a divisorial contraction $T_2 \longrightarrow X_2$ that makes the diagram on Figure 2 commute. Finally we play the two-rays game T_2/B_2. Since T_2/B_2 admits one divisor of type I and no divisor of type II (by our assumption on the covering gonality and by Lemma 3.15), the other side of the two-rays game must be a Mori fibre space, which gives the expected rank 1 fibration T'/B'. □

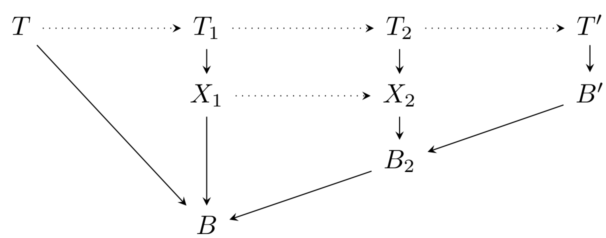


FIGURE 2

3.C. **Sarkisov links of conic bundles.** In this subsection, by applying Proposition 3.14 to the case $r = 2$, we classify Sarkisov links of conic bundles.

Lemma 3.17. *Let Y/B be a rank 2 fibration with $\dim B = \dim Y - 1$, and χ the associated Sarkisov link, well-defined up to taking inverse.*

(1) *χ has type* IV *if and only if B is not $\mathbf{Q}$-factorial.*

(2) *If B is $\mathbf{Q}$-factorial, let d_1 (resp. d_2) be the number of special divisors of type* I *(resp. of type* II*) for Y/B. Then*

- *χ has type* I *or* III *if and only if $(d_1, d_2) = (1, 0)$.*
- *χ has type* II *if and only if $(d_1, d_2) = (0, 1)$.*

Proof. (1) follows directly from the fact that the base of a terminal Mori fibre space if always $\mathbf{Q}$-factorial. To prove (1), first we observe that Proposition 3.14(5) gives $d_1 + d_2 = 1$, hence the two possibilities $(d_1, d_2) = (1, 0)$ or $(0, 1)$. Recall also from Proposition 3.14(1) that any divisor contracted by a birational contraction from Y over B must be of type I or II. By Lemma 3.10 the link χ is of type II if and only if there exist two birational contractions from Y contracting distinct prime divisors, and this is possible only in the case $(d_1, d_2) = (0, 1)$ where there is a pair of divisors of type II available. □

Corollary 3.18. *Let χ be a Sarkisov link of conic bundles of type* I*:*

$$\begin{array}{ccccc} Y_1 & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \rightarrow & X_2 \\ {\scriptstyle\pi_1}\downarrow & & & \nearrow & \downarrow{\scriptstyle\eta_2} \\ X_1 & & {\scriptstyle\chi} & & B_2 \\ & {\scriptstyle\eta_1}\searrow & & \swarrow & \\ & & B_1 & & \end{array}$$

Let E_1 be the exceptional divisor of the divisorial contraction π_1. Then $\eta_1 \circ \pi_1(E_1)$ has codimension at least 2 in B_1.

Proof. Follows from the fact that E_1 is a divisor of type I for Y_1/B_1. □

Remark 3.19. There are examples of link of type IV as in Lemma 3.17(1) only when $\dim B \geqslant 3$, hence $\dim Y \geqslant 4$. See the discussion on the two subtypes of type IV links in [HM13, p. 391 after Theorem 1.5]. For instance, take B_1 and B_2 that differ by a log-flip, and B the non **Q**-factorial target of the associated small contractions. Then take $Y = \mathbb{P}^1 \times B_1$.

Now we focus on the case of Sarkisov links of conic bundles of type II. First consider the following definition.

Definition 3.20. A *marked conic* is a triple $(X/B, \Gamma)$, where X, B, Γ are varieties, together with a morphism $X \longrightarrow B$ whose generic fibre is a geometrically irreducible conic defined over the function field $\mathbf{C}(B)$, and $\Gamma \subseteq B$ is an irreducible hypersurface, not contained in the discriminant locus of $X \longrightarrow B$ (i.e. the fibre of a general point of Γ is isomorphic to $\mathbb{P}^1$). The *marking* of the marked conic is defined to be Γ.

We say that two marked conics $(X/B, \Gamma)$, and $(X'/B', \Gamma')$ are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \\ B & \overset{\theta}{\dashrightarrow} & B' \end{array}$$

where ψ, θ are birational and such that the restriction of θ induces a birational map $\Gamma \dashrightarrow \Gamma'$ between the markings (in particular, if $(X/B, \Gamma)$, and $(X'/B', \Gamma')$ are equivalent, then the conic bundles X/B and X'/B' are equivalent).

For each variety Z, we denote by $\mathcal{C}_Z$ the set of equivalence classes of conic bundles X/B with X birational to Z and denote, for each class of conic bundles $C \in \mathcal{C}_Z$ by $\mathcal{M}_C$ the set of equivalence classes of marked conics $(X/B, \Gamma)$ where C is the class of X/B.

The next lemma explains how a Sarkisov link of conic bundles of type II gives rise to a marked conic.

Lemma 3.21. *Let χ be a Sarkisov link of conic bundles of type* II *between varieties of dimension $n \geqslant 2$. Recall that χ fits in a commutative diagram of the form*

$$\begin{array}{ccccc} Y_1 & & \overset{\varphi}{\cdots\cdots\cdots\cdots\cdots\cdots\rightarrow} & & Y_2 \\ {\scriptstyle\pi_1}\downarrow & & & & \downarrow{\scriptstyle\pi_2} \\ X_1 & & \overset{\chi}{\dashrightarrow} & & X_2 \\ & {\scriptstyle\eta_1}\searrow & & \swarrow{\scriptstyle\eta_2} & \\ & & B & & \end{array}$$

*where X_1, X_2, Y_1, Y_2 are **Q**-factorial terminal varieties of dimension n, B is a **Q**-factorial klt variety of dimension $n-1$, φ is a sequence of log-flips over B, and* *each π_i is a divisorial contraction with exceptional divisor $E_i \subseteq Y_i$ and centre $\Gamma_i = \pi_i(E_i) \subseteq X_i$.*

Then there exists an irreducible hypersurface $\Gamma \subseteq B$ (of dimension $n-2$) such that

(1) *for $i = 1, 2$, the centre $\Gamma_i = \pi_i(E_i)$ has codimension 2 in X_i, and the restriction $\eta_i|_{\Gamma_i} \colon \Gamma_i \longrightarrow \Gamma$ is birational. In particular, for each i we have $\eta_i \circ \pi_i(E_i) = \Gamma$, and the marked conics $(X_1/B, \Gamma)$ and $(X_2/B, \Gamma)$ are equivalent.*

(2) *Let Y be equal to Y_1, Y_2, or any one of the intermediate varieties in the sequence of log-flips φ. Then $E_1 \cup E_2$ is a pair of divisors of type* II *for Y/B.*

(3) *Γ is not contained in the discriminant locus of η_1, or equivalently of η_2, which means that a general fibre of $\eta_i \colon (\eta_i)^{-1}(\Gamma) \longrightarrow \Gamma$ is isomorphic to $\mathbb{P}^1$.*

(4) *At a general point $x \in \Gamma_i$, the fibre of X_i/B through x is transverse to Γ_i.*

Proof. (1) and (2): By Lemma 3.17, Y_1/B admits no divisor of type I, and exactly one pair of divisors of type II. By Lemma 3.10 we have $\varphi_* E_1 \neq E_2$, so the birational contractions $Y_1 \dashrightarrow X_1$ and $Y_1 \dashrightarrow X_2$ contract distinct divisors. It follows from Proposition 3.14 that the pair of divisors of type II is $E_1 \cup E_2$. So by definition E_1 and E_2 projects to the same hypersurface $\Gamma \subseteq B$.

(3) and (4) follow from Proposition 3.14(3). Indeed if Γ was in the discriminant locus of η_1 then the preimage in Y_1 of a general point $p \in B$ would have 3 irreducible components, instead of 2. Moreover writing $f_1 \cup f_2$ the fibre through x, with $f_i \subseteq E_i$, the fact that the fibre is transverse to Γ_i is equivalent to $f_1 \cdot E_2 = f_2 \cdot E_1 = 1$. □

Definition 3.22. By Lemma 3.21(1), to each Sarkisov link of conic bundles of type II $\chi \colon X_1 \dashrightarrow X_2$, we can associate the equivalence class of the marked conic $(X_1/B, \Gamma)$ given in this lemma. We define the *marking* of χ to be $\Gamma \subseteq B$. We say that two Sarkisov links of conic bundles of type II are *equivalent* if their corresponding marked conics are equivalent.

We also extend the notion of covering gonality (see §2.G) to Sarkisov links of conic bundles of type II.

Definition 3.23. Let χ be a Sarkisov of conic bundles of type II between varieties of dimension $n \geqslant 3$. We define $\mathrm{cov.gon}(\chi)$ to be $\mathrm{cov.gon}(\Gamma)$, where Γ is the marking of χ.

Remark 3.24. If two Sarkisov links of conic bundles of type II are equivalent, then their markings are birational to each other. In particular the number $\mathrm{cov.gon}(\chi)$ only depends on the equivalence class of χ.

The above definition makes sense if the varieties X_i have dimension $\geqslant 2$, but it is not a very good invariant if the dimension is 2, as the centre is always a point, and there is only one class of marked conics, given by a point in the base of a Hirzebruch surface. However, the analogue definition over $\mathbf{Q}$ or over a finite field, instead of over $\mathbf{C}$, is interesting even for surfaces.

4. Relations between Sarkisov links

The fact that one can give a definition of Sarkisov link in terms of relative Mori dream space of Picard rank 2 was independently observed in [AZ16, §2] and [LZ17, §2.3]. Our next aim is to extend this observation to associate some relations between Sarkisov links to each rank 3 fibration. For that we need to recall some preliminary material from [Kal13].

4.A. **Generation and relation in the Sarkisov program.** Here we present a summarised and to our needs simplified version of [Kal13]. The goal of this section is Proposition 4.8, which will allow us to define the homomorphisms of the main theorems.

For $i = 1, \dots, s$, let $\pi_i \colon X_i \longrightarrow B_i$ be Mori fibre spaces and $\theta_{ij} \colon X_i \dashrightarrow X_j$ birational maps. We pick a common resolution of all the θ_{ij}

$$f = (f_1, \dots, f_s) \colon Z \longrightarrow X_1 \times \cdots \times X_s$$

with Picard number $\rho(Z) \geqslant 4$.

For any sufficiently small ample $\mathbf{Q}$-divisors $A, A_1, \dots, A_n$ on Z, $K_Z + A + \Delta$ is not pseudo-effective for any convex combinations $\Delta = \sum a_i A_i$. We may assume that $N^1(Z)$ is generated by such Δ (after perhaps adding more ample divisors).

