The Rigid Pham-Brieskorn Threefolds

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Overview

Let $S = (a_0, \ldots, a_n) \in (\mathbb{N}^+)^{n+1}$. A **Pham-Brieskorn ring** is a ring of the following form:

$$B_S = B_{a_0,\ldots,a_n} = \mathbb{C}[X_0, \ldots, X_n]/\langle X_0^{a_0} + \cdots + X_n^{a_n}\rangle$$

where $a_0, \ldots, a_n \in \mathbb{N}^+$. Observe that if we let $L = \mathrm{lcm}(S)$ and define $w_i = \deg(X_i) = L/a_i$, then we can view $B_S = B_{a_0,\ldots,a_n}$ as an $\mathbb{N}$-graded ring and $\mathrm{Proj}\, B_S \subset \mathbb{P}(w_0, \ldots, w_n)$ as a weighted projective hypersurface.

Locally Nilpotent Derivations

- Let B be a commutative ring. A derivation $D : B \to B$ is *locally nilpotent* if for every $b \in B$ there exists $n \in \mathbb{N}^+$ such that $D^n(b) = 0$.
- **Example:** $B = \mathbb{C}[X_0, \ldots, X_n]$. For any i, the derivation $D = \frac{\partial}{\partial X_i}$ is a locally nilpotent derivation.
- A ring B is *rigid* if the only locally nilpotent derivation $D : B \to B$ is the zero derivation. (If B is a $\mathbf{k}$-domain, $\mathrm{Spec}\, B$ is *rigid* if it admits no non-trivial $\mathbb{G}_a$-actions.)

Rigidity Conjecture

Question: For which $(a_0, \ldots, a_n) \in (\mathbb{N}^+)^{n+1}$ is $B_{a_0,\ldots,a_n}$ rigid?

The following Conjecture has been open for over 25 years. [6]

Main Conjecture: Suppose that $n \geq 2$, $a_0 \leq \cdots \leq a_n$. Then,

$$B_{a_0,\ldots,a_n} \text{ is not rigid} \iff a_0 = 1 \text{ or } a_0 = a_1 = 2.$$

Remarks:

- $(\Leftarrow)$ is easy to prove.
- The $n = 2$ case of the conjecture is known to be true by Lemma 4 in [6].

Despite many partial results, there was no clear strategy for how to prove this conjecture in general.

A Reduction Theorem

The cotype of a tuple $(a_0, \ldots, a_n)$

- Given $S = (a_0, \ldots, a_n) \in (\mathbb{N}^+)^{n+1}$, let $S_i = (a_0, \ldots, a_{i-1}, \hat{a}_i, a_{i+1}, \ldots, a_n)$.
- Let $L = \mathrm{lcm}(S)$ and let $L_i = \mathrm{lcm}(S_i)$.
- Define $\mathrm{cotype}(S) = \#\{i : L_i \neq L\}$.
- Given $S = (a_0, \ldots, a_n) \in (\mathbb{N}^+)^{n+1}$ and $i \in \{0, \ldots, n\}$, define $g_i(S) = \gcd(a_i, \mathrm{lcm}(S_i))$.
- Let $S = (a_0, \ldots, a_n)$ and $S' = (a_0', \ldots, a_n')$ be elements of $(\mathbb{N}^+)^{n+1}$ and let $i \in \{0, \ldots, n\}$. We write $S \leq^i S'$ if and only if

  $$S_i = S_i' \quad \text{and} \quad g_i(S') \mid a_i \mid a_i'.$$

  We write $S <^i S'$ if and only if $S \leq^i S'$ and $S \neq S'$.

Example: $(2,3,3,2) <^3 (2,3,3,4) <^0 (10,3,3,4)$

Our first main result reduces the Main Conjecture to the $\mathrm{cotype}\ 0$ cases.

Reduction Theorem: To prove the Main Conjecture, it suffices to prove it for the cotype 0 cases.

Locally Nilpotent Derivations and Polar cylinders

Polar cylinders

- Let X be a normal projective $\mathbf{k}$-variety. A $\mathbb{Q}$-*divisor* is a sum $D = \sum_{i=1}^n a_i C_i$ where each C_i is a prime divisor and $a_i \in \mathbb{Q}^+$. Two $\mathbb{Q}$-divisors D and D' are $\mathbb{Q}$-*linearly equivalent* (write $D \sim_{\mathbb{Q}} D'$) if there exists $n \in \mathbb{N}^+$ such that nD and nD' are linearly equivalent divisors.
- Let X be normal projective $\mathbf{k}$-variety. An open subset $U \subset X$ is called a *cylinder* if $U \cong \mathbb{A}^1 \times Z$ for some affine variety Z.
- Let H be a $\mathbb{Q}$-divisor of X. A cylinder $U \subset X$ is H-*polar* if $U = X \setminus \mathrm{Supp}(D)$ for some effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} sH$ where $s \in \mathbb{Q}^+$.

There is an important connection between locally nilpotent derivations of graded rings and polar cylinders discovered by Kishimoto, Prokhorov and Zaidenberg in [7], and generalized in [5]. The theorem is as follows:

Cylinder Theorem: Assume $B = \bigoplus_{n\in\mathbb{N}} B_n$ satisfies $\mathrm{trdeg}(B/B_0) \geq 2$, and is a normal finitely generated $\mathbf{k}$-domain that is "saturated in codimension one". Let $X = \mathrm{Proj}\, B$ and let H be an ample $\mathbb{Q}$-divisor such that $B \cong \bigoplus_{i\in\mathbb{N}} H^0(X, \mathcal{O}_X(nH))$.

$$B \text{ is not rigid} \iff X \text{ contains an } H\text{-polar cylinder.}$$

Proof of the 3 dimensional case

Using the previously discussed material, we offer a proof of the 3-dimensional case of the Main Conjecture. To derive this result, we first state the following result which is a corollary of the Cylinder Theorem.

