The Abhyankar-Sathaye Epimorphism Conjecture

Neena Gupta
Indian Statistical Institute
Kolkata, India

6th May 2024
Algebraic Transformation Groups
Monte Verità Ascona, Switzerland

A.K. Dutta, S.M. Bhatwadekar and Avinash Sathaye

Challenging Problems on Polynomial Rings

Polynomial rings present several challenging open problems — easy to state (at least for mathematicians), difficult to solve.

Challenging Problems on Polynomial Rings

Polynomial rings present several challenging open problems — easy to state (at least for mathematicians), difficult to solve.

1. **Jacobian Conjecture** (O.H. Keller).
2. **Zariski Cancellation Problem**.
3. **$\mathbb{A}^n$-Fibration Problem** (Dolgachev-Weisfeiler).
4. **Epimorphism Problem** (Abhyankar-Sathaye).
5. **Linearisation Problem** (Kambayashi).
6. **Characterisation Problem**.
7. **$\mathbb{A}^n$-form Problem**.

Main themes involved in some of these problems:

• To determine whether a polynomial F is a *coordinate* of $k[X_1, \ldots, X_n]$, i.e., whether there exist $F_2, \ldots, F_n$ such that $k[X_1, \ldots, X_n] = k[F, F_2, \ldots, F_n]$.

• To examine whether $A \cong k[X_1, \ldots, X_n]$ for a given ring A.

Main themes involved in some of these problems:

• To determine whether a polynomial F is a *coordinate* of $k[X_1, \ldots, X_n]$, i.e., whether there exist $F_2, \ldots, F_n$ such that $k[X_1, \ldots, X_n] = k[F, F_2, \ldots, F_n]$.

• To examine whether $A \cong k[X_1, \ldots, X_n]$ for a given ring A.

Throughout my talk,
k: a field of any characteristic.

For a ring R,
$A = R^{[n]}$: A is a polynomial ring in n-indeterminates over R

Shreeram S. Abhyankar (1930-2012)

Polynomials and power series,
May they forever rule the world

Epimorphism Theorem (Abhyankar-Moh 1975)

k: field of characteristic $p \geq 0$.

$\phi : k[X, Y] \to k[T]$ an epimorphism (surjection).

Let $n = \deg_T \phi(X) \geq 1$, $m = \deg_T \phi(Y) \geq 1$.

Epimorphism Theorem (Abhyankar-Moh 1975)

k: field of characteristic $p \geq 0$.

$\phi : k[X, Y] \to k[T]$ an epimorphism (surjection).

Let $n = \deg_T \phi(X) \geq 1$, $m = \deg_T \phi(Y) \geq 1$.

Suppose $p \not| \mathrm{GCD}\{n, m\}$ (in particular, if $p = 0$).

Epimorphism Theorem (Abhyankar-Moh 1975)

k: field of characteristic $p \geq 0$.

$\phi : k[X, Y] \to k[T]$ an epimorphism (surjection).

Let $n = \deg_T \phi(X) \geq 1$, $m = \deg_T \phi(Y) \geq 1$.

Suppose $p \not| \operatorname{GCD}\{n, m\}$ (in particular, if $p = 0$).

Epimorphism Theorem. **Either $m|n$ or $n|m$.**

Epimorphism Theorem (Abhyankar-Moh 1975)

k: field of characteristic $p \geq 0$.

$\phi : k[X, Y] \to k[T]$ an epimorphism (surjection).

Let $n = \deg_T \phi(X) \geq 1$, $m = \deg_T \phi(Y) \geq 1$.

Suppose $p \not| \mathrm{GCD}\{n, m\}$ (in particular, if $p = 0$).

Epimorphism Theorem. **Either $m|n$ or $n|m$**.

i.e., if T can be expressed as a polynomial in $P(T)$ and $Q(T)$, then either $\deg_T P \mid \deg_T Q$ or $\deg_T Q \mid \deg_T P$.

Epimorphism Theorem (Abhyankar-Moh 1975)

k: field of characteristic $p \geq 0$.

$\phi : k[X, Y] \to k[T]$ an epimorphism (surjection).

Let $n = \deg_T \phi(X) \geq 1$, $m = \deg_T \phi(Y) \geq 1$.

Suppose $p \not| \, \mathrm{GCD}\{n, m\}$ (in particular, if $p = 0$).

Epimorphism Theorem. **Either** $m|n$ **or** $n|m$.

i.e., if T can be expressed as a polynomial in $P(T)$ and $Q(T)$, then either $\deg_T P \mid \deg_T Q$ or $\deg_T Q \mid \deg_T P$.

Theorem (Abhyankar-Moh). Let characteristic k be 0 and $F \in k[X, Y]$. Then

$$k[X, Y]/(F) = k^{[1]} \Rightarrow k[X, Y] = k[F]^{[1]}.$$

Epimorphism Theorem (Abhyankar-Moh 1975)

k: field of characteristic $p \geq 0$.

$\phi : k[X, Y] \to k[T]$ an epimorphism (surjection).

Let $n = \deg_T \phi(X) \geq 1$, $m = \deg_T \phi(Y) \geq 1$.

Suppose $p \not| \mathrm{GCD}\{n, m\}$ (in particular, if $p = 0$).

Epimorphism Theorem. **Either** $m|n$ **or** $n|m$.

i.e., if T can be expressed as a polynomial in $P(T)$ and $Q(T)$, then either $\deg_T P \mid \deg_T Q$ or $\deg_T Q \mid \deg_T P$.

Theorem (Abhyankar-Moh). Let characteristic k be 0 and $F \in k[X, Y]$. Then

$$k[X, Y]/(F) = k^{[1]} \Rightarrow k[X, Y] = k[F]^{[1]}.$$

Proved independently by Suzuki for $k = \mathbb{C}$.

Epimorphism Problem

Q. $\dfrac{k[X_1, \ldots, X_n]}{(F)} = k^{[n-1]} \implies k[X_1, \ldots, X_n] = k[F]^{[n-1]}$?

Epimorphism Problem

Q. $\dfrac{\mathbf{k[X_1, \ldots, X_n]}}{\mathbf{(F)}} = \mathbf{k^{[n-1]}} \implies \mathbf{k[X_1, \ldots, X_n] = k[F]^{[n-1]}}$?

$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki 1975)

$n = 2$: **NO** ch $k > 0$ (Segre 1957, Nagata 1972)

Epimorphism Problem

Q. $\dfrac{\mathbf{k[X_1, \ldots, X_n]}}{\mathbf{(F)}} = \mathbf{k^{[n-1]}} \implies \mathbf{k[X_1, \ldots, X_n] = k[F]^{[n-1]}}$?

$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki 1975)

$n = 2$: **NO** ch $k > 0$ (Segre 1957, Nagata 1972)

Abhyankar-Sathaye Conjecture: YES when ch $k = 0$.

Epimorphism Problem

Q. $\dfrac{\mathbf{k[X_1,\ldots,X_n]}}{\mathbf{(F)}} = \mathbf{k^{[n-1]}} \implies \mathbf{k[X_1,\ldots,X_n] = k[F]^{[n-1]}}$?

$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki 1975)

$n = 2$: **NO** ch $k > 0$ (Segre 1957, Nagata 1972)

Abhyankar-Sathaye Conjecture: YES when ch $k = 0$.

Question on Epimorphism problem can be asked even when ch $k \geq 0$ and F is of certain specified type.

Epimorphism Problem

Q. $\mathbf{\dfrac{k[X_1,\ldots,X_n]}{(F)} = k^{[n-1]} \implies k[X_1,\ldots,X_n] = k[F]^{[n-1]}}$?

$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki 1975)

$n = 2$: **NO** ch $k > 0$ (Segre 1957, Nagata 1972)

Abhyankar-Sathaye Conjecture: YES when ch $k = 0$.

