Factorization of holomorphic matrices

Frank Kutzschebauch, Universität Bern

Monte Verita — May 8th 2024

Outline

Any matrix $A \in SL_n(\mathbb{C})$ is a product of elementary matrices of the form

$$Id + a_{ij}E_{ij} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 0 & a_{ij} & 1 & & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & & 0 & & 1 \end{pmatrix}$$

or equivalently a product of upper and lower triangular unipotent matrices.

$$A = \begin{pmatrix} 1 & 0 \\ G_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2 \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_N \\ 0 & 1 \end{pmatrix}, \textit{where} \quad G_i \in \mathbb{C}^{n(n-1)/2}$$

Proof: Gauss elimination, it requires:
1.) Adding multiples of a row to another row
2.) Interchange of rows :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

3.) multiplication of rows by constants:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}-1 \\ 0 & 1 \end{pmatrix}$$

(Whitehead lemma)

What if the matrix A depends on a parameter x (continuously, polynomially, holomorphically)? Can the upper and lower triangular unipotent matrices be chosen depending well on the parameter?

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

Now the G_i are maps $G_i : X \to \mathbb{C}^{n(n-1)/2}$.

- Let $R = \{f : X \to \mathbb{C}\}$ denote the ring of continuous/ polynomial /holomorphic functions on a topological space/ algebraic variety / complex space X.
- In the language of K-theory we are asking about factorization of $SL_n(R)$ (special linear group over the ring R) as product of elementary matrices over that ring.
- Given $m \geq 2$ and an associative, commutative, unital ring R, let $E_n(R)$ denote the set of those $n \times n$ matrices which are representable as products of unipotent matrices with entries in R. We ask about the relation of $E_n(R)$ and $SL_n(R)$.
- The obstruction to this factorization is called the special K_1-group of the ring R, (more precise the n-th, where n is the size of the matrices).

Symplectic Notation

Let I_n denote the $(n \times n)$ identity matrix and 0_n the $(n \times n)$ zero matrix.
Recall $Sp_{2n}(\mathbb{C}) := \{A \in Gl_{2n}(\mathbb{C}) \ : \ A^T JA = J\}$, where
$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ is the standard symplectic form.

In the block notation

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{C}),$$

the symplectic condition $MJM^T = J$ is satisfied by the following matrices, we call them *elementary symplectic matrices*.

- (i): $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$, upper triangular with symmetric $B = B^T$.
- (ii): $\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$, lower triangular with symmetric $C = C^T$.

Algebraic results

- $SL_n(\mathbb{C}[z_1])$ factorizes, more generally for Euclidean rings R $SL_n(R)$ factorizes, however there is no universal bound on the number of factors! van der Kallen, W., $SL_3(\mathbb{C}[X])$ does not have bounded word length, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., 966, 357–361, 1982
- $SL_2(\mathbb{C}[z_1, z_2, \ldots, z_n])$ does not factorize for $n \geq 2$ counterexample found by Cohn (1966)

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

  P. M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 30, 5–53
- $SL_n(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 3$

  A. A. Suslin, The structure of the special linear group over rings of polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* 41 (1977), no. 2, 235–252, 477

Algebraic results

- $SL_n(\mathbb{C}[z_1])$ factorizes, more generally for Euclidean rings R $SL_n(R)$ factorizes, however there is no universal bound on the number of factors! van der Kallen, W., $SL_3(\mathbb{C}[X])$ does not have bounded word length, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., 966, 357–361, 1982
- $SL_2(\mathbb{C}[z_1, z_2, \ldots, z_n])$ does not factorize for $n \geq 2$ counterexample found by Cohn (1966)

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

P. M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 30, 5–53

- $SL_n(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 3$

A. A. Suslin, The structure of the special linear group over rings of polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* 41 (1977), no. 2, 235–252, 477