Denote by $\mathcal{C} \subset N^1(Z)$ the closure of the cone of pseudo-effective classes of the form

$$t\left(K_Z + A + \sum a_i A_i\right) \text{ where } t, a_i \geqslant 0$$

For each D in $\mathcal{C}$, we denote φ_D the rational map associated with the linear system $|nD|$ for $n \gg 0$. The image of φ_D is $Z_D := \operatorname{Proj}(\bigoplus_n H^0(Z, nD))$, where the sum is over all n such that nD is Cartier. Then $\mathcal{C}$ is a polyhedral cone with a finite Mori chamber decomposition

$$\mathcal{C} = \coprod_{i \in I} \mathcal{A}_i$$

given by the following equivalence relation: $D_1 \sim D_2$ if there is an isomorphism that makes the following diagram commutes.

$$\begin{array}{ccccc} & & Z & & \\ & \overset{\varphi_{D_1}}{\swarrow} & & \overset{\varphi_{D_2}}{\searrow} & \\ Z_{D_1} & & \xrightarrow{\ \simeq\ } & & Z_{D_2} \end{array}$$

In fact, $\mathcal{C}$ is the support of the ring $R(Z, K_Z + A, K_Z + A + A_1, \dots, K_Z + A + A_n)$, which is finitely generated [KKL16, Theorem 3.5], [Kal13, Proposition 3.1]. In particular, the φ_D exist and do not depend on the numerical choice of D, which justifies why we can work directly in $N^1(Z)$ instead of taking representatives in $\mathrm{Div}(Z)_{\mathbf{R}}$.

The Mori chamber decomposition yields a fan structure on $\mathcal{C}$, as follows. We call *dimension* of a chamber $\mathcal{A}_i$ the dimension of the smallest linear subspace containing $\mathcal{A}_i$. The closure of each chamber of maximal dimension $\rho(Z)$ is a polyhedral cone, which we call a *maximal facet*. The intersection of any finite collection of such maximal facets is a common facet of each, and the union of all maximal facets covers $\mathcal{C}$. Then $\mathcal{C}$ is a polyhedral complex for the union of all facets of these maximal facets. Observe that the relative interior of each facet is contained in a single Mori class, and conversely each Mori class is a finite union of relative interior of facets.

Example 4.1. We illustrate the definition of Mori chambers and facets on the simple example of the blow-up $Z \longrightarrow \mathbb{P}^2$ at two distinct points a and b (see Figure 3).

The open segment from $\mathbb{F}_{1,b}/\mathrm{pt}$ to $\mathbb{F}_0/\mathrm{pt}$ is a Mori chamber, where all $Z_D \simeq \mathbb{P}^1$. The closed segments from $\mathbb{F}_0/\mathrm{pt}$ to $Z/\mathbb{P}^1$, and from $Z/\mathbb{P}^1$ to $\mathbb{F}_{1,b}/\mathrm{pt}$, are two examples of codimension 1 facets. The vertices $\mathbb{F}_0/\mathrm{pt}$, $Z/\mathbb{P}^1$ and $\mathbb{F}_{1,b}/\mathrm{pt}$ are three

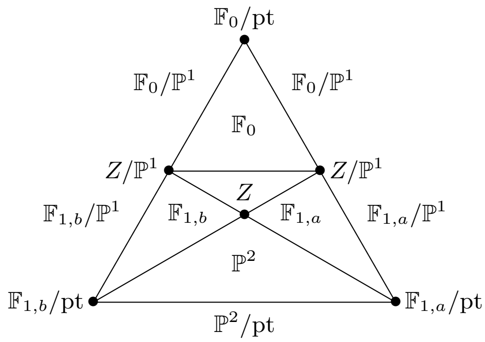


FIGURE 3. An affine section of the cone $\mathcal{C}$ from example 4.1.

examples of codimension 2 facets. The relation between codimension r facets in the boundary of $\mathcal{C}$ and rank r fibrations dominated by Z will be made precise in Proposition 4.3.

To simplify the notation, we write $\varphi_D = \varphi_i$ and $Z_i = Z_D$ for $D \in \mathcal{A}_i$. We have the following properties.

Proposition 4.2. *There exists a choice of ample divisors $A, A_1, \dots, A_n$ such that:*

(1) *For each $k = 1, \dots, s$ there are indexes i_1, i_2 such that $f_k = \varphi_{i_1}$ and $\pi_k \circ f_k = \varphi_{i_2}$, where $\pi_k \colon X_k \longrightarrow B_k$.*

(2) $\dim \bar{\mathcal{A}}_i = \rho(Z)$ *if and only if φ_i is birational and Z_i is $\mathbf{Q}$-factorial. In fact, then φ_i is the semiample model of any element of $\mathcal{A}_i$.*

(3) *If $\mathcal{A}_i \cap \bar{\mathcal{A}}_j \neq \emptyset$, then there exists a morphism $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ with connected fibres such that $\varphi_i = \varphi_{ji} \circ \varphi_j$. If $\mathcal{A}_i$ does not contain any big divisor, then φ_{ji} is the Itaka fibration associated to $(\varphi_j)_*(D_i)$.*

(4) *If there exists a morphism $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ such that $\varphi_i = \varphi_{ji} \circ \varphi_j$, then $\mathcal{A}_i \cap \bar{\mathcal{A}}_j \neq \emptyset$.*

(5) *If $\mathcal{A}_i$ is of maximal dimension, then the birational contraction φ_i is the result of $(K_Z + A)$-MMP with scaling Δ for some $K_Z + A + \Delta \in \mathcal{A}_i$ and $(Z, A + \Delta)$ klt. In particular, Z_i is $\mathbf{Q}$-factorial and terminal.*

(6) *If $\mathcal{A}_j$ has maximal dimension and $\mathcal{A}_i \cap \bar{\mathcal{A}}_j \neq \emptyset$, then $\rho(Z_j/Z_i)$ is equal to the codimension of $\mathcal{A}_i \cap \bar{\mathcal{A}}_j$ in $\bar{\mathcal{A}}_j$.*

(7) *Each φ_i is a finite composition of isomorphisms in codimension 1 and morphisms contracting a prime divisor, and their inverses.*

Proof. (1), (3), (5), (7) are [Kal13, Theorem 2.16, Proposition 2.18, Remark 3.4] since $\mathcal{C}$ contains the ample divisors A_i by construction, and (2) and (6) are [HM13, Theorem 3.3]. It remains to show (4). We pick $D_i \in \mathcal{A}_i$, $D_j \in \mathcal{A}_j$, and define $D_t := tD_j + (1-t)\varepsilon D_i$ for some $0 < \varepsilon \ll 1$. Then $D_t \in \mathcal{A}_j$ for $t > 0$ and $D_t \cdot C \geqslant 0$ for all curves contracted by φ_{ji} for ε small enough. Hence $D_0 \in \mathcal{A}_i \cap \bar{\mathcal{A}}_j$. □

Let $\partial^+\mathcal{C}$ be the set of non-big divisors in $\mathcal{C}$. Then for all $\mathcal{A}_i \subset \partial^+\mathcal{C}$, we have $\dim Z_i < \dim Z$. Moreover, $\partial^+\mathcal{C}$ is the cone over a disc or sphere of dimension $\rho(Z) - 2$ and it inherits a fan structure from the polyhedral structure on $\mathcal{C}$ as follows: Consider the set of maximal dimensional chambers $\mathcal{A}_j$ such that $\bar{\mathcal{A}}_j \cap \partial^+\mathcal{C}$

contains a codimension 1 facet $\mathcal{F}^1$ of the closed polytope $\bar{\mathcal{A}}_j$. More generally we shall denote $\mathcal{F}^k$ a (closed) codimension k facet of $\mathcal{C}$.

For a codimension 1 facet $\mathcal{F}^1$ of the chamber $\mathcal{A}_j$, D in the relative interior of $\mathcal{F}^1$ and sufficiently small $\varepsilon > 0$, the divisor $D' := D - \varepsilon(K + A)$ is in $\mathcal{A}_j$, and the images $Z_{D'} = \varphi_{D'}(Z)$ and $B_D = \varphi_D(Z)$ correspond to a Mori fibre space $Z_{D'}/B_D$ that depends only on $\mathcal{F}^1$ and not on the particular choice of D or ε [Kal13, Lemma 3.2].

More generally we have:

Proposition 4.3. *Let $\mathcal{F}^r \subset \mathcal{C}$ be a codimension r facet intersecting the relative interior of $\partial^+\mathcal{C}$. Let $\mathcal{A}_i$ be the Mori chamber containing the interior of $\mathcal{F}^r$. Then*

(1) *There exists a chamber $\mathcal{A}_j \subset \mathcal{C}$ of maximal dimension such that $\bar{\mathcal{A}}_j \cap \bar{\mathcal{A}}_i = \mathcal{F}^r$.*

(2) *The associated morphism $\varphi_{ji}\colon Z_j \longrightarrow Z_i$ is a rank r fibration.*

(3) *If $\mathcal{F}^t \subset \bar{\mathcal{A}}_k \cap \partial^+\mathcal{C}$ for some maximal dimensional chamber $\mathcal{A}_k$ and $\mathcal{F}^r \subset \mathcal{F}^t$, then the fibration associated to $\mathcal{F}^r$ factors through the fibration associated to $\mathcal{F}^t$.*

Proof. (1):

(2): By Proposition 4.2(6), the associated morphism $\varphi_{ji}\colon Z_j \longrightarrow Z_i$ has relative Picard rank equal to r.

We prove that Z_j/Z_i is a Mori dream space. Since Z is rationally connected, also Z_j is rationally connected and hence $N^1(Z_j)_{\mathbf{Q}} = \operatorname{Pic}(Z_j)_{\mathbf{Q}}$ by Lemma 2.7. The nef cone $\operatorname{Nef}(Z_j)$ is generated by finitely many semiample divisors. Indeed, let $D' \in \operatorname{Nef}(Z_j)$. Then Z_j is a minimal model of D', and it can be obtained by running a $(K_Z + A)$-MMP with scaling on Z. In particular, D' is a linear combination of $(\varphi_j)_*(K_Z + A + \Delta)$, which is semiample by [KKL16, Theorem 4.2], and $\tilde{A}_j, A_B$. Any variety pseudo-isomorphic to Z_j is an outcome of a $K_Z + A$-MMP with scaling on Z, and here are only finitely many facets of $\mathcal{C}$ that correspond to varieties pseudo-isomorphic to Z_j.

Let $D' \in \operatorname{Div}(Z_j)_{\mathbf{R}}$ be a divisor. We now show that the end-product of running any D'-MMP from Z_j over B can be obtained by running a $K_Z + A$-MMP with scaling from Z over B. Recall that $(\varphi_j)_*(D) = \eta^*(A_B)$ for some ample divisor A_B on B [KKL16, Theorem 4.2]. To run a D'-MMP from Z_j over B, we pick an ample divisor $A_j \in \operatorname{Div}(Z_j)_{\mathbf{Q}}$ and consider all pseudo-effective linear combinations $D'_t := \varepsilon(tD' + (1-t)\tilde{A}_j) + A_B$ for some $1 \gg \varepsilon > 0$. The set $(\varphi_j)^*(D'_t)$ is a segment in the relative neighbourhood of D inside $\mathcal{F}^r$. Therefore, any step in this D'-MMP over B can be obtained by running from Z a $(K_Z + A)$-MMP with scaling over B. In particular, the end result is $\mathbf{Q}$-factorial and terminal, so the condition on singularities in Definition 3.1(3) follows from Proposition 4.2(5).

Finally we show that $-K_{Z_j}$ is η-big. Indeed, as η is the Itaka fibration of $(\varphi_j)_*(D) = K_{Z_j} + (\varphi_i)_*(A + \Delta)$ for some scaling Δ by Proposition 4.2(5), and $(\varphi_j)_*(D)$ is η-trivial. Since φ_j is birational, and A and Δ are ample, $(\varphi_i)_*(A + \Delta)$ is big. Hence $K_{Z_j} + \text{big} = \eta$-trivial, which gives

$$-K_{Z_j} = \text{big} + \eta\text{-trivial} = \text{ample} + \eta\text{-trivial} + \text{effective} = \eta\text{-ample} + \text{effective}$$

hence $-K_{Z_j}$ is η-big by Lemma 2.1.