Question on Epimorphism problem can be asked even when ch $k \geq 0$ and F is of certain specified type.

Ex (Segre (1957), Nagata (1971)):
Let $g(Z,T) = Z^{p^e} + T + T^{sp} \in k[Z,T]$,
where $e, s \in \mathbb{N}$, $p^e \nmid sp$, $sp \nmid p^e$. Then

Epimorphism Problem

Q. $\mathbf{\frac{k[X_1,\ldots,X_n]}{(F)} = k^{[n-1]} \implies k[X_1,\ldots,X_n] = k[F]^{[n-1]}}$?

$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki 1975)

$n = 2$: **NO** ch $k > 0$ (Segre 1957, Nagata 1972)

Abhyankar-Sathaye Conjecture: YES when ch $k = 0$.

Question on Epimorphism problem can be asked even when ch $k \geq 0$ and F is of certain specified type.

Ex (Segre (1957), Nagata (1971)):
Let $g(Z,T) = Z^{p^e} + T + T^{sp} \in k[Z,T]$,
where $e, s \in \mathbb{N}$, $p^e \nmid sp$, $sp \nmid p^e$. Then

$k[Z,T]/(g(Z,T)) = k^{[1]}$ but $k[Z,T] \neq k[g(Z,T)]^{[1]}$.

Epimorphism Problem

Q. $\mathbf{\frac{k[X_1, \ldots, X_n]}{(F)} = k^{[n-1]} \implies k[X_1, \ldots, X_n] = k[F]^{[n-1]}}$?

$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki 1975)

$n = 2$: **NO** ch $k > 0$ (Segre 1957, Nagata 1972)

Abhyankar-Sathaye Conjecture: YES when ch $k = 0$.

Question on Epimorphism problem can be asked even when ch $k \geq 0$ and F is of certain specified type.

Ex (Segre (1957), Nagata (1971)):
Let $g(Z, T) = Z^{p^e} + T + T^{sp} \in k[Z, T]$,
where $e, s \in \mathbb{N}$, $p^e \nmid sp$, $sp \nmid p^e$. Then

$k[Z, T]/(g(Z, T)) = k^{[1]}$ but $k[Z, T] \neq k[g(Z, T)]^{[1]}$.

Defn. We say g is a *nontrivial line*.

Theorem on Linear Planes

Thm (Sathaye, Russell (1976)): Let ch $k \geq 0$ and

$$\frac{k[X,Y,Z]}{(G)} \cong k^{[2]}, \quad \text{where} \quad G = a(X,Z)Y - b(X,Z).$$

Then $k[X,Y,Z] = k[G]^{[2]}$ and there exists $X_1 \in k[X,Z]$ s.t.

$a(X,Z) = a_1(X_1), k[X,Z] = k[X_1]^{[1]}$ and $k[X,Y,Z] = k[X_1,G]^{[1]}$.

In particular, if $A = k^{[2]}$, then for any linear plane F in $A[Y]$, coordinates of A can be so chosen, such that

$$F = a(X)Y + b(X,Z).$$

Epimorphism Problem: Russell-Sathaye, Kaliman

Thm: k field of characteristic $p \geq 0$ and $F = aZ^n - b$, where $a, b \in k[X, Y]$ and $p \nmid n$. Then

$$\mathbf{k[X, Y, Z]/(F) \cong k^{[2]} \implies k[X, Y, Z] = k[F]^{[2]}},$$

k alg. closed (Wright (1978)); any k (Das–Dutta (2011)).

Thm (Russell-Sathaye (1979)): k field of characteristic zero and $F = a_n Z^n + a_{n-1} Z^{n-1} + \cdots + a_1 Z + a_0 \in k[X, Y, Z]$, where $a_0, \ldots, a_n \in k[X, Y]$ s.t. $\gcd(a_1, \ldots, a_n) \notin k$. Then

$$\mathbf{k[X, Y, Z]/(F) \cong k^{[2]} \implies k[X, Y, Z] = k[F]^{[2]}}.$$

Thm (Kaliman (2002)): Suppose that $G \in \mathbb{C}[X, Y, Z]$ s.t.

$$\frac{\mathbb{C}[\mathbf{X, Y, Z}]}{(\mathbf{G} - \lambda)} \cong \mathbb{C}^{[2]} \quad \text{for almost all} \quad \lambda \in \mathbb{C}.$$

Then $\mathbb{C}[\mathbf{X, Y, Z}] = \mathbb{C}[\mathbf{G}]^{[2]}$.

Thm (Kaliman, Vénéreau, Zaidenberg)

$\mathbf{F} = \mathbf{a}(\mathbf{X}, \mathbf{Y})\mathbf{W} - \mathbf{b}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \in \mathbb{C}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}]$, $\mathbf{a} \neq \mathbf{0}$ s.t.,

$$\mathbf{B} := \mathbb{C}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}]/(\mathbf{F}) = \mathbb{C}^{[3]}.$$

Then $\mathbb{C}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}] = \mathbb{C}[\mathbf{F}]^{[3]}$ in the following cases:

- $a \in \mathbb{C}[X]$.
- $\deg_Z b \leq 1$.
- b is of the form $b_0(X, Y) + b_2(X, Y, Z)Z^2$.
- There exists no irreducible factor a_1 of a such that $a_1 B \cap \mathbb{C}[X, Y]$ is a height two ideal.
- a is square free. In fact, it is enough to assume that for any irreducible factor a_1 of a such that $a_1 B \cap \mathbb{C}[X, Y]$ is a height one ideal, we have $a_1^2 \nmid a$.

Linear Affine Varieties

Recent developments in AAG have revealed the importance of study of the linear affine varieties defined by the polynomials of the form $F = aY - b$, where $a \in k[X]$ and $b \in k[X, Z, T]$, in many central problems like Affine Fibration Problem, Linearisation Problem and the Zariski Cancellation Problem.

Linear Affine Varieties

Recent developments in AAG have revealed the importance of study of the linear affine varieties defined by the polynomials of the form $F = aY - b$, where $a \in k[X]$ and $b \in k[X, Z, T]$, in many central problems like Affine Fibration Problem, Linearisation Problem and the Zariski Cancellation Problem.

• A crucial step in settling the Linearization Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ involved deciding whether certain threefolds defined by Koras and Russell is a polynomial ring, like whether the Russell cubic $A = \mathbb{C}[X, Y, Z, T]/(X^2Y + X + Z^2 + T^3)$ is $\mathbb{C}^{[3]}$.

Linear Affine Varieties

Recent developments in AAG have revealed the importance of study of the linear affine varieties defined by the polynomials of the form $F = aY - b$, where $a \in k[X]$ and $b \in k[X, Z, T]$, in many central problems like Affine Fibration Problem, Linearisation Problem and the Zariski Cancellation Problem.

• A crucial step in settling the Linearization Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ involved deciding whether certain threefolds defined by Koras and Russell is a polynomial ring, like whether the Russell cubic $A = \mathbb{C}[X, Y, Z, T]/(X^2Y + X + Z^2 + T^3)$ is $\mathbb{C}^{[3]}$.

Thm (Makar-Limanov, 1996): $A \neq \mathbb{C}^{[3]}$.

Linear Affine Varieties

Recent developments in AAG have revealed the importance of study of the linear affine varieties defined by the polynomials of the form $F = aY - b$, where $a \in k[X]$ and $b \in k[X, Z, T]$, in many central problems like Affine Fibration Problem, Linearisation Problem and the Zariski Cancellation Problem.

• A crucial step in settling the Linearization Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ involved deciding whether certain threefolds defined by Koras and Russell is a polynomial ring, like whether the Russell cubic $A = \mathbb{C}[X, Y, Z, T]/(X^2Y + X + Z^2 + T^3)$ is $\mathbb{C}^{[3]}$.

