When does the zero fiber of a moment map have rational singularities?

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Representation varieties

- Joint with H-C Herbig and C. Seaton, to appear in Geometry and Topology, arXiv:2108.07306. Same title. Call it [HSS].
- Let G be a semisimple complex Lie group.
- For $p > 1$, let π_p be the quotient of the free group on generators $a_1, b_1, a_2, \dots, a_p, b_p$ by the normal subgroup generated by

$$[a_1, b_1][a_2, b_2] \cdots [a_p, b_p]$$

where $[a_i, b_i]$ is the commutator $a_i b_i a_i^{-1} b_i^{-1}$.
- For C closed Riemann surface genus p, $\pi_1(C) \simeq \pi_p$.
- Let $Z_p := \mathrm{Hom}(\pi_p, G)$ denote the representation variety, the homomorphisms from π_p to G. Scheme structure as $\Phi^{-1}(e)$ where

$$\Phi\colon G^{2p} \to G, \quad (g_1, h_1, g_2, \dots, g_p, h_p) \mapsto [g_1, h_1] \cdots [g_p, h_p].$$

- Φ is G-equivariant where $g \in G$ acts on G and G^{2p} by conjugation on each component. Hence G acts on Z_p.

Representation varieties and growth sequence

- Let Γ be a group and $r_n(\Gamma)$ the number of n-dimensional irreducible complex representations of Γ (the growth sequence of Γ).
- Aizenbud and Avni, Invent. 2016 and Duke 2018.

Theorem

Let $p \geq 2$, let ρ_0 be trivial element of Z_p. Consider the following:

R1. *The tangent cone TC_{ρ_0} to Z_p at ρ_0 has rational singularities.*

R2. *Z_p has rational singularities.*

R3. *Assume G simple, simply connected, rank at least 2 and defined over $\mathbb{Z}$. Then $r_n(G(\mathbb{Z})) = o(n^{2p-2+\epsilon})$ for any $\epsilon > 0$.*

Then R1 $\iff$ R2 $\implies$ R3.

- R1 true for $p \geq 21$.

Bounds on p and tangent cone

- True for $p \geq 11$ Kapon (2019), $p \geq 4$ by Glazer, Hendel (2020).
- Budur (2019). If simple factors of G are of type A, then R1 holds for $p \geq 2$ (best possible).
- Proof uses quivers, criterion of Mustață for rational singularities in terms of jet schemes and Luna's slice theorem.
- [HSS]: $p \geq 2$ works for all G.
- R3: $r_n(G(\mathbb{Z})) = o(n^{2+\epsilon})$.
- The tangent cone to Z_p at ρ_0 is zeroes of map $(x_1, \dots, x_p, y_1, \dots, y_p) \in \mathfrak{g}^{2p} \mapsto \sum [x_i, y_i] \in \mathfrak{g}$. Case $p = 1$...
- Using Killing form this is a mapping from $\mathfrak{g}^p \oplus (\mathfrak{g}^p)^* \to \mathfrak{g}^*$, which is a moment mapping:

Moment mappings

- Let G be complex reductive and V a G-module.
- $U = V \oplus V^*$ has standard G-invariant symplectic form σ: $\sigma((v_1, v_1*), (v_2, v_2^*)) = v_1^*(v_2) - v_2^*(v_1)$. Then V and V^* are Lagrangian subspaces of U.
- Standard equivariant moment mapping $\mu \colon V \oplus V^* \to \mathfrak{g}^*$. For $v \in V$, $v^* \in V^*$, $A \in \mathfrak{g}$ we have $\mu(v, v^*)(A) = v^*(A(v))$.
- μ is unique equivar. moment map for (U, σ) with $\mu(0) = 0$.
- Denote $\mu^{-1}(0)$, the shell, by N. Only depends upon (V, G^0). Since μ is G-equivariant, N has a G-action.
- If $V = \mathfrak{g}^p$, then TC_{ρ_0} is isomorphic to N.

Properties of N and V

Main Question: For which V does N have rational singularities?

- Let $V_{(n)} = \{x \in V \mid \dim G_x = n\}$. We say V is k-modular, $k \geq 0$, if $\operatorname{codim}_V V_{(n)} \geq n + k$ for $n \geq 1$.
- (Panyushev) V is k-modular if and only if a certain cohomological condition holds for $\mathbb{C}[N]$.
- Let V' be a G-submodule of U which is Lagrangian with corresponding shell N'. Then $N \simeq N'$ G-equivariantly.
- V' is k-modular iff V is. Useful fact.

