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**Flexible varieties, images of affine spaces,  
and ellipticity after Mikhail Gromov**

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*There is no doubt that complex affine  $n$ -space  $\mathbb{A}^n = \mathbb{A}_{\mathbb{C}}^n$  is one of the basic objects in algebraic geometry. It is therefore surprising how little is known about its geometry and its symmetries.*

Hanspeter Kraft

Seminar Nicolas Bourbaki  
June 1995

## PART I

# Flexibility and Infinite Transitivity

## Multiple and Infinite Transitivity

### Definition

Let  $G$  be a group,  $X$  a set, and  $m$  a positive integer. An action  $G \times X \rightarrow X$  is *m-transitive* if for any two tuples  $(a_1, \dots, a_m)$  and  $(b_1, \dots, b_m)$  of pairwise distinct points on  $X$  there is  $g \in G$  such that  $(ga_1, \dots, ga_m) = (b_1, \dots, b_m)$ .

### Definition

An action  $G \times X \rightarrow X$  is *infinitely transitive* if it is *m-transitive* for any positive integer  $m$ .

### Example (of infinite transitivity)

- 1) Let  $X$  be an infinite set and  $G$  the group of all permutations of  $X$
- 2) Let  $X$  be an infinite set and  $G$  the group of all permutations with finite support of  $X$

# The Case of Finite Groups

Let  $X$  be a finite set with  $n$  elements.

- 1) The group  $S_n$  of all permutations of  $X$  is  **$n$ -transitive**
- 2) The group  $A_n$  of all even permutations of  $X$  is  **$(n - 2)$ -transitive**
- 3) All other finite permutation groups are at most **5-transitive**
- 4) **5-transitive** finite groups are precisely the Mathieu groups  $M_{12}$  and  $M_{24}$  and all **4-transitive** finite groups are precisely the Mathieu groups  $M_{11}$  and  $M_{23}$  (1861-1873)
- 5) There are infinitely many **3-transitive** finite permutation groups, for example,  $\text{PGL}_2(\mathbb{F}_q)$  acting on  $\mathbb{P}^1(\mathbb{F}_q)$

# The Case of Algebraic Groups

Let  $G$  be a connected algebraic group over an algebraically closed field  $\mathbb{K}$  acting on an algebraic variety  $X$ .

- 1) Such an action is at most **3-transitive**
- 2) The only **3-transitive** action is  $\mathrm{PGL}_2(\mathbb{K})$ -action on  $\mathbb{P}^1(\mathbb{K})$
- 3) The action of  $\mathrm{PGL}_{n+1}(\mathbb{K})$  on  $\mathbb{P}^n(\mathbb{K})$  for  $n \geq 2$  is **2-transitive**
- 4) There is a classification of **2-transitive actions**: Borel, Knop, Kramer,....

# The Case of Affine Spaces

## Theorem

*Over an infinite ground field  $K$ , the group  $\text{Aut}(\mathbb{A}^n)$  is infinitely transitive on  $\mathbb{A}^n$  for any  $n \geq 2$ .*

Idea ( $n = 2$ ): use parallel translations  $(x_1 + a, x_2)$ ,  $(x_1, x_2 + b)$  and their replicas  $(x_1 + af_1(x_2), x_2)$ ,  $(x_1, x_2 + bf_2(x_1))$ , where  $a, b \in K$ .

## Example

The group  $\text{Aut}(\mathbb{A}^1)$  is isomorphic to  $K^\times \ltimes K$ . It is 2-transitive, but not 3-transitive on  $\mathbb{A}^1$ .

## General Problems

Let  $X$  be an affine algebraic variety over an algebraically closed field  $\mathbb{K}$  of characteristic zero.

When the group  $\text{Aut}(X)$  of polynomial automorphisms of  $X$  is infinitely transitive on  $X$ ?

If  $X$  is singular, we ask this question for the smooth locus  $X^{\text{reg}}$ .

*Idea:* to use  $\mathbb{G}_a$ -subgroups in the group  $\text{Aut}(X)$  and their replicas. Here  $\mathbb{G}_a = (\mathbb{K}, +)$ .