Thm (Makar-Limanov, 1996): $A \neq \mathbb{C}^{[3]}$.

• "Asanuma threefold" in connection with the Affine Fibration Problem, Linearisation Problem and the Zariski Cancellation Problem.

Asanuma's Example (1987)

Ex: Let k be a field of characteristic $p > 0$ and

$$\mathbf{A} = \frac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^rY + Z^{p^2} + T + T^{sp}})}, \quad p \nmid s \quad .$$

Asanuma's Example (1987)

Ex: Let k be a field of characteristic $p > 0$ and

$$\mathbf{A} = \frac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^rY + Z^{p^2} + T + T^{sp}})}, \quad p \nmid s \quad .$$

Thm (Asanuma (1987)):
Let x denote the image of X in A. Then

- A is an $\mathbb{A}^2$-fibration (defined later) over $k[x]$.
- $\mathbf{A^{[1]} \cong_{k[x]} k[x]^{[3]} = k^{[4]}}$ but
- $\mathbf{A \ncong_{k[x]} k[x]^{[2]}}$.

Thus A is a nontrivial $\mathbb{A}^2$-fibration over the PID $k[x]$.

Asanuma's Example (1987)

Ex: Let k be a field of characteristic $p > 0$ and

$$\mathbf{A} = \frac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^r Y + Z^{p^2} + T + T^{sp}})}, \quad p \nmid s \quad .$$

Thm (Asanuma (1987)):
Let x denote the image of X in A. Then

- A is an $\mathbb{A}^2$-fibration (defined later) over $k[x]$.
- $\mathbf{A^{[1]} \cong_{k[x]} k[x]^{[3]} = k^{[4]}}$ but
- $\mathbf{A \ncong_{k[x]} k[x]^{[2]}}$.

Thus A is a nontrivial $\mathbb{A}^2$-fibration over the PID $k[x]$.

Q (Asanuma (1994)): Is $\mathbf{A \cong_k k^{[3]}}$?

Asanuma's Example (1987)

Ex: Let k be a field of characteristic $p > 0$ and

$$\mathbf{A} = \frac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^r Y + Z^{p^2} + T + T^{sp}})}, \quad p \nmid s \quad .$$

Thm (Asanuma (1987)):
Let x denote the image of X in A. Then

- A is an $\mathbb{A}^2$-fibration (defined later) over $k[x]$.
- $\mathbf{A^{[1]} \cong_{k[x]} k[x]^{[3]} = k^{[4]}}$ but
- $\mathbf{A \not\cong_{k[x]} k[x]^{[2]}}$.

Thus A is a nontrivial $\mathbb{A}^2$-fibration over the PID $k[x]$.

Q (Asanuma (1994)): Is $\mathbf{A \cong_k k^{[3]}}$?

If YES then Linearisation Prob has $-$ve soln for $k^{[3]}$ in $+$ve ch.
If NO then ZCP has $-$ve soln for $k^{[3]}$ in $+$ve ch.

P. Russell called this dichotomy: **Asanuma's Dilemma**.

T. Asanuma

Asanuma's Example (1987)

Ex: Let k be a field of characteristic $p > 0$ and

$$\mathbf{A} = \frac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^rY + Z^{p^2} + T + T^{sp}})}, \quad p \nmid s \quad .$$

Thm (Asanuma (1987)):
Let x denote the image of X in A. Then

- A is an $\mathbb{A}^2$-fibration over $k[x]$.
- $\mathbf{A^{[1]} \cong_{k[x]} k[x]^{[3]} = k^{[4]}}$ but
- $\mathbf{A \ncong_{k[x]} k[x]^{[2]}}$.

Thus A is a nontrivial $\mathbb{A}^2$-fibration over the PID $k[x]$.

Q (Asanuma (1994)): Is $\mathbf{A \cong_k k^{[3]}}$?
If YES then Linearisation Prob has $-$ve soln for $k^{[3]}$ in $+$ve ch.
If NO then ZCP has $-$ve soln for $k^{[3]}$ in $+$ve ch.

P. Russell called this dichotomy: **Asanuma's Dilemma**.

Thm (— 2014): $\mathbf{A \ncong k^{[3]}}$ for $r \geq 2$.

Oscar Zariski (1899-1986)
Brought rigour in classical algebraic geometry, laid the foundation of modern algebraic Geometry with A. Weil, connected it with commutative algebra

Zariski Cancellation Problem

Zariski Cancellation Problem: Is $\mathbb{A}_k^n$ cancellative as an affine variety? i.e., for an affine variety $\mathbb{V}$,

$$\mathbb{V} \times \mathbb{A}_{\mathbf{k}}^{\mathbf{1}} \cong \mathbb{A}_{\mathbf{k}}^{\mathbf{n+1}} \implies \mathbb{V} \cong \mathbb{A}_{\mathbf{k}}^{\mathbf{n}}?$$

More generally, is $k^{[n]}(= k[X_1, \dots, X_n])$ cancellative? i.e.,

$$\mathbf{A[W]} \cong_{\mathbf{k}} \mathbf{k[X_1, \dots, X_{n+1}]} \implies \mathbf{A} \cong_{\mathbf{k}} \mathbf{k[X_1, \dots, X_n]}?$$

$\mathbf{n = 1}$: **YES** (Abhyankar-Eakin-Heinzer 1972)

$\mathbf{n = 2}$: **YES** ch $k = 0$ (Fujita 1979, Miyanishi-Sugie 1980)

YES k perfect (Russell 1981)

YES ch $k \geq 0$, k any field (Bhatwadekar— (2015))

Zariski Cancellation Problem

Zariski Cancellation Problem: Is $\mathbb{A}^n_k$ cancellative as an affine variety? i.e., for an affine variety $\mathbb{V}$,

$$\mathbb{V} \times \mathbb{A}^{\mathbf{1}}_{\mathbf{k}} \cong \mathbb{A}^{\mathbf{n+1}}_{\mathbf{k}} \implies \mathbb{V} \cong \mathbb{A}^{\mathbf{n}}_{\mathbf{k}}?$$

More generally, is $k^{[n]}(= k[X_1, \ldots, X_n])$ cancellative? i.e.,

$$\mathbf{A[W]} \cong_{\mathbf{k}} \mathbf{k[X_1, \ldots, X_{n+1}]} \implies \mathbf{A} \cong_{\mathbf{k}} \mathbf{k[X_1, \ldots, X_n]}?$$

$\mathbf{n = 1}$: **YES** (Abhyankar-Eakin-Heinzer 1972)

$\mathbf{n = 2}$: **YES** ch $k = 0$ (Fujita 1979, Miyanishi-Sugie 1980)

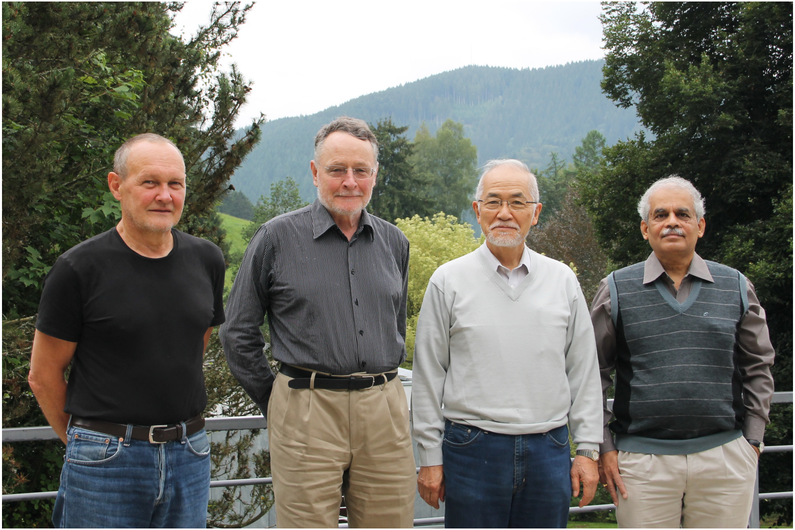
YES k perfect (Russell 1981)
YES ch $k \geq 0$, k any field (Bhatwadekar— (2015))