Theorem

Assume that V is 0-modular. Then

1. *N is a complete intersection of dimension $2\dim V - \dim G$.*
2. *N is reduced and irreducible if and only if V is 1-modular.*
3. *N is factorial ($\mathbb{C}[N]$ is a UFD) if and only if V is 2-modular.*

The generic case

Theorem (Generic case)

1. *Suppose that V is $(\dim G)$-modular. Then N is a complete intersection with rational singularities*
2. *Let $k \geq 0$ and G be semisimple. Consider G-modules V where $V^G = 0$ and all irreducible G-submodules $W \subset V$ are almost faithful. Then, up to isomorphism, there are only finitely many V which are not k-modular.*

More conditions on N and V

- Let X be an affine G-variety. Then there is the categorical quotient $\pi_X \colon X \to X/\!\!/G$ dual to the inclusion $\mathbb{C}[X]^G \subset \mathbb{C}[X]$.
- π_X is onto and each fiber of π_X contains a unique closed G-orbit.
- There is a principal isotropy group H such that the set of fibers of π_X whose closed orbits are G-isomorphic to G/H is an open and dense subset X_{pr} of X.
- We say X has FPIG (finite principal isotropy groups) if H is finite. Then X_{pr} consists of closed orbits.
- We say a G-variety is good if it is CI with FPIG and rational singularities.

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Theorem (HSS)

Let G^0 be a torus and V a G-module. If V is 1-modular, i.e., N is a variety, then N is good.

Symplectic slice representations of N

- Let $x \in N$ where Gx is closed. Then $H = G_x$ is reductive and $E = T_x(Gx) \simeq \mathfrak{g}/\mathfrak{h}$.
- $V \oplus V^* \simeq E \oplus E^* \oplus S$ where E and E^* are isotropic and σ induces a symplectic H-invariant form σ_S on S.
- Call (S, H) the symplectic slice representation at x.
- S admits an H-stable Lagrangian submodule W and σ_S is standard on $S = W \oplus W^*$. We have the standard moment mapping $S \to \mathfrak{h}^*$ with shell N_S.

Theorem (Symplectic slice theorem)

The following are G-equivariantly isomorphic: N near Gx and (the homogeneous bundle over G/H with fiber N_S) near G/H.

- *Corollary: Let (P) be one of the following conditions: reduced, smooth, normal or rational singularities. Then N satisfies (P) at x if and only if N_S satisfies (P) at 0.*

Symplectic slice representations satisfying $(*)$.

- The null cone $\mathcal{NC}(V)$ of a G-module V is $\pi_V^{-1}(\pi_V(0))$ and similarly $\mathcal{NC}(N) = \pi_N^{-1}(\pi_N(0))$.
- Define $\lambda(V) = \max\{\dim L \mid L \subset \mathcal{NC}(V), L \text{ is linear}\}$.
- $\lambda(V^*) = \lambda(V)$.
- Let $(S = W \oplus W^*, H)$ be symplectic slice representation of N. Write $S = S^H \oplus S_0$ as H-module.
- $N_S = S^H \times N_0$ where N_0 is shell of any Lagrangian H-submodule W_0 of S_0.

Definition

Let (S, H) be a symplectic slice representation of N and S_0, etc. as above. Then (S, H) satisfies $(*)$ if one of the following holds.

S1. H^0 is a torus and and N_0 is a variety.

S2. There is a Lagrangian H-submodule W_0 of S_0 such that $\lambda(W_0) < \dim W_0 - \dim H$.

Main Theorems

Theorem (Main Theorem)

Suppose that every symplectic slice representation of N satisfies $(*)$. Then N is good. Moreover, $N/\!\!/G$ has symplectic singularities and is (graded) Gorenstein.

Theorem (Main Theorem for $V = \mathfrak{g}^p$, $p > 1$.)