Algebraic results

- $SL_n(\mathbb{C}[z_1])$ factorizes, more generally for Euclidean rings R $SL_n(R)$ factorizes, however there is no universal bound on the number of factors! van der Kallen, W., $SL_3(\mathbb{C}[X])$ does not have bounded word length, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., 966, 357–361, 1982
- $SL_2(\mathbb{C}[z_1, z_2, \ldots, z_n])$ does not factorize for $n \geq 2$ counterexample found by Cohn (1966)

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

  P. M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 30, 5–53
- $SL_n(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 3$

  A. A. Suslin, The structure of the special linear group over rings of polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* 41 (1977), no. 2, 235–252, 477

Algebraic symplectic results

- $Sp_{2n}(\mathbb{Z}[z_1, z_2, \dots, z_m])$ does factorize for all m and all $n \geq 2$

  Grunewald, Fritz; Mennicke, Jens; Vaserstein, Leonid On symplectic groups over polynomial rings. Math. Z. 206 (1991)

- $Sp_{2n}(\mathbb{C}[z_1, z_2, \dots, z_m])$ does factorize for all m and all $n \geq 2$

  Kopeiko, V. I. On the structure of the symplectic group of polynomial rings over regular rings. (Russian) Fundam. Prikl. Mat. 1 (1995), no. 2, 545–548

  Kopeiko, V. I. Stabilization of symplectic groups over a ring of polynomials. (Russian) Mat. Sb. (N.S.) 106(148) (1978), no. 1, 94–107

Topological results

- $SL_n(Cont(\mathbb{R}^3))$ factorizes

  W. Thurston and L. Vaserstein, On K_1-theory of the Euclidean space, *Topology Appl.* 23 (1986), no. 2, 145–148

- A general observation:

$$A_t(x) = \begin{pmatrix} 1 & 0 \\ tG_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & tG_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & tG_N(x) \\ 0 & 1 \end{pmatrix} t \in [0,1]$$

gives a homotopy of the map $A : X \to SL_m(\mathbb{C})$ to a constant map. Such maps are called null-homotopic. **If a map factorizes, then it is necessarily null-homotopic.**

Topological results

- $SL_n(Cont(\mathbb{R}^3))$ factorizes

  W. Thurston and L. Vaserstein, On K_1-theory of the Euclidean space, *Topology Appl.* **23** (1986), no. 2, 145–148

- A general observation:

$$A_t(x) = \begin{pmatrix} 1 & 0 \\ tG_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & tG_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & tG_N(x) \\ 0 & 1 \end{pmatrix} t \in [0, 1]$$

gives a homotopy of the map $A : X \to SL_m(\mathbb{C})$ to a constant map. Such maps are called null-homotopic. **If a map factorizes, then it is necessarily null-homotopic.**

Continuous result

Theorem (Vaserstein)

For any natural number n and an integer $d \geq 0$ there is a natural number K such that for any finite dimensional normal topological space X of dimension d and null-homotopic continuous mapping $A\colon X \to SL_n(\mathbb{C})$ the mapping can be written as a finite product of no more than $K = K(d,n)$ unipotent matrices. That is, one can find continuous mappings $G_l\colon X \to \mathbb{C}^{n(n-1)/2}$, $1 \leq l \leq K$ such that

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

L. Vaserstein, Reduction of a matrix depending on parameters to a diagonal form by addition operations, *Proc. Amer. Math. Soc.* 103 (1988), no. 3, 741–746

Let

$$U_n(x_1, \ldots, x_{n(n+1)/2}) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \cdots & x_{n(n+1)/2} \end{pmatrix}.$$

Given a map $G\colon X \to \mathbb{C}^{n(n+1)/2}$ let $U_n(G(x)) = U_n(G_1(x), \ldots, G_{n(n+1)/2}(x))$ where the G_j's are components of the map G.

