On the Epimorphism Problem for linear hyperplanes

Parnashree Ghosh

Theoretical Statistics and Mathematics unit, Indian Statistical Institute Kolkata

Notation

k is a field and $\overline{k}$ is an algebraic closure of k.

For an integral domain R, $R^{[n]}$ denotes a polynomial ring in n variables over R.

The Epimorphism/Embedding Problem

Embedding Problem: Let m, n be two positive integers. Is every closed embedding $\mathbb{A}^m_k \hookrightarrow \mathbb{A}^n_k$ rectifiable?

The ring theoretic formulation of this problem is the following which is known as the *Epimorphism Problem*.

Epimorphism Problem: Let m, n be two positive integers. Then for a k-algebra epimorphism $\phi : k[X_1, \dots, X_n] \to k[Y_1, \dots, Y_m]$, does there exists a system of coordinates $\{F_1, \dots, F_n\}$ of $k[X_1, \dots, X_n]$ such that $\ker\phi = (F_1, \dots, F_{n-m})$?

The famous *Abhyankar-Sathaye Conjecture* asserts an affirmative answer to the Epimorphism Problem for $m = n - 1$ over fields of characteristic zero. In particular, it investigates the following version of the Epimorphism Problem.

Question 1. Let k be a field. For some integer $n \geqslant 2$, let $H \in k[X_1, \dots, X_n]$ be such that $\frac{k[X_1,\dots,X_n]}{(H)} = k^{[n-1]}$. Does it follow that $k[X_1, \dots, X_n] = k[H]^{[n-1]}$?

Some known answers to Question 1

YES; $ch.\, k = 0$ and $n = 2$ (Abhyankar-Moh (1975); Suzuki(1974)).

NO; $ch.\, k = p > 0$ and $n \geqslant 2$. (Segre (1956/1957); Nagata (1972)).

Some partial affirmative results:

Linear planes: (Sathaye (1976); Russell(1976))

$H = a(X_1, X_2)X_3 + b(X_1, X_2)$

Moreover, linear planes are shown to be of the form $H = a_0(Y)X_3 + b(Y, Z)$ where $k[X_1, X_2] = k[Y, Z]$.

Linear hyperplanes:

- $H = a(X_1)X_4 + b(X_1, X_2, X_3)$ over $\mathbb{C}$ (Kaliman, Venereau, Zaidenberg (2004)). Later extended over any field of characteristic zero (Kahoui, Essamaoui, Ouali (2020))
- $H = X_1^r X_4 + f(X_1, X_2, X_3)$, $r > 1$ and k is any field (Gupta, (2014)).
- $H = X_1^{r_1} \cdots X_{n-3}^{r_{n-3}} X_n - f(X_{n-2}, X_{n-1}) + X_1 \cdots X_{n-3} g(X_1, \dots, X_{n-1})$, $n \geqslant 4$, $r_i > 1$ and k is any field (Ghosh, Gupta (2023)).

Connection of linear hyperplanes with ZCP

B be an affine k-domain of the following form.

$$B = \frac{k[X_1, \dots, X_m, Y, Z, T]}{(G)}, \tag{1}$$

where $G := a(X_1, \dots, X_m)Y - F(X_1, \dots, X_m, Z, T)$.

Theorem A:(Gupta) Suppose that $a(X_1, \dots, X_m) = X_1^{r_1} \cdots X_m^{r_m}$, $r_i > 1$ and $F = f(Z, T)$ is a non-trivial line $\left(\text{i.e., } \frac{k[Z,T]}{(f)} = k^{[1]}, k[Z, T] \neq k[f]^{[1]}\right)$. Then $B^{[1]} = k^{[m+3]}$ but $B \neq k^{[m+2]}$.

Corollary: Over a field of positive characteristic, the ring B in Theorem A is counter example to the ZCP.

Problem: Investigate the necessary and sufficient conditions for $B = k^{[m+2]}$?

Theorem I (Ghosh, Gupta, Pal)

Let k be any field and B be an affine k-domain as in (1), with $m = 1$ and all roots of $a(X_1)$ in $\overline{k}$ are multiple roots. Then the following are equivalent.

(i) $k[X, Y, Z, T] = k[X, G]^{[2]}$.

(ii) $k[X, Y, Z, T] = k[G]^{[3]}$.

(iii) $B = k[x]^{[2]}$.

(iv) $B = k^{[3]}$.

(v) For every root λ of $a(X)$, $k(\lambda)[Z, T] = k(\lambda)[F(\lambda, Z, T)]^{[1]}$.

Polynomial of type A

$\alpha_0(X_1, \dots, X_m) = X_1^{r_1}\alpha_1(X_1, \dots, X_m)$, for some $\alpha_1 \in k^{[m]}, r_1 > 0$, with $X_1 \nmid \alpha_1$ and for any $i \in \{2, \dots, m\}$

$\alpha_i(X_i, \dots, X_m) := \alpha_{i-1}(0, X_i, \dots, X_m)/X_i^{r_i} \in k[X_i, \dots, X_m], r_i > 0$, with $X_i \nmid \alpha_i$.

i.e., for $i \in \{1, \dots, m-1\}$ there exist $\beta_i \in k^{[m-i+1]}$ such that

$$\begin{aligned}
\alpha_0 &= X_1^{r_1}\alpha_1(X_1, \dots, X_m)\\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + \alpha_1(0, X_2, \dots, X_m))\\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + X_2^{r_2}\alpha_2(X_2, \dots, X_m))\\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + X_2^{r_2}(X_2\beta_2(X_2, \dots, X_m) + X_3^{r_3}\alpha_3(X_3, \dots, X_m)))\\
&= \dots\\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + \dots + X_{m-1}^{r_{m-1}}(X_{m-1}\beta_{m-1}(X_{m-1}, X_m) + X_m^{r_m}\alpha_m(X_m)) \dots).
\end{aligned}$$

Example: $\alpha(X_1, \dots, X_m) = a_1(X_1) \cdots a_m(X_m) \in \overline{k}[X_1, \dots, X_m]$, where each $a_i(X_i)$ has a multiple root λ_i in $\overline{k}$.