Research on ZCP for $n = 2$ had led to:

- Topological characterisation of $\mathbb{C}^2$ (C.P. Ramanujam 1971)
- Algebraic characterisation of k^2 (M. Miyanishi 1975)

C.P. Ramanujam (1938-1974)

“He felt the spirit of Mathematics demanded of him not merely routine developments but the right theorem on any given topic.” – D. Mumford



M. Koras, P. Russell, M. Miyanishi and R.V. Gurjar

Zariski Cancellation Problem

ZCP. Is $k^{[n]}$ $(= k[X_1, \ldots, X_n])$ cancellative?

n = 1: **YES** (Abhyankar-Eakin-Heinzer 1972)

n = 2: **YES** ch $k = 0$ (Fujita 1979, Miyanishi-Sugie 1980)

YES k perfect (Russell 1981)
YES ch $k \geq 0$, k any field (Bhatwadekar— (2015))

Research on ZCP for $n = 2$ had led to:
- Topological characterisation of $\mathbb{C}^2$ (C.P. Ramanujam 1971)
- Algebraic characterisation of k^2 (M. Miyanishi 1975)

n ≥ 3: **NO** ch $k > 0$ (— 2014)

Zariski Cancellation Problem

ZCP. Is $k^{[n]}$ $(= k[X_1, \ldots, X_n])$ cancellative?

$\mathbf{n = 1}$: **YES** (Abhyankar-Eakin-Heinzer 1972)

$\mathbf{n = 2}$: **YES** ch $k = 0$ (Fujita 1979, Miyanishi-Sugie 1980)

YES k perfect (Russell 1981)
YES ch $k \geq 0$, k any field (Bhatwadekar— (2015))

Research on ZCP for $n = 2$ had led to:
- Topological characterisation of $\mathbb{C}^2$ (C.P. Ramanujam 1971)
- Algebraic characterisation of k^2 (M. Miyanishi 1975)

$\mathbf{n \geq 3}$: **NO** ch $k > 0$ (— 2014)

$\mathbf{n \geq 3}$: **OPEN** ch $k = 0$

The threefold $x^r y = F(x, z, t)$

I. Asanuma's example: $\dfrac{\mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]}{(\mathbf{X}^{\mathbf{r}}\mathbf{Y} + \mathbf{Z}^{\mathbf{p}^2} + \mathbf{T} + \mathbf{T}^{\mathbf{sp}})}$, $p \nmid s$, where $\mathbf{Z}^{\mathbf{p}^2} + \mathbf{T} + \mathbf{T}^{\mathbf{sp}}$ is a non-trivial line in $\mathbf{k}[\mathbf{Z}, \mathbf{T}]$.

The threefold $x^r y = F(x, z, t)$

I. Asanuma's example: $\dfrac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^rY + Z^{p^2} + T + T^{sp}})}$, $p \nmid s$,

where $\mathbf{Z^{p^2} + T + T^{sp}}$ is a non-trivial line in $\mathbf{k[Z, T]}$.

Question (**Russell**): Suppose that $\mathbf{f(Z, T)}$ is **any** non-trivial line in $\mathbf{k[Z, T]}$ and $\mathbf{A} = \dfrac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^rY + f(Z, T)})}$.

Is $\mathbf{A} \not\cong \mathbf{k^{[3]}}$?

The threefold $x^r y = F(x, z, t)$

I. Asanuma's example: $\dfrac{\mathbf{k[X,Y,Z,T]}}{(\mathbf{X^rY + Z^{p^2} + T + T^{sp}})}$, $p \nmid s$,

where $\mathbf{Z^{p^2} + T + T^{sp}}$ is a non-trivial line in $\mathbf{k[Z,T]}$.

Question (**Russell**): Suppose that $\mathbf{f(Z,T)}$ is **any** non-trivial line in $\mathbf{k[Z,T]}$ and $\mathbf{A} = \dfrac{\mathbf{k[X,Y,Z,T]}}{(\mathbf{X^rY + f(Z,T)})}$.

Is $\mathbf{A} \ncong \mathbf{k^{[3]}}$?

II. Russell-Koras example: $\dfrac{\mathbb{C}\mathbf{[X,Y,Z,T]}}{(\mathbf{X^2Y + X + Z^2 + T^3})}$.

The threefold $x^r y = F(x, z, t)$

I. Asanuma's example: $\dfrac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^r Y + Z^{p^2} + T + T^{sp}})}$, $p \nmid s$,

where $\mathbf{Z^{p^2} + T + T^{sp}}$ is a non-trivial line in $\mathbf{k[Z, T]}$.

Question (**Russell**): Suppose that $\mathbf{f(Z, T)}$ is **any** non-trivial line in $\mathbf{k[Z, T]}$ and $\mathbf{A} = \dfrac{\mathbf{k[X, Y, Z, T]}}{(\mathbf{X^r Y + f(Z, T)})}$.

Is $\mathbf{A} \ncong \mathbf{k^{[3]}}$?

II. Russell-Koras example: $\dfrac{\mathbb{C}\mathbf{[X, Y, Z, T]}}{(\mathbf{X^2 Y + X + Z^2 + T^3})}$.

Led us to consider the more general threefold:

$$\mathbf{A = k[X, Y, Z, T]/(X^r Y - F(X, Z, T))}$$

over *any* field k and any polynomial $F(X, Z, T) \in k[X, Z, T]$.

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,

$G := X^r Y - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,
$G := X^r Y - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

$$\mathbf{A} = \mathbf{k[X, Y, Z, T]/(X^rY - F(X, Z, T))}, \quad \text{where} \quad \mathbf{r > 1}.$$

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,
$G := X^r Y - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

$$\mathbf{A} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{r}}\mathbf{Y} - \mathbf{F}(\mathbf{X}, \mathbf{Z}, \mathbf{T})), \quad \text{where} \quad \mathbf{r} > \mathbf{1}.$$

Then the following statements are equivalent:

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,

$G := X^r Y - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

$$\mathbf{A} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{r}}\mathbf{Y} - \mathbf{F}(\mathbf{X}, \mathbf{Z}, \mathbf{T})), \quad \text{where} \quad \mathbf{r} > \mathbf{1}.$$

Then the following statements are equivalent:

- $A \cong k^{[3]}$.

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,
$G := X^r Y - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

$$\mathbf{A} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{r}}\mathbf{Y} - \mathbf{F}(\mathbf{X}, \mathbf{Z}, \mathbf{T})), \quad \text{where} \quad \mathbf{r} > \mathbf{1}.$$

Then the following statements are equivalent:

- $A \cong k^{[3]}$.
- $A \cong k[x]^{[2]}$, where x denotes the image of X in A.

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,
$G := X^r Y - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

$$\mathbf{A} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{r}}\mathbf{Y} - \mathbf{F}(\mathbf{X}, \mathbf{Z}, \mathbf{T})), \quad \text{where} \quad \mathbf{r} > \mathbf{1}.$$

Then the following statements are equivalent:

- $A \cong k^{[3]}$.
- $A \cong k[x]^{[2]}$, where x denotes the image of X in A.
- $f(Z, T)$ is a coordinate in $k[Z, T]$.

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,
$G := X^r Y - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

$$\mathbf{A} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{r}}\mathbf{Y} - \mathbf{F}(\mathbf{X}, \mathbf{Z}, \mathbf{T})), \quad \text{where} \quad \mathbf{r} > \mathbf{1}.$$

Then the following statements are equivalent:

- $A \cong k^{[3]}$.
- $A \cong k[x]^{[2]}$, where x denotes the image of X in A.
- $f(Z, T)$ is a coordinate in $k[Z, T]$.
- $k[X, Y, Z, T] = k[G]^{[3]}$.