(3): Analogous to [LZ17, Proposition 3.10 (2)]. ☐

Corollary 4.4. *Suppose that Z is rationally connected. If the intersection $\mathcal{F}_i^1 \cap \mathcal{F}_j^1$ of non-big codimension* 1 *facets has codimension* 1 *in $\mathcal{F}_i^1$, then there is a link between the corresponding Mori fibre spaces.*

Proof. By Proposition 4.3 there is a rank 2 fibration corresponding to the codimension 2 facet $\mathcal{F}^2 := \mathcal{F}_i^1 \cap \mathcal{F}_j^1$ that factorises through the rank 1 fibrations associated to $\mathcal{F}_i^1$ and $\mathcal{F}_j^1$. This is exactly our definition of Sarkisov link (see Definition 3.8). □

Proposition 4.5. *Let T/B be a rank* 3 *fibration. Then there are only finitely many Sarkisov links χ_i dominated by T/B, and they fit in a relation*

$$\chi_s \circ \cdots \circ \chi_1 = \mathrm{id}.$$

Proof. By assumption $N^1(T/B) \simeq \mathbf{R}^3$, and $\mathrm{Eff}(T/B) \subset N^1(T/B)$ is a convex cone (not necessarily strictly convex). In particular, the intersection of $\mathrm{Eff}(T/B)$ with a sphere centered at the origin is homeomorphic to a disc $\mathcal{D}$, with a chamber structure inherited from the fan structure on $\mathrm{Eff}(T/B)$. By Proposition 4.3, the finite number of codimension 1 facets (resp. codimension 2 facets) at the boundary of $\mathcal{D}$ are in one-to-one correspondence with the rank 1 fibrations (resp. the rank 2 fibrations) dominated by T/B, and the claim follows. □

Definition 4.6. In the situation of Proposition 4.5, we say that

$$\chi_s \circ \cdots \circ \chi_1 = \mathrm{id}$$

is an *elementary relation* between Sarkisov links, coming from the rank 3 fibration X/B. Observe that the elementary relation is uniquely defined by X/B, up to taking the inverse, cyclic permutations and insertion of isomorphisms.

Lemma 4.7. *Let $\mathcal{F}^3$ be a facet in $\partial^+\mathcal{C}$ of codimension* 3 *and T/B be the associated rank* 3 *fibration, as given in Proposition 4.3. Then the elementary relation associated to T/B corresponds to the finite collection of codimension 1 facets $\mathcal{F}_1^1, \dots, \mathcal{F}_k^1$ containing $\mathcal{F}^2$ in their closure, and ordered such that $\mathcal{F}_j^1$ and $\mathcal{F}_{j+1}^1$ share a codimension* 2 *facet for all j (where indexes are in $\mathbf{Z}/k\mathbf{Z}$).*

Proof. Analogously to [LZ17, Corollary 3.13]. □

Let X/B a Mori fibre space. We denote by $\mathrm{BirMori}(X)$ the groupoid of birational maps between Mori fibre spaces birational to X. The group of birational selfmaps $\mathrm{Bir}(X)$ is a subgroupoid of $\mathrm{BirMori}(X)$. The motivation for introducing the notion of elementary relations is the following result. The first part is a reformulation of [HM13, Theorem 1.1]. The second part is essentially [Kal13, Theorem 1.3], observe however that our notion of elementary relation is more restrictive.

In the statement we use the formalism of presentations by generators and relations for groupoids. This is very similar to the more familiar setting of groups: we have natural notions of a free groupoid, and of a normal subgroupoid generated by a set of elements. We refer to [Bro06, §8.2 and 8.3] for details.

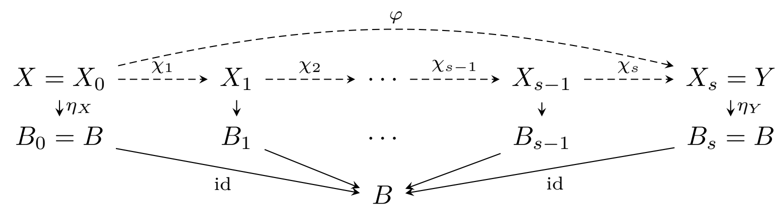
Proposition 4.8. *Let X/B be a Mori fibre space and suppose that X is rationally connected.*

(1) *The groupoid* $\mathrm{BirMori}(X)$ *is generated by Sarkisov links (and automorphisms).*
(2) *Any relation between Sarkisov links in* $\mathrm{BirMori}(X)$ *is generated by elementary relations.*

Proof. Analogous to [LZ17, Proposition 3.14, Proposition 3.15]; the analogons of their ingredients in higher dimension are Corollary 4.4, Proposition 4.5 and Lemma 4.7. □

For birational maps between conic bundles, we have the following statement, , slightly more precise than Proposition 4.8(1).

Lemma 4.9. *Let $\eta_X: X \longrightarrow B$ and $\eta_Y: Y \longrightarrow B$ be two conic bundles. Every birational map $\varphi\colon X \dashrightarrow Y$ define over B decomposes into a sequence of Sarkisov links of conic bundles over B. More precisely, we have a commutative diagram*



such that χ_i is a Sarkisov links of conic bundles, X_i/B_i is a conic bundle and B_i/B is a birational morphism for $i = 1, \dots, s$.

Proof. In the setting of Proposition 4.2, let $s = 2$, $X_1 = X$, $X_s = Y$ and $B = B_1 = B_s$ and $\theta_{ij} = \varphi$. Then $\varphi_B := \pi_k \circ f_k\colon Z \to B$ does not depend on $k \in \{1, \dots, s\}$. Let A_B be the pullback of an ample divisor on B via φ_B. We replace A by the ample divisor $\varepsilon A + A_b$ for some sufficiently small $\varepsilon > 0$. Then all $\mathcal{A}_i$ contain the same facet $\mathcal{F}_B$ associated to B. In particular, any φ_i associated to a $\mathcal{A}_i$ in the decomposition of $\mathcal{C}$ contracts only curves contracted by φ_B, i.e. all φ_i are results of $(K_Z + \text{ample})$-MMP with scaling over B. Analogous to Proposition 4.8, φ is a composition of Sarkisov links over B with contraction morphisms $X_i \longrightarrow B_i \longrightarrow B$. It follows from $\dim X - 1 = \dim X_i - 1 = \dim B_i \geqslant \dim B = \dim X - 1$, that $\dim B_i = \dim B$ and $B_i \longrightarrow B$ is birational. □

5. Constraints on relations involving Sarkisov links of conic bundles of type II

This section is devoted to the study of the elementary relations between Sarkisov links that involve Sarkisov links of conic bundles χ of type II which are complicated enough (meaning the covering gonality cov.gon(χ) is large enough). We give some restriction on such relations that allow us to prove Theorem C. Firstly the case of relations over a base of small dimension (dimension $\leqslant n-2$, where n is the dimension of the Mori fibre spaces) is done in Proposition 5.2, using the BAB conjecture and then working with Sarkisov links of large enough covering gonality. Secondly, the case of relations over a base of dimension $n-1$ is handled in Proposition 5.3, using only the fact that the covering gonality is > 1.

5.A. **A consequence of the BAB conjecture.** The following is a consequence of the BAB conjecture, which was recently established in arbitrary dimension by C. Birkar.

Proposition 5.1. *Let n be an integer, and let $\mathcal{Q}$ be the set of weak Fano terminal varieties of dimension n. There are integers $d, l, m \geqslant 1$, depending only on n, such that for each $X \in \mathcal{Q}$ the following hold:*

(1) $\dim(H^0(-mK_X)) \leqslant l$;

(2) *The linear system* $|-mK_X|$ *is base-point free;*

(3) *The morphism* $\varphi\colon X \xrightarrow{|-mK_X|} \mathbb{P}^{\dim(H^0(-mK_X))-1}$ *is birational onto its image and contracts only curves* $C \subseteq X$ *with* $C \cdot K_X = 0$;

(4) $\deg \varphi(X) \leqslant d$.

Proof. By [Bir16b, Theorem 1.1], varieties in $\mathcal{Q}$ form a bounded family. In particular, by [Bir16a, Lemma 2.24], the Cartier index of such varieties is uniformly bounded. Then [Kol93, Theorem 1.1] gives the existence of $m = m(n)$ such that $|-mK_X|$ is base-point free for each $X \in \mathcal{Q}$. By [Bir16a, Theorem 1.2], we can increase m if needed, and assume that the associated morphism

$$\varphi\colon X \xrightarrow{|-mK_X|} \mathbb{P}^{\dim(H^0(-mK_X))-1}$$

is birational onto its image. As it is a morphism, this implies that it contracts only curves $C \subseteq X$ with $C \cdot K_X = 0$. Finally, since $\mathcal{Q}$ is a bounded family, the two integers $\dim(H^0(-mK_X))$ and $\deg(\varphi(X))$ are bounded. □

Proposition 5.2. *For each dimension* n*, there exist an integer* $d \geqslant 1$ *depending only on* n *such that the following holds. If* χ *is a Sarkisov link of conic bundles of type* II *that arises in an elementary relation induced by a rationally connected rank 3 fibration* T/B *with* $\dim(T) = n$ *and* $\dim(B) \leqslant n-2$*, then* $\operatorname{cov.gon}(\chi) \leqslant d$.

Proof. The integer d will be the maximum of the integers d given by Proposition 5.1 for the dimensions $2, \dots, n$.

The Sarkisov link χ, which is dominated by T/B by assumption, has the form

$$\begin{array}{ccccc} Y_1 & \cdots\cdots & & \cdots\cdots & Y_2 \\ \downarrow & & & & \downarrow \\ X_1 & & \overset{\chi}{\dashrightarrow} & & X_2 \\ & \searrow & & \swarrow & \\ & & B' & & \end{array}$$

where X_1, X_2, Y_1, Y_2 have dimension n and B' has dimension $n-1$. Since $\dim B \leqslant n-2$, we have $\rho(B'/B) \geqslant 1$, and on the other hand $\rho(Y_i/B) \leqslant 3$, for $i = 1, 2$, which implies that $\rho(B'/B) = 1$, and that the birational contractions $T \dashrightarrow Y_1$, $T \dashrightarrow Y_2$ are pseudo-isomorphisms. Moreover, $Y_1 \longrightarrow X_1$ contracts a divisor E_1 onto a variety $\Gamma_1 \subseteq X_1$ of dimension $n-2$, birational to its image $\Gamma' \subseteq B'$ via the morphism $X_1 \longrightarrow B'$ (Lemma 3.21). We need to check that $\operatorname{cov.gon}(\Gamma_1) = \operatorname{cov.gon}(\Gamma') \leqslant d$, where d is given by Proposition 5.1 for one of the dimensions $2, \dots, n$. We may then assume that $\operatorname{cov.gon}(\Gamma') > 1$.

We work with the commutative diagram

$$\begin{array}{ccccc} Y_1 & \cdots\cdots & T & \cdots\cdots & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & & \downarrow & & X_2 \\ & \searrow & \downarrow & \swarrow & \\ & & B' & & \\ & & \downarrow & & \\ & & B & & \end{array}$$

where $B' \longrightarrow B$ is a klt Mori fibre space by Lemma 3.11. We take a general point $p \in B$, and consider the fibres over p in X_1, Y_1, that we denote by X_p, Y_p, and which are varieties of dimension

$$n_0 = n - \dim(B) \in \{2, \dots, n\}.$$

By Lemma 3.4 and Corollary 3.5 the two varieties X_p and Y_p are pseudo-isomorphic to weak Fano terminal varieties $\hat{X}$ and $\hat{Y}$ of dimension n_0.

