On Seshadri stratifications

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Algebraic Transformation Groups

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joint work with Rocco Chirivì and Xin Fang

- report on joint work:
  Rocco Chirivì (Lecce), Xin Fang (Aachen),
- $\mathbb{K}$ algebraically closed field, and
- $X \subseteq \mathbb{P}(V)$ embedded projective irreducible variety,
- smooth in codimension one (singular set has codimension at least two).

What is a Seshadri stratification?

basics:

a *Seshadri stratification* on $X \subseteq \mathbb{P}(V)$ is a collection of

- subvarieties $X_p \subseteq X$, $p \in A$,
- A a finite indexing set, and
- homogeneous functions $f_p \in \mathbb{K}[X]$, $p \in A$,
- have to satisfy certain compatibility conditions.

The f_p are called the *extremal functions* of the stratification.

What is a Seshadri stratification good for?

Rough idea:

• $X \longrightarrow$ combinatorial objects (semigroups, NO-body type objects)

• hope: can be used to get information about X.

• X admits a Seshadri stratification $\Rightarrow\ \exists$ flat degeneration of X into a reduced union of projective toric varieties X_0; (toric: not necessarily normal!)

- *in nice cases*: semigroups $\rightarrow$ standard monomial theory on $\mathbb{K}[X]$

- *Example: flag variety*: $G/B \hookrightarrow \mathbb{P}(V(\lambda))$ combinatorics recovers path model of representations + standard monomial theory...

...after this quick survey, let us be more precise...

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What is it? More about the subvarieties.

Recall: Seshadri stratification on X is a collection of subvarieties X_p, $p \in A, \ldots$

In this way A is partially ordered: $p \leq q$ if $X_p \subseteq X_q$.

The condition (S1):

- like X itself, the X_p, $p \in A$, are smooth in codimension one;
- X is an element in this collection (so A has unique max. element);
- the subvarieties corresponding to minimal elements are points;
- inclusions $X_q \subsetneq X_p$ can always be extended to a "full flag"

$$X_q = X_{q_1} \subset X_{q_2} \subset \ldots \subset X_{q_s} = X_p,$$

i.e. all inclusions are of codimension one, $q_1, \ldots, q_s \in A$.

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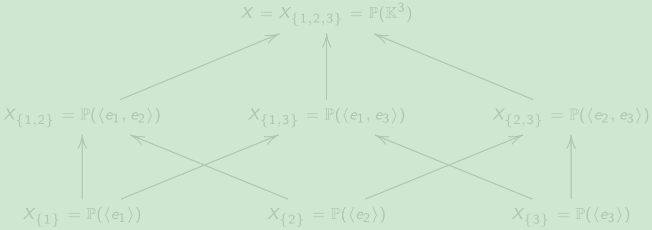
Example: $\mathbb{P}^2$ and the coordinate hyperplanes

Example (1)

$X = \mathbb{P}^2 = \mathbb{P}(\mathbb{K}^3)$, $\{e_1, e_2, e_3\}$ standard basis of $\mathbb{K}^3$.

$A =$ subsets p of $\{1, 2, 3\}$ different from $\emptyset$

Collection of subvarieties: $\{X_p \mid p \in A\}$.



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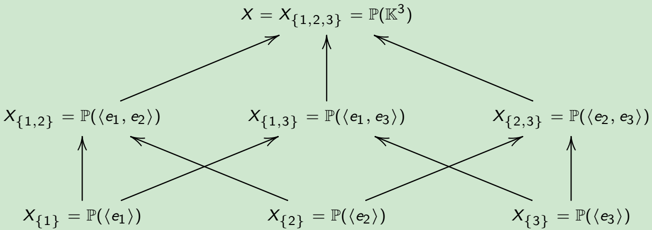

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Examples: toric varieties, Schubert varieties

Example (2)

T a torus, $X \hookrightarrow \mathbb{P}(V)$ embedded normal projective toric variety for T.

P = moment polytope, A = set of faces of the polytope $\leftrightarrow$ T-orbits in X.

Collection of subvarieties: $X_F = \overline{O_F}$, F face of P, $O_F = T$-orbit.