*Notation:*  $\text{SAut}(X)$  is the subgroup of  $\text{Aut}(X)$  generated by all  $\mathbb{G}_a$ -subgroups.



# Locally Nilpotent Derivations

## Definition

A derivation  $D: A \rightarrow A$  of an algebra  $A$  is *locally nilpotent* if for any  $a \in A$  there is a positive integer  $k$  such that  $D^k(a) = 0$ .

Locally nilpotent derivations on  $\mathbb{K}[X] \Leftrightarrow \mathbb{G}_a$ -subgroups in  $\text{Aut}(X)$ :

$$D \in \text{LND}(\mathbb{K}[X]) \iff \exp(\mathbb{K}D) \subseteq \text{Aut}(X)$$

If  $D \in \text{LND}(A)$  and  $f \in \text{Ker}(D)$ , then  $fD \in \text{LND}(A)$ .

$\mathbb{G}_a$ -subgroups corresponding to LNDs of the form  $fD$  are *replicas* of the  $\mathbb{G}_a$ -subgroup corresponding to  $D$ .

# Flexibility vs Infinite Transitivity

## Definition

An affine variety  $X$  is *flexible* if the tangent space  $T_x(X)$  at any smooth point  $x \in X^{\text{reg}}$  is generated by velocity vectors to orbits of  $\mathbb{G}_a$ -subgroups passing through  $x$ .

## Theorem (A.-Flenner-Kaliman-Kutzschebauch-Zaidenberg'2013)

Let  $X$  be an irreducible affine variety of dimension  $\geq 2$ . The following conditions are equivalent:

- (a) the group  $\text{SAut}(X)$  acts transitively on  $X^{\text{reg}}$ ;
- (b) the group  $\text{SAut}(X)$  acts infinitely transitively on  $X^{\text{reg}}$ ;
- (c) the variety  $X$  is flexible.

## Examples of Flexible Varieties

- Suspensions  $\text{Susp}(X, f)$  given by  $\{uv = f(x)\}$ ,  $f \in \mathbb{K}[X] \setminus \mathbb{K}$ , in  $\mathbb{A}^2 \times X$  over a flexible variety  $X$ ;
- Non-degenerate  $(\mathbb{K}[X]^\times = \mathbb{K}^\times)$  affine toric varieties;
- Non-degenerate horospherical varieties of reductive groups;
- Homogeneous spaces  $G/F$ , where  $G$  is semisimple and  $F$  is reductive;
- Normal affine  $\text{SL}(2)$ -embeddings;
- Affine cones over flag varieties and over del Pezzo surfaces of degree  $\geq 4$ .

# The Gromov-Winkellmann Theorem

Theorem (Flenner-Kaliman-Zaidenberg'2016)

*Let  $X$  be a flexible quasiprojective variety and  $Z \subseteq X$  be a closed subvariety with  $\text{codim}_X Z \geq 2$ . Then  $X \setminus Z$  is flexible.*

## PART II

# Infinite Transitivity and Finite Generation

## Finite generation

**Conjecture A.** Any flexible affine variety  $X$  admits a finite collection  $H_1, \dots, H_k$  of  $\mathbb{G}_a$ -subgroups in  $\text{Aut}(X)$  such that the group  $G = \langle H_1, \dots, H_k \rangle$  acts infinitely transitively on  $X^{\text{reg}}$ .

Plan of a possible proof:

*Step 1.* Find  $G = \langle H_1, \dots, H_s \rangle$  that acts transitively on  $X^{\text{reg}}$

*Step 2.* Prove that the closure  $\overline{G}$  of the subgroup  $G$  in  $\text{Aut}(X)$  in the ind-topology contains 'many other'  $\mathbb{G}_a$ -subgroups

*Step 3.* Prove that  $\overline{G}$  acts infinitely transitively on  $X^{\text{reg}}$

*Step 4.* Prove that  $G$  acts infinitely transitively on  $X^{\text{reg}}$

Implication Step 3  $\Rightarrow$  Step 4 turns out to be true in general.