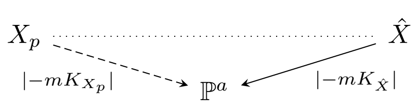
Observe that $\Gamma' \subseteq B'$ is a hypersurface and that $\Gamma' \longrightarrow B$ is dominant. Indeed, otherwise Γ' would be the preimage of a divisor on B, hence by Lemma 3.11 we would have $\operatorname{cov.gon}(\Gamma') = 1$, in contradiction with our assumption.

We then denote by $\Gamma_p \subseteq X_p$ the codimension 2 subscheme $\Gamma_p = \Gamma_1 \cap X_p$, which is the fibre of $\Gamma_1 \longrightarrow B$ over p, and which is not necessarily irreducible. Observe that $Y_p \longrightarrow X_p$ is the blow-up of Γ_p, as $Y_1 \longrightarrow X_1$ is the blow-up of Γ_1 and the fibre over p is transverse to Γ_1 (Lemmas 2.5 and 3.21(4)).

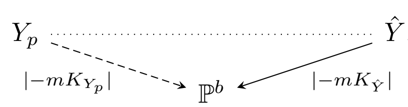
Suppose first that $n_0 = 2$, which corresponds to $\dim(\Gamma_1) = \dim(B)$. In this case, $X_p \simeq \hat{X}$ and $Y_p \simeq \hat{Y}$ are smooth del Pezzo surfaces, because by Corollary 3.5 the locus covered by curves trivial againts the canonical divisor has codimension 2, hence is empty in the case $n_0 = 2$. Moreover Γ_p is a disjoint union of r points, where r is the degree of the field extension $\mathbf{C}(B) \subseteq \mathbf{C}(\Gamma_1)$. As Y_p is obtained from X_p by blowing-up Γ_p, the degree of the field extension is at most 8, which implies that $\operatorname{cov.gon}(\Gamma_1) \leqslant 8 \cdot \operatorname{cov.gon}(B)$ (Lemma 2.18(1)). Moreover, B is rationally connected, so $\operatorname{cov.gon}(B) = 1$, which yields $\operatorname{cov.gon}(\Gamma_1) \leqslant 8$.

We now consider the case $n_0 \geqslant 3$, which implies that Γ_p has dimension $n_0 - 2 \geqslant 1$. By Lemma 2.18(2), each irreducible component F_i of Γ_p satisfies $\operatorname{cov.gon}(F_i) \geqslant \operatorname{cov.gon}(\Gamma_1) > 1$.

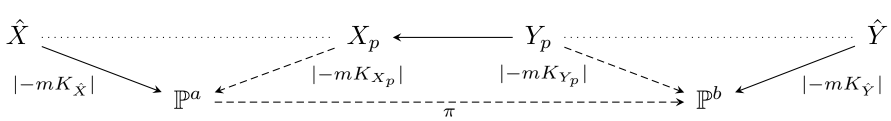
The base locus of the pseudo-isomorphism $X_p \dashrightarrow \hat{X}$ is covered by rational curves: this follows by applying [Kaw91, Theorem 1] to the small contractions associated to the sequence of log-flips $X_p \dashrightarrow \hat{X}$. We deduce that the image of Γ_p in $\hat{X}$ is a (maybe reducible) reduced scheme $\hat{\Gamma}_p$ birational to Γ_p, because all components F_i of Γ_p satisfy $\operatorname{cov.gon}(F_i) > 1$. We choose the integers $d, l, m \geqslant 1$ associated to the dimension n_0 in Proposition 5.1, and want to show that $d \geqslant \operatorname{cov.gon}(\Gamma_1)$. We write $a = \dim(H^0(-mK_{\hat{X}})) - 1$ and $b = \dim(H^0(-mK_{\hat{Y}})) - 1$. Using the pseudo-isomorphisms $X_p \dashrightarrow \hat{X}$ and $Y_1 \dashrightarrow \hat{Y}$, we also have $a = \dim(H^0(-mK_{X_p})) - 1$ and $b = \dim(H^0(-mK_{Y_1})) - 1$, and we get commutative diagrams



and



The rational morphisms $|-mK_{\hat{X}}|$ and $|-mK_{\hat{Y}}|$ are birational onto their images (Proposition 5.1) and are moreover pseudo-isomorphisms onto their images. Indeed, the morphism cannot contract any divisor, as the image would then be not terminal by Lemma 2.2 (because each contracted curve has trivial intersection with $-mK$ by Proposition 5.1). Since $Y_p \longrightarrow X_p$ is the divisorial contraction of a (non-necessarily irreducible) divisor onto Γ_p, each effective divisor equivalent to $-mK_{Y_p}$ is the strict transform of an effective divisor equivalent to $-mK_{X_p}$ passing through Γ_p (with some multiplicity). In particular, we have $b \leqslant a$ and obtain a commutative diagram



where π is a linear projection away from a linear subspace $\mathcal{L} \simeq \mathbb{P}^r$ of $\mathbb{P}^a$ containing the image of Γ_p. This latter is birational to Γ_p and $\hat{\Gamma}_p$, by the same argument as before. We write $\varphi\colon \hat{X} \longrightarrow \mathbb{P}^a$ the morphism given by $|-mK_{\hat{X}}|$. The variety $\varphi(\hat{X})$ is a variety of dimension n_0 and of degree $\leqslant d$ (Proposition 5.1) which is not contained in a hyperplane section.

We now prove that there is no (irreducible) variety $S \subseteq \varphi(\hat{X}) \cap \mathcal{L}$ of dimension $n_0 - 1$ (recall that $\varphi(\hat{\Gamma}_p) \subseteq \varphi(\hat{X}) \cap \mathcal{L}$ has dimension $n_0 - 2$). Indeed, otherwise the strict transform of S on X_p would be a variety $S' \subseteq X_p$ birational to S, so its strict transform in Y_p and in $\mathbb{P}^b$ is again birational to S (as the birational map from Y_p to its image in $\mathbb{P}^b$ is a pseudo-isomorphism). The rational map $X_p \dashrightarrow \mathbb{P}^b$ is given by a subsystem of the linear system associated to $X_p \dashrightarrow \mathbb{P}^a$ (which is a pseudo-isomorphism onto its image), and as a general hyperplane of $\mathbb{P}^b$ does not contain the image of S', the image S of S' in $\mathbb{P}^a$ cannot be contained in $\mathcal{L}$.

Now, the fact that $\varphi(\hat{X}) \cap \mathcal{L} \subseteq \mathbb{P}^a$ does not contain any variety of dimension $\geqslant n_0 - 1$ implies, by Bézout Theorem, that all its irreducible components of dimension $n_0 - 2$ have degree $\leqslant d$. Therefore, each of the irreducible components of $\varphi(\hat{\Gamma}_p)$ (birational to Γ_p) has degree $\leqslant d$, and then by Theorem 2.17 we obtain $d \geqslant \operatorname{cov.gon}(\Gamma_1)$ as desired. ☐

5.B. Elementary relations of length 4.

Proposition 5.3. *Let χ_1 be a Sarkisov link of conic bundles of type* II *with* $\operatorname{cov.gon}(\chi_1) > 1$. *Let T/B be a rank 3 fibration with $\dim B = \dim T - 1$, which factorises through the Sarkisov link χ_1. Then, the elementary relation associated to T/B has the form*

$$\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id},$$

where χ_3 is a Sarkisov link of conic bundles of type II *which is equivalent to χ_1.*

Proof. The Sarkisov link χ_1 is given by a diagram

$$\begin{array}{ccccc} Y_1 & \cdots\cdots\cdots & & \cdots\cdots\cdots & Y_2 \\ {\scriptstyle \pi_1}\downarrow & & & & \downarrow{\scriptstyle \pi_2} \\ X_1 & & \overset{\chi_1}{\dashrightarrow} & & X_2 \\ & \searrow & & \swarrow & \\ & & \hat{B} & & \end{array}$$

where X_1, X_2, Y_1, Y_2 are varieties of dimension n, and $\dim \hat{B} = n - 1$. Denote by $E_1 \subseteq Y_1$ and $E_2 \subseteq Y_2$ the respective exceptional divisors of the divisorial contractions π_1 and π_2. We denote again by $E_1, E_2 \subseteq T$ the strict transforms of these divisors, under the birational contractions $T \dashrightarrow Y_1$ and $T \dashrightarrow Y_2$. Then by Lemma 3.21(2), $E_1 \cup E_2$ is a pair of divisors of type II for $Y_1/\hat{B}$, hence also for T/B by Lemma 3.15. By Proposition 3.14(5), we are in one of the following mutually exclusive three cases:

(1) B is $\mathbf{Q}$-factorial, and there exists a divisor G of type I for T/B.
(2) B is not $\mathbf{Q}$-factorial.
(3) B is $\mathbf{Q}$-factorial, and there exists another pair $F_1 \cup F_2$ of divisors of type II for T/B.

We denote $\{X_i/B_i\}$ the finite collection of the rank 1 fibrations dominated by T/B. In each case we are going to show that this collection has cardinal 4.

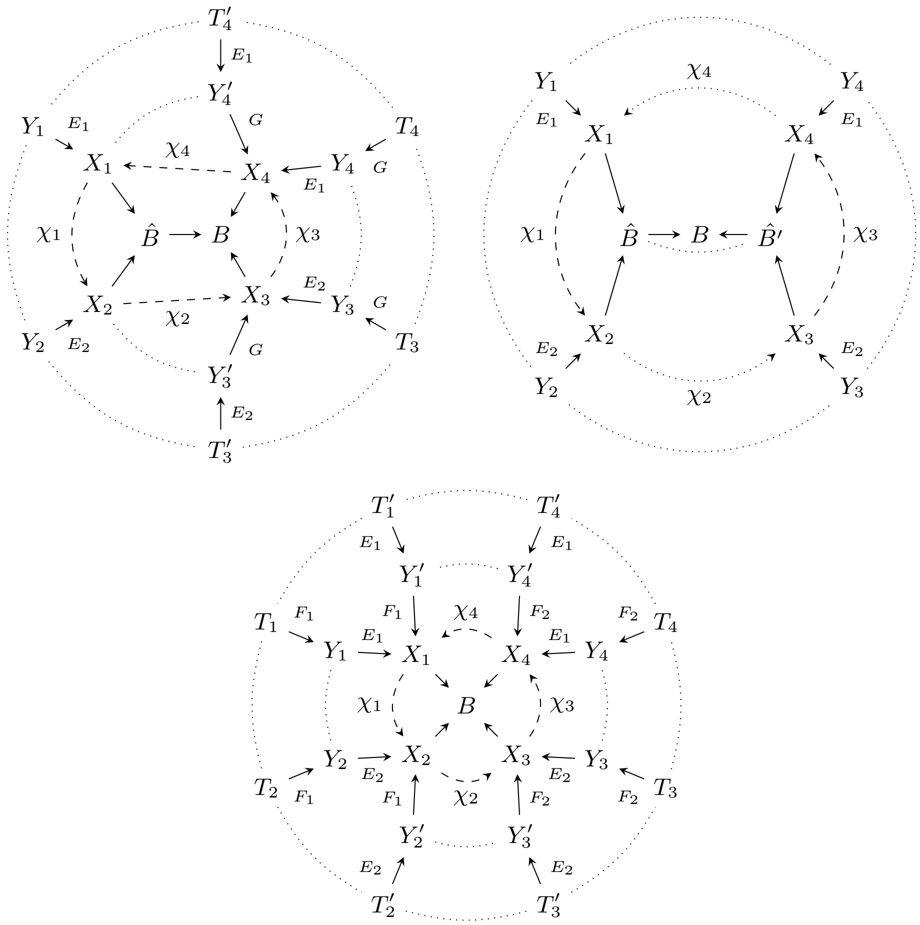


FIGURE 4. The elementary relation associated to T/B in cases (1), (2) and (3) of the proof of Proposition 5.3. Varieties are organized in circles according to their Picard rank over B.