Symplectic continuous result

Theorem (Ivarsson, Kutzschebauch, Løw, 2020)

Let X be a d-dimensional normal topological space and $f: X \to \mathrm{Sp}_{2n}(\mathbb{C})$ be a continuous mapping that is null-homotopic. Then there exist a natural number $K = K_{cont}(n, d)$ and continuous mappings $G_1, \ldots, G_K : X \to \mathbb{C}^{n(n+1)/2}$ such that

$$f(x) = \begin{pmatrix} I_n & 0_n \\ U_n(G_1(x)) & I_n \end{pmatrix} \begin{pmatrix} I_n & U_n(G_2(x)) \\ 0_n & I_n \end{pmatrix} \ldots \begin{pmatrix} I_n & U_n(G_K(x)) \\ 0_n & I_n \end{pmatrix}$$

Ivarsson, B.; Kutzschebauch, F.; Løw, Erik Factorization of symplectic matrices into elementary factors. Proc. Amer. Math. Soc. 148 (2020), no. 5, 1963–1970.

The first holomorphic result: Gromov-Vaserstein problem for SL_n

Theorem

Let X be a finite dimensional reduced Stein space and $A\colon X \to SL_n(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number $K = K(dimX, n)$ and holomorphic mappings $G_1, \dots, G_K\colon X \to \mathbb{C}^{n(n-1)/2}$ such that A can be written as a product of upper and lower diagonal unipotent matrices

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

Ivarsson, B., Kutzschebauch, F. *Holomorphic factorization of mappings into $SL_n(\mathbb{C})$*, Ann. of Math. (2)

The new symplectic holomorphic result: The symplectic Gromov-Vaserstein problem

Theorem (Schott 2022, proved in the smallest dimension for n=2 by Ivarsson, Kutzschebauch, Løw)

Let X be a d-dimensional reduced Stein space and $f : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number $K_{symp} = K_{symp}(n, d)$ and holomorphic mappings $G_1, \dots, G_K : X \to \mathbb{C}^{n(n-1)}$ such that

$$f(x) = \begin{pmatrix} I_n & 0_n \\ U_n(G_1(x)) & I_n \end{pmatrix} \begin{pmatrix} I_n & U_n(G_2(x)) \\ 0_n & I_n \end{pmatrix} \dots \begin{pmatrix} I_n & U_n(G_K(x)) \\ 0_n & I_n \end{pmatrix}$$

Ivarsson, B.; Kutzschebauch, F.; Løw, Erik Holomorphic factorization of mappings into $Sp_{\mathbf{4}}(\mathbb{C})$, Anal. PDE, 16, 2023, 1, 233–277
Schott, J. Holomorphic factorization of mappings into $Sp_{\mathbf{2}n}(\mathbb{C})$, arXiv:2207.05389, to appear in J. Eur. Math. Soc.

Number of factors for the special linear group

Analytic techniques (Ivarsson-K.) can be used to show:

$$K(2,1) = 4 \quad \text{and} \quad K(2,2) = 5$$

K-theoretic arguments (Dennis, Vaserstein, Vavilov, Smolenskii, Sury, generalized by Huang, Kutzschebauch, Schott) guarantee $K(n,d) \geq K(n+1,d)$, the optimal bounds satisfy

$$\begin{aligned} K(n,1) &= 4 \ \forall n, \\ 4 \leq K(n,2) &\leq 5 = K(2,2) \ \forall n, \\ \text{for each } d, \ \text{there exists } n(d) \text{ such that } K(n,d) &\leq 6 \text{ for all } n \geq n(d), \end{aligned}$$

K-theorists use another symplectic form

There is another simple type of symplectic matrices, namely $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ block diagonal with $A \in \mathrm{GL_n}(\mathbb{C})$ and $D = (A^{-1})^T$
If block A is upper triangular, then $D = (A^{-1})^T$ is lower triangular. In fact, A and D are simultaneously upper or lower triangular in another basis, obtained from the old one by reversing the order of the last n basis elements, giving a Gramian matrix

$$\tilde{J} = \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix},$$

where L is the $n \times n$ matrix with 1 along the skew-diagonal. Matrices of type (i) and (ii) remain upper or lower triangular with respect to $\tilde{J}$, respectively. If we allow these matrices too the number of factors will be denoted by $\tilde{K}_{symp}(n, d)$