X_F normal, so smooth in codim 1, X = unique maximal,
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Example (3)

Flag varieties: for simplicity char $\mathbb{K} = 0$ and
G simple alg. group, B Borel subgroup, λ regular dominant weight.

$$G/B \quad \hookrightarrow \quad \mathbb{P}(V(\lambda))$$

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Examples: Generic hyperplane stratification

Example (4)

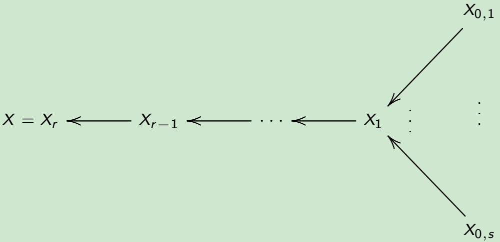
$X \subseteq \mathbb{P}(V)$ proj. variety, $\dim X = r$, smooth in codim. 1.
Bertini: there exist generic hyperplanes $H_1, \ldots, H_r$ in $\mathbb{P}(V)$ such that

$$X_r := X, \quad X_{r-1} := X_r \cap H_r, \quad \ldots \quad , \quad X_1 := X_2 \cap H_2,$$

reduced, irreducible subvarieties, smooth in codimension one.

$X_0 := X_1 \cap H_1 = X_{0,1} \cup \ldots \cup X_{0,s}$ is a finite union of points, $s =$ degree of X.

Collection of subvarieties: $\{X_r, \ldots X_1, X_{0,1}, \ldots, X_{0,s}\}$. Hasse diagram = broom:



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What is it? The conditions on the functions

(S1) conditions on the subvarieties

Next: *extremal functions*, conditions concern their vanishing behavior.

(S2) for any $q \in A$ and any $p \not\leq q$, f_p vanishes identically on X_q;

(S3) for $p \in A$ set theoretically we have:

$$\{\text{zero set of } f_p\} \cap X_p = \bigcup_{\substack{q \in A \\ X_q \text{ codim one in } X_p}} X_q.$$

Definition

A *Seshadri stratification* on X is a collection of subvarieties $X_p \subseteq X$, $p \in A$ a finite indexing set, and a collection of homogeneous functions $f_p \in \mathbb{K}[V]$, $p \in A$, satisfying the conditions (S1), (S2), (S3).

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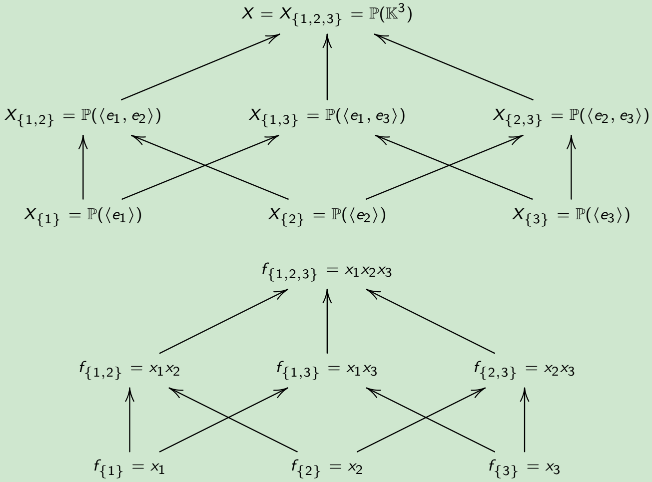
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Example with functions: toric varieties

Example (2)

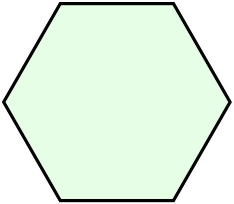
T a torus, M character lattice, $P \subseteq M_{\mathbb{R}}$ full dimensional normal lattice polytope, $X \hookrightarrow \mathbb{P}(V)$ embedded normal projective toric variety.

$A =$ set of faces of P.

Fixed: collection of subvarieties: $X_F = \overline{O_F}$, F face of P, $O_F = T$-orbit.

face F: fix a rational point μ_F in the relative interior of F.

$m_F \in \mathbb{N}$, positive, such that $\lambda_F = m_F \mu_F$ is a character, fix $f_F \in \mathbb{K}[X]_{m_F}$
The collection $\{X_F, f_F\}_{F \in A}$ is a Seshadri stratification, and all Seshadri stratifications are of this form.



Barycentric subdivision

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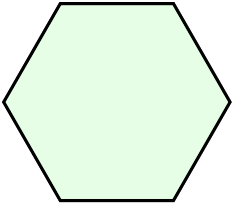

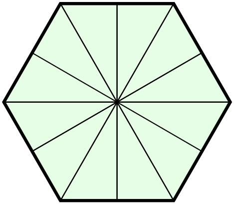

Barycentric subdivision

Example with functions: flag varieties

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collection of subvarieties: $A = W =$ Weyl group

Schubert varieties $X(w) \subseteq G/B \subseteq \mathbb{P}(V(\lambda))$,

$v_\lambda \in V(\lambda)$ a highest weight vector,
$f_\lambda \in V(\lambda)^*$ dual vector to v_λ.