Suppose first that (1) holds. By Proposition 3.14(1)(4) and Lemma 3.16, we can obtain such an X_i/B_i by a birational contraction contracting one the following four sets of divisors: $\{E_1\}$, $\{E_2\}$, $\{E_1, G\}$ or $\{E_2, G\}$. Moreover X_i/B_i is determined up to isomorphism by such a choice of contracted divisors:

- If $T \dashrightarrow X_i$ contracts $\{E_1, G\}$ or $\{E_2, G\}$, then $\rho(X_i) = \rho(T) - 2$ hence $\rho(B_i/B) = 0$, that is, $B_i \longrightarrow B$ is an isomorphism (a bijective morphism between normal varieties is an isomorphism). Then X_i is uniquely determined by Corollary 2.13.
- If $T \dashrightarrow X_i$ contracts $\{E_1\}$ or $\{E_2\}$, then $\rho(B_i/B) = 1$, and $B_i \dashrightarrow B$ is a birational contraction contracting the image of the divisor G. Then such a B_i is uniquely determined by Corollary 2.14.

In conclusion the relation given by Proposition 4.5 has the form

$$\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \text{id},$$

and more precisely, up to a cyclic permutation exchanging the role of χ_1 and χ_3, we have a commutative diagram as in Figure 4, top left, where χ_2 and χ_4 have respectively type III and I, and χ_1 and χ_3 are equivalent Sarkisov links of type II.

Now consider Case (2). As $\hat{B}$ is $\mathbf{Q}$-factorial (Proposition 2.3), we have $\hat{B} \neq B$, hence $\rho(\hat{B}/B) = 1$ and the morphism $\hat{B} \longrightarrow B$ is a small contraction. By unicity of log-flip, there are exactly two small contractions to B. Denote $\hat{B}' \longrightarrow B$ the other one. Then for each X_i/B_i, we have $B_i \simeq \hat{B}$ or $\hat{B}'$, and $\rho(X_i/B) = 2$. Hence the birational contraction $T \dashrightarrow X_i$ contracts exactly one divisor, which must be E_1 or E_2. Again this gives four possibilities. The actual existence of $X_3/\hat{B}'$ and $X_4/\hat{B}'$ arises from the two-rays games X_1/B and X_2/B. We get a relation as in Figure 4, top right, with χ_1, χ_3 of type II and χ_2, χ_4 of type IV.

Finally consider Case (3). Then by Proposition 3.14(1)(4), each birational contraction $T \dashrightarrow X_i$ contracts one divisor among $E_1 \cup E_2$, and another one among $F_1 \cup F_2$. Again this gives four possibilities. In each case $\rho(B_i/B) = 0$ hence B_i is isomorphic to B, and then Corollary 2.13 says that X_i is determined up to isomorphism by such a choice. We obtain a relation with four links of type II, as on Figure 4, bottom. □

The following example illustrates why the assumption on the covering gonality is necessary in Proposition 5.3.

Example 5.4. Consider $\mathbb{F}_1 \longrightarrow \mathbb{P}^2$ the blow-up of a point, with $\Gamma \subseteq \mathbb{F}_1$ the exceptional section. In $\mathbb{P}^1 \times \mathbb{F}_1$, denote $D = \mathbb{P}^1 \times \Gamma$, and $C = \{0\} \times \Gamma$. Let T be the blow-up of C, with exceptional divisor E. Then $T/\mathbb{P}^2$ is a rank 3 fibration, and we now describe the associated elementary relation (see Figure 5). We let the reader verify the following assertions (since all varieties are toric, one can for instance use the associated fans).

First the two-rays game $T/\mathbb{F}_1$ gives a link of type II

$$\chi_1 \colon \mathbb{P}^1 \times \mathbb{F}_1 \dashrightarrow \mathbb{P}^1 \boxtimes \mathbb{F}_1,$$

where $\mathbb{P}^1 \boxtimes \mathbb{F}_1$ denotes a Mori fibre space over $\mathbb{F}_1$ with all fibres isomorphic to $\mathbb{P}^1$, but which is not a direct product. The link χ_1 involves the pair $D \cup E$ of divisors of type II for $T/\mathbb{F}_1$.

The divisor D on T can be contracted in two ways to a curve $\mathbb{P}^1$, that is, T dominates a flop between $\mathbb{P}^1 \boxtimes \mathbb{F}_1$ and another variety X. This variety X admits a divisorial contraction to $\mathbb{P}^1 \times \mathbb{P}^2$, with exceptional divisor the strict transform of E, which here is a divisor of type I for $X/\mathbb{P}^2$. This corresponds to a link of type III

$$\chi_2 \colon \mathbb{P}^1 \boxtimes \mathbb{F}_1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2.$$

Finally the two-rays game $\mathbb{P}^1 \times \mathbb{F}_1/\mathbb{P}^2$, which factorises via $\mathbb{F}_1$ and $\mathbb{P}^1 \times \mathbb{P}^2$, gives a link of type I

$$\chi_3 \colon \mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{F}_1.$$

In conclusion we get an elementary relation $\chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id}$.

In contrast with Lemma 3.15, observe that D and E are divisors of type II for $T/\mathbb{F}_1$, but divisors of type I for $T/\mathbb{P}^2$.

5.C. **Proof of Theorem C.** In order to prove Theorem C, we use the generators and relations of $\mathrm{BirMori}(X)$ which are given in Proposition 4.8. The key results are then Propositions 5.2 and 5.3.

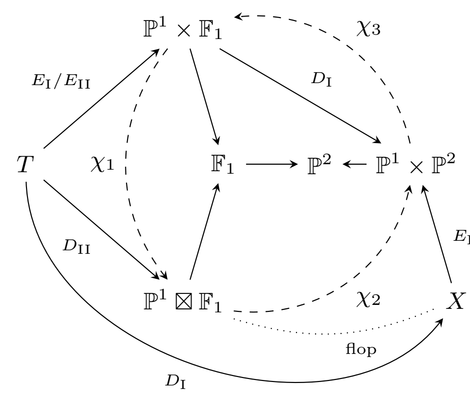


Figure 5. The elementary relation from Example 5.4. We indicate the type of contracted divisors in index.

Proof of Theorem C. We choose the integer d associated to the dimension n by Proposition 5.2. By Theorem 4.8(1), the groupoid $\mathrm{BirMori}(X)$ is generated by Sarkisov links and automorphisms of Mori fibre spaces. By Theorem 4.8(2), the relations are generated by elementary relations, so it suffices to show that every elementary relation corresponds to the identity in $\underset{C\in\mathcal{C}_X}{\ast}\left(\bigoplus_{\mathcal{M}_C}\mathbf{Z}/2\right)$.

Let $\chi_s\circ\cdots\circ\chi_1=\mathrm{id}$ be an elementary relation, coming from a rank 3 fibration T/B. We may assume that one of the χ_i is a Sarkisov link of conic bundles of type II with $\mathrm{cov.gon}(\chi_i)\geqslant d$, otherwise the relation is sent onto the identity as all χ_i are sent to the identity. We may moreover conjugate the relation and assume that χ_1 is a Sarkisov link of conic bundles of type II with $\mathrm{cov.gon}(\chi_1)\geqslant d$. By Proposition 5.2, we have $\dim(B)=n-1$. Then, Proposition 5.3 implies that $s=4$ and that χ_1 and χ_3 are equivalent Sarkisov link of conic bundles of type II. Applying the same argument to the relation $\chi_1\circ\chi_4\circ\chi_3\circ\chi_2=\mathrm{id}$ we either find that both χ_2 and χ_4 are sent to the identity or are equivalent Sarkisov links of conic bundles of type II (again by Proposition 5.3). Moreover, all the conic bundles involved in this relation are equivalent. This proves the existence of the groupoid homomorphism.

The fact that it restricts to a group homomorphism from $\mathrm{Bir}(X)$ is immediate, and the fact that it restricts as a group homomorphism

$$\mathrm{Bir}(X/B)\longrightarrow\bigoplus_{\mathcal{M}_{X/B}}\mathbf{Z}/2$$

is given by Lemma 4.9. □

6. Image of the group homomorphism given by Theorem C

In this section, we study the image of $\mathrm{Bir}(X)$ under the group homomorphism $\mathrm{Bir}(X)\longrightarrow\underset{C\in\mathcal{C}_X}{\ast}(\bigoplus_{\mathcal{M}_C}\mathbf{Z}/2)$ given by Theorem C, and more precisely the image of $\mathrm{Bir}(X/B)\longrightarrow\bigoplus_{\mathcal{M}_{X/B}}\mathbf{Z}/2$ for some conic bundles X/B. To simplify the notation, we will identify an equivalence class of marked conics in $\mathcal{M}_{X/B}$ with the generator of $\mathbf{Z}/2$ associated to it. We can then speak about sums of elements of $\mathcal{M}_{X/B}$, that we see in $\bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$, twice the same class being equal to zero.

6.A. **A criterion.**

Definition 6.1. Let $(X/B, \Gamma)$ be a marked conic, and $\varphi\colon X/B \dashrightarrow Y/B$ a birational map over B between conic bundles. For a general point $p \in \Gamma$ and a general irreducible curve $C \subseteq B$ containing p, let $b \in \mathbf{N}$ be the number of base-points of the birational map $\eta_X^{-1}(C) \dashrightarrow \eta_Y^{-1}(C)$ induced by φ that are infinitely near to a point of the fibre of p. We call the class $\bar{b} \in \mathbf{Z}/2$ *the parity of* φ *along* Γ.

The following lemma shows that this definition does not depend on the choice of p or C. It will also help to compute the image of the group homomorphism of Theorem C by studying locally a birational map near a hypersurface Γ of the base.

Lemma 6.2. *Let $\eta_X : X \longrightarrow B$ and $\eta_Y : Y \longrightarrow B$ be two conic bundles, $\varphi\colon X \dashrightarrow Y$ a birational map over B, and $\Gamma \subseteq B$ an irreducible hypersurface not contained in the discriminant locus of X/B.*

For any decomposition $\varphi = \chi_s \circ \cdots \circ \chi_1$ as in Lemma 4.9, the parity of φ along Γ is equal to the parity of the number of indexes $i \in \{1, \dots, s\}$ such that χ_i is a Sarkisov link of type II *whose marking $\Gamma_i \subseteq B_i$ is sent to Γ via B_i/B.*

Proof. Fix a decomposition $\varphi = \chi_s \circ \cdots \circ \chi_1$ as in Lemma 4.9, a general point $p \in \Gamma$ and a general irreducible curve $C \subseteq B$ containing p. For $i = 0, \dots, s$, we denote by $\eta_i\colon X_i \longrightarrow B$ the morphism given by the composition $X_i \longrightarrow B_i \longrightarrow B$.

