# Automorphisms of the Ring of Invariants of the Binary Quintic Representation of $SL_2$

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## Introduction

We describe the main results of [4].

Let $k$ be an algebraically closed field of characteristic zero. For $n \geq 1$, let $V_n \subset k[x,y]$ be the vector space of homogeneous polynomials of degree $n$, which is of dimension $n+1$. The natural action of $SL_2(k)$ on $V_n$ gives rise to the irreducible representation of $SL_2(k)$ of degree $n$ on $V_n$ together with its ring of invariants $Q_n := k[V_n]^{SL_2(k)}$, where $k[V_n] = k^{[n+1]}$. Of particular interest are the groups $\mathrm{Aut}_{SL_2}(V_n)$ of equivariant automorphisms of $k^{[n+1]}$. For $k = \mathbb{C}$, Kurth [8] showed $\mathrm{Aut}_{SL_2}(V_n) \cong \mathbb{C}^*$ when $1 \leq n \leq 4$, and gave a criterion for higher $n$ (see below).

We study the case $n = 5$. The irreducible representation of degree 5 is called the **binary quintic representation** of $SL_2(k)$. It was shown by Hermite (1854) that $Q_5$ has the form $Q_5 = k[x,y,z,w]$ where $16w^2 = F$ for

$$F = x^2z^3 - 2xy^2z^2 + y^4z - 72x^2yz + 8xy^3 - 432x^3 .$$

Our main results imply that

$$\mathrm{Aut}_k(Q_5) \cong k^* \text{ and } \mathrm{Aut}_{SL_2}(V_5) \cong k^*.$$

By comparison, note that $Q_1 \cong k$, $Q_2 \cong Q_3 \cong k^{[1]}$ and $Q_4 \cong k^{[2]}$, and their automorphism groups are known.

## Main Results

Let $R = \bigoplus_{d\in\mathbb{N}} R_d$ be the $\mathbb{N}$-grading of $R = k[x,y,z] \cong k^{[3]}$ where $x \in R_3$, $y \in R_2$ and $z \in R_1$. Consider the following conditions on an element $f \in R_9$.

- (C1) $f$ is irreducible.
- (C2) The curve $\mathrm{Proj}(R/fR)$ has at least two singular points.
- (C3) $f(x,y,0) = x(px^2 + qy^3)$ for $p, q \in k^*$.
- (C4) $f_z(x,y,0) = y(rx^2 + sy^3)$ for $r, s \in k$ where $\det \left(\begin{smallmatrix} 3p & r \\ q & s \end{smallmatrix}\right) \neq 0$.

**Theorem 1.** *If $f \in R_9$ satisfies conditions* (C1)-(C3) *then* $|f|_R \geq 2$.

The inequality $|f|_R \geq 2$ means that there is no nonzero locally nilpotent derivation $\delta$ of $R$ for which $\delta^2 f = 0$.

Recall that an integral domain $A$ of characteristic zero is said to be *rigid* if the only locally nilpotent derivation $\delta : A \to A$ is the zero derivation. The proof of *Theorem (1)* uses recent results on rigidity of graded rings, due to the first author and Chitayat [2, 3]. The first of these papers generalizes earlier results of Kishimoto, Prokhorov and Zaidenberg [7] that relate the cylindricity of a polarized projective variety $X$ to the existence of nontrivial $\mathbb{G}_a$-actions on the affine cone over $X$. Whereas the result of [7] requires the varieties involved to be normal, the result that we use (quoted below) does not. This is crucial to our argument, since the rings that we consider are not generally normal.

**Theorem 2.** *Let $f \in R_9$ satisfy conditions* (C1)-(C3) *and let $R[W] = R^{[1]}$. For every integer $n \geq 2$ with $n \notin 3\mathbb{Z}$, the quotient ring $R[W]/(W^n + f)$ is a rigid rational UFD.*

The proof of *Theorem (2)* uses *Theorem (1)* together with results due to the second author and Moser-Jauslin [6] which characterize the homogeneous locally nilpotent derivations of $R[W](W^n - r)$, $r \in R$, in terms of the homogeneous locally nilpotent derivations of $R$.