For $\tau \in A$ set $f_\tau := \tau(f_\lambda)$ (extremal weight vectors).

$$\begin{cases} \text{collection of subvarieties} = \text{Schubert varieties} & X(\tau), \tau \in A; \\ \text{extremal function} = \text{extremal weight vectors} & f_\tau, \tau \in A. \end{cases}$$

The extremal weight vectors satisfy (S2) and (S3).

So this defines a *Seshadri stratification* on $G/B \subseteq \mathbb{P}(V(\lambda))$.

Example (4)

$X \subset \mathbb{P}(V)$ embedded projective variety of dimension r, smooth in codimension one, generic hyperplanes $H_r, \dots, H_1$:

$$X_r := X, \quad X_{r-1} := X_r \cap H_r, \quad \dots \quad , \quad X_1 := X_2 \cap H_2,$$

Let $f_1, \dots, f_r$ be the linear function on V defining $H_1, \dots, H_r$.

"Exercise": find functions $f_{0,j}$, $j = 1, \dots, s$ such that the collection of

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Every embedded projective variety $X \subseteq \mathbb{P}(V)$, smooth in codimension one, admits a Seshadri stratification. □

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For every maximal chain $\mathfrak{C} \subseteq A$ a valuation on $\mathbb{K}[X]$.

1. step: maximal chain $\mathfrak{C} : p_r > \ldots > p_1 > p_0$ in A

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(use j instead of p_j, recall: f_j vanishes on X_{j-1})

valuation $\nu_{\mathfrak{C}}$ on $\mathbb{K}[\hat{X}]$ ($\hat{X}$ = *affine cone over...*), rough idea:

- $h \in \mathbb{K}[\hat{X}_r]$, a_r = vanishing multiplicity at $\hat{X}_{r-1}$
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Define: $\nu_{\mathfrak{C}}(h) := (a_r, a_{r-1}, \ldots, a_0) \in \mathbb{Z}^{\mathfrak{C}}$ cheating !

Correct: divide the a_j by b_j = vanishing multiplicity of f_j on X_{j-1}:

$$\mathcal{V}_{\mathfrak{C}}(h) := \left(\frac{a_r}{b_r}, \frac{a_{r-1}}{b_{r-1}}, \ldots, \frac{a_0}{b_0}\right) \in \mathbb{Q}^{\mathfrak{C}}$$

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Seshadri stratification and a quasi-valuation

2. step: look at all $\nu_{\mathfrak{C}}$ at once, $\mathfrak{C} \subseteq A$ maximal chain. To do this:

– fix a total order on A refining the given partial order,

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$$\begin{array}{rcccl} \mathcal{V} & : & \mathbb{K}[X] - \{0\} & \to & \mathbb{Q}^A_{\geq 0} \\ & & h & \mapsto & \min\{\nu_{\mathfrak{C}}(h) \mid \mathfrak{C} \text{ maximal chain}\} \end{array}$$

is a quasi-valuation.

The non-negativity is an important point (Rees Valuation Theorem).

If $h \in \mathbb{K}[X]$ is homogeneous and $\mathcal{V}(h) = (c_p)_{p \in A} \in \mathbb{Q}^A_{\geq 0}$, then

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$X \subseteq \mathbb{P}(V)$, smooth in codim. 1, fix Seshadri stratification

Theorem

The quasi-valuation $\mathcal{V}$ induces a filtration on $\mathbb{K}[X]$ such that:

– filtration has at most one-dimensional leaves; indexed by
$\Gamma = \{\mathcal{V}(h) \mid h \in \mathbb{K}[X] \text{ homogeneous}\} \subseteq \bigcup_{\mathfrak{C}} \mathbb{Q}^{\mathfrak{C}}_{\geq 0} \subseteq \mathbb{Q}^{A}_{\geq 0}$;

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– Let $\mathbb{K}[\Gamma]$ = fan algebra, then $gr_{\mathcal{V}}\mathbb{K}[X] \simeq \mathbb{K}[\Gamma]$.
In particular: it is finitely generated and reduced;

– there exists a flat family over $\mathbb{A}^1$ with generic fibre X and special fibre $X_0 = Proj(gr_{\mathcal{V}}\mathbb{K}[X]) = Proj(\mathbb{K}[\Gamma])$.
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Simplices and simplicial complexes

$\Gamma_{\mathfrak{C}}$ has degree function $\rightarrow$ Newton-Okounkov body $D_{\mathfrak{C}}$,
– can be identified with a simplex in some $\mathbb{R}^r$ with rational vertices. We call it a simplex with a rational structure.

– Do it for all maximal chains:

Proposition

Get a Newton-Okounkov simplicial complex with a rational structure.

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The degree of the embedded variety $X \hookrightarrow \mathbb{P}(V)$ is equal to

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Example: $X = G/B \hookrightarrow \mathbb{P}(V(\lambda))$ and the path model

For simplicity: $\operatorname{char}\mathbb{K} = 0$, G simple, B Borel subgroup, λ regular dominant, $X = G/B \hookrightarrow \mathbb{P}(V(\lambda))$.