Number of factors for the symplectic group

$K_{symp}(n, d)$ is not well understood, only recently the optimal number of factors, namely 5, over the field of complex numbers was established

Jin, Pengzhan and Lin, Zhangli and Xiao, Bo, *Optimal unit triangular factorization of symplectic matrices*, Linear Algebra Appl., 650, 2022, 236–247

For $\tilde{K}_{symp}(n, d)$ we can again use K-theory to get:

$$\tilde{K}_{symp}(n, d) \geq \tilde{K}_{symp}(n+1, d)$$

$$\tilde{K}_{symp}(n, 1) = 4 \ \forall n,$$

$$4 \leq \tilde{K}_{symp}(n, 2) \leq 5 = \tilde{K}_{symp}(1, 2) \ \forall n$$

G. Huang, F. Kutzschebauch, J. Schott, Factorization of Holomorphic Matrices and Kazhdan's property (T), Bull. Sci. Math. 190 (2024)

Number of factors for the symplectic group, continued

Theorem (Tran 2024)

$K_{symp}(2,d) \leq 4\tilde{K}_{symp}(2,d)$, $K_{symp}(3,d) \leq 4\tilde{K}_{symp}(3,d)$,
$6 \leq K_{symp}(n,d) \leq 8\tilde{K}_{symp}(n,d) \quad \forall n \geq 4$.

We can conclude

$$K_{symp}(N,d) \leq 8K_{symp}(n,d) \quad \forall N \geq n+1$$

$$5 \leq K_{symp}(n,1) \leq 16 \quad \textit{for} \quad n = 2,3$$

$$6 \leq K_{symp}(n,1) \leq 32 \; \forall n$$

$$5 \leq K_{symp}(n,2) \leq 20 \quad \textit{for} \quad n = 2,3$$

$$6 \leq K_{symp}(n,2) \leq 40 \; \forall n$$

Definitions

Let R be a commutative unital ring. An element $(x_1, \ldots, x_k) \in R^k$ is called *unimodular* if

$$\sum_{j=1}^{k} x_j R = R.$$

Let $U_k(R)$ the set of all unimodular elements in R^k.
An element $x = (x_1, \ldots, x_{k+1}) \in U_{k+1}(R)$ is called *reducible* if there exists $(y_1, \ldots, y_k) \in R^k$ such that

$$(x_1 + y_1 x_{k+1}, \ldots, x_k + y_k x_{k+1}) \in U_k(R).$$

The *Bass stable rank* of R, denoted by $\mathrm{bsr}(R)$ is the least $k \in \mathbb{N}$ such that every $x \in U_{k+1}(R)$ is reducible. If there is no such $k \in \mathbb{N}$, then we set $\mathrm{bsr}(R) = \infty$.

A sufficient condition for factorization

Remark

The identity $\mathrm{bsr}(R) = 1$ is equivalent to the following property: For any $x_1, x_2 \in R$ such that $x_1 R + x_2 R = R$, there exist $y \in R$ such that $x_1 + y x_2 \in R^*$.

Theorem (Vavilov, Smolenskii, Sury)

Let R be a unital commutative ring and $n \geq 2$. If $\mathrm{bsr}(R) = 1$, then $E_n(R) = SL_n(R)$ and $\mathrm{Ep}_{2n}(R) = Sp_{2n}(R)$. Moreover the number of elementary factors is 4.