**Theorem 3.** *Let $f \in R_9$ satisfy conditions* (C1)-(C4) *and let $A = R[W]/(W^n+f)$ where $R[W] = R^{[1]}$ and $n \geq 2$ is such that $n \notin 3\mathbb{Z}$. Let $w$ be the image of $W$ in $A$, and let $\mathfrak{g}$ be the $\mathbb{N}$-grading of $A$ defined by declaring that $x \in A_{3n}$, $y \in A_{2n}$, $z \in A_n$ and $w \in A_9$. Let $T(\mathfrak{g})$ be the subtorus of $\mathrm{Aut}_k(A)$ determined by $\mathfrak{g}$. Then $\mathrm{Aut}_k(A) = T(\mathfrak{g}) \cong k^*$.*

This description of the automorphism group relies on *Theorem (2)* and the theorem of Arzhantsev and Gaifullin [1] which asserts that the group of algebraic automorphisms of a rigid affine $k$-variety contains a unique maximal torus. Uniqueness of the maximal torus was first proved for rigid surfaces by Flenner and Zaidenberg [5].

Since the polynomial $F$ above satisfies conditions (C1)-(C4), *Theorem (3)* has the following special case:

**Corollary 1.** $\mathrm{Aut}_k(Q_5) \cong k^*$

Kurth also showed the following criterion.

**Theorem 4** (Kurth [8])**.** *Let $k = \mathbb{C}$. Given $n \geq 1$, let $\Delta \in Q_n$ be the discriminant, and let $T \subset \mathrm{Aut}_{SL_2}(V_n)$ be the torus $T \cong \mathbb{C}^*$ defined by the standard grading of $k^{[n+1]}$. If $\varphi \in \mathrm{Aut}_{SL_2}(V_n)$ is such that $\varphi^*(\Delta) = c\Delta$ for $c \in \mathbb{C}^*$, then $\varphi \in T$.*

By *Corollary (1)* it follows that $\mathrm{Aut}_{SL_2}(V_5) \cong \mathbb{C}^*$ when $k = \mathbb{C}$. We show that the hypothesis $k = \mathbb{C}$ can be removed.

**Corollary 2** (2)**.** *For any algebraically closed field $k$ of characteristic zero,* $\mathrm{Aut}_{SL_2}(V_5) \cong k^*$.

## Corollary to the Daigle-Chitayat Theorem

Let $B$ be a $\mathbb{Z}$-graded integral domain. An element $\xi \in B$ is **cylindrical** if it is nonzero, homogeneous, $\deg \xi \neq 0$ and $(B_\xi)_0$ is a polynomial ring in one variable over a subring, where $B_\xi$ is the localization of $B$ at $\{\xi^n\}_{n\in\mathbb{N}}$.

**Theorem 5** (Daigle and Chitayat [2])**.** *Let $B = \bigoplus_{i\in\mathbb{Z}} B_i$ be a $\mathbb{Z}$-graded affine domain over a field of characteristic zero. Assume that one of the following conditions* (i), (ii) *is satisfied:*

- (i) the transcendence degree of $B$ over $B_0$ is at least 2;
- (ii) there exist $i, j$ such that $i < 0 < j$, $B_i \neq 0$ and $B_j \neq 0$.

*Then the following are equivalent:*

- (a) There exists $d \geq 1$ such that $B^{(d)}$ is non-rigid, where $B^{(d)} = \bigoplus_{i\in\mathbb{Z}} B_{id}$.
- (b) $B$ has a cylindrical element.

We obtain:

**Corollary 3.** *Let $R$ be a 3-dimensional affine $k$-domain and let $f$ be a prime element of $R$. Then* (a)⇒(b)⇒(c)*, where:*

- (a) *There exists a nonzero locally nilpotent derivation $\delta$ of $R$ such that $\delta(f) = 0$.*
- (b) *The ring $R/fR$ is not rigid.*
- (c) *For each $\mathbb{N}$-grading $R = \bigoplus_{d\in\mathbb{N}} R_d$ such that $f$ is homogeneous and $R_0 = k$, $\mathrm{Proj}(R/fR)$ is a unicuspidal rational curve.*

Recall that a ***unicuspidal rational curve*** is a projective curve that contains an open set isomorphic to $\mathbb{A}^1_k$.

## References

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