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WHAT ARE THE SEMIGROUPS $\Gamma_{\mathfrak{C}}$?

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$$\Gamma_{\mathfrak{C}} = \left\{ v = \begin{pmatrix} a_r \\ \vdots \\ a_0 \end{pmatrix} \in \mathbb{Q}^{\mathfrak{C}}_{\geq 0} \;\middle|\; \begin{array}{r} b_r a_r \in \mathbb{Z} \\ b_{r-1}(a_r + a_{r-1}) \in \mathbb{Z} \\ \dots \\ b_1(a_r + a_{r-1} + \ldots + a_1) \in \mathbb{Z} \\ \text{(degree:) } a_0 + a_1 + \ldots + a_r \in \mathbb{Z} \end{array} \right\}$$

Example: $X = G/B \hookrightarrow \mathbb{P}(V(\lambda))$ and the path model

For simplicity: $\operatorname{char} \mathbb{K} = 0$, G simple, B Borel subgroup, λ regular dominant, $X = G/B \hookrightarrow \mathbb{P}(V(\lambda))$.

Seshadri stratification: collection of Schubert varieties $X(\tau)$, $\tau \in W$, collection of functions: extremal weight vectors $f_\tau = \tau(f_\lambda) \in V(\lambda)^*$.

WHAT ARE THE SEMIGROUPS $\Gamma_{\mathfrak{C}}$?

maximal chain: $\mathfrak{C} : \tau_r = w_0 > \tau_{r-1} \cdots > \tau_1 > \tau_0 = id$ decreasing sequence of Weyl group elements

vanishing multiplicity of $f_{\tau_j}|_{X(\tau_j)}$ at $X(\tau_{j-1})$: $b_j = \langle \tau_j(\lambda), \beta^\vee \rangle$ Pieri-Chevalley formula, β positive root such that $s_\beta \tau_j = \tau_{j-1}$

Proposition

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Example: $G/B \hookrightarrow \mathbb{P}(V(\lambda))$ and path model

linearly ordered sequences of Weyl group elements + rational numbers + integrality conditions, this may sound familiar....

Corollary

$\Gamma = \bigcup \Gamma_{\mathfrak{C}} = LS$ - path model [L94] for all representations $V(m\lambda)$, $m \geq 0$.

Algebraic geometric interpretation of path model !
Collects successive vanishing multiplicities of functions/sections.

Special: $\mathcal{V}$ depends on choice of total order on A, but Γ NOT!

Example for a normal ($= \Gamma_{\mathfrak{C}}$ saturated) and balanced Seshadri stratification: notion of indecomposable in $\Gamma_{\mathfrak{C}}$, decomposition in indecomposables is unique, get a

Standard Monomial Theory

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STANDARD MONOMIAL THEORY

Newton-Okounkov simplicial complex

What about the Newton-Okounkov simplicial complex?
"Newton-Okounkov type" interpretation of Raika Dehy's realization of the path model as integral points in a simplicial complex [1998].

Multi-projective version

Consider as an example

$$G/B \hookrightarrow \mathbb{P}(V(\omega_1)) \times \cdots \times \mathbb{P}(V(\omega_n)).$$

Henrik Müller (phd-student) has developed a multi-projective version of Seshadri stratifications.

In particular, in the situation above, he recovers Young tableaux in the case SL_n and certain Lakshmibai-Seshadri tableaux in the general case as elements of the fan of semigroups.

still in the process of writing up...

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Miscellaneous

- Seshadri stratifications and standard monomial theory; Rocco Chirivì, Xin Fang, PL; Invent. Math. 234 (2023), no. 2, 489–572.
- Seshadri stratification for Schubert varieties and standard monomial theory; Rocco Chirivì, Xin Fang, PL; Proc. Indian Acad. Sci. Math. Sci. 132 (2022), no. 2, Paper No. 74.
- Seshadri stratifications and Schubert varieties: a geometric construction of a standard monomial theory; Rocco Chirivì, Xin Fang, PL; Pure and Applied Mathematics Quarterly Volume 20, Number 1, 139–169, 2024.
- On normal Seshadri stratifications, Rocco Chirivì, Xin Fang, PL; Pure and Applied Mathematics Quarterly, 2024.

Thank's a lot for your attention!

A quote

A quote from
Standard Monomial Theory - A historical account,
Collected papers of C. S. Seshadri. Volume 2, (2012):

I have felt that a good understanding of Standard Monomial Theory would be via a cellular Riemann-Roch formula as the definition of LS paths could be formulated geometrically in terms of the canonical cellular decomposition of G/B.