RIGIDITY OF PERIODIC POINTS FOR LOXODROMIC AUTOMORPHISMS OF AFFINE SURFACES

Marc Abboud, institut de mathématiques, Université de Neuchâtel
5-8 May 2024

1. Dynamical degree

Let X_0 be an affine surface over a field $\mathbf{K}$ and let $f \in \mathrm{Aut}(X_0)$. A completion X of X_0 is a projective surface with an open embedding $X_0 \hookrightarrow X$. Let H be an ample divisor over X. The *dynamical degree* of f is defined as the following limit

$$\lambda(f) := \lim_n ((f^n)^* H \cdot H)^{1/n}. \quad (1)$$

It does not depend on X nor on H.
An automorphism is called *loxodromic* if $\lambda(f) > 1$.

2. Rigidity of periodic points

Main theorem

Theorem 1 ([Abb24a]) *Let X_0 be a normal affine surface over a field $\mathbf{K}$ and let $f, g \in \mathrm{Aut}(X_0)$ be loxodromic automorphisms, then*

$$\mathrm{Per}(f) \cap \mathrm{Per}(g) \text{ Zariski dense } \Leftrightarrow \mathrm{Per}(f) = \mathrm{Per}(g). \quad (2)$$

3. Dynamics of a loxodromic automorphism

Theorem 2 ([Abb23]) *If f is a loxodromic automorphisms of X_0, there exists a completion X of X_0 and $p_+, p_- \in X \setminus X_0(\mathbf{K})$ such that*

1. *$p_+ \neq p_-$.*
2. *$f(p_+) = p_+, f^{-1}(p_-) = p_-$.*
3. *There exists $N_0 \geqslant 1$, $f^{\pm N_0}(X \setminus X_0) = p_\pm$.*
4. *$f^{\pm 1}$ at $p_\pm$ is locally attracting for any absolute value over $\mathbf{K}$ for the Euclidian topology.*

If v is an absolute value on $\mathbf{K}$, write $\mathbf{C}_v$ is the completion of the algebraic closure of $\mathbf{K}$ with respect to v.

4. Construction of dynamical Green functions

Theorem 3 *Let v be an absolute value over $\mathbf{K}$ with an embedding $X_0 \hookrightarrow \mathbf{C}_v^n$. The functions*

$$\forall p \in X_0(\mathbf{C}_v), \quad G_v^\pm = \lim_N \frac{1}{\lambda(f)^N} \log^+ \left\| f^{\pm N}(p) \right\| \quad (3)$$

are well defined over $X_0(\mathbf{C}_v)$ and satisfy the following properties:

1. *$G_v^\pm$ is continuous, $\geqslant 0$, plurisubharmonic and pluriharmonic over the set $\{G_v^\pm > 0\}$.*
2. *$G_v^\pm(p) = 0$ if and only if $\{f^{\pm n}(p)\}_{n \geqslant 0}$ is bounded.*
3. *$G_v^\pm \left(f^{\pm 1}(p)\right) = \lambda(f) p$.*

The measure $\mu_v := dd^c G_v^+ \wedge dd^c G_v^-$ is called the *equilibrium measure* of f at v.

5. Equidistribution of periodic points

Theorem 4 ([YZ23]) *Suppose $\mathbf{K}$ is a number field. Let (p_n) be a generic sequence of periodic points of f, then for every absolute value v of $\mathbf{K}$,*

$$\frac{1}{\#\mathrm{Gal}(p_n)} \sum_{q \in \mathrm{Gal}(p_n)} \delta_q \to \mu_v. \quad (4)$$

This implies $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *Zariski dense* $\Rightarrow \forall v, \quad \mu_{f,v} = \mu_{g,v}$.

6. Canonical heights

Theorem 5 *Suppose $\mathbf{K}$ is a number field and $\mathscr{M}(\mathbf{K})$ is the set of (normalised) absolute values of $\mathbf{K}$. The canonical height of f is defined as*

$$\forall p \in X_0(\overline{\mathbf{K}}), \quad h_f(p) = \frac{1}{\#\mathrm{Gal}(p)} \sum_{v \in \mathscr{M}(\mathbf{K})} \sum_{q \in \mathrm{Gal}(p)} G_v(q) \quad (5)$$

where $G_v := G_v^+ + G_v^-$. And we have $p \in \mathrm{Per}(f) \Leftrightarrow h_f(p) = 0$.