Let G be semisimple, $V = \mathfrak{g}^p$ for $p > 1$ and $N = N_V$. Let (S, H) be a symplectic slice representation of N. Then

1. *(S, H) satisfies $(*)$, hence N is good.*

Suppose that $p > 2$ or that G has no simple factor of rank 1. Then

2. *N_S is factorial. So N is factorial and V is 2-modular.*
3. $\mathbb{C}[N_S]^* = \mathbb{C}^\times = \mathbb{C}[N]^*$.
4. *If H has trivial character group, then $N_S/\!\!/H$ is factorial. In particular, $N/\!\!/G$ is factorial.*

When is $\lambda(V) < \dim V - \dim G$?

- It is not easy to determine $\lambda(V)$ in general.

Proposition

Suppose that V is orthogonal.

1. $\lambda(V) = (1/2)(\dim V - \dim V^T)$ *where T is a maximal torus of G.*
2. $\lambda(V) < \dim V - \dim G$ *if and only if* $\dim G < \frac{1}{2}(\dim V + \dim V^T)$.

- If $V = \mathfrak{g}$, due to Gerstenhaber for $G = \mathrm{SL}_n$ (1958). For general G simple this is due to Meshulam and Radwan (1998). Also treated by Draisma-Kraft-Kuttler [2006]. Latter gave idea we used to prove proposition.

Applications to representation and character varieties

- Recall that $Z_p = \mathrm{Hom}(\pi_p, G)$ is a representation variety and $Y_p = Z_p /\!\!/ G$ is the corresponding character variety. Here $p \geq 2$.
- Let $z \in Z_p$ such that Gz is closed with isotropy group H. W. Goldman calculated TC_z in the 80's.
- [HSS]: Up to a vector space, $TC_z \simeq N_W$ where $W = \mathfrak{h}^p \oplus (\mathfrak{g}/\mathfrak{h})^{p-1}$.
- $(*)$ holds, so that TC_z has rational singularities. It follows that Z_p has rational singularities. Not AA proof.

Theorem

1. *Z_p is good of pure dimension $(2p-1)\dim G$.*
2. *Y_p has dimension $2(p-1)\dim G$ and symplectic singularities.*

Suppose that $p > 2$ or G has no factor of rank 1.

3. *Z_p is locally factorial with singularities in codimension four.*
4. *The component of Y_p containing the image of ρ_0 is locally factorial with singularities in codimension 4.*

Classical representations of classical groups

- Let G be a classical group and V a direct sum of copies of the defining representation (and its dual).
- Here are NASC conditions on V to be 1-modular, i.e., for N to be a variety;

$(V, G) = (k\,\mathbb{C}^n \oplus \ell\,(\mathbb{C}^n)^*, \mathrm{GL}_n(\mathbb{C}))$, $k + \ell \geq 2n$.
$(V, G) = (k\,\mathbb{C}^n \oplus \ell\,(\mathbb{C}^n)^*, \mathrm{SL}_n(\mathbb{C}))$, $k + \ell \geq 2n - 1$.
$(V, G) = (k\,\mathbb{C}^{2n}, \mathrm{Sp}_{2n})$, $k \geq 2n + 1$.
$(V, G) = (k\mathbb{C}^n, \mathrm{SO}_n(\mathbb{C}))$, $n \geq 2$, $k \geq n - 1$.
$(V, G) = (k\,\mathbb{C}^7, G_2)$, $k \geq 4$.
$(V, G) = (k\,\mathbb{C}^8, \mathrm{Spin}_7(\mathbb{C}))$, $k \geq 5$.

Theorem

Let (V, G) be as above. If N is a variety, then it is good.

- Techniques are case by case.

Jet schemes of N

- Let V be a G-module. We use a criterion of Mustață to establish that N has rational singularities.
- Let $\vec{x} = (x_0, \dots, x_m) \in V^{m+1}$ and $\vec{\xi} = (\xi_0, \dots, \xi_m) \in (V^*)^{m+1}$. The mth jet scheme N_m of N is given by the following equations where A runs through a basis of $\mathfrak{g}$.

$$\xi_0(A(x_0)) = 0, \tag{e.0}$$
$$\xi_0(A(x_1)) + \xi_1(A(x_0)) = 0, \tag{e.1}$$
$$\xi_0(A(x_2)) + \xi_1(A(x_1)) + \xi_2(A(x_0)) = 0, \tag{e.2}$$
$$\xi_0(A(x_3)) + \xi_1(A(x_2)) + \xi_2(A(x_1)) + \xi_3(A(x_0)) = 0, \tag{e.3}$$
$$\vdots$$
$$\xi_0(A(x_m)) + \xi_1(A(x_{m-1})) + \dots + \xi_m(A(x_0)) = 0. \tag{e.m}$$

- Let $\rho_m \colon N_m \to N$ be the projection.