Theorem (— 2014)

k: a field of ANY characteristic, $B = k[X, Y, Z, T]$,
$G := X^rY - F(X, Z, T) \in B$ and $f(Z, T) := F(0, Z, T)$.

$$\mathbf{A} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{r}}\mathbf{Y} - \mathbf{F}(\mathbf{X}, \mathbf{Z}, \mathbf{T})), \quad \text{where} \quad \mathbf{r} > \mathbf{1}.$$

Then the following statements are equivalent:

- $A \cong k^{[3]}$.
- $A \cong k[x]^{[2]}$, where x denotes the image of X in A.
- $f(Z, T)$ is a coordinate in $k[Z, T]$.
- $k[X, Y, Z, T] = k[G]^{[3]}$.
- $k[X, Y, Z, T] = k[X, G]^{[2]}$.

Investigations on $x_1^{r_1} \cdots x_m^{r_m} y = F(x_1, \ldots, x_m, z, t)$

Let k be a field and

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

Problem: To obtain a criterion on F for which

- $\mathbf{A \cong k[V_1, \ldots, V_{m+2}]}$.
- $\mathbf{A[W] \cong k[V_1, \ldots, V_{m+3}]}$.
- $(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})$ is a coordinate in $\mathbf{k[X_1, \ldots, X_m, Y, Z, T]}$.

Let k be a field and

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \le i \le m$.

Problem: To obtain a criterion on F for which

- $\mathbf{A \cong k[V_1, \ldots, V_{m+2}]}$.
- $\mathbf{A[W] \cong k[V_1, \ldots, V_{m+3}]}$.
- $\mathbf{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T))}$ is a coordinate in $\mathbf{k[X_1, \ldots, X_m, Y, Z, T]}$.

Details are in Parnashree Ghosh's Poster Presentation.
Sketch follows:

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$,

$G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$,
$G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where
$F = f(Z, T) + (X_1 \cdots X_m) g(X_1, \ldots, X_m, Z, T)$.

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$, $G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where $F = f(Z, T) + (X_1 \cdots X_m) g(X_1, \ldots, X_m, Z, T)$.

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

The following statements are equivalent:

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$,
$G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where
$F = f(Z, T) + (X_1 \cdots X_m) g(X_1, \ldots, X_m, Z, T)$.

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

The following statements are equivalent:

- $A \cong k^{[m+2]}$.

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$, $G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where $F = f(Z, T) + (X_1 \cdots X_m) g(X_1, \ldots, X_m, Z, T)$.

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

The following statements are equivalent:

- $A \cong k^{[m+2]}$.
- $f(Z, T)$ is a coordinate in $k[Z, T]$.

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$,
$G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where
$F = f(Z, T) + (X_1 \cdots X_m) g(X_1, \ldots, X_m, Z, T)$.

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

The following statements are equivalent:

- $A \cong k^{[m+2]}$.
- $f(Z, T)$ is a coordinate in $k[Z, T]$.
- $B = k[G]^{[m+2]}$.

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$, $G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where $F = f(Z, T) + (X_1 \cdots X_m) g(X_1, \ldots, X_m, Z, T)$.

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

The following statements are equivalent:

- $A \cong k^{[m+2]}$.
- $f(Z, T)$ is a coordinate in $k[Z, T]$.
- $B = k[G]^{[m+2]}$.
- $B = k[X_1, X_2, \ldots, X_m, G]^{[2]}$.

Thm (Ghosh—, 2023)

k: a field of ANY characteristic, $B = k[X_1, \ldots, X_m, Y, Z, T]$,
$G := X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T) \in B$ where
$F = f(Z, T) + (X_1 \cdots X_m) g(X_1, \ldots, X_m, Z, T)$.

$$\mathbf{A} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{(\mathbf{X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)})} \text{ where } r_i > 1,$$

for each i, $1 \leq i \leq m$.

The following statements are equivalent:

- $A \cong k^{[m+2]}$.
- $f(Z, T)$ is a coordinate in $k[Z, T]$.
- $B = k[G]^{[m+2]}$.
- $B = k[X_1, X_2, \ldots, X_m, G]^{[2]}$.
- $A \cong k[x_1, \ldots, x_m]^{[2]}$, where x_i'=s images of X_i in A.

Linear affine varieties

k: any field,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, such that $f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Linear affine varieties

k: any field,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, such that $f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Q. (i) Under what condition $A = k^{[m+2]}$?

Linear affine varieties

k: any field,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, such that $f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Q. (i) Under what condition $A = k^{[m+2]}$?

(ii) Does $A = k^{[m+2]} \implies B = k[H]^{[m+2]}$?

Linear affine varieties

k: any field,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, such that $f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Q. (i) Under what condition $A = k^{[m+2]}$?

(ii) Does $A = k^{[m+2]} \implies B = k[H]^{[m+2]}$?

(iii) Does $A = k^{[m+2]} \implies B = k[X_1, \ldots, X_m, H]^{[2]}$?

Thm A (Ghosh — Pal 2024)

k: field of characteristic zero,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, s.t.
$f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Thm A (Ghosh — Pal 2024)

k: field of characteristic zero,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z,T) - h(X_1, \ldots, X_m, Z, T)$, s.t.
$f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z,T) - h(X_1, \ldots, X_m, Z, T))}.$$

Suppose $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$
and that $k[Z,T]/(f)$ is a regular ring. Then

$$k[Z,T] = k[f]^{[1]}$$

and

$$B = k[X_1, \ldots, X_m, H]^{[2]}$$

.

Thm B (Ghosh — Pal 2024)

k: field of characteristic zero,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, such that $f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Let $\alpha = \prod_{i=1}^{n} p_i^{s_i}$ be a prime factorization of α in B and one of the following conditions is satisfied.

I. $s_i = 1$ for some i.
II. $s_i > 1$ for every i and at least one of the following holds.
   (a) $p_j^2 \mid h$ for some j.
   (b) $(p_j, (p_j)_{X_1}, \ldots, (p_j)_{X_m})k[X_1, \ldots, X_m]$ is a proper ideal for some j.
   (c) If $n \geqslant 2$, then $(p_l, p_j)k[X_1, \ldots, X_m]$ is a proper ideal for some $l \neq j$.

Let $x_1, \ldots, x_m$ be the images of $X_1, \ldots, X_m$ in A respectively. Then the following statements are equivalent:

- $B = k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.
- $k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]}$.
- $A = k[x_1, \ldots, x_m]^{[2]}$.
- $A = k^{[m+2]}$.
- $k[Z, T] = k[f(Z, T)]^{[1]}$.
- A is an $\mathbb{A}^2$-fibration over $k[x_1, \ldots, x_m]$.
- $A^{[l]} = k^{[m+l+2]}$ for some $l \geqslant 0$.

The hypersurfaces defined by

$$H = \alpha(X_1, \ldots, X_m)Y - f(Z, T) \in k[X_1, \ldots, X_m, Y, Z, T], \alpha \notin k$$

are contained in the family of hypersurfaces mentioned in Theorem B.