Theorem II (Ghosh, Gupta, Pal)

Let k be any field and B be an affine domain of the following form as in (1) such that

$$G := \alpha_0(X_1, \dots, X_m)Y - f(Z, T) - h(X_1, \dots, X_m, Z, T)$$

be such that

(a) α_0 is a polynomial of type A with respect to $(r_1, \dots, r_m) \in \mathbb{Z}^m_{>1}$ in the system of coordinates $X_1 - \lambda_1, \dots, X_m - \lambda_m$, for some $\lambda_i \in \overline{k}$, .

(b) Every prime divisor of α_0 divides h in $k[X_1, \dots, X_m, Z, T]$.

(c) For each i, λ_i is separable over k i.e. $k_1 := k(\lambda_1, \dots, \lambda_m)$ is separable over k.

Then the following are equivalent.

(i) $k[X_1, \dots, X_m, Y, Z, T] = k[X_1, \dots, X_m, G]^{[2]}$.

(ii) $k[X_1, \dots, X_m, Y, Z, T] = k[G]^{[m+2]}$.

(iii) $B = k[x_1, \dots, x_m]^{[2]}$.

(iv) $B = k^{[m+2]}$.

(v) $k[Z, T] = k[f(Z, T)]^{[1]}$.

Note: The family of hyperplanes defined by

$$G = a_1(X_1) \cdots a_m(X_m)Y - f(Z, T) + h(X_1, \dots, X_m, Z, T)$$

are contained in the family of hyperplanes in Theorem B, if $a_i(X_i)$ has a separable multiple root in $\overline{k}$ and every prime divisor of $a_i(X_i)$ divides h.

Theorem III (Ghosh, Gupta, Pal)

Let k be a field of characteristic zero and B be an affine domain as in (1). If

$$G = a(X_1, \dots, X_m)Y - f(Z, T),$$

then the following are equivalent.

(i) $k[X_1, \dots, X_m, Y, Z, T] = k[X_1, \dots, X_m, G]^{[2]}$.

(ii) $k[X_1, \dots, X_m, Y, Z, T] = k[G]^{[m+2]}$.

(iii) $B = k[x_1, \dots, x_m]^{[2]}$.

(iv) $B = k^{[m+2]}$.

(v) $k[Z, T] = k[f(Z, T)]^{[1]}$.

(vi) B is an A^2-fibration over $k[x_1, \dots, x_m]$.

(vii) $B^{[l]} = k^{[m+l+2]}$ for some $l \geqslant 0$.

A new Family of Counter Examples to ZCP (Ghosh, Pal)

$$A = \frac{k[X_1, \dots, X_m, Y, Z, T]}{(a_1(X_1) \cdots a_m(X_m)Y - f(Z, T))},$$

where $f(Z, T)$ is a non-trivial line, each a_i has only multiple roots and some a_i has at least two distinct roots in $\overline{k}$.

Consequences

(i) The equivalence of satemements (ii) and (iv) in Theorems I, II, III solves the Epimorphism Problem for the concerned hyperplanes and thus provides affirmative answers to the Abhyankar-Sathaye Conjecture.

(ii) In Theorem II, over a field of positive characteristic if $f(Z, T)$ is a non-trivial line, B can be shown to be stably isomorphic to $k^{[m+3]}$. But the equivalence of the satements (iv) and (v) yields that $B \neq k^{[m+2]}$. Thus the theory connects the Epimorphism and Zariski Cancellation Problem.

References

S. S. Abhyankar and T. T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. 276 (1975), 148–166.

P. Ghosh and N. Gupta, *On the triviality of a family of linear hyperplanes*, Adv. Math. 428 (2023) 109166.

P. Ghosh, N. Gupta and A. Pal, *On Abhyankar-Sathaye Conjecture for a family of linear hypersurfaces in $\mathbb{A}^4_k$*, https://arxiv.org/html/2403.17397v1.

P. Ghosh, N. Gupta and A. Pal, *On the Epimorphism Problem for a family of linear hypersurfaces in $\mathbb{A}^4_k$*, preprint.

N. Gupta, *On the family of affine threefolds $x^m y = F(x, z, t)$*, Compositio Math. 150 (2014), 979–998.

N. Gupta, *On Zariski's Cancellation Problem in positive characteristic*, Adv. Math. 264 (2014), 296–307.

S. Kaliman, S. Vénéreau and M. Zaidenberg, *Simple birational extensions of the polynomial algebra $\mathbb{C}^{[3]}$*, Trans. Amer. Math. Soc. 356(2) (2004), 509–555.

P. Russell, *Simple birational extensions of two dimensional affine rational domains*, Compositio Math. 33(2) (1976), 197–208.

A. Sathaye, *On linear planes*, Proc. Amer. Math. Soc. 56 (1976), 1–7.

M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace ^{2}*, J. Math. Soc. Japan 26 (1974), 241–257.