Plan of proof in the number field case

$$\mathrm{Per}(f) \cap \mathrm{Per}(g) \text{ Zariski dense} \quad \Rightarrow \quad \forall v \in \mathscr{M}(\mathbf{K}), \quad \mu_{f,v} = \mu_{g,v} \quad \Rightarrow \quad \forall v \in \mathscr{M}(\mathbf{K}), \quad \{G_{v,f} = 0\} = \{G_{v,g} = 0\} \quad \Rightarrow \quad \mathrm{Per}(f) = \mathrm{Per}(g).$$

7. The character variety of the punctured torus

Let $\mathbb{T}_1$ be the once punctured torus, we have $\pi_1(\mathbb{T}_1) = F_2 = \langle a, b \rangle$. Let X be the character variety

$$\mathsf{X} := \mathrm{Hom}(\pi_1(\mathbb{T}_1), \mathrm{SL}_2(\mathbf{C})) // \mathrm{SL}_2(\mathbf{C}). \quad (6)$$

By a theorem of Fricke and Klein, we have the following isomorphism

$$[\rho] \in \mathsf{X} \mapsto (\mathrm{Tr}\,\rho(a), \mathrm{Tr}\,\rho(b), \mathrm{Tr}\,\rho(ab)) =: (x, y, z) \in \mathbf{C}^3. \quad (7)$$

And we have the following relation in $\mathrm{SL}_2(\mathbf{C})$:

$$x^2 + y^2 + z^2 = xyz + \mathrm{Tr}\,\rho(aba^{-1}b^{-1}) + 2. \quad (8)$$

The generalised Mapping class group $\mathrm{Mod}(\mathbb{T}_1)^* = \mathrm{GL}_2(\mathbf{Z})$ acts on X and preserves the regular function $\mathrm{Tr}\,\rho(aba^{-1}b^{-1})$.

8. The family of Markov surfaces

Let $D \in \mathbf{C}$, the Markov surface M_D of parameter D is the hypersurface in $\mathbf{C}^3$ defined by the equation

$$x^2 + y^2 + z^2 = xyz + D. \quad (11)$$

The group homomorphism

$$\mathrm{Mod}(\mathbb{T}_1^*) \supset \mathrm{SL}_2(\mathbf{Z}) \to \mathrm{Aut}(M_D) \quad (12)$$

is with finite kernel and the image is of finite index.

9. The parameter $D = 4$

It corresponds to reducible representations. We have a 2:1 cover

$$(u, v) \in \mathbb{G}_m^2 \mapsto \left(u + \frac{1}{u}, v + \frac{1}{v}, uv + \frac{1}{uv}\right). \quad (9)$$

and this cover is $\mathrm{SL}_2(\mathbf{Z})$-equivariant with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (u, v) = \left(x^a y^b, x^c y^d\right). \quad (10)$$

10. Strong rigidity of periodic points

Theorem 6 ([DF17; Abb24b; Abb24a]) *If $X_0 = \mathbf{A}^2_{\mathbf{C}}$ or $X_0 = M_D$ with D transcendental or $D = 0, \cos \frac{2\pi}{q}, q \geqslant 2$, then*

$$\mathrm{Per}(f) \cap \mathrm{Per}(g) \text{ Zariski dense } \Leftrightarrow \exists N, M \in \mathbf{Z} \setminus \{0\}, f^N = g^M. \quad (13)$$

11. Counterexamples and Conjecture

If $A, B \in \mathrm{SL}_2(\mathbf{Z})$, then they induce automorphisms f_A, f_B over $\mathbb{G}_m^2$ and we have

$$\mathrm{Per}(f_A) = \mathbb{U} \times \mathbb{U} = \mathrm{Per}(f_B) \quad (14)$$

and this also holds over M_4.

Conjecture 7 *If $\mathrm{char}\,\mathbf{K} = 0$ and X_0 is a normal affine surface over $\mathbf{K}$, then for $f, g \in \mathrm{Aut}(X_0)$ loxodromic one has*

$$\mathrm{Per}(f) = \mathrm{Per}(g) \Leftrightarrow \exists N, M \in \mathbf{Z} \setminus \{0\}, \quad f^N = g^M \quad (15)$$

unless $X_0 = \mathbb{G}_m^2$ or M_4.

References

[Abb23] Marc Abboud. *On the Dynamics of Endomorphisms of Affine Surfaces*. Nov. 2023.

[Abb24a] Marc Abboud. *Rigidity of Periodic Points for Loxodromic Automorphisms of Affine Surfaces, In Preparation*. 2024.

[Abb24b] Marc Abboud. *Unlikely Intersections Problem for Automorphisms of Markov Surfaces*. Jan. 2024.

[DF17] Romain Dujardin and Charles Favre. "The Dynamical Manin–Mumford Problem for Plane Polynomial Automorphisms". In: *Journal of the European Mathematical Society* 19.11 (Oct. 2017), pp. 3421–3465.

[YZ23] Xinyi Yuan and Shou-Wu Zhang. *Adelic Line Bundles on Quasi-Projective Varieties*. Feb. 2023.