Type A

$(0 \neq)\alpha \in k^{[m]}$ is called of *type A with respect to* $(r_1, \ldots, r_m) \in \mathbb{Z}^m_{>0}$ if there exists a system of coordinates $X_1, \ldots, X_m$ of $k^{[m]}$ such that $\alpha(X_1, \ldots, X_m) = X_1^{r_1}\alpha_1(X_1, \ldots, X_m),$ where $\alpha_1 \in k^{[m]}$ with $X_1 \nmid \alpha_1,$ for any $i \in \{2, \ldots, m\}$ $\alpha_i(X_i, \ldots, X_m) := \dfrac{\alpha_{i-1}(0, X_i, \ldots, X_m)}{X_i^{r_i}} \in k[X_i, \ldots, X_m]$ with $X_i \nmid \alpha_i$ $\alpha_{m+1} := \alpha_m(0) \in k^*$, i.e., $\exists$ $\beta_i \in k^{[m-i+1]}$,$i \in \{1, \ldots, m\}$, s.t.

$$\begin{array}{rcl}
\alpha & = & X_1^{r_1}\alpha_1(X_1, \ldots, X_m) \\
& = & X_1^{r_1}(X_1\beta_1(X_1, \ldots, X_m) + \alpha_1(0, X_2, \ldots, X_m)) \\
& = & X_1^{r_1}(X_1\beta_1(X_1, \ldots, X_m) + X_2^{r_2}\alpha_2(X_2, \ldots, X_m)) \\
& = & X_1^{r_1}(X_1\beta_1(X_1, \ldots, X_m) + X_2^{r_2}(X_2\beta_2(X_2, \ldots, X_m) + X_3^{r_3}\alpha_3(X_3, \\
& = & \ldots \\
& = & X_1^{r_1}(X_1\beta_1(X_1, \ldots, X_m) + \cdots + X_{m-1}^{r_{m-1}}(X_{m-1}\beta_{m-1}(X_{m-1}, X_m) + \\
& = & X_1^{r_1}(X_1\beta_1(X_1, \ldots, X_m) + \cdots + X_{m-1}^{r_{m-1}}(X_{m-1}\beta_{m-1}(X_{m-1}, X_m) +
\end{array}$$

Prop (Ghosh — Pal 2024)

k: infinite field of any characteristic,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - X_1\beta(X_1, \ldots, X_m, Z, T)$,
such that $f \neq 0$ and α is **type A** w.r.t. $(r_1, \ldots, r_m) \in \mathbb{Z}^m_{>1}$ in $\{X_1, \ldots, X_m\}$ and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - X_1\beta(X_1, \ldots, X_m, Z, T))}.$$

Suppose either $\mathrm{ML}(A) = k$ or $\mathrm{DK}(A) = A$. Then there exists Z_1, T_1 of $k[Z, T]$ and $a_0, a_1 \in k^{[1]}$ such that

$$k[Z, T] = k[Z_1, T_1]$$

and

$$f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1.$$

Examples

I. Let

$$H := X^2(1+X)^2Y + Z^2 + T^3 + Xa(X, Z, T)$$

$X^2(1+X)^2$ is Type A w.r.t. (2) in $\{X\}$ and
$f(Z, T) = Z^2 + T^3$ which cannot be a linear polynomial.

Examples

I. Let

$$H := X^2(1+X)^2 Y + Z^2 + T^3 + Xa(X, Z, T)$$

$X^2(1+X)^2$ is Type A w.r.t. (2) in $\{X\}$ and
$f(Z, T) = Z^2 + T^3$ which cannot be a linear polynomial.

Thus $A = k[X, Y, Z, T]/(H)$ is NOT a polynomial ring, even if A is a 'nice' ring.

Examples

I. Let

$$H := X^2(1+X)^2Y + Z^2 + T^3 + Xa(X,Z,T)$$

$X^2(1+X)^2$ is Type A w.r.t. (2) in $\{X\}$ and
$f(Z,T) = Z^2 + T^3$ which cannot be a linear polynomial.

Thus $A = k[X,Y,Z,T]/(H)$ is NOT a polynomial ring, even if A is a 'nice' ring.

II. Let

$$H := X_1X_2^2(X_1 + X_2^2)^2Y + Z^2 + T^3 - X_2\beta(X_1,X_2,Z,T)$$

$X_1X_2^2(X_1+X_2^2)^2$ is Type A w.r.t. $(2,2)$ in $\{X_2,X_1\}$ and
$f(Z,T) = Z^2 + T^3$ which cannot be a linear polynomial.

Examples

I. Let

$$H := X^2(1+X)^2Y + Z^2 + T^3 + Xa(X, Z, T)$$

$X^2(1+X)^2$ is Type A w.r.t. (2) in $\{X\}$ and
$f(Z, T) = Z^2 + T^3$ which cannot be a linear polynomial.

Thus $A = k[X, Y, Z, T]/(H)$ is NOT a polynomial ring, even if A is a 'nice' ring.

II. Let

$$H := X_1X_2^2(X_1 + X_2^2)^2Y + Z^2 + T^3 - X_2\beta(X_1, X_2, Z, T)$$

$X_1X_2^2(X_1 + X_2^2)^2$ is Type A w.r.t. $(2, 2)$ in $\{X_2, X_1\}$ and
$f(Z, T) = Z^2 + T^3$ which cannot be a linear polynomial.

Thus $A = k[X_1, X_2, Y, Z, T]/(H)$ is NOT a polynomial ring even if A is a 'nice' ring.

Thm C (Ghosh — Pal 2024)

k: field of any characteristic,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, such that $f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Suppose α is a polynomial of **type A** with respect to $(r_1, \ldots, r_m) \in \mathbb{Z}^m_{>1}$ in the system of coordinates $\{X_1 - \lambda_1, \ldots, X_m - \lambda_m\}$, for some $\lambda_i \in \overline{k}$ s.t. $k_1 := k(\lambda_1, \ldots, \lambda_m)$ is separable over k.

Thm C (Ghosh — Pal 2024)

k: field of any characteristic,
$B := k[X_1, \ldots, X_m, Y, Z, T]$,
$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T)$, such that $f \neq 0$ and every prime divisor of α divides h and

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T))}.$$

Suppose α is a polynomial of **type A** with respect to $(r_1, \ldots, r_m) \in \mathbb{Z}^m_{>1}$ in the system of coordinates $\{X_1 - \lambda_1, \ldots, X_m - \lambda_m\}$, for some $\lambda_i \in \overline{k}$ s.t. $k_1 := k(\lambda_1, \ldots, \lambda_m)$ is separable over k.

Let $x_1, \ldots, x_m$ be the images of $X_1, \ldots, X_m$ in A respectively. Then the following statements are equivalent:

- $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.
- $k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]}$.
- $A = k[x_1, \ldots, x_m]^{[2]}$.
- $A = k^{[m+2]}$.
- $k[Z, T] = k[f(Z, T)]^{[1]}$.
- $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$ and $\mathrm{ML}(A) = k$.
- $f(Z, T)$ is a line in $k[Z, T]$ and $\mathrm{ML}(A) = k$.
- A is an $\mathbb{A}^2$-fibration over E and $\mathrm{ML}(A) = k$.
- $A \otimes_k k_1$ is a UFD, $\mathrm{ML}(A) = k$ and $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$.
- $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$ and $\mathrm{DK}(A) = A$.
- $f(Z, T)$ is a line in $k[Z, T]$ and $\mathrm{DK}(A) = A$.
- A is an $\mathbb{A}^2$-fibration over E and $\mathrm{DK}(A) = A$.
- $A \otimes_k k_1$ is a UFD, $\mathrm{DK}(A) = A$ and $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$.

The family of hypersurfaces given by

$$(X_1^{r_1+1} + X_1^{r_1} X_2^{r_2+1} + \cdots + X_1^{r_1} \ldots X_{m-1}^{r_{m-1}} X_m^{r_m+1})Y - f(Z, T),$$

for $r_i \geqslant 2$, $1 \leqslant i \leqslant m$ and

The family of hypersurfaces given by

$$(X_1^{r_1+1} + X_1^{r_1}X_2^{r_2+1} + \cdots + X_1^{r_1}\ldots X_{m-1}^{r_{m-1}}X_m^{r_m+1})Y - f(Z,T),$$

for $r_i \geqslant 2$, $1 \leqslant i \leqslant m$ and

$$a_1(X_1)\cdots a_m(X_m)Y - f(Z,T) - h(X_1,\ldots,X_m,Z,T),$$

where every prime divisor of $a_1(X_1)\cdots a_m(X_m)$ in $k[X_1,\ldots,X_m]$ divides h, and every $a_i(X_i)$ has a separable multiple root λ_i over k are included in the family of hypersurfaces mentioned in Theorem C.