It suffices to prove, for $i = 0, \dots, s$, that the following holds:

(a) The morphism $\eta_i^{-1}(C) \longrightarrow C$ has general fibre $\mathbb{P}^1$, and the fibre over p is $\mathbb{P}^1$ (corresponds to Γ is not in the discriminant locus).

(b) If $i \geqslant 1$, then $\chi_i \circ \cdots \circ \chi_1$ induces a birational map over C

$$\eta_0^{-1}(C) = \eta_X^{-1}(C) \dashrightarrow \eta_i^{-1}(C)$$

and the number of base-points that are infinitely near to a point of the fibre of p has the same parity as the number of integers $j \in \{1, \dots, i\}$ such that χ_j is a Sarkisov link of type II with marking $\Gamma_j \subseteq B_j$, sent to Γ via B_j/B.

We prove this by induction on i. If $i = 0$, then only (a) needs to be proven, and follows from the fact Γ is not contained in the discriminant locus of X/B.

For $i \geqslant 1$, the birational map χ_i induces a birational map over C

$$\theta_i\colon \eta_{i-1}^{-1}(C) \dashrightarrow \eta_i^{-1}(C).$$

If χ_i is a Sarkisov link of type II with marking $\Gamma_i \subseteq B_i$, sent to Γ via B_i/B, then θ_i is the composition of the blow-up a point on the fibre of p, the contraction of the strict transform of the fibre and of a birational map that is an isomorphism over an open subset of C that contains the fibre of p. Indeed, this follows from the fact that θ_i is the restriction of χ_i and of the description of χ_i given in Lemma 3.21. This achieves the proof of (a) and (b) in this case, using the induction hypothesis.

If χ_i is a Sarkisov link of type II with a marking not sent to Γ or if χ_i is a Sarkisov link of type I or III, then the restriction θ_i of χ_i is an isomorphism over an open subset of C that contains the fibre of p. This follows again from Lemma 3.21 if the Sarkisov link is of type II and from Corollary 3.18 if it is of type I or III. As before, this gives the result by applying the induction hypothesis. □

Corollary 6.3. *Let X be a rationally connected variety of dimension $\geqslant 3$, let X/B be a conic bundle, let $\varphi \in \mathrm{Bir}(X/B)$. The image of φ under the group homomorphism*

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$$

of Theorem C is equal to the sum of equivalence classes of marked conics $(X/B, \Gamma)$ with $\mathrm{cov.gon}(\Gamma) \geqslant d$ such that the parity of φ along Γ is odd.

Proof. Using Lemma 4.9, we decompose φ as $\varphi = \chi_s \circ \cdots \circ \chi_1$ where each χ_i is a Sarkisov link of conic bundles from X_{i-1}/B_{i-1} to X_i/B_i. Denote by $\Delta \subseteq \{1, \ldots, s\}$ the subset of indexes i such that the Sarkisov links χ_i is of type II and satisfies $\mathrm{cov.gon}(\chi_i) \geqslant d$. By definition of the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$$

of Theorem C, the image of φ is the sum of the equivalence classes of marked conics of χ_i where i runs over Δ. For each $i \in \Delta$, the marked conic of χ_i is equal to $(X_i/B_i, \hat{\Gamma}_i)$ for some irreducible hypersurface $\hat{\Gamma}_i \subseteq B_i$ with $\mathrm{cov.gon}(\hat{\Gamma}_i) \geqslant d$. Hence, $(X_i/B_i, \hat{\Gamma}_i)$ is equivalent to $(X/B, \Gamma_i)$, where $\Gamma_i \subseteq B$ is the image of $\hat{\Gamma}_i \subseteq B_i$ via B_i/B. This implies that the image of φ is the sum of the classes of $(X/B, \Gamma_i)$, where i runs over Δ.

By Lemma 6.2, this sum is equal to the sum of equivalence classes of marked conics $(X/B, \Gamma)$ with $\mathrm{cov.gon}(\Gamma) \geqslant d$ and such that the parity of φ along Γ is odd. □

6.B. **The case of trivial conic bundles and the proof of Theorem A.** For a variety B, let $X = \mathbb{P}^1 \times B$, and X/B be given by the second projection. The group $\mathrm{Bir}(X/B)$ is then canonically isomorphic to $\mathrm{PGL}_2(\mathbf{C}(B))$, via the action

$$\begin{array}{ccc} \mathrm{PGL}_2(\mathbf{C}(B)) \times X & \dashrightarrow & X \\ \left(\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, ([u:v], t) \right) & \mapsto & ([a(t)u + b(t)v : c(t)u + d(t)v], t). \end{array}$$

Since $\mathrm{PSL}_2(\mathbf{C}(B))$ is a simple group and has trivial centraliser inside the group $\mathrm{PGL}_2(\mathbf{C}(B))$, we obtain that any surjective group homomorphism $\mathrm{PGL}_2(\mathbf{C}(B)) \twoheadrightarrow G$ that is not an isomorphism must factorise through the quotient

$$\mathrm{PGL}_2(\mathbf{C}(B))/\,\mathrm{PSL}_2(\mathbf{C}(B)) \simeq \mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2,$$

where the isomorphism corresponds to the determinant. In particular G must be abelian of exponent 2. Write $\mathrm{div}\colon \mathbf{C}(B)^* \longrightarrow \mathrm{Div}(B)$ the classical group homomorphism that sends a rational function onto its divisor of poles and zeroes, and whose image is the group of principal divisors of B. Denoting by $\mathcal{P}_B$ the set of prime divisors of B (or equivalently the set of irreducible hypersurfaces), the group homomorphism div naturally gives a group homomorphism

$$\mathrm{PGL}_2(\mathbf{C}(B))/\,\mathrm{PSL}_2(\mathbf{C}(B)) \simeq \mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2 \longrightarrow \bigoplus_{\mathcal{P}_B} \mathbf{Z}/2.$$

We project on the sum of prime divisors with large enough covering gonality and identify the ones which are equivalent up to a birational map of B (this identification correponds to take orbits of the action of $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}(B))$ on $\mathbf{C}(B)$). This yields a group homomorphism that extends to $\mathrm{Bir}(X)$, if B is rationally connected, as the following lemma shows.

Note that for $M \in \mathrm{PGL}_2(\mathbf{C}(B))$, the parity of the multiplicity of $\det(M)$ as pole or zero along an irreducible hypersurface $\Gamma \subseteq B$ does not depend on the class of $\det(M)$ in $\mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2$

Lemma 6.4. *Let B be a rationally connected variety of dimension $\geqslant 2$, let $X = \mathbb{P}^1 \times B$, and let $\varphi_M \in \mathrm{Bir}(X/B) \simeq \mathrm{PGL}_2(\mathbf{C}(B))$ be the birational map*

$$\varphi_M \colon ([u:v],t) \dashrightarrow ([a(t)u + b(t)v : c(t)u + d(t)v], t),$$

where

$$M = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}(B)).$$

The image of φ_M under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$$

of Theorem C is equal to the sum of the equivalence classes of marked conics $(X/B, \Gamma)$ such that $\Gamma \subseteq B$ is a irreducible hypersurfaces of B with $\mathrm{cov.gon}(\Gamma) \geqslant d$ and such that the multiplicity of $\det(M)$ along Γ is odd.

Proof. We first observe that the image of $\mathrm{PSL}_2(\mathbf{C}(B)) \subseteq \mathrm{PGL}_2(\mathbf{C}(B)) \simeq \mathrm{Bir}(X/B)$ under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$$

is trivial, since $\mathrm{PSL}_2(\mathbf{C}(B))$ is simple and not abelian. The image of an element $\varphi \in \mathrm{Bir}(X/B) \simeq \mathrm{PGL}_2(\mathbf{C}(B))$ is then uniquely determined by its determinant $\delta \in \mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2$, and is the same as the image of the diagonal element

$$\psi_\delta \colon ([u:v],t) \longmapsto ([\delta(t)u : v], t).$$

We then only need to prove the result for diagonal elements of $\mathrm{PGL}_2(\mathbf{C}(B))$.

We denote as before by $\mathcal{P}_B$ the set of prime divisors of B, or equivalently the set of irreducible hypersurfaces of B. For $\delta \in \mathbf{C}(B)^*$ and $\Gamma \in \mathcal{P}_B$, we denote by $m_\delta(\Gamma) \in \mathbf{Z}$ the multiplicity of δ at Γ. Then

$$\mathrm{div}(\delta) = \sum_{\Gamma \in \mathcal{P}_B} m_\delta(\Gamma) \cdot \Gamma.$$

We also denote by $P_\delta(\Gamma) \in \{0,1\}$ the parity of ψ_δ along Γ as defined in Definition 6.1 and Lemma 6.2.

For each $\delta \in \mathbf{C}(B)^*$, the image of the diagonal element $\psi_\delta \in \mathrm{Bir}(X/B)$ under the group homomorphism $\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$ is equal to the sum of equivalence classes of marked conics $(X/B, \Gamma)$ such that $\Gamma \subseteq B$ is an irreducible hypersurfaces of B with $\mathrm{cov.gon}(\Gamma) \geqslant d$ and such that $P_\delta(\Gamma)$ is odd (Corollary 6.3). To prove the result, it suffices to show that $P_\delta(\Gamma)$ and $m_\delta(\Gamma)$ have the same parity.

For all $\delta, \delta' \in \mathbf{C}(B)^*$, we have

$$m_\delta(\Gamma) + m_{\delta'}(\Gamma) = m_{\delta\cdot\delta'}(\Gamma) \quad \text{and} \quad P_\delta(\Gamma) + P_{\delta'}(\Gamma) \equiv P_{\delta\cdot\delta'}(\Gamma) \pmod 2.$$

Indeed, the first equality follows from the definition of the multiplicity and the second follows from Lemma 6.2, since $\psi_\delta \circ \psi_{\delta'} = \psi_{\delta\cdot\delta'}$. The local ring $\mathcal{O}_\Gamma(B)$ being a DVR, the group $\mathbf{C}(B)^*$ is generated by elements $\delta \in \mathbf{C}(B)^*$ with $m_\delta(\Gamma) = 0$, and by one single element δ_0 which satisfies $m_{\delta_0}(\Gamma) = 1$. It therefore suffices to consider the case where $m_\delta(\Gamma) \in \{0, 1\}$.