Mustață's theorem

Theorem (Mustață, Invent. 2001)

Let N be a local complete intersection variety. The following are equivalent for $m \geq 1$ and each implies that $\dim N_m = (m+1)\dim N$.

1. N_m *is irreducible.*
2. $\dim \rho_m^{-1}(N_{\mathrm{sing}}) < (m+1)\dim N$.

Moreover, N has rational singularities if and only if (1) or (2) holds for all $m \geq 1$.

$(*)$ implies Main Theorem

- Assume $(*)$ holds for all symplectic slice representations of N.
- Can show that N has FPIG, is CI and normal.
- Can assume by induction over slice representations that $N \setminus \mathcal{NC}(N)$ has rational singularities.
- Can assume $V^G = 0$. If G^0 is a torus N is good.
- Otherwise can assume $\lambda(V) < \dim V - \dim G$.
- Let N'_m be closure of $\rho_m^{-1}(N \setminus \mathcal{NC}(N)_{\mathrm{sing}})$.
- N'_m is irreducible. Show $N_m = N'_m$.

Points in N_m lying over $\mathcal{NC}(N)$

- Fix $\vec{x} = (x_0, \ldots, x_m) \in V^{m+1}$.
- Let r_j be the rank of the $(\dim G)$ equations (e.j) on $(V^*)^{m+1}$.
- Let $Y \subset (V^*)^{m+1}$ be solutions to (e.0)–(e.m) with our fixed $\vec{x}$.
- Y is linear of dimension $(m+1)\dim V^* - \sum_{j=0}^{m} r_j$.

Lemma

The projection $\vec{\xi} \mapsto \xi_0$ maps Y onto a linear subspace $F_m \subset V^*$ of dimension $\dim V^* - r_m \geq \dim V^* - \dim G$.

- By $(*)$, $\dim F_m > \lambda(V^*) = \lambda(V)$ so that $F_m \not\subset \mathcal{NC}(V^*)$.
- If $(x_0, \xi_0) \in N$ and $\xi_0 \notin \mathcal{NC}(V^*)$, then $(x_0, \xi_0) \notin \mathcal{NC}(N)$.
- Hence $\rho_m^{-1}(\mathcal{NC}(N)) \cap (\{\vec{x}\} \times Y)$ is not dense in $\{\vec{x}\} \times Y$.
- So $N'_m \cap (\{\vec{x}\} \times Y)$ is dense in $\{\vec{x}\} \times Y$.
- Since $\vec{x}$ is arbitrary, N'_m is dense in N_m.
- So N_m is irreducible and N has rational singularities.

References

A Aizenbud and N Avni, *Representation growth and rational singularities of the moduli space of local systems*, Invent. Math. 204 (2016) 245–316.

A Aizenbud and N Avni, *Counting points of schemes over finite rings and counting representations of arithmetic lattices*, Duke Math. J. 167 (2018) 2721–2743.

N Budur, *Rational singularities, quiver moment maps, and representations of surface groups*, Int. Math. Res. Not. IMRN (2021) 11782–11817.

J Draisma, H Kraft, J Kuttler, *Nilpotent subspaces of maximal dimension in semi-simple Lie algebras*, Compos. Math. 142 (2006) 464–476.

M Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices. I*, Amer. J. Math. 80 (1958) 614–622.

References

I Glazer, Y I Hendel, *On singularity properties of word maps and applications to probabilistic Waring type problems* (2020) arXiv:1912.12556v2 .

G Kapon, *Singularity Properties of Graph Varieties* (2019) arXiv:1905.05847.

R Meshulam, N Radwan, *On linear subspaces of nilpotent elements in a Lie algebra*, Linear Algebra Appl. 279 (1998) 195–199.

M Mustaţă, *Jet schemes of locally complete intersection canonical singularities*, Invent. Math. 145 (2001) 397–424.

D I Panyushev, *The Jacobian modules of a representation of a Lie algebra and geometry of commuting varieties*, Compositio Math. 94 (1994) 181–199.