The family of hypersurfaces given by

$$(X_1^{r_1+1} + X_1^{r_1}X_2^{r_2+1} + \cdots + X_1^{r_1}\ldots X_{m-1}^{r_{m-1}}X_m^{r_m+1})Y - f(Z,T),$$

for $r_i \geqslant 2$, $1 \leqslant i \leqslant m$ and

$$a_1(X_1)\cdots a_m(X_m)Y - f(Z,T) - h(X_1,\ldots,X_m,Z,T),$$

where every prime divisor of $a_1(X_1)\cdots a_m(X_m)$ in $k[X_1,\ldots,X_m]$ divides h, and every $a_i(X_i)$ has a separable multiple root λ_i over k are included in the family of hypersurfaces mentioned in Theorem C.

Unified treatment of several apparently different-looking questions which have been of long interest to mathematicians (including Cancellation, Epimorphism and Fibration problems).

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n) g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n)g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

- A is a polynomial ring in $m + 2$ variables over k if and only if $f(Z, T)$ is a coordinate in $k[Z, T]$.

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \le i \le m$, $m \ge 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n) g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

- A is a polynomial ring in $m + 2$ variables over k if and only if $f(Z, T)$ is a coordinate in $k[Z, T]$.

Provides a general framework for understanding the non-triviality of Russell-Koras threefold $x^2y + x + z^2 + t^3 = 0$ and the generalised Asanuma varieties.

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n)g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

- A is a polynomial ring in $m + 2$ variables over k if and only if $f(Z, T)$ is a coordinate in $k[Z, T]$.

Provides a general framework for understanding the non-triviality of Russell-Koras threefold $x^2y + x + z^2 + t^3 = 0$ and the generalised Asanuma varieties.

- If A is isomorphic to $k^{[m+2]}$, then $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, G]^{[2]}$.

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n) g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

- A is a polynomial ring in $m + 2$ variables over k if and only if $f(Z, T)$ is a coordinate in $k[Z, T]$.

Provides a general framework for understanding the non-triviality of Russell-Koras threefold $x^2y + x + z^2 + t^3 = 0$ and the generalised Asanuma varieties.

- If A is isomorphic to $k^{[m+2]}$, then $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, G]^{[2]}$.

Proves a partial case of the Abhyankar-Sathaye Conjecture.
Extends partially Sathaye-Russell theorem to the case $n \geq 3$.

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n) g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n)g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

- A is isomorphic to $k^{[m+2]}$ if and only if $A = k[x_1, \ldots, x_m]^{[2]}$.

Consequences

k: a field of ANY characteristic, $\mathbf{r_i > 1}$, for $1 \leq i \leq m$, $m \geq 1$.

$$A := K[X_1, \ldots, X_m, Y, Z, T]/(aY - F),$$

where $a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$ is a type A polynomial w.r.t $(r_1, \ldots, r_m)$, π's primes
$F := f(Z, T) + (\pi_1 \cdots \pi_n) g(X_1, \ldots, X_m, Z, T)$, $H = aY - F$.

- A is isomorphic to $k^{[m+2]}$ if and only if $A = k[x_1, \ldots, x_m]^{[2]}$.
- A is a non-trivial $\mathbb{A}^2$-fibration over $k[X_1, \ldots, X_m]$ if and only if $f(Z, T)$ is a non-trivial line.

Thm (— 2014)

let R be a ring, $\pi_1, \pi_2, \ldots, \pi_n \in R$ $\pi := \pi_1\pi_2 \cdots \pi_n$ and $G(Z,T) \in R[Z,T]$ be such that

$$R[Z,T]/(\pi, G(Z,T)) \cong (R/\pi)^{[1]}.$$

Let

$$D := R[Z,T,Y]/({\pi_1}^{s_1}{\pi_2}^{s_2} \cdots {\pi_n}^{s_n} Y - G(Z,T))$$

for any set of positive integers $s_1, \ldots, s_n$. Then

$$D^{[1]} = R^{[3]}.$$

Thm (Ghosh – Pal) 2024

Let k be a field,

$$a = {\pi_1}^{s_1} \dots {\pi_n}^{s_n} \in k[X_1, \dots, X_m]$$

be a type A polynomial w.r.t $(r_1, \dots, r_m)$, $\mathbf{r_i} > \mathbf{1}$ and

$$F = f(Z, T) + (\pi_1 \cdots \pi_m) g(X_1, \dots, X_m, Z, T),$$

where $f(Z, T)$ is a line in $k[Z, T]$. Let

$$A = K[X_1, \dots, X_m, Y, Z, T]/(aY - F).$$

Then $A^{[1]} = k^{[m+3]}$.

Thm (Ghosh – Pal) 2024

Let k be a field,

$$a = {\pi_1}^{s_1} \ldots {\pi_n}^{s_n} \in k[X_1, \ldots, X_m]$$

be a type A polynomial w.r.t $(r_1, \ldots, r_m)$, $\mathbf{r_i > 1}$ and

$$F = f(Z, T) + (\pi_1 \cdots \pi_m) g(X_1, \ldots, X_m, Z, T),$$

where $f(Z, T)$ is a line in $k[Z, T]$. Let

$$A = K[X_1, \ldots, X_m, Y, Z, T]/(aY - F).$$

Then $A^{[1]} = k^{[m+3]}$.
Further if $f(Z, T)$ is a non-trivial line then $A \not\cong k^{[m+2]}$.

Thm (Ghosh – Pal) 2024

Let k be a field,

$$a = {\pi_1}^{s_1} \dots {\pi_n}^{s_n} \in k[X_1, \dots, X_m]$$

be a type A polynomial w.r.t $(r_1, \dots, r_m)$, $\mathbf{r_i > 1}$ and

$$F = f(Z, T) + (\pi_1 \cdots \pi_m) g(X_1, \dots, X_m, Z, T),$$

where $f(Z, T)$ is a line in $k[Z, T]$. Let

$$A = K[X_1, \dots, X_m, Y, Z, T]/(aY - F).$$

Then $A^{[1]} = k^{[m+3]}$.
Further if $f(Z, T)$ is a non-trivial line then $A \ncong k^{[m+2]}$.

Thus, if $f(Z, T)$ is a non-trivial line, then A gives rise to a counter-example to the Zariski Cancellation Problem.

Theorem (Ghosh—, 2023)

k: a field of positive characteristic,

$$\mathbf{A(r_1, \ldots, r_m, f)} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{\mathbf{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T))}},$$

where $\mathbf{r_i > 1}$ for each i, $1 \leq i \leq m$ and $f(Z, T)$ is any non-trivial line in $k[Z, T]$.
Then:

Theorem (Ghosh—, 2023)

k: a field of positive characteristic,

$$\mathbf{A(r_1, \ldots, r_m, f)} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{\mathbf{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T))}},$$

where $\mathbf{r_i > 1}$ for each i, $1 \leq i \leq m$ and $f(Z, T)$ is any non-trivial line in $k[Z, T]$.
Then:

- $A(r_1, \ldots, r_m, f) \cong A(s_1, \ldots, s_m, g)$ iff $(r_1, \ldots, r_m)$ is equal to $(s_1, \ldots, s_m)$ up to permutation and f and g are equivalent.