Vavilov, N. A.; Smolenskii, A. V.; Sury, B. Unitriangular factorizations of Chevalley groups. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 388 (2011), Voprosy Teorii Predstavlenii Algebr i Grupp. 21, 17–47, 309–310; translation in J. Math. Sci. (N.Y.) 183 (2012), no. 5, 584–599

A sufficient condition for factorization

Remark

The identity $\mathrm{bsr}(R) = 1$ is equivalent to the following property: For any $x_1, x_2 \in R$ such that $x_1R + x_2R = R$, there exist $y \in R$ such that $x_1 + yx_2 \in R^*$.

Theorem (Vavilov, Smolenskii, Sury)

Let R be a unital commutative ring and $n \geq 2$. If $\mathrm{bsr}(R) = 1$, then $E_n(R) = SL_n(R)$ and $\mathrm{Ep}_{2n}(R) = Sp_{2n}(R)$. Moreover the number of elementary factors is 4.

Vavilov, N. A.; Smolenskii, A. V.; Sury, B. Unitriangular factorizations of Chevalley groups. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 388 (2011), Voprosy Teorii Predstavlenii Algebr i Grupp. 21, 17–47, 309–310; translation in J. Math. Sci. (N.Y.) 183 (2012), no. 5, 584–599

Examples

- Holomorphic functions $\mathcal{O}(X)$ on open Riemann surfaces X Herta Florack 1948: $\mathrm{bsr}(\mathcal{O}(X)) = 1$.
- Jones, Marshall and Wolff (1986) and independently Corach and Suárez (1985): $\mathrm{bsr}(A(\mathbb{D})) = 1$
- for an open Riemann surface Ω with boundary, Leiterer (2021) $\mathrm{bsr}(A(\Omega)) = 1$
- Treil (1992): $\mathrm{bsr}(H^\infty(\mathbb{D})) = 1$,
- Tolokonnikov (1995): $\mathrm{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface G and for certain infinitely connected planar domains G (Behrens domains).

Examples

- Holomorphic functions $\mathcal{O}(X)$ on open Riemann surfaces X
  Herta Florack 1948: $\mathrm{bsr}(\mathcal{O}(X)) = 1$.
- Jones, Marshall and Wolff (1986) and independently Corach and Suárez (1985): $\mathrm{bsr}(A(\mathbb{D})) = 1$
- for an open Riemann surface Ω with boundary, Leiterer (2021) $\mathrm{bsr}(A(\Omega)) = 1$
- Treil (1992): $\mathrm{bsr}(H^\infty(\mathbb{D})) = 1$,
- Tolokonnikov (1995): $\mathrm{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface G and for certain infinitely connected planar domains G (Behrens domains).

Examples

- Holomorphic functions $\mathcal{O}(X)$ on open Riemann surfaces X
  Herta Florack 1948: $\mathrm{bsr}(\mathcal{O}(X)) = 1$.
- Jones, Marshall and Wolff (1986) and independently Corach and Suárez (1985): $\mathrm{bsr}(A(\mathbb{D})) = 1$
- for an open Riemann surface Ω with boundary, Leiterer (2021) $\mathrm{bsr}(A(\Omega)) = 1$
- Treil (1992): $\mathrm{bsr}(H^\infty(\mathbb{D})) = 1$,
- Tolokonnikov (1995): $\mathrm{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface G and for certain infinitely connected planar domains G (Behrens domains).

Examples

- Holomorphic functions $\mathcal{O}(X)$ on open Riemann surfaces X Herta Florack 1948: $\operatorname{bsr}(\mathcal{O}(X)) = 1$.
- Jones, Marshall and Wolff (1986) and independently Corach and Suárez (1985): $\operatorname{bsr}(A(\mathbb{D})) = 1$
- for an open Riemann surface Ω with boundary, Leiterer (2021) $\operatorname{bsr}(A(\Omega)) = 1$
- Treil (1992): $\operatorname{bsr}(H^\infty(\mathbb{D})) = 1$,
- Tolokonnikov (1995): $\operatorname{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface G and for certain infinitely connected planar domains G (Behrens domains).