# A Conjecture on Locally Nilpotent Derivations

To Step 2:

**Conjecture B.** Let  $X$  be an affine variety and  $A = \mathbb{K}[X]$ . Consider the group  $G = \langle H_1, \dots, H_k \rangle$  generated by a finite collection of  $\mathbb{G}_a$ -subgroups  $H_i = \exp(\mathbb{K}D_i) \subseteq \text{SAut}(X)$ , where  $D_i \in \text{LND}(A)$ . Then the  $\mathbb{G}_a$ -subgroup

$$H = \exp(\mathbb{K}D) \subseteq \text{SAut}(X),$$

where  $D \in \text{LND}(A)$ , is contained in  $\overline{G} \iff D \in \text{Lie} \langle D_1, \dots, D_k \rangle$ .

# Kraft-Zaidenberg's Theorem

## Theorem (Kraft-Zaidenberg'2024)

*Let  $X$  be an affine variety. A subgroup  $G \subseteq \text{Aut}(X)$  generated by a family of connected algebraic subgroups  $G_i$  is algebraic if and only if the Lie algebras  $\text{Lie}(G_i) \subseteq \text{Vec}(X)$  generate a finite-dimensional Lie subalgebra in  $\text{Vec}(X)$ .*

*In particular,  $G = \langle H_1, \dots, H_k \rangle$  is a linear algebraic group if and only if  $\text{Lie} \langle D_1, \dots, D_k \rangle$  is finite-dimensional.*



## Root Subgroups and Demazure Roots

Let  $X$  be a variety with an action of a torus  $T$ . A  $\mathbb{G}_a$ -subgroup  $H$  in  $\text{Aut}(X)$  is called a *root subgroup* if  $H$  is normalized in  $\text{Aut}(X)$  by the torus  $T$ . In this case  $T$  acts on  $H$  by some character  $e$ . Such a character is called a *root* of the  $T$ -variety  $X$ .

Assume  $X$  is toric with acting torus  $T$ . What are the roots of  $X$ ?

Let  $p_1, \dots, p_s$  be the primitive lattice vectors on rays of the fan  $\Sigma_X$ .

### Definition

A *Demazure root* of the fan  $\Sigma_X$  in a character  $e \in M$  such that there exists  $1 \leq i \leq s$  with  $\langle e, p_i \rangle = -1$  and  $\langle e, p_j \rangle \geq 0$  for  $j \neq i$ .

### Theorem (Demazure'1970)

*Let  $X$  be a complete or affine toric variety. Then root subgroups on  $X$  are in bijection with Demazure roots of the fan  $\Sigma_X$ .*

## The Toric Case

Theorem (A.-Kuyumzhiyan-Zaidenberg'2019)

*For any non-degenerate affine toric variety  $X$  of dimension at least 2, which is smooth in codimension 2, one can find root subgroups  $H_1, \dots, H_k$  such that the group  $G = \langle H_1, \dots, H_s \rangle$  acts infinitely transitively on the smooth locus  $X^{\text{reg}}$ .*

In the proof we use Cox rings and the quotient presentation  $\pi: \mathbb{A}^s \rightarrow X$  by an action of a quasitorus.

# Finite Generation for Affine Spaces - I

## Theorem (Bodnarchuk'2001)

*For any  $n \geq 3$  and any triangular  $h \in \text{Aut}(\mathbb{A}^n) \setminus \text{Aff}_n$  we have  $\langle \text{Aff}_n, h \rangle = \text{Tame}_n$ .*

## Corollary

*For any  $n \geq 3$  and any non-affine root subgroup  $H$  in  $\text{Aut}(\mathbb{A}^n)$  the group  $\langle \text{Aff}_n, H \rangle$  acts on  $\mathbb{A}^n$  infinitely transitively. In particular, one can find  $n + 2$  root subgroups which generate a subgroup acting infinitely transitively on  $\mathbb{A}^n$ .*

## Theorem (Andrist'2019, A.-Kuyumzhiyan-Zaidenberg'2019)

*For any  $n \geq 2$  one can find  $\mathbb{G}_a$ -subgroups  $H_1, H_2, H_3$  in  $\text{Aut}(\mathbb{A}^n)$  such that  $G = \langle H_1, H_2, H_3 \rangle$  acts infinitely transitively on  $\mathbb{A}^n$ .*