Theorem (Ghosh—, 2023)

k: a field of positive characteristic,

$$\mathbf{A(r_1, \ldots, r_m, f)} := \frac{\mathbf{k[X_1, X_2, \ldots, X_m, Y, Z, T]}}{\mathbf{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T))}},$$

where $\mathbf{r_i} > \mathbf{1}$ for each i, $1 \le i \le m$ and $f(Z, T)$ is any non-trivial line in $k[Z, T]$.

Then:

- $A(r_1, \ldots, r_m, f) \cong A(s_1, \ldots, s_m, g)$ iff $(r_1, \ldots, r_m)$ is equal to $(s_1, \ldots, s_m)$ up to permutation and f and g are equivalent.

Thus, over a field k of positive characteristic, there is an infinite family of non-isomorphic rings which are stably isomorphic to $k^{[m+2]}$.

Generalised Epimorphism Theorems over Rings

Thm (Bhatwadekar (1988)): Let R be a Noetherian ring of characteristic zero and $F \in R[X, Y]$. Then

$$\frac{R[X, Y]}{(F)} = R^{[1]} \implies R[X, Y] = R[F]^{[1]}$$

whenever R contains $\mathbb{Q}$ or R is seminormal domain.

Generalised Epimorphism Theorems over Rings

Thm (Bhatwadekar (1988)): Let R be a Noetherian ring of characteristic zero and $F \in R[X, Y]$. Then

$$\frac{R[X, Y]}{(F)} = R^{[1]} \implies R[X, Y] = R[F]^{[1]}$$

whenever R contains $\mathbb{Q}$ or R is seminormal domain.

Example (Asanuma-Dutta (2021)): The hypotheses are necessary.

Thm (Bhatwadekar (1988)): Let R be a Noetherian ring of characteristic zero and $F \in R[X, Y]$. Then

$$\frac{R[X,Y]}{(F)} = R^{[1]} \implies R[X,Y] = R[F]^{[1]}$$

whenever R contains $\mathbb{Q}$ or R is seminormal domain.

Example (Asanuma-Dutta (2021)): The hypotheses are necessary.

Thm (Bhatwadekar-Dutta (1994)): Let R be a DVR with uniformizing parameter T,
$\kappa := R/(T)$ and $K := R[1/T]$. Let $G = aY - b \in R[X,Z][Y]$ be a linear polynomial such that $R[X,Z,Y]/(G) = R^{[2]}$. Then there exists $X_0 \in R[X,Z]$ such that $K[X,Z] = K[X_0]^{[1]}$, $a \in R[X_0]$ and $\bar{X}_0 \notin \kappa$. Further, $R[X,Z,Y] = R[G]^{[2]}$, whenever $T \nmid a$ or if $a = T^n$ for some integer n.

An Example of Bhatwadekar-Dutta (1993)

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

Then

- A is an $\mathbb{A}^2$-fibration over B,

An Example of Bhatwadekar-Dutta (1993)

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

Then

- A is an $\mathbb{A}^2$-fibration over B,
- $\mathbf{A}^{[1]} = \mathbf{B}^{[3]}$ and

An Example of Bhatwadekar-Dutta (1993)

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

Then

- A is an $\mathbb{A}^2$-fibration over B,
- $\mathbf{A}^{[1]} = \mathbf{B}^{[3]}$ and
- $\mathbf{A}/(\mathbf{F}) = \mathbf{R}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}]/(\mathbf{F}) = \mathbf{R}^{[2]} = \mathbb{C}^{[3]}$.

Thus F is a linear hyperplane in $\mathbb{C}^{[4]}$.

An Example of Bhatwadeker-Dutta (1993)

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

Then

- A is an $\mathbb{A}^2$-fibration over B,
- $\mathbf{A^{[1]} = B^{[3]}}$ and
- $\mathbf{A/(F) = R[X, Y, Z]/(F) = R^{[2]}} = \mathbb{C}^{[3]}$.

Thus F is a linear hyperplane in $\mathbb{C}^{[4]}$.

Q. Is $\mathbf{A = B^{[2]}}(= \mathbb{C}[\mathbf{T}, \mathbf{F}]^{[2]})$?

An Example of Bhatwadekar-Dutta (1993)

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

Then

- A is an $\mathbb{A}^2$-fibration over B,
- $\mathbf{A}^{[1]} = \mathbf{B}^{[3]}$ and
- $\mathbf{A}/(\mathbf{F}) = \mathbf{R}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}]/(\mathbf{F}) = \mathbf{R}^{[2]} = \mathbb{C}^{[3]}$.

Thus F is a linear hyperplane in $\mathbb{C}^{[4]}$.

Q. Is $\mathbf{A} = \mathbf{B}^{[2]} (= \mathbb{C}[\mathbf{T}, \mathbf{F}]^{[2]})$? At least is $\mathbf{A} = \mathbb{C}[\mathbf{F}]^{[3]}$?

An Example of Bhatwadeker-Dutta (1993)

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

Then

- A is an $\mathbb{A}^2$-fibration over B,
- $\mathbf{A^{[1]} = B^{[3]}}$ and
- $\mathbf{A/(F) = R[X, Y, Z]/(F) = R^{[2]}} = \mathbb{C}^{[3]}$.

Thus F is a linear hyperplane in $\mathbb{C}^{[4]}$.

Q. Is $\mathbf{A = B^{[2]}} (= \mathbb{C}[\mathbf{T}, \mathbf{F}]^{[2]})$? At least is $\mathbf{A} = \mathbb{C}[\mathbf{F}]^{[3]}$?

If **NO**, then it is a counter-example to the following problems:

An Example of Bhatwadekar-Dutta (1993)

$R = \mathbb{C}[T]$,
$A = \mathbb{C}[T, X, Y, Z]$,
$B = \mathbb{C}[T, F] \subset A$, where $F = TX^2Z + X + T^2Y + TXY^2$.

Then

- A is an $\mathbb{A}^2$-fibration over B,
- $\mathbf{A^{[1]} = B^{[3]}}$ and
- $\mathbf{A/(F) = R[X, Y, Z]/(F) = R^{[2]}} = \mathbb{C}^{[3]}$.

Thus F is a linear hyperplane in $\mathbb{C}^{[4]}$.

Q. Is $\mathbf{A = B^{[2]}} (= \mathbb{C}[\mathbf{T}, \mathbf{F}]^{[2]})$? At least is $\mathbf{A} = \mathbb{C}[\mathbf{F}]^{[3]}$?

If **NO**, then it is a counter-example to the following problems:

- $\mathbb{A}^2$-fibration Problem over $\mathbb{C}^{[2]}$;
- Cancellation Problem over $\mathbb{C}^{[1]}$;
- Epimorphism Problem for $\mathbb{C}^{[4]} \twoheadrightarrow \mathbb{C}^{[3]}$, $R^{[3]} \twoheadrightarrow R^{[2]}$.

References

- H. Kraft *Challenging problems on affine n-space* Astérisque, **237** (1996), Séminaire Bourbaki 802, 295–317.
- P. Russell and A. Sathaye, *Forty years of the epimorphism theorem*, Newsletter EMS, **90** (December 2013), 12–17.
- A. K. Dutta and N. Gupta, *The Epimorphism Theorem and its generalisations*, J. Algebra Appl. **14(9)** (2015), 15400101–30.
- N. Gupta, *The Zariski Cancellation Problem and related problems in Affine Algebraic Geometry*, Proc. Int. Cong. Math. 2022, Vol. 3, pp. 1578–1598.
- P. Ghosh and N. Gupta, *On the triviality of a family of linear hyperplanes*, Adv. Math. **428** (2023) 109166.
- P. Ghosh, N. Gupta and A. Pal, *On Epimorphism and related problems for linear affine varieties*, preprint.

S.M. Bhatwadekar with his students (including A.K. Dutta) and grandstudents (including N. Gupta), and G. Freudenburg at IIT Bombay