Examples

- Holomorphic functions $\mathcal{O}(X)$ on open Riemann surfaces X
  Herta Florack 1948: $\mathrm{bsr}(\mathcal{O}(X)) = 1$.
- Jones, Marshall and Wolff (1986) and independently Corach and Suárez (1985): $\mathrm{bsr}(A(\mathbb{D})) = 1$
- for an open Riemann surface Ω with boundary, Leiterer (2021) $\mathrm{bsr}(A(\Omega)) = 1$
- Treil (1992): $\mathrm{bsr}(H^\infty(\mathbb{D})) = 1$,
- Tolokonnikov (1995): $\mathrm{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface G and for certain infinitely connected planar domains G (Behrens domains).

- Let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle. Given a function $f \in H^\infty(\mathbb{D})$, it is well-known that the radial limit $\lim_{r\to 1-} f(r\zeta)$ exists for almost all $\zeta \in \mathbb{T}$ with respect to Lebesgue measure on $\mathbb{T}$. So, let $H^\infty(\mathbb{T})$ denote the space of the corresponding radial values. It is known that $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is an algebra,
  Mortini, Wick: $\mathrm{bsr}(H^\infty(\mathbb{T}) + C(\mathbb{T})) = 1$
- Now, let B denote a Blaschke product in $\mathbb{D}$. Then $\mathbb{C} + BH^\infty(\mathbb{D})$ is an algebra.
  Mortini, Sasane, Wick: $\mathrm{bsr}(\mathbb{C} + BH^\infty(\mathbb{D})) = 1$.
- Brudnyi (2019): for Stein spaces: $\mathrm{bsr}(\mathcal{O}(X)) = \lfloor \frac{1}{2} \dim X \rfloor + 1$

- Let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle. Given a function $f \in H^\infty(\mathbb{D})$, it is well-known that the radial limit $\lim_{r\to 1-} f(r\zeta)$ exists for almost all $\zeta \in \mathbb{T}$ with respect to Lebesgue measure on $\mathbb{T}$. So, let $H^\infty(\mathbb{T})$ denote the space of the corresponding radial values. It is known that $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is an algebra,
  Mortini, Wick: $\mathrm{bsr}(H^\infty(\mathbb{T}) + C(\mathbb{T})) = 1$
- Now, let B denote a Blaschke product in $\mathbb{D}$. Then $\mathbb{C} + BH^\infty(\mathbb{D})$ is an algebra.
  Mortini, Sasane, Wick: $\mathrm{bsr}(\mathbb{C} + BH^\infty(\mathbb{D})) = 1$.
- Brudnyi (2019): for Stein spaces: $\mathrm{bsr}(\mathcal{O}(X)) = \lfloor \frac{1}{2} \dim X \rfloor + 1$

- Let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle. Given a function $f \in H^\infty(\mathbb{D})$, it is well-known that the radial limit $\lim_{r\to 1-} f(r\zeta)$ exists for almost all $\zeta \in \mathbb{T}$ with respect to Lebesgue measure on $\mathbb{T}$. So, let $H^\infty(\mathbb{T})$ denote the space of the corresponding radial values. It is known that $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is an algebra,
  Mortini, Wick: $\mathrm{bsr}(H^\infty(\mathbb{T}) + C(\mathbb{T})) = 1$
- Now, let B denote a Blaschke product in $\mathbb{D}$. Then $\mathbb{C} + BH^\infty(\mathbb{D})$ is an algebra.
  Mortini, Sasane, Wick: $\mathrm{bsr}(\mathbb{C} + BH^\infty(\mathbb{D})) = 1$.
- Brudnyi (2019): for Stein spaces: $\mathrm{bsr}(\mathcal{O}(X)) = \lfloor \frac{1}{2} \dim X \rfloor + 1$

Let R be a m-convex Frechet algebra with 1.
When is a matrix $F \in GL_n(R)$ representable as a product of exponentials, that is, $F = \exp G_1 \ldots \exp G_k$ with $G_j \in M_n(R)$?