## Finite Generation for Affine Spaces - II

Let  $H$  be the  $\mathbb{G}_a$ -subgroup of  $\text{Aut}(\mathbb{A}^n)$  given by

$$(x_1 + ax_2^2, x_2, \dots, x_n).$$

**Theorem (A.-Kuyumzhiyan-Zaidenberg'2019)**

*Consider the action of the symmetric group  $S_n$  on  $\mathbb{A}^n$  by permutations. Then for any  $n \geq 3$  the subgroup*

$$G = \langle H, S_n \rangle \subset \text{Aut}(\mathbb{A}^n)$$

*acts infinitely transitively in  $\mathbb{A}^n \setminus \{0\}$ .*

## Finite Generation for Affine Plane - I

Let  $L_k$  and  $R_s$  be the  $\mathbb{G}_a$ -subgroups of  $\text{Aut}(\mathbb{A}^2)$  given by

$$(x_1 + ax_2^k, x_2) \text{ and } (x_1, x_2 + bx_1^s), \text{ respectively.}$$

Let  $G_{k,s} = \langle L_k, R_s \rangle$ . We claim that if  $ks \neq 2$  then  $G_{k,s}$  can not be 2-transitive. Indeed,

if  $k = 0$  or  $s = 0$ , then there are only parallel translations along one coordinate;

if  $k = s = 1$ , then  $G_{1,1}$  is the group  $\text{SL}(2)$  preserving colinearity;

if  $ks > 2$ , we take a root of unity  $\omega$  of degree  $ks - 1$  and consider

$$S = \{(P, Q) \in \mathbb{A}^2 \times \mathbb{A}^2 \mid P = (x_1, x_2), Q = (\omega x_1, \omega^s x_2)\}$$

$$P' = (x_1 + ax_2^k, x_2), Q' = (\omega x_1 + a(\omega^s x_2)^k, \omega^s x_2) = (\omega(x_1 + ax_2^k), \omega^s x_2)$$

$$P'' = (x_1, x_2 + bx_1^s), Q'' = (\omega x_1, \omega^s x_2 + b(\omega x_1)^s) = (\omega x_1, \omega^s(x_2 + bx_1^s))$$

## Finite Generation for Affine Plane - II

Theorem (Lewis-Perry-Straub'2019)

*The group  $G_{1,2}$  generated by two subgroups*

$$(x_1 + ax_2, x_2) \text{ and } (x_1, x_2 + bx_1^2)$$

*acts infinitely transitively on  $\mathbb{A}^2 \setminus \{0\}$ .*

The proof is based on a detailed study of the Polydegree Conjecture for plane polynomial automorphisms.

Chistopolskaya-Taroyan: [arxiv.org/abs/2202.02214](https://arxiv.org/abs/2202.02214), a simpler proof

## Tits Type Alternative - I

**Conjecture C.** Let  $X$  be an affine variety and  $G = \langle H_1, \dots, H_k \rangle$ , where  $H_1, \dots, H_k$  are  $\mathbb{G}_a$ -subgroups in  $\text{Aut}(X)$ . Then the group  $G$  is either a unipotent linear algebraic group, or contains the free group  $F_2$ .

### Corollary

*If  $G$  is 2-transitive then  $G$  contains  $F_2$  and is of exponential growth.*

## Tits Type Alternative - II

### Theorem (A.-Zaidenberg'2022)

*Let  $X$  be an affine toric variety and  $G = \langle H_1, \dots, H_k \rangle$ , where  $H_1, \dots, H_k$  are root  $\mathbb{G}_a$ -subgroups in  $\text{Aut}(X)$ . Then the group  $G$  is either a unipotent linear algebraic group, or contains  $F_2$ .*

### Theorem (A.-Zaidenberg'2023)

*Let  $X$  be an affine surface and  $G = \langle H_1, \dots, H_k \rangle$ , where  $H_1, \dots, H_k$  are  $\mathbb{G}_a$ -subgroups in  $\text{Aut}(X)$ . Then the group  $G$  is either a metabelian unipotent linear algebraic group, or contains  $F_2$ .*