We take a general point $p \in \Gamma$, a general irreducible curve $C \subseteq B$ containing p, and compute the number of base-points of the birational map $\theta \colon \mathbb{P}^1 \times C \dashrightarrow \mathbb{P}^1 \times C$ given by $([u : v], t) \longmapsto ([\delta(t)u : v], t)$ that are infinitely near to a point of the fibre of p. If $m_\delta(\Gamma) = 0$, then δ is well defined on p, so the birational map θ induces an isomorphism $\mathbb{P}^1 \times \{p\} \longrightarrow \mathbb{P}^1 \times \{p\}$, which implies that $P_\delta(\Gamma) = 0$. If $m_\delta(\Gamma) = 1$, then δ has a zero of multiplicity one at p, so θ has exactly one base-point on $\mathbb{P}^1 \times \{p\}$, namely $([1 : 0], p)$. The composition of θ with the blow-up of $Z \longrightarrow \mathbb{P}^1 \times C$ of $([1 : 0], p)$ yields a birational map $Z \dashrightarrow \mathbb{P}^1 \times C$ with no more base-point on the exceptional divisor, as the multiplicity of both δ and v/u at the point is 1, so $P_\delta(\Gamma) = 1$. This achieves the proof. □

Corollary 6.5. *Let B be a rationally connected variety of dimension $\geqslant 2$, let $X = \mathbb{P}^1 \times B$. The image of the subgroup of* $\mathrm{Bir}(X/B)$ *given by*

$$\{([u : v], t) \mapsto ([\delta(t)u : v], t) \mid \delta \in \mathbf{C}(B)^*\}$$

under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$$

of Theorem C is infinite. Moreover, for each subfield $\mathbf{k} \subseteq \mathbf{C}$ over wich X is defined, the image of elements of $\mathrm{Bir}(X/B)$ *defined over $\mathbf{k}$ is also infinite.*

Proof. We fix a subfield $\mathbf{k} \subseteq \mathbf{C}$ over which B is defined. We can assume that B is smooth, by replacing it with a smooth variety birational to it (over $\mathbf{k}$), and take a closed embedding $B \hookrightarrow \mathbb{P}^r$ defined over $\mathbf{k}$ for some positive r. We can assume that B is not contained in a hyperplane section of $\mathbb{P}^r$ (otherwise we lower the integer r). We then fix the hyperplane $H \subseteq \mathbb{P}^r$ given by $x_0 = 0$, so that $\mathbb{P}^r \smallsetminus H$ is isomorphic to $\mathbb{A}^r$. We fix an integer D_0 big enough, and choose, for each $D \geqslant D_0$, a general homogeneous polynomial $P_D \in \mathbf{k}[x_0, \ldots, x_n]$ of degree D, so that the intersection of $P_D = 0$ with B is a smooth irreducible hypersurface $V_D \subseteq B$ of degree $\deg(V_D) = D \cdot \deg(B)$, where $\deg(B)$ and $\deg(V_D)$ correspond to the degree of B and V_D viewed in $\mathbb{P}^r$. We then consider the rational function $\delta_D = \frac{P_D}{x_0^D} \in \mathbf{C}(B)^*$, whose poles lie on $B \cap H$ and zeroes on V_D. If $\mathrm{cov.gon}(V_D) \geqslant d$, then Lemma 6.4 implies that the image of the element of $\varphi_{P_D} \in \mathrm{Bir}(X/B)$ given by

$$\varphi_{P_D} \colon ([u : v], t) \longmapsto ([P(x)u : v], t)$$

in $\bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$ is equal to the sum of $(X/B, V_D)$ with a finite sum of equivalence classes $(X/B, \Gamma)$ with $\Gamma \subseteq B \cap H$.

Taking D big enough, this provides countably many distinct elements in the image, since $D \cdot \deg(B) \geqslant \mathrm{cov.gon}(V_D) \geqslant D \cdot \deg(B) - r + 1$ (Theorem 2.17). □

Proof of Theorem A. We denote by $\mathrm{Dil}_{\mathbf{k}}$ the subgroup of birational dilatations

$$\begin{aligned}\mathrm{Dil}_{\mathbf{k}} &= \{(x_1, \ldots, x_n) \longmapsto (x_1\alpha(x_2, \cdots, x_n), x_2, \ldots, x_n) \mid \alpha \in \mathbf{k}(x_2, \ldots, x_n)^*\} \\ &\subseteq \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \simeq \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}(x_1, \ldots, x_n)).\end{aligned}$$

Any field isomorphism $\kappa\colon \mathbf{k}\longrightarrow \mathbf{k}'$ induces an isomorphism

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)\simeq \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}(x_1,\dots,x_n)) \xrightarrow{\simeq} \mathrm{Bir}_{\mathbf{k}'}(\mathbb{P}^n)\simeq \mathrm{Aut}_{\mathbf{k}'}(\mathbf{k}'(x_1,\dots,x_n)),$$

and also an isomorphism between the subgroups $\mathrm{Dil}_{\mathbf{k}}$ and $\mathrm{Dil}_{\mathbf{k}'}$. We can thus assume that $\mathbf{k}$ is a subfield of $\mathbf{C}$. We then choose $B=\mathbb{P}^{n-1}$ and use the birational map (defined over $\mathbf{k}$)

$$\begin{array}{ccc} X=\mathbb{P}^1\times B & \dashrightarrow & \mathbb{P}^n \\ ([u:1],[t_1,\dots,t_{n-1}:1]) & \longmapsto & [1:u:t_1:\cdots:t_{n-1}] \end{array}$$

that conjugates $\mathrm{Bir}(X)$ to $\mathrm{Bir}(\mathbb{P}^n)$, sending elements of the form

$$\{([u:v],t)\mapstochar\dashrightarrow([\alpha(t)u:v],t)\mid \alpha\in\mathbf{C}(B)^*\}$$

onto element locally given by $(x_1,\dots,x_n)\longmapsto(x_1\alpha(x_2,\cdots,x_n),x_2,\dots,x_n)$.

We then consider an integer D big enough and consider the set $\mathcal{H}_D$ of irreducible hypersurfaces of $\mathbb{P}^{n-1}$ of degree D. For each element $h\in\mathcal{H}_D$, we consider an irreducible polynomial $P_h\in\mathbf{k}[x_0,\dots,x_n]$ of degree D defining the hypersurface h, choose $\alpha_h\in\mathbf{k}(\mathbb{P}^{n-1})$ to be P_h/x_0^D and associate to h the element $\varphi_{\alpha_h}\in\mathrm{Bir}(X/B)$ given by

$$\varphi_{\alpha_h}\colon([u:v],t)\mapstochar\dashrightarrow([\alpha_h(t)u:v],t).$$

By Lemma 6.4, the image of φ_{α_h} under the group homomorphism

$$\mathrm{Bir}(X/B)\longrightarrow\bigoplus_{\mathcal{M}_{X/B}}\mathbf{Z}/2$$

of Theorem C is the unique marked conic $(X/B,\mathcal{H}_D)$ (as the hypersurface $H_0\subseteq B$ given by $x_0=0$ satisfies $\mathrm{cov.gon}(H_0)=1$). It remains to observe that we have enough elements of $\mathcal{H}_D$, up to birational maps of $\mathbb{P}^{n-1}$, namely as much as in the field $\mathbf{k}$. Indeed, if we take two general hypersurfaces $H_1,H_2\subseteq\mathbb{P}^{n-1}$ of degree d, then every birational map $H_1\dashrightarrow H_2$ extends to a linear automorphism of $\mathbb{P}^n$; this can be shown by taking the Veronese embedding of $\mathbb{P}^n$ of the degree needed such that the canonical divisors of H_1 and H_2 are hyperplane sections. The dimension of $\mathrm{PGL}_{n+1}(\mathbf{k})$ being bounded, for a degree D big enough we obtain a quotient of $\mathcal{H}_D$ by $\mathrm{PGL}_{n+1}(\mathbf{k})$ which has dimension >1. □

Remark 6.6. As all elements explicitely described in the proof of Theorem A above belong to the Jonquières subgroup of elements preserving a pencil of lines, the image of the Jonquières subgroup by the group homomorphism of $\mathrm{Bir}(\mathbb{P}^n)\longrightarrow\bigoplus_{\mathbf{k}}\mathbf{Z}/2$ provided by the proof of Theorem A is the whole group $\bigoplus_{\mathbf{k}}\mathbf{Z}/2$. We will in fact need other conic bundles structures on rational varieties to obtain Theorem B.

Remark 6.7. Using the same ideas as in the present article, it is possible to prove that for a ground field $\mathbf{k}$ over which there exists irreducible polynomials in $\mathbf{k}[T]$ of arbitrary large degree (for instance $\mathbf{k}=\mathbb{Q}$, or $\mathbf{k}$ a finite field), and for some integer d large enough (in fact $d=9$ works) the determinant morphism induces a group homomorphism

$$\mathrm{PGL}_2(\mathbf{k}(T))\twoheadrightarrow\mathrm{PGL}_2(\mathbf{k}(T))/\mathrm{PSL}_2(\mathbf{k}(T))\simeq\mathbf{k}(T)^*/(\mathbf{k}(T)^*)^2\twoheadrightarrow\bigoplus_{D\geqslant d}\mathbf{Z}/2$$

which extends to group homomorphism from $\mathrm{Bir}(\mathbb{P}^2)$. The last homomorphism sends an irreducible polynomial of degree d onto the generator of $\mathbf{Z}/2$ indexed by d.

6.C. **The case of non-trivial conic bundles and the proof of Theorem B.** We now use a construction of involution on conic bundles given by the first two authors in [BL15, Lemma 4.4], and compute the image of these involution under the group homomorphism of Theorem C.

Lemma 6.8. *Let* $\eta\colon X \longrightarrow B$ *be a conic bundle over a rationally connected normal variety* B*, given by the restriction of a (Zariski locally trivial)* $\mathbb{P}^2$*-bundle* $\hat{\eta}\colon P \longrightarrow B$*.*

Let $s\colon B \dashrightarrow P$ *be a rational section (i.e. a rational map, birational to its image, such that* $\hat{\eta}\circ s = \mathrm{id}_B$*), whose image is not contained in* X*. We define* $\iota \in \mathrm{Bir}(X/B)$ *to be the birational involution whose restriction to a general fibre* $\eta^{-1}(p)$ *is the involution induced by the projection from the point* $s(p)$*: it is the involution of the plane* $\hat{\eta}^{-1}(p)$ *which preserves the conic* $\eta^{-1}(p)$ *and fixes* $s(p)$*.*

For each irreducible hypersurface $\Gamma \subseteq B$ *not contained in the discriminant locus of* η*, the parity of* ι *along* Γ *(in the sense of Definition 6.1) is the parity of the multiplicity of* $F(s)$ *along* Γ*, where* F *is the equation of* X *in* P *(which is locally a degree 2 equation).*

Proof. We choose a dense open subset of B which intersects Γ and trivialises the $\mathbb{P}^2$-bundle, and view X locally inside of $\mathbb{P}^2 \times B$, given by $F \in \mathbf{C}(B)[x,y,z]$, homogeneous of degree 2 in x,y,z. The fibre of $\eta\colon X \longrightarrow B$ over a general point of Γ (respectively of B) is a smooth conic. The section s corresponds to $[\alpha:\beta:\gamma]$, where $\alpha,\beta,\gamma \in \mathbf{C}(B)$ are not all zero and are uniquely determined by s, up to multiplication by an element of $\mathbf{C}(B)^*$.

We then write $a = F(\alpha,\beta,\gamma) \in \mathbf{C}(B)$ the "evaluation" of F at s which is uniquely determined by s, up to multiplication by the square of an element of $\mathbf{C}(B)^*$. The parity of the multiplicity of a along Γ is then well defined. The statement of this lemma consists in showing this parity is equal to the parity of ι along Γ.

We can choose that neither α,β,γ have a pole at Γ and that not all three are zero on Γ (by multiplying all three with a suitable power of a local equation of Γ). The restriction of α,β,γ give then an element of $\mathbf{C}(\Gamma)^3 \smallsetminus \{0\}$, that is the first row of an element of $\mathrm{GL}_3(\mathbf{C}(\Gamma))$, and then extend it to a matrix of $\mathrm{GL}_3(\mathbf{C}(B))$ having α,β,γ as it first row and having all entries which are defined on Γ, and a determinant which is not zero on Γ. Applying the corresponding element of $\mathrm{PGL}_3(\mathbf{C}(B))$, we can assume that $(\alpha,\beta,\gamma) = (1,0,0)$. We then write the equation of X as

$$F = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$$

where $a,b,c,d,e,f \in \mathbf{C}(B)$ have all no pole at Γ and are not all together zero on Γ, and obtain that $F(\alpha,\beta,\gamma) = a$. With these coordinates, the involution ι is given by

$$[x:y:z] \longmapsto [-(x + \frac{b}{a}y + \frac{c}{a}z) : y : z].$$

In particular, if a does not vanish on Γ, then ι is an isomorphism on a general point of Γ, and the multiplicity of ι at Γ is equal to zero. This achieves the proof in this case.