Corollary 1: Let $B = B_{a_0,a_1,a_2,a_3}$ where $\mathrm{cotype}(a_0, a_1, a_2, a_3) = 0$, let $\alpha = L - \sum_i \deg(x_i)$, and let $X = \mathrm{Proj}\, B$. Then, $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(\alpha)$ and

(a) if $\alpha > 0$ then B is rigid $\iff$ X doesn't contain a K_X-polar cylinder;
(b) if $\alpha = 0$ then X is not birationally ruled, so B is rigid;
(c) if $\alpha < 0$ then B is rigid $\iff$ X doesn't contain a $-K_X$-polar cylinder.

In view of Corollary 1, there are 2 cases to consider:

the $\alpha > 0$ case, and the $\alpha < 0$ case.

The $\alpha > 0$ case.

We want to prove that if $X = \mathrm{Proj}\, B_{a_0,a_1,a_2,a_3}$, $\mathrm{cotype}(a_0, a_1, a_2, a_3) = 0$ and $\alpha > 0$, then X doesn't contain a K_X-polar cylinder. But in fact, by a generalization of an argument in [2], X doesn't contain any cylinder, and so the $\alpha > 0$ case is complete by Corollary 1 (a).

The $\alpha < 0$ case.

If $X = \mathrm{Proj}\, B_{a_0,a_1,a_2,a_3}$, $\mathrm{cotype}(a_0, a_1, a_2, a_3) = 0$ and $\alpha < 0$, then X is a singular del Pezzo surface with quotient singularities. A number theoretic argument (Lemma 4.2.4 in [4]) shows that only 8 cases are possible. They are as follows:

- $(2,3,3,6)$, $(2,3,6,6)$, $(2,4,4,4)$, $(3,3,3,3)$, $(3,3,4,4)$, $(3,3,5,5)$
- $(2,3,4,12)$
- $(2,3,5,30)$

The first six cases are resolved by Cheltsov, Park and Won's classification of anti-canonical polar cylinders in del Pezzo surfaces with Du Val singularities [3] and by computations of log canonical threshholds [1]. It now remains to prove the $(2,3,4,12)$ and $(2,3,5,30)$ cases.

The (2,3,5,30) case.

By Corollary 1 (c), it suffices to show that $X = \mathrm{Proj}\, B_{2,3,5,30}$ does not contain a $-K_X$-polar cylinder. Unfortunately, we cannot apply the classification in [3] because X does not have Du Val singularities.

Facts about $X = \mathrm{Proj}\, B_{2,3,5,30}$

- X has 3 singular points $\{P_2, P_3, P_5\}$ of type $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,1)$, $\frac{1}{5}(1,1)$;
- the curve $\Delta = V_+(x_3) \subset X$ is isomorphic to $\mathbb{P}^1$ and satisfies $\mathrm{mult}_{P_2}(\Delta) = \mathrm{mult}_{P_3}(\Delta) = \mathrm{mult}_{P_5}(\Delta) = 1$;
- if $\tilde{X} \to X$ is the minimal resolution of singularities of X, then the exceptional locus consists of three curves $\tilde{E}_2$, $\tilde{E}_3$ and $\tilde{E}_5$, each isomorphic to $\mathbb{P}^1$.

Lemma: If X contains a $-K_X$-polar cylinder $U = X \setminus \mathrm{Supp}(D)$ where $D = \sum_{i\in I} a_i C_i$, then $\Delta \subseteq \mathrm{Supp}(D)$.

We now assume for the sake of a contradiction, that X contains a $-K_X$-polar cylinder $U = X \setminus \mathrm{Supp}(D)$ where D is an effective anti-canonical $\mathbb{Q}$-divisor. By the Lemma, D contains Δ in its support, one can show that $\tilde{X}$ contains a $-K_{\tilde{X}}$-polar cylinder whose complement contains $\tilde{\Delta}$, $\tilde{E}_2$ and $\tilde{E}_3$ in its support. Moreover, these three curves can be contracted via a morphism $\tau : \tilde{X} \to \hat{X}$. Since these three curves are contained in the complement of the $-K_{\tilde{X}}$-polar cylinder, it turns out that $\hat{X}$ is a smooth degree 1 del Pezzo surface which too contains a $-K_{\hat{X}}$-polar cylinder. But this contradicts Cheltsov, Park and Won's classification of del Pezzo surfaces that contain anti-canonical polar cylinders established in [3]. We thus conclude that X itself could not have contained an anti-canonical polar cylinder to begin with. We obtain:

Proposition: The surface $X = \mathrm{Proj}\, B_{2,3,5,30}$ does not contain an anti-canonical polar cylinder. Consequently, $B_{2,3,5,30}$ is rigid.

Remark: The proof that $B_{2,3,4,12}$ is rigid is done in the same way. We conclude:

Main Theorem

Suppose that $a_0 \leq a_1 \leq a_2 \leq a_3$. Then,

$$B_{a_0,a_1,a_2,a_3} \text{ is not rigid} \iff a_0 = 1 \text{ or } a_0 = a_1 = 2.$$

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