Example

$$A(z) = \begin{pmatrix} 1 & 1 \\ 0 & e^{4\pi i z} \end{pmatrix}, z \in \Delta.$$

is not an exponential.

Lemma

Let $X \in SL_n(R)$ be a unipotent matrix. Then X is an exponential. Conjugates of exponentials are exponentials.

Let R be a m-convex Frechet algebra with 1.
When is a matrix $F \in GL_n(R)$ representable as a product of exponentials, that is, $F = \exp G_1 \dots \exp G_k$ with $G_j \in M_n(R)$?

Example

$$A(z) = \begin{pmatrix} 1 & 1 \\ 0 & e^{4\pi i z} \end{pmatrix}, z \in \Delta.$$

is not an exponential.

Lemma

Let $X \in SL_n(R)$ be a unipotent matrix. Then X is an exponential. Conjugates of exponentials are exponentials.

Let R be a m-convex Frechet algebra with 1.
When is a matrix $F \in GL_n(R)$ representable as a product of exponentials, that is, $F = \exp G_1 \ldots \exp G_k$ with $G_j \in M_n(R)$?

Example

$$A(z) = \begin{pmatrix} 1 & 1 \\ 0 & e^{4\pi i z} \end{pmatrix}, z \in \Delta.$$

is not an exponential.

Lemma

Let $X \in SL_n(R)$ be a unipotent matrix. Then X is an exponential. Conjugates of exponentials are exponentials.

Corollary (Doubtsov, Kutzschebauch)

Let X be a Stein space of dimension d and let $A \in GL_n(\mathcal{O}(X))$. Then there exists a number $e(n,d)$ such that the following properties are equivalent:

(i) *A is null-homotopic;*

(ii) *A is a product of $e(n,d)$ exponentials.*

Theorem (Kutzschebauch, Studer 2019)

Any null-homotopic holomorphic map from an open Riemann surface to the linear group $GL_2(\mathbb{C})$ is a product of two exponentials.

E. Doubtsov, F. Kutzschebauch: Factorization by elementary matrices, null-homotopy and products of exponentials for invertible matrices over rings, F. Anal. Math. Phys. (2019).

Corollary (Doubtsov, Kutzschebauch)

Let X be a Stein space of dimension d and let $A \in GL_n(\mathcal{O}(X))$. Then there exists a number $e(n,d)$ such that the following properties are equivalent:

(i) *A is null-homotopic;*

(ii) *A is a product of $e(n,d)$ exponentials.*

Theorem (Kutzschebauch, Studer 2019)

Any null-homotopic holomorphic map from an open Riemann surface to the linear group $GL_2(\mathbb{C})$ is a product of two exponentials.

E. Doubtsov, F. Kutzschebauch: Factorization by elementary matrices, null-homotopy and products of exponentials for invertible matrices over rings, F. Anal. Math. Phys. (2019).

Product of symplectic Exponentials

Theorem (Huang, Kutzschebauch, Schott)

Let X be a reduced Stein space of dimension d and let $f : X \to \mathrm{Sp}_{2n}(\mathbb{C})$ be a null-homotopic holomorphic mapping with $n \geq 2$. Then

- (a) *There exist a natural number $e = e_{symp}(n, d)$ and holomorphic mappings $A_1, ..., A_e : X \to \mathfrak{sp}_{2n}(\mathbb{C})$ such that*

$$f(x) = \exp(A_1(x)) \cdots \exp(A_e(x)).$$

- (b) $2 \leq e_{symp}(n, 1) \leq e_{symp}(n, 2) \leq 3 \quad \forall n.$

G. Huang, F. Kutzschebauch, J. Schott, Factorization of Holomorphic Matrices and Kazhdan's property (T), Bull. Sci. Math. 190 (2024)

THANK YOU!
ALL THE BEST TO YOU HANSPETER!