We can then assume that a is zero on Γ, which is implies that either b or c is not zero on Γ. We denote by $E \subseteq P$ the preimage of Γ, which is an irreducible hypersurface such that the general fibre of E/Γ is isomorphic to $\mathbb{P}^1$ (because Γ is not contained in the discriminant of X/B). The variety E is contracted by ι. We then denote by $\hat{\iota}, \nu \in \mathrm{Bir}(P/B)$ the birational maps given locally by

$$[x:y:z] \longmapsto [-(x + by + cz) : y : z] \text{ and } [x:y:z] \longmapsto [ax:y:z]$$

respectively, and observe that $\iota = \nu^{-1} \circ \hat{\iota} \circ \nu$. Moreover, the variety $\hat{Q} = \nu(Q) \subseteq P$ is given by the zero set of

$$\hat{F} = x^2 + bxy + cxz + ady^2 + aeyz + afz^2.$$

In particular, the preimage of Γ in $\hat{Q}$ is a union of two distinct hypersurfaces of $E_1, E_2 \subseteq \hat{Q}$, given respectively by $x = 0$ and $x + by + cz = 0$ (as explained before, b and c are not both zero on Γ, so we indeed get $E_1 \neq E_2$). For $i = 1, 2$, a general fibre of $E_i \longrightarrow \Gamma$, for $i = 1, 2$, is isomorphic to $\mathbb{P}^1$. Moreover, the restriction of ν induces a birational map $E \dashrightarrow E_1$ over Γ, corresponding on a general fibre to the projection from a smooth conic to a line, via a point of the conic. The birational involution $\hat{\iota}$ induces a birational transformation of $\hat{Q}$ whose restriction gives a birational map $E_1 \dashrightarrow E_2$ over Γ. We denote by $r \geqslant 1$ the multiplicity of a along r and write a as $a = a_1 \cdot a_2 \cdots a_r$, where each $a_i \in \mathbf{C}(B)$ has multiplicity 1 along Γ. We then can write ν^{-1} as $\nu^{-1} = \chi_r \circ \cdots \circ \chi_1$, where χ_i is given by $[x : y : z] \longmapsto [x : a_i y : a_i z]$. We write $Q_1 = \chi_1(\hat{Q})$ and write $Q_i = \chi_i(Q_{i-1})$ for $i = 2, \ldots, r$. In particular, $Q_r = Q$. The equation of Q_i is given by

$$F_i = a_1 \cdots a_i x^2 + bxy + cxz + a_{i+1} \cdots a_r dy^2 + a_{i+1} \cdots a_r eyz + a_{i+1} \cdots a_r fz^2.$$

For $i = 1, \ldots, r-1$, the preimage of Γ in Q_i is the union of $E_{1,i}$ and $E_{i,2}$ given by $x = 0$ and $by + cz = 0$. Writing $E_1 = E_{1,0}$ and $E_2 = E_{2,0}$, we obtain that χ_i contracts $E_{2,i-1}$ onto a rational section of Γ contained in $E_{2,i} \setminus E_{1,i}$, for $i = 1, \ldots, r$. If we denote by $C \subseteq \Gamma$ a general curve passing through a general point $p \in \Gamma$ the restriction of the birational map ι to the preimage of C can be computed as then be restriction of ι to the preimage of a surface obtained by taking $\nu^{-1} \circ \hat{\iota} \circ \nu$; the first maps $\hat{\iota} \circ \nu$ is a local isomorphism at the point p and the map ν^{-1} then corresponds to a sequence of r elementary links. The multiplicity of ι along Γ is then r as desired. ☐

In order to prove Theorem B, we will use the following simplest example of a conic bundle on a rational variety whose generic fibre is not $\mathbb{P}^1$.

Lemma 6.9. *Let $n \geqslant 3$, let $B = \mathbb{P}^{n-1}$, and let X be given by*

$$X = \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1 : \cdots : y_{n-1}]) \in \mathbb{P}^2 \times B \;\middle|\; \sum_{i=0}^{2} x_i^2 y_i = 0 \right\}.$$

Then, X is a smooth variety of dimension n, rational over $\mathbf{Q}$. Moreover, the second projection gives a conic bundle X/B whose ramification locus is given by $y_0 y_1 y_2 = 0$ and whose generic fibre is not $\mathbb{P}^1$ (equivalently, X/B is not equivalent to $(\mathbb{P}^1 \times B)/B$).

Proof. The fact that X is a smooth hypersurface of $\mathbb{P}^2 \times B$ can be checked by taking the local charts and derivatives. The $\mathbf{Q}$-rationality of X is simply given by the birational map $X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^{n-2}$ given by

$$(x, [y_0 : \cdots : y_{n-1} : y_n]) \dashrightarrow (x, [y_0 : \cdots : y_{n-1}]).$$

The discriminant locus is computed by computing the discriminant of the polynomial $\sum x_i^2 y_i$. Since all fibres are conics, and moreover the divisor over any of the three lines in the discriminant locus is irreducible, we obtain that the relative Picard rank is 1, so X/B is a conic bundle in the sens of Definition 2.4.

We now show that the generic fibre of X/B is not $\mathbb{P}^1$, or equivalently that X/B does not have any rational section. Suppose for contradiction the existence of a rational section, which is then given by three homogeneous polynomials $P_0, P_1, P_2 \in \mathbf{k}[y_0, \dots, y_n]$ of the same degree and without any common factor, such that $\sum_{i=0}^{2} P_i^2 y_i = 0$. We denote by $R_1, R_2 \in \mathbf{k}[y_1, \dots, y_n]$ the polynomials such that $P_1 \equiv R_1, P_2 \equiv R_2 \pmod{y_0}$, and obtain $R_1^2 y_1 = -R_2^2 y_2$. By counting the parity of the multiplicity of y_1 or y_2 on both sides, we find that that $R_1 = R_2 = 0$. This implies that y_0 divides P_1 and P_2, so y_0 does not divide P_0. This is impossible as $y_0 P_0^2 = -y_1 P_1^2 - y_2 P_2^2$ is divisible by y_0^2. This contradiction achieves the proof. □

Lemma 6.10. *Let $B = \mathbb{P}^2$, let $X = \{([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \sum_{i=0}^{2} x_i^2 y_i = 0\}$, and let X/B be the conic bundle given by the second projection. For each $d \geqslant 0$, we denote by $\Gamma \subseteq B$ the smooth curve given by $\sum_{i=0}^{2} y_i^{2d+1} = 0$. Then, taking d big enough such that* cov.gon$(\Gamma) \geqslant d$, *there is a birational involution $\iota \in \mathrm{Bir}(X/B)$ whose image under the group homomorphism of Theorem C is the unique class $(X/B, \Gamma)$ where $\Gamma \subseteq B$ is given by $\sum_{i=0}^{2} y_i^{2d+1} = 0$.*

Proof. Writing $P = \mathbb{P}^2 \times B$, the second projection $\hat{\eta}\colon P \longrightarrow B$ gives a (trivial) $\mathbb{P}^2$-bundle P/B. Let $\eta\colon X \longrightarrow B$ be its restriction, which gives a conic bundle structure X/B with no rational section (Lemma 6.9). For d big enough, we denote by $C \subseteq X$ the curve given by

$$C = \left\{ ([y_0^d : y_1^d : y_2^d], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times B \,\middle|\, \sum_{i=0}^{2} y_i^{2d+1} = 0 \right\}.$$

The restriction of η to C gives an isomorphism $C \longrightarrow \Gamma$, whose inverse is the restriction of the section $s\colon \mathbb{P}^2 \dashrightarrow P = \mathbb{P}^2 \times \mathbb{P}^2$ given by $s([y_0 : y_1 : x_2]) = ([y_0^d : y_1^d : y_2^d], [y_0 : y_1 : y_2])$.

Following Lemma 6.8, we consider the involution $\iota \in \mathrm{Bir}(X/B)$ whose restriction to a general fibre $\eta^{-1}(p)$ is the involution induced by the projection from the point $s(p)$: it is the involution of the plane $\hat{\eta}^{-1}(p)$ which preserves the conic $\eta^{-1}(p)$ and fixes $s(p)$.

By Lemma 6.8, the parity of ι along Γ is one, and the parity of ι along any other irreducible hypersurface of $\mathbb{P}^2$ is zero. Hence, the image of ι in under the group

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathcal{M}_{X/B}} \mathbf{Z}/2$$

of Theorem C is the equivalence class of $(X/B, \Gamma)$, if d is large enough such that cov.gon$(\Gamma) \geqslant d$ (which exists by Theorem 2.17). □

Proof of Theorem B. We take a subfield $\mathbf{k}$ of $\mathbf{C}$, and take a subset $S \subset \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^3)$ of bounded degree degree. We then prove that the normal subgroup $G \subseteq \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^3)$ generated by S, by $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^3)$ and by all Jonquières elements is a strict subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^3)$.

To do this, we use the group homomorphism $\mathrm{Bir}(\mathbb{P}^3) \longrightarrow \mathop{\ast}_{C \in \mathcal{C}_{\mathbb{P}^3}} (\bigoplus_{\mathcal{M}_C} \mathbf{Z}/2)$ given by Theorem C (associated to some fixed d big enough).

Firstly, we observe that S being of bounded degree, the set of hypersurfaces of $\mathbb{P}^3$ which are contracted by an element of S is of bounded degree, and then of bounded covering gonality (follows from the easy direction of Theorem 2.17). The image of S under the group homomorphism then involves Sarkisov links of bounded gonality

We then choose r big enough such that the curve $\Gamma \subseteq \mathbb{P}^2$ defined by $\sum_{i=0}^{2} x_i^{2r+1} = 0$ satisfies $\text{cov.gon}(\Gamma) \geqslant d$ (which exists by Theorem 2.17). We then augment r again such that the image of every element of S involves only links with covering gonality smaller than $\text{cov.gon}(\Gamma)$. We then take an involution $\iota \in \text{Bir}(X/B)$ as in Lemma 6.10, whose image under the group homomorphism of Theorem C is the class of $(X/B, \Gamma)$. As X is rational over $\mathbf{Q}$, there is a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^3$ defined over $\mathbf{Q}$, and then over $\mathbf{k}$, which conjugates ι to an element of $\text{Bir}_{\mathbf{k}}(\mathbb{P}^3)$. This latter is not in the group G, as the image of any element of G does not involve the class of $(X/B, \Gamma)$:

(1) Elements of $\text{Aut}_{\mathbf{k}}(\mathbb{P}^3)$ are in the kernel;

(2) The image of Jonquières elements are sums of pairs $(X'/B', \Gamma')$ where X'/B' is equivalent to $(\mathbb{P}^1 \times \mathbb{P}^2)/\mathbb{P}^2$, and then not to X/B (Lemma 6.9);

(3) The image of elements of S are sent onto pairs of covering gonality smaller than $\text{cov.gon}(X/B, \Gamma)$.

□

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