## PART III

# Unirationality and Images of Affine Spaces

## A Key Lemma

### Lemma

Let  $X$  be a flexible variety. Then there are  $\mathbb{G}_a$ -subgroups  $H_1, \dots, H_m$  in  $\text{Aut}(X)$  such that

- (a)  $H_1 \cdot H_2 \cdot \dots \cdot H_m \cdot x = X^{\text{reg}}$  for any  $x \in X^{\text{reg}}$ ;
- (b)  $T_x(X) = \langle h_1, \dots, h_m \rangle$ , where  $h_i$  is a tangent vector to the orbit  $H_i \cdot x$  at  $x$ .

*Remark:* properties (a) and (b) do not imply each other.

# Unirationality vs Flexibility

Theorem (A.-Flenner-Kaliman-Kutzschebauch-Zaidenberg'2013)

*Any flexible variety  $X$  is unirational.*

Proof.

The morphism  $\varphi: \mathbb{A}^m \rightarrow X$ ,

$$(a_1, a_2, \dots, a_m) \mapsto H_1(a_1) \cdot H_2(a_2) \cdot \dots \cdot H_m(a_m) \cdot x,$$

is dominant for any  $x \in X^{\text{reg}}$ . □

*Remark:* there are homogeneous spaces  $SL_n/F$ , where  $F$  is a finite subgroup, that are not stably rational.

# Bogomolov's Conjecture

## Definition

An irreducible variety  $X$  is *stably birationally infinitely transitive* if for some  $m > 0$  the variety  $X \times \mathbb{A}^m$  is birational to an affine flexible variety.

**Conjecture** (Bogomolov'2013) An irreducible variety  $X$  is unirational if and only if it is stably birationally infinitely transitive.

In Bogomolov-Karzhemanov-Kuyumzhiyan'2013, this conjecture is proved for some classes of (unirational non-rational) varieties.

## Unirationality vs Images of Affine Spaces

*Observation.* If a flexible variety  $X$  is smooth, then the morphism  $\varphi: \mathbb{A}^m \rightarrow X$  is surjective.

### Definition

An algebraic variety  $X$  is an *A-image*, if there is a surjective morphism  $\mathbb{A}^m \rightarrow X$  for some  $m \in \mathbb{Z}_{>0}$ .

### Corollary

*Any smooth flexible variety  $X$  is an A-image.*

### Example

The morphism  $\mathbb{A}^1 \rightarrow \mathbb{P}^1$ ,  $x \mapsto [x : 1 + x^2]$  is surjective.

## Three Necessary Conditions

If  $X$  is an A-image, then

(1)  $X$  is irreducible;

(2)  $\mathbb{K}[X]^\times = \mathbb{K}^\times$ ;

(3)  $X$  is unirational

**Question.** Do (1)-(3) imply that  $X$  is an A-image?

## Three Results on $A$ -Images

### Definition

An algebraic variety  $X$  is  $A$ -covered if  $X = \cup_i U_i$ , and each  $U_i$  is isomorphic to  $\mathbb{A}^n$ .

### Problem

*Which homogeneous spaces  $G/H$  are  $A$ -covered?*

### Theorem (A.'2023)

- 1) *Any  $A$ -covered variety is an  $A$ -image.*
- 2) *A toric variety  $X$  is an  $A$ -image  $\Leftrightarrow \mathbb{K}[X]^\times = \mathbb{K}^\times$ .*
- 3) *A homogeneous space  $X$  of a linear algebraic group  $G$  is an  $A$ -image  $\Leftrightarrow \mathbb{K}[X]^\times = \mathbb{K}^\times$ .*

# Infinite Transitivity for Endomorphisms

## Definition

Let  $X$  be an algebraic variety. The monoid  $\text{End}(X)$  acts on  $X$  *infinitely transitively* if for any finite subset  $Z \subseteq X$  and any map  $\psi: Z \rightarrow X$  there is a morphism  $\varphi: X \rightarrow X$  such that  $\varphi|_Z = \psi$ .

