ETH zürich

D-MATH

Finitely generated subgroups of algebraic elements of Cremona groups are bounded

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1 Groups of birational transformations

To an algebraic surface S over a field k we associate its group of birational transformations $\mathrm{Bir}(S)$.
Let G be an algebraic group and $G \times S \dashrightarrow S$ a faithful rational action on S. This induces an injection $G(k) \to \mathrm{Bir}(S)$, whose image we call an *algebraic group*.

Definition
An element $g \in \mathrm{Bir}(S)$ is *algebraic* if it is contained in an algebraic subgroup of $\mathrm{Bir}(S)$.

2 A question by Favre

A subgroup of $\mathrm{Bir}(S)$ such that all its elements are algebraic does not need to be contained in an algebraic subgroup. Consider for example the following subgroup H of $\mathrm{Bir}(\mathbb{A}^2)$:

$$H = \{(x + p(y), y) \mid p \in k[y]\}.$$

However, in [2] Charles Favre asked whether a **finitely generated** subgroup of $\mathrm{Bir}(S)$ consisting only of algebraic elements is always contained in an algebraic subgroup. This subtle question remained open for more than a decade.

The question can be seen as an analogue of the Burnside problem, which has a positive answer for $\mathrm{Bir}(S)$:

Theorem (Cantat [1])
Let S be a regular surface over a field k and let $G < Bir(S)$ be a finitely generated subgroup such that every element of G has **finite order**. Then G is **finite**.

3 Main Result

In our preprint [3] we give a positive answer to Favre's question:

Main Theorem (LPU)
Let S be a regular surface over a field k and let $G < Bir(S)$ be a finitely generated subgroup such that every element of G is algebraic. Then G is contained in an algebraic group.

The most interesting and difficult case is the case, where S is a rational surface. In this case, $\mathrm{Bir}(S)$ is isomorphic to the *Cremona group* $\mathrm{Cr}_2(k) := \mathrm{Bir}(\mathbb{P}^2_k)$ over the field k.

5 Strategy of the proof

The proof of the Main Theorem has various ingredients.

Ingredient 1: Cantat showed in [1] the main theorem for the case, where G **does not** preserve rational fibration $\pi\colon S \dashrightarrow C$. This reduces the proof of the main theorem to the case, where G preserves a rational fibration, i.e., G is a subgroup of the *Jonquières group*.

Ingredient 2: If only finitely many fibres of the rational fibration are contracted by elements of G, we can construct a locally finite dimensional CAT(0) cube complex on which G acts by isometries, which implies the main theorem for this specific case.

Ingredient 3: We reduce the case, where G preserves a rational fibration to the case, where G preserves a rational fibration fiberwise by exploiting the dynamics of G on the base-curve C.

4 Degree growth of finitely generated groups

To a transformation $f \in \mathrm{Cr}_2(k)$ we associate its degree $\deg(f)$. The growth of $\deg(f^n)$ as $n \to \infty$ has been studied extensively. Recall that the Cremona group acts by isometries on an infinite dimensional hyperbolic space $\mathbb{H}^\infty$ and the degree growth is closely linked to the dynamics of this action.

Our Main Theorem allows us to describe the **degree growth of finitely generated subgroups** of $\mathrm{Cr}_2(k)$ instead of just single elements. Let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup with a finite generating set T. Denote by $B_T(n)$ the set of all elements in G of word length l_T at most n and define $D_T(n) := \max_{f\in B_T(n)}\{\deg(f)\}$. For two functions, φ and ψ on $\mathbb{N}$, we write $\varphi \asymp \psi$ if $\varphi(x) \le a\psi(bx)$ and $\psi(x) \le c\varphi(dx)$ for some $a, b, c, d > 0$.

Corollary (LPU)
Let k be algebraically closed and let $G \subset \mathrm{Cr}_2(k)$ be a finitely generated subgroup. Then one of the following is true:

- All elements in G are algebraic, $D_T(n)$ is bounded, and G is contained in an algebraic subgroup.
- The group G preserves a rational fibration and $D_T(n) \asymp n$.
- The group G preserves an elliptic fibration and $D_T(n) \asymp n^2$.
- The group G preserves no fibration and $D_T(n) \asymp \lambda^n$.

References

[1] Serge Cantat. Sur les groupes de transformations birationnelles des surfaces. *Ann. of Math. (2)*, 174(1):299–340, 2011.

[2] Charles Favre. Le groupe de Cremona et ses sous-groupes de type fini. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011.

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