## Theorem (Kaliman-Zaidenberg'2023)

*Let  $X$  be an affine variety, which is an  $A$ -image. Then the monoid  $\text{End}(X)$  acts on  $X$  infinitely transitively.*



# Proof

Let  $Z = \{z_1, \dots, z_k\}$ . Given  $\psi: Z \rightarrow X$ , fix  $a_1, \dots, a_k \in \mathbb{A}^m$  with  $\pi(a_i) = \psi(z_i)$ .

$$\begin{array}{ccc} & \mathbb{A}^m & \\ \tilde{\varphi} \nearrow & & \searrow \pi \\ X \supseteq Z & \xrightarrow{\psi} & X \end{array}$$

Consider  $h_j \in \mathbb{K}[Z]$ ,  $j = 1, \dots, m$ , where  $h_j(z_i)$  is the  $j$ th coordinate of  $a_i$ . There are  $f_j \in \mathbb{K}[X]$  with  $f_j|_Z = h_j$  and the functions  $(f_1, \dots, f_m)$  define the morphism  $\tilde{\varphi}: X \rightarrow \mathbb{A}^m$  with  $\tilde{\varphi}(z_i) = a_i$ . Then with  $\varphi := \pi \circ \tilde{\varphi}: X \rightarrow X$  we have  $\varphi|_Z = \psi$ .

## The main result

Theorem (A.-Kaliman-Zaidenberg'2024)

*Let  $X$  be a complete variety. Then  $X$  is an  $A$ -image if and only if  $X$  is unirational.*

The proof is based on the ellipticity property.

## PART IV

# Ellipticity after Mikhail Gromov

## Elliptic Varieties: Gromov'1989

### Definition

A *spray* of rank  $r$  over a smooth algebraic variety  $X$  is a triple  $(E, p, s)$ , where  $p: E \rightarrow X$  is a vector bundle of rank  $r$  with zero section  $Z$ , and  $s: E \rightarrow X$  is a morphism such that  $p|_Z = s|_Z$ .

### Definition

A spray  $(E, p, s)$  is *dominant* at  $x \in X$  if the map  $s_x := s|_{E_x}: E_x \rightarrow X$  is dominant at zero, i.e.  $ds_x: E_x \rightarrow T_x X$  is surjective.

### Definition

A smooth algebraic variety  $X$  is *elliptic* if there is a spray  $(E, p, s)$ , which is dominant at every  $x \in X$ .

## Ellipticity vs Flexibility

### Example

Any smooth flexible variety  $X$  is elliptic: take  $E = \mathbb{A}^m \times X$  and  $s_x: \mathbb{A}^m \rightarrow X$ ,  $s_x(a_1, \dots, a_m) = H_1(a_1) \cdot \dots \cdot H_m(a_m) \cdot x$ .

*Remark:* elliptic  $\implies$  unirational

# Locally Elliptic Varieties

## Definition

A *local spray* on a smooth algebraic variety  $X$  is a tuple  $(U, E, p, s)$ , where  $U \subseteq X$  is open,  $p: E \rightarrow U$  is a vector bundle with  $s: E \rightarrow X$  and  $p|_Z = s|_Z$ .

## Definition

A smooth algebraic variety  $X$  is *locally elliptic* if for every  $x \in X$  there is a local spray  $(U, E, p, s)$  with  $x \in U$  that is dominant at  $x$ .

# Subelliptic Varieties

## Definition

A smooth algebraic variety  $X$  is *subelliptic* if there is a finite collection  $(E_i, p_i, s_i)$  of sprays such that  $T_x X = \sum_i (ds_i)_x((E_i)_x)$  for all  $x \in X$ .

## Theorem (Kaliman-Zaidenberg'2023)

Let  $X$  be a smooth algebraic variety. Then

*elliptic*  $\Leftrightarrow$  *locally elliptic*  $\Leftrightarrow$  *subelliptic*.

# Uniformly rational varieties

## Definition

An irreducible variety  $X$  is *uniformly rational* if  $X = \cup_i U_i$  and every  $U_i$  is isomorphic to an open subset of  $\mathbb{A}^n$ .

## Theorem (Gromov,...)

*Uniform rationality is preserved under blow ups of smooth subvarieties.*

## Question (Gromov):

Is any smooth rational variety  $X$  uniformly rational?



# Uniform Rationality vs Ellipticity

## Theorem (A.-Kaliman-Zaidenberg'2024)

*Let  $X$  be a smooth complete uniformly rational variety. Then*

- (a)  $X$  is elliptic;*
- (b) if  $X$  is projective and  $Y$  is an affine cone over  $X$ , then  $Y \setminus \{0\}$  is elliptic.*

## Ellipticity vs A-Image

### Theorem (Kusakabe'2022)

*Any elliptic variety  $X$  is an A-image. Moreover, if  $\dim X = n$  then there is a surjective morphism  $\mathbb{A}^{n+1} \rightarrow X$ .*

### Proof.

Let  $(E, p, s)$  be a dominant spray of rank  $r$  on  $X$ .  $\exists x_1, \dots, x_k \in X$  with  $s(E_{x_1}) \cup \dots \cup s(E_{x_k}) = X$ . Since  $X$  is unirational  $\Rightarrow$  rationally connected  $\Rightarrow \exists \gamma: \mathbb{A}^1 \rightarrow X$ ,  $x_1, \dots, x_k \in \gamma(\mathbb{A}^1)$ . Lift  $E$  to  $\mathbb{A}^1 \Rightarrow$  trivial vector bundle, so  $\mathbb{A}^r \times \mathbb{A}^1 \rightarrow X$  is surjective. Take  $V_i \subseteq E_{x_i}$ ,  $\dim V_i = n$ , such that  $ds_{x_i}: V_i \rightarrow T_{x_i}(X)$  is surjective. We can assume that  $s(V_1) \cup \dots \cup s(V_k) = X$ . Fix  $y_i \in \mathbb{A}^1$ ,  $\gamma(y_i) = x_i$ . As  $\gamma^*(E) \cong \mathbb{A}^r \times \mathbb{A}^1$ , take linear operators  $L_i$  on  $\gamma^*(E)_{y_i}$  that map a fixed subspace  $\mathbb{A}^n$  in  $\mathbb{A}^r$  to the preimage of  $V_i$ . Then  $L: \mathbb{A}^n \times \mathbb{A}^1 \rightarrow \mathbb{A}^r \times \mathbb{A}^1$ ,  $L(v, a) = (\sum_i \xi_i(a)L_i(v), a)$ , where  $\xi_i(y_j) = \delta_{ij}$ . Then  $s \circ \gamma_* \circ L: \mathbb{A}^{n+1} \rightarrow X$  is surjective. □

## Return to the result

Theorem (A.-Kaliman-Zaidenberg'2024)

*Let  $X$  be a complete variety. Then  $X$  is an A-image if and only if  $X$  is unirational.*

Let us prove that a complete unirational variety  $X$  is an A-image.

By Chow's Lemma,  $\exists$  a birational surjection  $X' \rightarrow X$  with  $X'$  projective  $\Rightarrow$  we assume further that  $X$  is projective.

## Proof

$X$  is unirational  $\Rightarrow \exists$  a dominant rational map  $h$  from  $\mathbb{P}^n$  to  $X$ .  
By Hironaka's Theorem on elimination of indeterminacy, we have

$$\begin{array}{ccc} & \tilde{X} & \\ f \swarrow & & \searrow g \\ \mathbb{P}^n & \overset{h}{\dashrightarrow} & X, \end{array}$$

where  $f$  is a composition of blowups with smooth centers and  $g$  is a generically finite morphism, which is birational if  $h$  is.

So  $\tilde{X}$  is uniformly rational  $\Rightarrow$  elliptic  $\Rightarrow$  A-image  
 $\Rightarrow X$  is an A-image.

## Improvement

Forstnerič'2017: if  $\mathbb{K} = \mathbb{C}$ , Kusakabe's result on a surjective morphism  $\mathbb{A}^{n+1} \rightarrow X$  for elliptic  $X$  with  $\dim X = n$  can be improved to a surjective morphism  $\mathbb{A}^n \rightarrow X$

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